APPLICATIONS OF TOPONOGOV'S COMPARISON THEOREMS FOR OPEN TRIANGLES

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Abstract

Recently we generalized Toponogov's comparison theorem to a complete Riemannian manifold with smooth convex boundary, where a geodesic triangle was replaced by an open (geodesic) triangle standing on the boundary of the manifold, and a model surface was replaced by the universal covering surface of a cylinder of revolution with totally geodesic boundary. The aim of this article is to prove splitting theorems of two types as an application. Moreover, we establish a weaker version of our Toponogov comparison theorem for open triangles, because the weaker version is quite enough to prove one of the splitting theorems.

1. Introduction

Words have fully expressed a matter of great importance for Toponogov's comparison theorem. However that may be, we can not stop telling the importance in Riemannian geometry. The comparison theorem has played a vital role in the comparison geometry, that is, the theorem gives us some techniques originating from Euclidean geometry. Such techniques, drawing a circle or a geodesic polygon, and joining two points by a minimal geodesic segment, are very powerful in the geometry. One may find concrete examples of such techniques in proofs of the maximal diameter theorem and the splitting theorem by Toponogov ([17], [18]), the structure theorem with positive sectional curvature by Gromoll and Meyer ([4]), the soul theorem with non-negative sectional curvature by Cheeger and Gromoll ([3]), the diameter sphere theorem by Grove and Shiohama ([5]), etc.

From the standpoint of the radial curvature geometry, we very recently generalized the Toponogov comparison theorem to a complete Riemannian manifold with smooth convex boundary, where a geodesic triangle was replaced by an open (geodesic) triangle standing on the boundary of the manifold, and a model surface was replaced by the universal covering surface of a cylinder of revolution with totally geodesic boundary ([12, Theorem 8.4], which will be stated as Theorem 2.5 in this article).

The aim of our article is to prove splitting theorems of two types as an application of Toponogov's comparison theorem for open triangles and a weaker version of the comparison theorem (Theorem 2.12), respectively. The weaker version will be proved in this article.

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Now we will introduce the radial curvature geometry for manifolds with boundary: We first introduce our model, which will be later employed as a reference surface of comparison theorems in complete Riemannian manifolds with boundary. Let $\tilde{M} := (\mathbb{R}, d\tilde{x}^2) \times_m (\mathbb{R}, d\tilde{y}^2)$ be a warped product of two 1-dimensional Euclidean lines $(\mathbb{R}, d\tilde{x}^2)$ and $(\mathbb{R}, d\tilde{y}^2)$, where the warping function $m \colon \mathbb{R} \to (0, \infty)$ is a positive smooth function satisfying $m(0) = 1$ and $m'(0) = 0$. Then we call

$$\tilde{X} := \{\tilde{p} \in \tilde{M} \mid \tilde{x}(\tilde{p}) \geq 0\}$$

a *model surface*. Since $m'(0) = 0$, the boundary $\partial \tilde{X} := \{\tilde{p} \in \tilde{X} \mid \tilde{x}(\tilde{p}) = 0\}$ of $\tilde{X}$ is *totally geodesic*. The metric $\tilde{g}$ of $\tilde{X}$ is expressed as

$$(1.1) \qquad\qquad\qquad \tilde{g} = d\tilde{x}^2 + m(\tilde{x})^2 \, d\tilde{y}^2$$

on $[0, \infty) \times \mathbb{R}$. The function $G \circ \tilde{\mu} \colon [0, \infty) \to \mathbb{R}$ is called the *radial curvature function* of $\tilde{X}$, where we denote by G the Gaussian curvature of $\tilde{X}$, and by $\tilde{\mu}$ any ray emanating perpendicularly from $\partial \tilde{X}$ (note that such a $\tilde{\mu}$ will be called a $\partial \tilde{X}$-ray). Remark that $m \colon [0, \infty) \to \mathbb{R}$ satisfies the differential equation $m''(t) + G(\tilde{\mu}(t))m(t) = 0$ with initial conditions $m(0) = 1$ and $m'(0) = 0$. Note that the n-dimensional model surfaces are defined similarly, and, as seen in [10], we may completely classify them by taking half spaces of spaces in [13, Theorem 1.1].

Hereafter, let $(X, \partial X)$ denote a complete Riemannian n-dimensional manifold X with smooth boundary ∂X. We say that ∂X is *convex*, if all eigenvalues of the shape operator A_ξ of ∂X are non-negative in the inward vector ξ normal to ∂X. Note that our sign of A_ξ differs from [14]. That is, for each $p \in \partial X$ and $v \in T_p \partial X$, $A_\xi(v) = -(\nabla_v N)^\top$ holds. Here, we denote by N a local extension of ξ, and by ∇ the Riemannian connection on X.

For a positive constant l, a unit speed geodesic segment $\mu \colon [0, l] \to X$ emanating from ∂X is called a ∂X-*segment*, if $d(\partial X, \mu(t)) = t$ on $[0, l]$. If $\mu \colon [0, l] \to X$ is a ∂X-segment for all $l > 0$, we call μ a ∂X-*ray*. Here, we denote by $d(\partial X, \cdot)$ the distance function to ∂X induced from the Riemannian structure of X. Note that a ∂X-segment is orthogonal to ∂X by the first variation formula, and so a ∂X-ray is too.

$(X, \partial X)$ is said to have the *radial curvature* (*with respect to* ∂X) *bounded from below by that of* $(\tilde{X}, \partial \tilde{X})$ if, for every ∂X-segment $\mu \colon [0, l] \to X$, the sectional curvature K_X of X satisfies

$$K_X(\sigma_t) \geq G(\tilde{\mu}(t))$$

for all $t \in [0, l)$ and all 2-dimensional linear spaces σ_t spanned by $\mu'(t)$ and a tangent vector to X at $\mu(t)$. For example, if the Riemannian metric of $\tilde{X}$ is $d\tilde{x}^2 + d\tilde{y}^2$, or $d\tilde{x}^2 + \cosh^2(\tilde{x}) \, d\tilde{y}^2$, then $G(\tilde{\mu}(t)) = 0$, or $G(\tilde{\mu}(t)) = -1$, respectively. Furthermore, *the radial curvature may change signs wildly*. Examples of model surfaces admitting such a crazy behavior of radial curvature are found in [16, Theorems 1.3 and 4.1].

Our main theorems in this article are now stated as follows:

Theorem 1.1. *Let $(X, \partial X)$ be a complete non-compact connected Riemannian manifold X with smooth convex boundary ∂X whose radial curvature is bounded from below by that of a model surface $(\tilde{X}, \partial \tilde{X})$ with its metric* (1.1). *Assume that X admits at least one ∂X-ray.*
(ST-1) *If $(\tilde{X}, \partial \tilde{X})$ satisfies*

$$\int_0^\infty \frac{1}{m(t)^2} \, dt = \infty,$$

then X is isometric to $[0, \infty) \times_m \partial X$. In particular, ∂X is the soul of X, and the number of connected components of ∂X is one.
(ST-2) *If $(\tilde{X}, \partial \tilde{X})$ satisfies $\liminf_{t \to \infty} m(t) = 0$, then X is diffeomorphic to $[0, \infty) \times \partial X$. In particular, the number of connected components of ∂X is one.*

Toponogov's comparison theorem for open triangles in a weak form (Theorem 2.12) will be applied in the proof of Theorem 1.1 (see Section 4). The assumption on the existence of a ∂X-ray is very natural, because we may find at least one ∂X-ray if ∂X is compact. If the model $\tilde{X}$ is Euclidean (i.e., $m \equiv 1$), then the (ST-1) holds. Hence, Theorem 1.1 extends one of Burago and Zalgaller' splitting theorems to a wider class of metrics than those described in [2, Theorem 5.2.1], i.e., we mean that they assumed that sectional curvature is *non-negative everywhere*.

Theorem 1.2. *Let $(X, \partial X)$ be a complete connected Riemannian manifold X with disconnected smooth compact convex boundary ∂X whose radial curvature is bounded from below by 0. Then, X is isometric to $[0, l] \times \partial X_1$ with Euclidean product metric of $[0, l]$ and ∂X_1, where ∂X_1 denotes a connected component of ∂X. In particular, ∂X_1 is the soul of X.*

Toponogov's comparison theorem for open triangles (Theorem 2.5) will be applied in the proof of Theorem 1.2 (see Section 5). Note that non-negative radial curvature *does not always mean* non-negative sectional curvature (cf. [11, Example 5.6]). Although Theorem 1.2 extends one of Burago and Zalgaller' splitting theorems to a wider class of metrics than those described in [2, Theorem 5.2.1], Ichida [6] and Kasue [9] obtain the same conclusion of the theorem under weaker assumptions, i.e., the mean curvature (with respect to the inner normal direction) of boundary are non-negative, and that Ricci curvature is non-negative everywhere.

In the following sections, all geodesics will be normalized, unless otherwise stated.

2. Toponogov's theorems for open triangles

Throughout this section, let $(X, \partial X)$ denote a complete connected Riemannian manifold X with smooth *convex* boundary ∂X whose radial curvature is bounded from below

by that of a model surface $(\tilde{X}, \partial\tilde{X})$ with its metric (1.1).

DEFINITION 2.1 (Open triangles).   For any fixed two points $p, q \in X \setminus \partial X$, an *open triangle*

$$\mathrm{OT}(\partial X, p, q) = (\partial X, p, q; \gamma, \mu_1, \mu_2)$$

in X is defined by two ∂X-segments $\mu_i \colon [0, l_i] \to X$, $i = 1, 2$, a minimal geodesic segment $\gamma \colon [0, d(p,q)] \to X$, and ∂X such that $\mu_1(l_1) = \gamma(0) = p$, $\mu_2(l_2) = \gamma(d(p,q)) = q$.

REMARK 2.2.   In this article, we always use the symbol γ as the segment emanating from p to q, which is called the *opposite side of* an open triangle $\mathrm{OT}(\partial X, p, q) = (\partial X, p, q; \gamma, \mu_1, \mu_2)$ in X, and the ∂X-segments μ_1, μ_2 always denote sides of the $\mathrm{OT}(\partial X, p, q)$ emanating from ∂X to p, q, respectively.

DEFINITION 2.3.   We call the set $\tilde{X}(\theta) := \tilde{y}^{-1}((0, \theta))$ a sector in $\tilde{X}$ for each constant number $\theta > 0$.

REMARK 2.4.   Since a map $(\tilde{x}, \tilde{y}) \to (\tilde{x}, \tilde{y} + c)$, $c \in \mathbb{R}$, over $\tilde{X}$ is an isometry, a sector $\tilde{X}(\theta)$ is isometric to $\tilde{y}^{-1}(c, c + \theta)$ for all $c \in \mathbb{R}$.

Toponogov's comparison theorem for open triangles is stated as follows:

Theorem 2.5 ([12, Theorem 8.4])**.**   *Let $(X, \partial X)$ be a complete connected Riemannian manifold X with smooth convex boundary ∂X whose radial curvature is bounded from below by that of a model surface $(\tilde{X}, \partial\tilde{X})$ with its metric* (1.1). *Assume that $\tilde{X}$ admits a sector $\tilde{X}(\theta_0)$ which has no pair of cut points. Then, for every open triangle $\mathrm{OT}(\partial X, p, q) = (\partial X, p, q; \gamma, \mu_1, \mu_2)$ in X with $d(\mu_1(0), \mu_2(0)) < \theta_0$, there exists an open triangle $\mathrm{OT}(\partial\tilde{X}, \tilde{p}, \tilde{q}) = (\partial\tilde{X}, \tilde{p}, \tilde{q}; \tilde{\gamma}, \tilde{\mu}_1, \tilde{\mu}_2)$ in $\tilde{X}(\theta_0)$ such that*

$$(2.1) \qquad d(\partial\tilde{X}, \tilde{p}) = d(\partial X, p), \quad d(\tilde{p}, \tilde{q}) = d(p, q), \quad d(\partial\tilde{X}, \tilde{q}) = d(\partial X, q)$$

and that

$$(2.2) \qquad \angle p \geq \angle\tilde{p}, \quad \angle q \geq \angle\tilde{q}, \quad d(\mu_1(0), \mu_2(0)) \geq d(\tilde{\mu}_1(0), \tilde{\mu}_2(0)).$$

Furthermore, if $d(\mu_1(0), \mu_2(0)) = d(\tilde{\mu}_1(0), \tilde{\mu}_2(0))$ holds, then

$$\angle p = \angle\tilde{p}, \quad \angle q = \angle\tilde{q}$$

hold. Here $\angle p$ denotes the angle between two vectors $\gamma'(0)$ and $-\mu_1'(d(\partial X, p))$ in $T_p X$.

REMARK 2.6. In Theorem 2.5, we do not assume that ∂X is connected. Moreover, the opposite side γ of $\mathrm{OT}(\partial X, p, q)$ does not meet ∂X (see [12, Lemma 6.1]). In [13], they treat a pair (M, N) of a complete connected Riemannian manifold M and a compact connected totally geodesic hypersurface N of M such that the radial curvature with respect to N is bounded from below by that of the model $((a, b) \times_m N, N)$, where (a, b) denotes an interval, in their sense. Note that the radial curvature with respect to N is bounded from below by that of our model $([0, \infty), d\tilde{x}^2) \times_m (\mathbb{R}, d\tilde{y}^2)$, if it is bounded from below by that of their model $((a, b) \times_m N, N)$. Thus, Theorem 2.5 is *applicable to* the pair (M, N).

In the following, we will prove the Toponogov comparison theorem for open triangles in a weak form (Theorem 2.12), where we do not demand any assumption on a sector. To do so, we need to introduce definitions and a key lemma:

DEFINITION 2.7 (Generalized open triangles). A generalized open triangle

$$\mathrm{GOT}(\partial \tilde{X}, \hat{p}, \hat{q}) = (\partial \tilde{X}, \hat{p}, \hat{q}; \hat{\gamma}, \hat{\mu}_1, \hat{\mu}_2)$$

in $\tilde{X}$ is defined by two $\partial \tilde{X}$-segments $\hat{\mu}_i \colon [0, l_i] \to \tilde{X}$, $i = 1, 2$, and a geodesic segment $\hat{\gamma}$ emanating from $\hat{p}$ to $\hat{q}$ such that $\hat{\mu}_1(l_1) = \hat{\gamma}(0) = \hat{p}$, $\hat{\mu}_2(l_2) = \hat{\gamma}(d(\hat{p}, \hat{q})) = \hat{q}$, and that $\hat{\gamma}$ is a shortest arc joining $\hat{p}$ to $\hat{q}$ in the compact domain bounded by $\hat{\mu}_1$, $\hat{\mu}_2$, and $\hat{\gamma}$.

DEFINITION 2.8 (The injectivity radius). The *injectivity radius* $\mathrm{inj}(\tilde{p})$ of a point $\tilde{p} \in \tilde{X}$ is the supremum of $r > 0$ such that, for any point $\tilde{q} \in \tilde{X}$ with $d(\tilde{p}, \tilde{q}) < r$, there exists a unique minimal geodesic segment joining $\tilde{p}$ to $\tilde{q}$.

REMARK 2.9. For each point $\tilde{p} \in \tilde{X} \setminus \partial \tilde{X}$, $\mathrm{inj}(\tilde{p}) > d(\partial \tilde{X}, \tilde{p})$ holds, if $\tilde{p}$ is sufficiently close to $\partial \tilde{X}$.

DEFINITION 2.10 (Thin open triangle). An open triangle $\mathrm{OT}(\partial X, p, q)$ in X is called a *thin open triangle*, if
(TOT-1) the opposite side γ of $\mathrm{OT}(\partial X, p, q)$ to ∂X emanating from p to q is contained in a normal convex neighborhood in $X \setminus \partial X$, and
(TOT-2) $L(\gamma) < \mathrm{inj}(\tilde{q}_s)$ for all $s \in [0, d(p, q)]$,
where $L(\gamma)$ denotes the length of γ, and $\tilde{q}_s$ denotes a point in $\tilde{X}$ with $d(\partial \tilde{X}, \tilde{q}_s) = d(\partial X, \gamma(s))$ for each $s \in [0, d(p, q)]$.

Then, we have the key lemma to prove the weaker version of Toponogov's comparison theorem for open triangles.

Lemma 2.11 ([12, Lemma 5.8]). *For every thin open triangle* $\mathrm{OT}(\partial X, p, q)$ *in* X, *there exists an open triangle* $\mathrm{OT}(\partial \tilde{X}, \tilde{p}, \tilde{q})$ *in* $\tilde{X}$ *such that*

$$(2.3) \qquad d(\partial \tilde{X}, \tilde{p}) = d(\partial X, p), \quad d(\tilde{p}, \tilde{q}) = d(p, q), \quad d(\partial \tilde{X}, \tilde{q}) = d(\partial X, q)$$

and that

$$(2.4) \qquad\qquad\qquad \angle p \geq \angle \tilde{p}, \quad \angle q \geq \angle \tilde{q}.$$

Now, the weaker version of Toponogov's comparison theorem for open triangles is stated as follows:

Theorem 2.12. *Let* $(X, \partial X)$ *be a complete connected Riemannian manifold* X *with smooth convex boundary* ∂X *whose radial curvature is bounded from below by that of a model surface* $(\tilde{X}, \partial \tilde{X})$. *Then, for every open triangle* $\mathrm{OT}(\partial X, p, q) = (\partial X, p, q; \gamma, \mu_1, \mu_2)$ *in* X, *there exists a generalized open triangle* $\mathrm{GOT}(\partial \tilde{X}, \hat{p}, \hat{q}) = (\partial \tilde{X}, \hat{p}, \hat{q}; \hat{\gamma}, \hat{\mu}_1, \hat{\mu}_2)$ *in* $\tilde{X}$ *such that*

$$(2.5) \qquad\qquad d(\partial \tilde{X}, \hat{p}) = d(\partial X, p), \quad d(\partial \tilde{X}, \hat{q}) = d(\partial X, q),$$

and

$$(2.6) \qquad d(\partial X, q) - d(\partial X, p) \leq d(\hat{p}, \hat{q}) \leq L(\hat{\gamma}) \leq d(p, q),$$

and that

$$(2.7) \qquad\qquad\qquad \angle p \geq \angle \hat{p}, \quad \angle q \geq \angle \hat{q}.$$

Here $L(\hat{\gamma})$ *denotes the length of* $\hat{\gamma}$.

Proof.   Let $s_0 := 0 < s_1 < \cdots < s_{k-1} < s_k := d(p, q)$ be a subdivision of $[0, d(p, q)]$ such that, for each $i \in \{1, \ldots, k\}$, the open triangle $\mathrm{OT}(\partial X, \gamma(s_{i-1}), \gamma(s_i))$ is thin. It follows from Lemma 2.11 that, for each triangle $\mathrm{OT}(\partial X, \gamma(s_{i-1}), \gamma(s_i))$, there exists an open triangle $\tilde{\triangle}_i := \mathrm{OT}(\partial \tilde{X}, \tilde{\gamma}(s_{i-1}), \tilde{\gamma}(s_i))$ in $\tilde{X}$ such that

$$(2.8) \qquad\qquad d(\partial \tilde{X}, \tilde{\gamma}(s_{i-1})) = d(\partial X, \gamma(s_{i-1})),$$

$$(2.9) \qquad\qquad d(\tilde{\gamma}(s_{i-1}), \tilde{\gamma}(s_i)) = d(\gamma(s_{i-1}), \gamma(s_i)),$$

$$(2.10) \qquad\qquad d(\partial \tilde{X}, \tilde{\gamma}(s_i)) = d(\partial X, \gamma(s_i)),$$

and that

$$(2.11) \qquad\qquad \angle(\partial X, \gamma(s_{i-1}), \gamma(s_i)) \geq \angle(\partial \tilde{X}, \tilde{\gamma}(s_{i-1}), \tilde{\gamma}(s_i)),$$

$$(2.12) \qquad\qquad \angle(\partial X, \gamma(s_i), \gamma(s_{i-1})) \geq \angle(\partial \tilde{X}, \tilde{\gamma}(s_i), \tilde{\gamma}(s_{i-1})).$$

Here $\angle(\partial X, \gamma(s_{i-1}), \gamma(s_i))$ denotes the angle between two sides joining $\gamma(s_{i-1})$ to ∂X and $\gamma(s_i)$ forming the triangle $\mathrm{OT}(\partial X, \gamma(s_{i-1}), \gamma(s_i))$. Under this situation, draw $\tilde{\triangle}_1 = \mathrm{OT}(\partial \tilde{X}, \tilde{p}, \tilde{\gamma}(s_1))$ in $\tilde{X}$ satisfying (2.8), (2.9), (2.10), (2.11), (2.12) for $i = 1$. Inductively, we draw an open triangle $\tilde{\triangle}_{i+1} = \mathrm{OT}(\partial \tilde{X}, \tilde{\gamma}(s_i), \tilde{\gamma}(s_{i+1}))$ in $\tilde{X}$, which is adjacent to $\tilde{\triangle}_i$ so as to have the $\partial \tilde{X}$-segment to $\tilde{\gamma}(s_i)$ as a common side. Since

$$\angle(\partial X, \gamma(s_i), \gamma(s_{i-1})) + \angle(\partial X, \gamma(s_i), \gamma(s_{i+1})) = \pi,$$

for each $i = 1, 2, \ldots, k - 1$, we get, by (2.11) and (2.12),

$$(2.13) \qquad \angle(\partial \tilde{X}, \tilde{\gamma}(s_i), \tilde{\gamma}(s_{i-1})) + \angle(\partial \tilde{X}, \tilde{\gamma}(s_i), \tilde{\gamma}(s_{i+1})) \leq \pi$$

and

$$(2.14) \qquad \angle p \geq \angle(\partial \tilde{X}, \tilde{\gamma}(s_0), \tilde{\gamma}(s_1)), \quad \angle q \geq \angle(\partial \tilde{X}, \tilde{\gamma}(s_k), \tilde{\gamma}(s_{k-1})).$$

Then, we get a domain $\mathcal{D}$ bounded by two $\partial \tilde{X}$-segments $\tilde{\mu}_0, \tilde{\mu}_k$ to $\tilde{\gamma}(s_0), \tilde{\gamma}(s_k)$, respectively, and $\tilde{\eta}$, where $\tilde{\eta}$ denotes the broken geodesic consisting of the opposite sides of $\tilde{\triangle}_i$ $(i = 1, 2, \ldots, k)$ to $\partial \tilde{X}$. Since the domain $\mathcal{D}$ is locally convex by (2.13), there exists a minimal geodesic segment $\hat{\gamma}$ in the closure of $\mathcal{D}$ joining $\tilde{\gamma}(s_0)$ to $\tilde{\gamma}(s_k)$. From (2.14), it is clear that the generalized open triangle $(\partial \tilde{X}, \tilde{\gamma}(s_0), \tilde{\gamma}(s_0); \hat{\gamma}, \tilde{\mu}_0, \tilde{\mu}_k)$ has the required properties in our theorem. $\qquad \square$

3. Definitions and notations for Sections 4 and 5

Throughout this section, let $(X, \partial X)$ denote a complete connected Riemannian manifold X with smooth boundary ∂X. Our purpose of this section is to recall the definitions of ∂X-Jacobi fields, focal loci of ∂X, and cut loci of ∂X, which will appear in Sections 4 and 5.

DEFINITION 3.1 (∂X-Jacobi field).   Let $\mu \colon [0, \infty) \to X$ be a geodesic emanating perpendicularly from ∂X. A Jacobi field $J_{\partial X}$ along μ is called a ∂X-*Jacobi field*, if $J_{\partial X}$ satisfies $J_{\partial X}(0) \in T_{\mu(0)}\partial X$ and $J'_{\partial X}(0) + A_{\mu'(0)}(J_{\partial X}(0)) \in (T_{\mu(0)}\partial X)^{\perp}$. Here J' denotes the covariant derivative of J along μ, and $A_{\mu'(0)}$ denotes the shape operator of ∂X.

DEFINITION 3.2 (Focal locus of ∂X).   A point $\mu(t_0)$, $t_0 \neq 0$, is called a *focal point of* ∂X along a geodesic $\mu \colon [0, \infty) \to X$ emanating perpendicularly from ∂X, if there exists a non-zero ∂X-Jacobi field $J_{\partial X}$ along μ such that $J_{\partial X}(t_0) = 0$. The *focal locus* $\mathrm{Foc}(\partial X)$ *of* ∂X is the union of the focal points of ∂X along all of the geodesics emanating perpendicularly from ∂X.

Definition 3.3 (Cut locus of ∂X).   Let $\mu: [0, l_0] \to X$ be a ∂X-segment. The end point $\mu(l_0)$ of $\mu([0, l_0])$ is called a *cut point of ∂X along μ*, if any extended geodesic $\bar{\mu}: [0, l_1] \to X$ of μ, $l_1 > l_0$, is not a ∂X-segment anymore. The *cut locus* $\mathrm{Cut}(\partial X)$ *of ∂X* is the union of the cut points of ∂X along all of the ∂X-segments.

4. Proof of Theorem 1.1

From the similar argument in the proof of [15, Lemma 3.1], one may prove

Lemma 4.1. *Let*

$$f''(t) + K(t)f(t) = 0, \quad f(0) = 1, \quad t \in [0, \infty),$$

$$m''(t) + G(t)m(t) = 0, \quad m(0) = 1, \quad m'(0) = 0, \quad t \in [0, \infty),$$

be two ordinary differential equations with $K(t) \geq G(t)$ on $[0, \infty)$.
(L-1) If $f > 0$ on $(0, \infty)$, $f'(0) = 0$, and

$$\int_0^\infty \frac{1}{m(t)^2} \, dt = \infty,$$

then $K(t) = G(t)$ on $[0, \infty)$.
(L-2) If $m > 0$ on $(0, \infty)$, $f'(0) < 0$, and

$$\int_0^\infty \frac{1}{m(t)^2} \, dt = \infty,$$

then there exists $t_0 \in (0, \infty)$ such that $f > 0$ on $[0, t_0)$ and $f(t_0) = 0$.

Hereafter, let $(X, \partial X)$ be a complete non-compact connected Riemannian n-manifold X with smooth *convex* boundary ∂X whose radial curvature is bounded from below by that of a model surface $(\tilde{X}, \partial \tilde{X})$ with its metric (1.1). Moreover, we denote by

$$\mathcal{I}_{\partial X}^l(V, W) := I_l(V, W) - \langle A_{\mu'(0)}(V(0)), W(0) \rangle$$

the index form with respect to a ∂X-segment $\mu: [0, l] \to X$ for piecewise C^∞ vector fields V, W along μ, where we set

$$I_l(V, W) := \int_0^l \{ \langle V', W' \rangle - \langle R(\mu', V)\mu', W \rangle \} \, dt,$$

which is a symmetric bilinear form. Furthermore,

we assume that X admits at least one ∂X-ray.

By Lemma 4.1, we have

Lemma 4.2. *Let $\mu\colon [0,\infty) \to X$ be a ∂X-ray. If $(\tilde{X}, \partial\tilde{X})$ satisfies*

$$\int_0^\infty \frac{1}{m(t)^2}\, dt = \infty,$$

then, $\mu(0)$ is the geodesic point in ∂X, i.e., the second fundamental form vanishes at the point.

Proof.   Let E be a unit parallel vector field along μ such that

(4.1) $$A_{\mu'(0)}(E(0)) = \lambda E(0),$$

(4.2) $$E(t) \perp \mu'(t).$$

Here λ denotes an eigenvalue of the shape operator $A_{\mu'(0)}$ of ∂X. Since ∂X is convex, $\lambda \geq 0$ holds. Consider a smooth vector field $Y(t) := f(t)E(t)$ along μ satisfying

$$f''(t) + K_X(\mu'(t), E(t))f(t) = 0,$$

with initial conditions

(4.3) $$f(0) = 1, \quad f'(0) = -\lambda.$$

Here $K_X(\mu'(t), E(t))$ denotes the sectional curvature with respect to the 2-dimensional linear space spanned by $\mu'(t)$ and $E(t)$ at $\mu(t)$. Note that Y satisfies $Y(0) \in T_{\mu(0)}\partial X$ and $Y'(0) + A_{\mu'(0)}(Y(0)) = 0 \in (T_{\mu(0)}\partial X)^\perp$, by (4.1), (4.2), and (4.3). Suppose that $\lambda > 0$. Since $f'(0) < 0$ and

$$\int_0^\infty \frac{1}{m(t)^2}\, dt = \infty,$$

it follows from (L-2) in Lemma 4.1 that there exists $t_0 \in (0, \infty)$ such that $f > 0$ on $[0, t_0)$ and

(4.4) $$f(t_0) = 0,$$

i.e.,

(4.5) $$Y(t) \neq 0, \quad t \in [0, t_0)$$

and $Y(t_0) = 0$. Since $\langle R(\mu'(t), Y(t))\mu'(t), Y(t)\rangle = f(t)^2 \langle R(\mu'(t), E(t))\mu'(t), E(t)\rangle = -f''(t)f(t)$, we have, by (4.3) and (4.4),

(4.6) $$I_{t_0}(Y, Y) = \int_0^{t_0} \frac{d}{dt}(ff')\, dt = f(t_0)f'(t_0) - f(0)f'(0) = \lambda.$$

Thus, by (4.1), (4.3), and (4.6),

$$(4.7) \qquad \mathcal{I}_{\partial X}^{t_0}(Y, Y) = I_{t_0}(Y, Y) - \langle A_{\mu'(0)}(Y(0)), Y(0) \rangle = \lambda - \lambda = 0.$$

On the other hand, since ∂X has no focal point along μ, for any non-zero vector field Z along μ satisfying $Z(0) \in T_{\mu(0)} \partial X$ and $Z(t_0) = 0$,

$$(4.8) \qquad \mathcal{I}_{\partial X}^{t_0}(Z, Z) > 0$$

holds (cf. Lemma 2.9 in [14, Chapter III]). Thus, by (4.7) and (4.8), $Y \equiv 0$ on $[0, t_0]$. This is a contradiction to (4.5). Therefore, $\lambda = 0$, i.e., $\mu(0)$ is the geodesic point in ∂X.
□

Here we want to go over some fundamental tools on $(\tilde{X}, \partial \tilde{X})$: A geodesic $\tilde{\gamma} : [0, a) \to \tilde{X}$ $(0 < a \leq \infty)$ is expressed by $\tilde{\gamma}(s) = (\tilde{x}(\tilde{\gamma}(s)), \tilde{y}(\tilde{\gamma}(s))) =: (\tilde{x}(s), \tilde{y}(s))$. Then, there exists a non-negative constant ν depending only on $\tilde{\gamma}$ such that

$$(4.9) \qquad \nu = m(\tilde{x}(s))^2 |\tilde{y}'(s)| = m(\tilde{x}(s)) \sin \angle \left(\tilde{\gamma}'(s), \left(\frac{\partial}{\partial \tilde{x}} \right)_{\tilde{\gamma}(s)} \right).$$

This (4.9) is a famous formula—the *Clairaut relation*. The constant ν is called the *Clairaut constant of* $\tilde{\gamma}$. Remark that, by (4.9), $\nu > 0$ *if and only if* $\tilde{\gamma}$ *is not a* $\partial \tilde{X}$*-ray, or its subarc*. Since $\tilde{\gamma}$ is unit speed, we have, by (4.9),

$$(4.10) \qquad \tilde{x}'(s) = \pm \frac{\sqrt{m(\tilde{x}(s))^2 - \nu^2}}{m(\tilde{x}(s))}.$$

By (4.10), we see that $\tilde{x}'(s) = 0$ *if and only if* $m(\tilde{x}(s)) = \nu$. Moreover, by (4.10), we have that, for a geodesic $\tilde{\gamma}(s) = (\tilde{x}(s), \tilde{y}(s))$, $s_1 \leq s \leq s_2$, with the Clairaut constant ν,

$$(4.11) \qquad s_2 - s_1 = \phi(\tilde{x}'(s)) \int_{\tilde{x}(s_1)}^{\tilde{x}(s_2)} \frac{m(t)}{\sqrt{m(t)^2 - \nu^2}} \, dt,$$

if $\tilde{x}'(s) \neq 0$ on (s_1, s_2). Here, $\phi(\tilde{x}'(s))$ denotes the sign of $\tilde{x}'(s)$. Furthermore, we have a lemma with respect to the length $L(\tilde{\gamma})$ of $\tilde{\gamma}$:

Lemma 4.3. *Let* $\tilde{\gamma} : [0, s_0] \to \tilde{X} \setminus \partial \tilde{X}$ *denote a geodesic segment with the Clairaut constant* ν. *Then,* $L(\tilde{\gamma})$ *is not less than*

$$(4.12) \qquad t_2 - t_1 + \frac{\nu^2}{2} \int_{t_1}^{t_2} \frac{1}{m(t)\sqrt{m(t)^2 - \nu^2}} \, dt,$$

where we set $t_1 := \tilde{x}(0)$ *and* $t_2 := \tilde{x}(s_0)$.

Proof.   We may assume that $t_2 > t_1$, otherwise (4.12) is non-positive. Let $[s_1, s_2]$ be a sub-interval of $[0, s_0]$ such that $\tilde{x}'(s) \neq 0$ on (s_1, s_2). By (4.11),

$$L(\tilde{\gamma}|_{[s_1, s_2]}) = s_2 - s_1 = \left| \int_{\tilde{x}(s_1)}^{\tilde{x}(s_2)} \frac{m(t)}{\sqrt{m(t)^2 - v^2}} \, dt \right|.$$

Since $\tilde{x}'(s) \neq 0$ for all $s \in (s_1, s_2)$ with $\tilde{x}(s) \in [t_1, t_2]$, we may choose the numbers s_1 and s_2 in such a way that $\tilde{x}(s_1) = t_1$ and $\tilde{x}(s_2) = t_2$ or that $\tilde{x}(s_1) = t_2$ and $\tilde{x}(s_2) = t_1$. Thus, we see that

$$(4.13) \qquad\qquad L(\tilde{\gamma}) \geq \int_{t_1}^{t_2} \frac{m(t)}{\sqrt{m(t)^2 - v^2}} \, dt.$$

Since

$$\frac{m(t)}{\sqrt{m(t)^2 - v^2}} \geq 1 + \frac{v^2}{2m(t)\sqrt{m(t)^2 - v^2}},$$

we have, by (4.13),

$$L(\tilde{\gamma}) \geq t_2 - t_1 + \frac{v^2}{2} \int_{t_1}^{t_2} \frac{1}{m(t)\sqrt{m(t)^2 - v^2}} \, dt. \qquad\qquad \square$$

The next lemma is well-known in the case of the cut locus of a point (see [1]). Although it can be proved similarly, we here give a proof of the lemma totally different from it.

Lemma 4.4.   *For any $q \in \mathrm{Cut}(\partial X) \cap (X \setminus \partial X)$ and any $\varepsilon > 0$, there exists a point in $\mathrm{Cut}(\partial X) \cap B_\varepsilon(q)$ which admits at least two ∂X-segments.*

Proof.   Suppose that the cut point q admits a unique ∂X-segment μ_q to q. Then, q is the first focal point of ∂X along μ_q. For each $p \in \partial X$, we denote by v_p the inward pointing unit normal vector to ∂X at $p \in \partial X$. And let $\mathcal{U}$ be a sufficiently small open neighborhood around $d(\partial X, q)\mu_q'(0)$ in the normal bundle $\mathcal{N}_{\partial X}$ of ∂X, so that there exists a number $\lambda(v_p) \in (0, \infty)$ such that $\exp^\perp(\lambda(v_p)v_p)$ is the first focal point of ∂X for each $\lambda(v_p)v_p \in \mathcal{U}$. Set $k := \liminf_{v_p \to \mu_q'(0)} \nu(v_p)$, where $\nu(v_p) := \dim \ker(d \exp^\perp)_{\lambda(v_p)v_p}$. Since $\mathcal{U}$ is sufficiently small, we may assume that $\nu(v_p) \geq k$ on $\mathcal{S} := \{w/\|w\| \mid w \in \mathcal{U}\}$, which is open in the unit sphere normal bundle of ∂X. It is clear that, for each integer $m \geq 0$, the set $\{v_p \in \mathcal{S} \mid \mathrm{rank}(d \exp^\perp)_{\lambda(v_p)v_p} \geq m\}$ is open in $\mathcal{S}$. Hence, by [8, Lemma 1], λ is smooth on the open set $\{v_p \in \mathcal{S} \mid \nu(v_p) \leq k\} = \{v_p \in \mathcal{S} \mid \nu(v_p) = k\} \subset \mathcal{S}$. Since $(d \exp^\perp)_{\lambda(v_p)v_p} : T_{\lambda(v_p)v_p}\mathcal{N}_{\partial X} \to T_{\exp^\perp(\lambda(v_p)v_p)}X$ is a linear map depending smoothly on $v_p \in \mathcal{S}$, there exists a *non-zero* vector field W on $\mathcal{S}$ such that $W_{v_p} \in$

$\ker(d \exp^\perp)_{\lambda(v_p)v_p}$ on $\mathcal{S}$. Here, we assume that $\ker(d \exp^\perp)_{\lambda(v_p)v_p} \subset T_{v_p}\mathcal{S}$ by the natural identification.

Assume that there exists a sequence $\{\mu_i \colon [0, l_i] \to X\}$ of ∂X-segments convergent to μ_q such that $\mu_i(l_i) \in \mathrm{Cut}(\partial X)$ and $\mu_i(l_i) \notin \mathrm{Foc}(\partial X)$ along μ_i. Then it is clear that each $\mu_i(l_i)$ admits at least two ∂X-segments. Hence, we have proved our lemma in this case.

Assume that $\exp^\perp(\lambda(v_p)v_p) \in \mathrm{Cut}(\partial X)$ for all $v_p \in \mathcal{S}$. Let $\sigma(s)$, $s \in (-\delta, \delta)$, be the local integral curve of W on $\mathcal{S}$ with $\mu_q(0) = \sigma(0)$. Hence, $(d \exp^\perp)_{\lambda(\sigma(s))\sigma(s)}(\sigma'(s)) = 0$ on $(-\delta, \delta)$. By [7, Lemma 1], $\exp^\perp(\lambda(\sigma(s))\sigma(s)) = \exp^\perp(\lambda(\sigma(0))\sigma(0)) = q$ holds. Hence q is a point in $\mathrm{Cut}(\partial X)$ admitting at least two ∂X-segments. $\qquad\square$

REMARK 4.5.   Lemma 4.4 holds without curvature assumption on $(X, \partial X)$.

Proposition 4.6.   *Let $\mu_0 \colon [0, \infty) \to X$ be a ∂X-ray guaranteed by the assumption above. If $(\tilde{X}, \partial \tilde{X})$ satisfies*

(4.14)
$$\int_0^\infty \frac{1}{m(t)^2} \, dt = \infty,$$

or

(4.15)
$$\liminf_{t \to \infty} m(t) = 0,$$

then, any point of X lies in a unique ∂X-ray. In particular, ∂X is totally geodesic in the case where (4.14) is satisfied.

Proof.   Choose any point $q \in X \setminus \partial X$ not lying on μ_0. Let $\mu_1 \colon [0, d(\partial X, q)] \to X$ denote a ∂X-segment with $\mu_1(d(\partial X, q)) = q$. For each $t > 0$, let $\gamma_t \colon [0, d(q, \mu_0(t))] \to X$ denote a minimal geodesic segment emanating from q to $\mu_0(t)$. From Theorem 2.12 and the triangle inequality, it follows that there exists a generalized open triangle

$$\mathrm{GOT}(\partial \tilde{X}, \hat{\mu}_0(t), \hat{q}) = (\partial \tilde{X}, \hat{\mu}_0(t), \hat{q}; \hat{\gamma}_t, \hat{\mu}_0^{(t)}, \hat{\mu}_1)$$

in $\tilde{X}$ corresponding to the triangle $\mathrm{OT}(\partial X, \mu_0(t), q) = (\partial X, \mu_0(t), q; \gamma_t, \mu_0|_{[0,t]}, \mu_1)$ in X such that

(4.16)
$$d(\partial \tilde{X}, \hat{\mu}_0(t)) = t, \quad d(\partial \tilde{X}, \hat{q}) = d(\partial X, q),$$

and

(4.17)
$$L(\hat{\gamma}_t) \le d(\mu_0(t), q) \le t + d(q, \mu_0(0))$$

and that

(4.18)
$$\angle(\partial X, q, \mu_0(t)) \ge \angle(\partial \tilde{X}, \hat{q}, \hat{\mu}_0(t)).$$

Here $\angle(\partial X, q, \mu_0(t))$ denotes the angle between two sides μ_1 and γ_t joining q to ∂X and $\mu_0(t)$ forming the triangle $OT(\partial X, \mu_0(t), q)$. From Lemma 4.3, (4.16), and (4.17), we get

$$(4.19) \qquad \begin{aligned} t + d(q, \mu_0(0)) &\geq L(\hat{\gamma}_t) \\ &\geq t - d(\partial X, q) + \frac{v_t^2}{2} \int_{d(\partial X, q)}^{t} \frac{1}{m(t)\sqrt{m(t)^2 - v_t^2}} \, dt. \end{aligned}$$

where v_t denotes the Clairaut constant of $\hat{\gamma}_t$. By (4.19),

$$(4.20) \qquad d(\partial X, q) + d(q, \mu_0(0)) \geq \frac{v_t^2}{2} \int_{d(\partial X, q)}^{t} \frac{1}{m(t)^2} \, dt.$$

First, assume that $(\tilde{X}, \partial \tilde{X})$ satisfies (4.14). Then, it is clear from (4.20) that $\lim_{t \to \infty} v_t = 0$. Hence, by (4.9), we have

$$(4.21) \qquad \lim_{t \to \infty} \angle(\partial \tilde{X}, \hat{q}, \hat{\mu}_0(t)) = \pi.$$

By (4.18) and (4.21), $\gamma_\infty := \lim_{t \to \infty} \gamma_t$ is a ray emanating from q such that

$$\angle(\gamma_\infty'(0), -\mu_1'(d(\partial X, q))) = \pi.$$

This implies that q lies on a unique ∂X-segment. Therefore, by Lemma 4.4, q lies on a ∂X-ray. Now, it is clear from Lemma 4.2 that ∂X is totally geodesic.

Second, assume that $(\tilde{X}, \partial \tilde{X})$ satisfies (4.15). Then, there exists a divergent sequence $\{t_i\}_{i \in \mathbb{N}}$ such that

$$(4.22) \qquad \lim_{t \to \infty} m(t_i) = 0.$$

From (4.9), we see

$$(4.23) \qquad v_i \leq m(t_i),$$

where v_i denotes the Clairaut constant of $\hat{\gamma}_{t_i}$. Hence, by (4.22) and (4.23), $\liminf_{t \to \infty} v_t = 0$ holds. Now, it is clear that there exist a limit geodesic γ_∞ of $\{\gamma_{t_i}\}$ such that γ_∞ is a ray emanating from q and satisfies $\angle(\gamma_\infty'(0), -\mu_1'(d(\partial X, q))) = \pi$. Therefore, by Lemma 4.4, q lies on a ∂X-ray. $\qquad \square$

By Proposition 4.6, there does not exist a cut point of ∂X. Therefore, it is clear that

Corollary 4.7. *If $(\tilde{X}, \partial \tilde{X})$ satisfies (4.14), or (4.15), then X is diffeomorphic to* $[0, \infty) \times \partial X$.

Furthermore, we may reach stronger conclusion than Corollary 4.7:

Theorem 4.8. *If $(\tilde{X}, \partial\tilde{X})$ satisfies*

$$\int_0^\infty \frac{1}{m(t)^2}\, dt = \infty,$$

then, for every ∂X-ray $\mu\colon [0, \infty) \to X$, the radial curvature K_X satisfies

(4.24) $$K_X(\sigma_t) = G(\tilde{\mu}(t))$$

for all $t \in [0, \infty)$ and all 2-dimensional linear space σ_t spanned by $\mu'(t)$ and a tangent vector to X at $\mu(t)$. In particular, X is isometric to the warped product manifold $[0, \infty) \times_m \partial X$ of $[0, \infty)$ and $(\partial X, g_{\partial X})$ with the warping function m. Here $g_{\partial X}$ denotes the induced Riemannian metric from X.

Proof.   Take any point $p \in \partial X$, and fix it. By Proposition 4.6, we may take a ∂X-ray $\mu\colon [0, \infty) \to X$ emanating from $p = \mu(0)$. Suppose that

(4.25) $$K_X(\sigma_{t_0}) > G(\tilde{\mu}(t_0))$$

for some linear plane σ_{t_0} spanned by $\mu'(t_0)$ and a unit tangent vector v_0 orthogonal to $\mu'(t_0)$. If we denote by $E(t)$ the parallel vector field along μ satisfying $E(t_0) = v_0$, then $E(t)$ is unit and orthogonal to $\mu'(t_0)$ for each t. We define a non-zero vector field $Y(t)$ along μ by $Y(t) := f(t)E(t)$, where f is the solution of the following differential equation

(4.26) $$f''(t) + K_X(\mu'(t), E(t))f(t) = 0$$

with initial condition $f(0) = 1$ and $f'(0) = 0$. Here $K_X(\mu'(t), E(t))$ denotes the sectional curvature of the plane spanned by $\mu'(t)$ and $E(t)$. It follows from (4.25) and (L-1) in Lemma 4.1 that there exists $t_1 > 0$ such that $f(t_1) = 0$. From (4.26), we get

(4.27) $$I_{t_1}(Y, Y) = \int_0^{t_1} \frac{d}{dt}(ff')\, dt = 0.$$

Since ∂X is totally geodesic by Proposition 4.6, $A_{\mu'(0)}(E(0)) = 0$. Thus, by (4.27), $\mathcal{I}_{\partial X}^{t_1}(Y, Y) = 0$ holds. On the other hand, $\mathcal{I}_{\partial X}^{t_1}(Y, Y) > 0$ holds, since there is no focal point of ∂X along μ. This is a contradiction. Therefore, we get the first assertion (4.24).

Now it is clear that the map $\varphi\colon [0, \infty) \times_m \partial X \to X$ defined by $\varphi(t, q) := \exp^\perp(t v_q)$ gives an isometry from $[0, \infty) \times_m \partial X$ onto X. Here v_q denotes the inward pointing unit normal vector to ∂X at $q \in \partial X$.   □

5.   Proof of Theorem 1.2

Throughout this section, let $(X, \partial X)$ be a complete connected Riemannian manifold X with *disconnected* smooth compact *convex* boundary ∂X whose radial curvature is bounded from below by 0. Under the hypothesis, we may assume

$$\partial X = \bigcup_{i=1}^{k} \partial X_i, \quad k \geq 2.$$

Here each ∂X_i denotes a connected component of ∂X and is compact. Set

$$l := \min\{d(\partial X_i, \partial X_j) \mid 1 \leq i, j \leq k, \ i \neq j\}.$$

Then let ∂X_1, ∂X_2 denote the connected components of ∂X satisfying

$$d(\partial X_1, \partial X_2) = l.$$

Lemma 5.1.   *Let μ denote a minimal geodesic segment in X emanating from ∂X_1 to ∂X_2. Then, there does not exist any other ∂X-segment to $\mu(l/2)$ than $\mu|_{[0,l/2]}$ and $\mu|_{[l/2,l]}$. Furthermore, each midpoint $\mu(l/2)$ is not a focal point of ∂X along μ.*

Proof.   Note that the μ is not tangent to another connected components ∂X_i ($3 \leq i \leq k$) of ∂X, since ∂X is convex. Suppose that there exists a ∂X-segment μ_{i_0}: $[0, l/2] \to X$ for a number $i_0 \in \{3, 4, \ldots, k\}$ such that μ_{i_0} emanates from ∂X_{i_0} to $\mu(l/2)$. Since $\angle(\partial X_{i_0}, \mu(l/2), \partial X_i) < \pi$ ($i = 1$, or 2), we have, by the triangle inequality, $l = d(\partial X_{i_0}, \mu(l/2)) + d(\mu(l/2), \partial X_i) > d(\partial X_{i_0}, \partial X_i)$. This is a contradiction to the definition of l.

Suppose that there exists a ∂X-segment η: $[0, l/2] \to X$ emanating from ∂X_i ($i = 1$, or 2) to $\mu(l/2)$ satisfying $\eta(0) \neq \mu(0)$ if $i = 1$, and $\eta(0) \neq \mu(l)$ if $i = 2$. In the case where $i = 1$: Let $\angle(\partial X_1, \eta(l/2), \partial X_2)$ denote the angle between two segments η and μ joining $\mu(l/2)$ to ∂X_1 and ∂X_2, respectively. Then $\angle(\partial X_1, \eta(l/2), \partial X_2) < \pi$ holds. By the triangle inequality, we have $l = d(\eta(0), \eta(l/2)) + d(\mu(l/2), \mu(l)) = d(\partial X_1, \eta(l/2)) + d(\eta(l/2), \partial X_2) > d(\partial X_1, \partial X_2)$. In the case where $i = 2$, we also have $l > d(\partial X_1, \partial X_2)$. Thus, both cases are also a contradiction to the definition of l.

Hence, the first assertion of this lemma has been proved. In the proof of the second assertion of this lemma, suppose that $\mu(l/2)$ is a focal point of ∂X along μ. Then, we may get a contradiction to the definition of l by the similar way above.   □

Hereafter, the half plane

$$\mathbb{R}^2_+ := \{\tilde{p} \in \mathbb{R}^2 \mid \tilde{x}(\tilde{p}) \geq 0\}$$

with Euclidean metric $d\tilde{x}^2 + d\tilde{y}^2$ will be used as the model surface for $(X, \partial X)$.

Lemma 5.2. *Any point in X lies on a minimal geodesic segment emanating from ∂X_1 to ∂X_2 of length l. In particular, ∂X consists of ∂X_1 and ∂X_2.*

Proof. Since X is connected, it is sufficient to prove that the subset $\mathcal{O}$ of X is open and closed, where $\mathcal{O}$ denotes the set of all points $r \in X$ which lies on a minimal geodesic segment emanating from ∂X_1 to ∂X_2 of length l. Since it is trivial that $\mathcal{O}$ is closed, we will prove that $\mathcal{O}$ is open.

Choose any point $r \in \mathcal{O}$, and fix it. Thus, r lies on a minimal geodesic segment $\mu_1 \colon [0,l] \to X$ emanating from ∂X_1 to ∂X_2. Set $p := \mu_1(l/2)$. Let S be the equidistant set from ∂X_1 and ∂X_2, i.e.,

$$(5.1) \qquad\qquad S := \{q \in X \mid d(\partial X_1, q) = d(\partial X_2, q)\}.$$

It follows from Lemma 5.1 that $S \cap B_{\varepsilon_1}(p) \subset \mathrm{Cut}(\partial X)$, if $\varepsilon_1 > 0$ is chosen sufficiently small. Choose any point $q \in S \cap B_{\varepsilon_1}(p) \setminus \{p\}$, and also fix it. Let η_i, $i = 1, 2$, denote a ∂X-segment to q such that $\eta_1(0) \in \partial X_1$ and $\eta_2(0) \in \partial X_2$, respectively. Moreover, let $\gamma \colon [0, d(p,q)] \to X$ denote a minimal geodesic segment emanating from p to q. Since

$$\angle\left(\gamma'(0), -\mu_1'\left(\frac{l}{2}\right)\right) + \angle\left(\gamma'(0), \mu_1'\left(\frac{l}{2}\right)\right) = \pi,$$

we may assume, without loss of generality, that

$$(5.2) \qquad\qquad \angle\left(\gamma'(0), -\mu_1'\left(\frac{l}{2}\right)\right) \leq \frac{\pi}{2}.$$

It follows from Theorem 2.5 that there exists an open triangle

$$\mathrm{OT}(\partial\mathbb{R}_+^2, \tilde{p}, \tilde{q}) = (\partial\mathbb{R}_+^2, \tilde{p}, \tilde{q}; \tilde{\gamma}, \tilde{\mu}_1, \tilde{\eta}_1)$$

in $\mathbb{R}_+^2$ corresponding to the triangle $\mathrm{OT}(\partial X_1, p, q) = (\partial X_1, p, q; \gamma, \mu_1|_{[0,l/2]}, \eta_1)$ such that

$$(5.3) \qquad d(\partial\mathbb{R}_+^2, \tilde{p}) = \frac{l}{2}, \quad d(\tilde{p}, \tilde{q}) = d(p, q), \quad d(\partial\mathbb{R}_+^2, \tilde{q}) = d(\partial X_1, q),$$

and

$$(5.4) \qquad\qquad \angle\left(\gamma'(0), -\mu_1'\left(\frac{l}{2}\right)\right) = \angle p \geq \angle\tilde{p}, \quad \angle q \geq \angle\tilde{q}.$$

By (5.2) and $\angle p \geq \angle\tilde{p}$ of (5.4), we have

$$(5.5) \qquad\qquad \angle\tilde{p} \leq \frac{\pi}{2}.$$

Since our model is $\mathbb{R}_+^2$, it follows from the two equations $d(\partial\tilde{X}, \tilde{p}) = l/2$, $d(\partial\mathbb{R}_+^2, \tilde{q}) = d(\partial X_1, q)$ of (5.3), and (5.5) that

$$(5.6) \qquad\qquad d(\partial X_1, q) = d(\partial\mathbb{R}_+^2, \tilde{q}) \leq \frac{l}{2}.$$

On the other hand, the broken geodesic segment defined by combining η_1 and η_2 is a curve joining ∂X_1 to ∂X_2. This implies that length of the broken geodesic segment is not less than that of μ_1. Thus,

$$(5.7) \qquad\qquad 2L(\eta_1) = L(\eta_1) + L(\eta_2) \geq l,$$

where $L(\,\cdot\,)$ denotes the length of a curve. Since $L(\eta_1) = d(\partial X_1, q)$, we have, by (5.7), that

$$(5.8) \qquad\qquad d(\partial X_1, q) \geq \frac{l}{2}.$$

By (5.6) and (5.8), $d(\partial X_1, q) = d(\partial X_2, q) = l/2$. Therefore, we have proved that any point $q \in S \cap B_{\varepsilon_1}(p)$ is the midpoint of a minimal geodesic segment emanating from ∂X_1 to ∂X_2 of length l. Furthermore, by Lemma 5.1, each point of $S \cap B_{\varepsilon_1}(p)$ is not a focal point of ∂X. It is therefore clear that any point sufficiently close to the point $r \in \mathcal{O}$ is a point of $\mathcal{O}$, i.e, $\mathcal{O}$ is open. $\qquad\square$

REMARK 5.3.   From Lemmas 5.1 and 5.2, it is clear that

$$(5.9) \qquad\qquad \mathrm{Cut}(\partial X) = \left\{ p \in X \;\middle|\; d(\partial X, p) = \frac{l}{2} \right\} = S$$

and that

$$(5.10) \qquad\qquad d(\partial X, p) \leq \frac{l}{2}$$

for all $p \in X$. Here S is the equidistant set defined by (5.1). Thus, from the proof of Lemma 5.2, we see that $\angle p = \angle q = \pi/2$ holds for all $p, q \in \mathrm{Cut}(\partial X)$.

Lemma 5.4. $\mathrm{Cut}(\partial X)$ *is totally geodesic.*

Proof.   Let p, q be any mutually distinct points of $\mathrm{Cut}(\partial X)$, and fix them. Moreover, let $\gamma\colon [0, d(p, q)] \to X$ denote a minimal geodesic segment emanating from p to q. If we prove that $\gamma(t) \in \mathrm{Cut}(\partial X)$ for all $t \in [0, d(p, q)]$, then our proof is complete.
   Suppose that

$$(5.11) \qquad\qquad \gamma(t_0) \notin \mathrm{Cut}(\partial X)$$

for some $t_0 \in (0, d(p, q))$. By (5.9), we have that

$$(5.12) \qquad\qquad d(\partial X, \gamma(t_0)) \neq \frac{l}{2},$$

and that

$$(5.13) \qquad\qquad d(\partial X, p) = d(\partial X, q) = \frac{l}{2}.$$

The equations (5.10) and (5.12) imply that

$$(5.14) \qquad\qquad d(\partial X, \gamma(t_0)) < \frac{l}{2}.$$

Without loss of generality, we may assume that

$$(5.15) \qquad d(\partial X, \gamma(t_0)) = \min\{d(\partial X, \gamma(t)) \mid 0 \leq t \leq d(p, q)\}.$$

By Remark 5.3, (5.11), and (5.15), we obtain the open triangle $\mathrm{OT}(\partial X, p, \gamma(t_0))$ satisfying

$$(5.16) \qquad\qquad \angle p = \frac{\pi}{2}, \quad \angle \gamma(t_0) = \frac{\pi}{2}.$$

From Theorem 2.5, (5.13), (5.14), and (5.16), we thus get an open triangle $\mathrm{OT}(\partial \mathbb{R}_+^2, \tilde{p}, \tilde{\gamma}(t_0))$ in $\mathbb{R}_+^2$ corresponding to the triangle $\mathrm{OT}(\partial X, p, \gamma(t_0))$ such that

$$d(\partial \mathbb{R}_+^2, \tilde{p}) = \frac{l}{2}, \quad d(\partial \mathbb{R}_+^2, \tilde{\gamma}(t_0)) < \frac{l}{2},$$

and that

$$\angle \tilde{p} \leq \frac{\pi}{2}, \quad \angle \tilde{\gamma}(t_0) \leq \frac{\pi}{2}.$$

This is a contradiction, since our model is $\mathbb{R}_+^2$. Therefore, $\gamma(t) \in \mathrm{Cut}(\partial X)$ holds for all $t \in [0, d(p, q)]$. $\qquad\square$

Lemma 5.5. *For each $t \in (0, l/2)$, the level set $H_i(t) := \{p \in X \mid d(\partial X_i, p) = t\}$, $i = 1, 2$, is totally geodesic, and $H_1(t)$ is totally geodesic for all $t \in (0, l)$.*

Proof. Take any $t \in (0, l/2)$, and fix it. Let p, q be any mutually distinct points in $H_1(t)$, and also fix them. Let $\mu_1, \mu_2 \colon [0, l] \to X$ denote minimal geodesic segment emanating from ∂X_1 to ∂X_2 and passing through $\mu_1(t) = p$, $\mu_2(t) = q$, respectively. Thus, we have an open triangle $\mathrm{OT}(\partial X_1, p, q) = (\partial X_1, p, q; \gamma_t, \mu_1|_{[0,t]}, \mu_2|_{[0,t]})$, where

$\gamma_t \colon [0, d(p, q)] \to X$ denotes a minimal geodesic segment emanating from p to q. If we prove

$$\angle p = \angle q = \frac{\pi}{2}, \tag{5.17}$$

then we see, by similar argument in the proof of Lemma 5.4, that $H_1(t)$ is totally geodesic. Thus, we will prove (5.17) in the following.

By Theorem 2.5, there exists an open triangle

$$\text{OT}(\partial \mathbb{R}^2_+, \tilde{p}, \tilde{q}) = (\partial \mathbb{R}^2_+, \tilde{p}, \tilde{q}; \tilde{\gamma}_t, \tilde{\mu}_1|_{[0,t]}, \tilde{\mu}_2|_{[0,t]})$$

in $\mathbb{R}^2_+$ corresponding to the triangle $\text{OT}(\partial X_1, p, q)$ such that

$$d(\partial \mathbb{R}^2_+, \tilde{p}) = d(\partial \mathbb{R}^2_+, \tilde{q}) = t, \quad d(\tilde{p}, \tilde{q}) = d(p, q) \tag{5.18}$$

and that

$$\angle p \geq \angle \tilde{p}, \quad \angle q \geq \angle \tilde{q}. \tag{5.19}$$

Since our model is $\mathbb{R}^2_+$, the equation $d(\partial \mathbb{R}^2_+, \tilde{p}) = d(\partial \mathbb{R}^2_+, \tilde{q})$ of (5.18) implies that

$$\angle \tilde{p} = \angle \tilde{q} = \frac{\pi}{2}. \tag{5.20}$$

Thus, by (5.19) and (5.20), we have

$$\angle p \geq \frac{\pi}{2}, \quad \angle q \geq \frac{\pi}{2}. \tag{5.21}$$

On the other hand, by Lemma 5.4, $\text{Cut}(\partial X)$ is totally geodesic, i.e., all eigenvalues of the shape operator of $\text{Cut}(\partial X)$ are 0 in the vector normal to $\text{Cut}(\partial X)$. Since the radial vector of any $\text{Cut}(\partial X)$-segment is parallel to that of a ∂X-segment, $\text{Cut}(\partial X)$ has also non-negative radial curvature. Therefore, we can apply Theorem 2.5 to the open triangle

$$\text{OT}(\text{Cut}(\partial X), p, q) = (\text{Cut}(\partial X), p, q; \gamma_t, \mu_1|_{[t,l/2]}, \mu_2|_{[t,l/2]}).$$

Thus, by Theorem 2.5, there exists an open triangle

$$\text{OT}(\partial \mathbb{R}^2_+, \hat{p}, \hat{q}) = (\partial \mathbb{R}^2_+, \hat{p}, \hat{q}; \tilde{\gamma}_t, \tilde{\mu}_1|_{[t,l/2]}, \tilde{\mu}_2|_{[t,l/2]})$$

in $\mathbb{R}^2_+$ corresponding to the triangle $OT(\mathrm{Cut}(\partial X), p, q)$ such that

$$(5.22) \qquad d(\partial \mathbb{R}^2_+, \hat{p}) = d(\partial \mathbb{R}^2_+, \hat{q}) = \frac{l}{2} - t, \quad d(\hat{p}, \hat{q}) = d(p, q)$$

and that

$$(5.23) \qquad \pi - \angle p \geq \angle \hat{p}, \quad \pi - \angle q \geq \angle \hat{q}.$$

As well as above, the equations (5.22) and (5.23) imply $\pi - \angle p \geq \pi/2$ and $\pi - \angle q \geq \pi/2$, since our model is $\mathbb{R}^2_+$. Thus, we have

$$(5.24) \qquad \angle p \leq \frac{\pi}{2}, \quad \angle q \leq \frac{\pi}{2}.$$

By (5.21) and (5.24), we therefore get (5.17). By the same argument above, one may prove that $H_2(t)$ is also totally geodesic for all $t \in (0, l/2)$. Since $H_1(t) = H_2(l - t)$, $H_1(t)$ is totally geodesic for all $t \in (0, l)$. $\qquad \square$

Theorem 5.6. *Let $(X, \partial X)$ be a complete connected Riemannian manifold X with disconnected smooth compact convex boundary ∂X whose radial curvature is bounded from below by 0. Then, X is isometric to $[0, l] \times \partial X_1$ with Euclidean product metric of $[0, l]$ and ∂X_1, where ∂X_1 denotes a connected component of ∂X. In particular, ∂X_1 is the soul of X.*

Proof. Let $\Phi \colon [0, l] \times \partial X_1 \to X$ denote the map defined by $\Phi(t, p) := \exp^\perp(t v_p)$, where v_p denotes the inward pointing unit normal vector to ∂X_1 at $p \in \partial X_1$. We will prove that the Φ is an isometry. From Lemma 5.2, it is clear that Φ is a diffeomorphism.

Let $\mu_1 \colon [0, l] \to X$ denote any minimal geodesic segment emanating from ∂X_1 to ∂X_2, and fix it. Choose a minimal geodesic segment $\mu_2 \colon [0, l] \to X$ emanating from ∂X_1 to ∂X_2 sufficiently close μ_1, so that, for each $t \in (0, l)$, $\mu_1(t)$ is joined with $\mu_2(t)$ by a unique minimal geodesic segment γ_t. Since each level hypersurface $H_1(t)$ is totally geodesic by Lemma 5.5, γ_t meets μ_1 and μ_2 perpendicularly at $\mu_1(t)$ and $\mu_2(t)$, respectively. Therefore, by the first variation formula,

$$\frac{d}{dt} d(\mu_1(t), \mu_2(t)) = 0,$$

holds for all $t \in (0, l)$. Thus, $d(\mu_1(t), \mu_2(t)) = d(\mu_1(0), \mu_2(0))$ holds for all $t \in [0, l]$. This implies that

$$(5.25) \qquad \left\| d\Phi_{(t, p)}\left(\frac{\partial}{\partial x_i}\right) \right\| = \left\| d\Phi_{(0, p)}\left(\frac{\partial}{\partial x_i}\right) \right\|$$

for all $t \in [0, l]$. Here $(x_1, x_2, \ldots, x_{n-1})$ denotes a system of local coordinates around $p := \mu_1(0)$ with respect to ∂X_1. Since

$$d\Phi_{(0,p)}\left(\frac{\partial}{\partial x_i}\right) = \left(\frac{\partial}{\partial x_i}\right)_{(0,p)},$$

we get, by (5.25),

$$(5.26) \qquad \left\|d\Phi_{(t,p)}\left(\frac{\partial}{\partial x_i}\right)\right\| = \left\|\left(\frac{\partial}{\partial x_i}\right)_{(0,p)}\right\| = \left\|\left(\frac{\partial}{\partial x_i}\right)_{p}\right\|.$$

It is clear that

$$(5.27) \qquad d\Phi_{(t,p)}\left(\frac{\partial}{\partial x_i}\right) \perp d\Phi_{(t,p)}\left(\frac{\partial}{\partial x_0}\right), \quad i = 1, 2, \ldots, n-1,$$

and

$$(5.28) \qquad \left\|d\Phi_{(t,p)}\left(\frac{\partial}{\partial x_0}\right)\right\| = 1$$

for all $t \in [0, l]$. Here x_0 denotes the standard local coordinate system for $[0, l]$. By (5.26), (5.27), (5.28), Φ is an isometry. $\square$

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