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ON A MORELLI TYPE EXPRESSION OF COHOMOLOGY CLASSES OF TORUS ORBIFOLDS

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Abstract

Let X be a complete toric variety of dimension n and Δ the fan in a lattice N associated to X. For each cone σ of Δ there corresponds an orbit closure $V(\sigma)$ of the action of complex torus on X. The homology classes $\{[V(\sigma)] \mid \dim\sigma = k\}$ form a set of specified generators of $H_{n-k}(X,\mathbb{Q})$. Then any $x \in H_{n-k}(X,\mathbb{Q})$ can be written in the form

$$x = \sum_{\sigma\in\Delta_X,\ \dim\sigma=k} \mu(x,\sigma)[V(\sigma)].$$

A question occurs whether there is some canonical way to express $\mu(x,\sigma)$. Morelli [12] gave an answer when X is non-singular and at least for $x = \mathcal{T}_{n-k}(X)$ the Todd class of X. However his answer takes coefficients in the field of rational functions of degree 0 on the Grassmann manifold $G_{n-k+1}(N_{\mathbb{Q}})$ of $(n-k+1)$-planes in $N_{\mathbb{Q}}$. His proof uses Baum–Bott's residue formula for holomorphic foliations applied to the action of complex torus on X

On the other hand there appeared several attempts for generalizing non-singular toric varieties in topological contexts [4, 10, 7, 11, 9, 2]. Such generalized manifolds of dimension $2n$ acted on by a compact n dimensional torus T are called by the names quasi-toric manifolds, torus manifolds, toric manifolds, toric origami manifolds, topological toric manifolds and so on. Similarly torus orbifold can be considered. To a torus orbifold X a simplicial set Δ_X called multi-fan of X is associated. A question occurs whether a similar expression to Morelli's formula holds for torus orbifolds. It will be shown the answer is yes in this case too at least when the rational cohomology ring $H^*(X)_{\mathbb{Q}}$ is generated by $H^2(X)_{\mathbb{Q}}$. Under this assumption the equivariant cohomology ring with rational coefficients $H_T^*(X,\mathbb{Q})$ is isomorphic to $H_T^*(\Delta_X,\mathbb{Q})$, the face ring of the multi-fan Δ_X, and the proof is carried out on $H_T^*(\Delta_X,\mathbb{Q})$ by using completely combinatorial terms.

1. Introduction

Let X be a complete toric variety of dimension n and Δ_X the fan associated to X. Δ_X is a collection of rational convex cones in $N_{\mathbb{R}} = N \otimes \mathbb{R}$ where N is a lattice of rank n. For each k-dimensional cone σ in Δ_X, let $V(\sigma)$ be the corresponding orbit closure of dimension $n-k$ and $[V(\sigma)] \in A_{n-k}(X)$ be its Chow class. Then the Todd

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class $\mathcal{T}_{n-k}(X)$ of X can be written in the form

$$\mathcal{T}_{n-k}(X) = \sum_{\sigma\in\Delta_X,\ \dim\sigma=k} \mu_k(\sigma)[V(\sigma)]. \tag{1}$$

However, since the $[V(\sigma)]$ are not linearly independent, the coefficients $\mu_k(\sigma) \in \mathbb{Q}$ are not determined uniquely. Danilov [3] asks if $\mu_k(\sigma)$ can be chosen so that it depends only on the cone σ not depending on a particular fan in which it lies.

The equality (1) has a close connection with the number $\#(P)$ of lattice points contained in a convex lattice polytope P in $M_{\mathbb{R}}$ where M is the dual lattice of N. For a positive integer ν the number $\#(\nu P)$ is expanded as a polynomial in ν (called Ehrhart polynomial):

$$\#(\nu P) = \sum_k a_k(P)\nu^{n-k}.$$

A convex lattice polytope P in $M_{\mathbb{R}}$ determines a complete toric variety X and an invariant Cartier divisor D on X. There is a one-to-one correspondence between the cells $\{\sigma\}$ of Δ_X and the faces $\{P_\sigma\}$ of P. Then the coefficient $a_k(P)$ has an expression

$$a_k(P) = \sum_{\dim\sigma=k} \mu_k(\sigma)\operatorname{vol} P_\sigma \tag{2}$$

with the same $\mu_k(\sigma)$ as in (1).

Hereafter we shall use notation $H_T^*(\ )_{\mathbb{Q}}$ to mean $H_T^*(\ )\otimes\mathbb{Q}$ and so on.

We shall restrict ourselves to the case where X is non-singular. Put $D_i = [V(\sigma_i)]$ for the one dimensional cone σ_i, and let $x_i \in H^2(X)$ denote the Poincaré dual of D_i. The divisor D is written in the form $D = \sum_i d_i D_i$ with positive integers d_i. Put $\xi = \sum_i d_i x_i$. It is known that

$$a_k(P) = \int_X e^\xi \mathcal{T}^k(X)$$

and

$$\operatorname{vol} P_\sigma = \int_X e^\xi x_\sigma,$$

where $\mathcal{T}^k(X) \in H^{2k}(X)_{\mathbb{Q}}$ is the k-th component of the Todd cohomology class, the Poincaré dual of $\mathcal{T}_{n-k}(X)$, and $x_\sigma \in H^{2k}(X)$ is the Poincaré dual of $[V(\sigma)]$. The cohomology class x_σ can also be written as $x_\sigma = \prod_j x_j$ where the product runs over such j that σ_j is an edge of σ. Then the equality (2) can be rewritten as

$$\int_X e^\xi \mathcal{T}^k(X) = \sum_{\dim\sigma=k} \mu_k(\sigma)\int_X e^\xi x_\sigma. \tag{3}$$

The reader is referred to [5] Section 5.3 for details and Note 17 there for references.

In his paper [12] Morelli gave an answer to Danilov's question. Let $Rat(G_{n-k+1}(N_{\mathbb{Q}}))_0$ denote the field of rational functions of degree 0 on the Grassmann manifold of $(n-k+1)$-planes in $N_{\mathbb{Q}}$. For a cone σ of dimension k in $N_{\mathbb{R}}$ he associates a rational function $\mu_k(\sigma) \in Rat(G_{n-k+1}(N_{\mathbb{Q}}))_0$. With this $\mu_k(\sigma)$, the right hand side of (1) belongs to

$$Rat(G_{n-k+1}(N_{\mathbb{Q}}))_0 \otimes_{\mathbb{Q}} A_{n-k}(X)_{\mathbb{Q}},$$

and the equality (1) means that the rational function with values in $A_{n-k}(X)_{\mathbb{Q}}$ in the right hand side is in fact a constant function equal to $\mathcal{T}_{n-k}(X)$ in $A_{n-k}(X)_{\mathbb{Q}}$. In other words this means that

$$\sum_{\sigma \in \Delta_X,\ \dim \sigma = k} \mu_k(\sigma)(E)[V(\sigma)] = \mathcal{T}_{n-k}(X)$$

for any generic $(n-k+1)$-plane E in $N_{\mathbb{Q}}$.

Morelli gives an explicit formula for $\mu_k(\sigma)$ when the toric variety is non-singular using Baum–Bott's residue formula for singular foliations [1] applied to the action of $(\mathbb{C}^*)^n$ on X. He then shows that the function $\mu_k(\sigma)$ is additive with respect to non-singular subdivisions of the cone σ. This fact leads to (1) in its general form.

One can ask a similar question about general classes other than the Todd class whether it is possible to define $\mu(x,\sigma) \in Rat(G_{n-k+1}(N_{\mathbb{Q}}))_0$ for $x \in A_{n-k}(X)$ in a canonical way to satisfy

$$x = \sum_{\sigma \in \Delta_X,\ \dim \sigma = k} \mu(x,\sigma)[V(\sigma)]. \tag{4}$$

When X is non-singular, one can expect that $\mu(x,\sigma)$ satisfies a formula analogous to (3)

$$\int_X e^{\xi} x = \sum_{\dim \sigma = k} \mu(x,\sigma) \int_X e^{\xi} x_\sigma \tag{5}$$

for any cohomology class $\xi = \sum_i d_i x_i$. In this sense the formula does not explicitly refer to convex polytopes. Fulton and Sturmfels [6] used Minkowski weights to describe intersection theory of toric varieties. For complete non-singular varieties or $\mathbb{Q}$-factorial varieties X the Minkowski weight $\gamma_x \colon H^{2(n-k)}(X) \to \mathbb{Q}$ corresponding to $x \in H^{2k}(X)$ is defined by $\gamma_x(y) = \int_X xy$. Thus, if the d_i are considered as variables in ξ, the formula (5) is considered as describing γ_x as a linear combination of the Minkowski weights of γ_{x_σ}.

On the other hand topological analogues of toric variety were discussed by several authors [4, 10, 7, 11, 9, 2]. Most general one would be torus orbifold [7]. To a torus orbifold X a multi-fan Δ_X is associated. Multi-fan is a generalized notion of fan. Its cohomology reflects the cohomology of the torus orbifold.

The purpose of the present paper is to establish the formula (5) by showing an explicit formula for $\mu(x,\sigma)$ when X is a torus orbifold. Moreover our proof is based on a simple combinatorial argument carried on the associated multi-fan Δ_X. Topologically the formula concerns equivariant cohomology classes on torus orbifolds. This would suggest that actions of compact tori equipped with some nice conditions admit topological residue formulas similar to Baum–Bott' formula.

In Section 2 we recall the definition of multi-fans and torus orbifolds together with relevant facts. The definition of $\mu(x, \sigma)$ is given for multi-fans and consequently for torus orbifolds. Theorem 3.1 states that the formula (1) holds for any torus orbifolds. Furthermore Corollary 3.2 ensures that the formula (2) holds for torus orbifolds. Finally Corollary 3.3 states that (4) holds for a torus orbifolds X such that $H^*(X)_{\mathbb{Q}}$ is generated by $H^2(X)_{\mathbb{Q}}$.

2. Multi-fans and torus orbifolds

The notion of multi-fan and multi-polytope were introduced in [10]. In this article we shall be concerned only with simplicial multi-fans. See [10, 7, 8] for details.

Let N be a lattice of rank n. A *simplicial multi-fan* in N is a triple $\Delta = (\Sigma, C, w)$ where $\Sigma = \bigsqcup_{k=0}^{n} \Sigma^{(k)}$ is an (augmented) simplicial complex, $\Sigma^{(k)}$ being the set of $k-1$ simplices, C is a map from $\Sigma^{(k)}$ into the set of k-dimensional strongly convex rational polyhedral cones in the vector space $N_{\mathbb{R}} = N \otimes \mathbb{R}$ for each k, and w is a map $\Sigma^{(n)} \to \mathbb{Z}$. $\Sigma^{(0)}$ consists of a single element $o =$ the empty set. (The definition in [10] and [7] requires additional restriction on w.) We assume that any $J \in \Sigma$ is contained in some $I \in \Sigma^{(n)}$ and $\Sigma^{(n)}$ is not empty.

The map C is required to satisfy the following condition; if $J \in \Sigma$ is a face of $I \in \Sigma$, then $C(J)$ is a face of $C(I)$, and for any I, the map C restricted on $\Sigma(I) = \{J \in \Sigma \mid J \subset I\}$ is an isomorphism of ordered sets onto the set of faces of $C(I)$. It follows that $C(I)$ is necessarily a simplicial cone and $C(o) = 0$. A simplicial fan is considered as a simplicial multi-fan such that the map C on Σ is injective and $w \equiv 1$.

For each $K \in \Sigma$ we set

$$\Sigma_K = \{J \in \Sigma \mid K \subset J\}.$$

It inherits the partial ordering from Σ and becomes a simplicial set where $\Sigma_K^{(j)} \subset \Sigma^{(j+|K|)}$. K is the unique element in $\Sigma_K^{(0)}$. Let N_K be the minimal primitive sublattice of N containing $N \cap C(K)$, and N^K the quotient lattice of N by N_K. For $J \in \Sigma_K$ we define $C_K(J)$ to be the cone $C(J)$ projected on $N^K \otimes \mathbb{R}$. We define a function

$$w\colon \Sigma_K^{(n-|K|)} \subset \Sigma^{(n)} \to \mathbb{Z}$$

to be the restrictions of w to $\Sigma_K^{(n-|K|)}$. The triple $\Delta_K = (\Sigma_K, C_K, w)$ is a multi-fan in N^K and is called the *projected multi-fan* with respect to $K \in \Sigma$. For $K = o$, the projected multi-fan Δ_o is nothing but Δ itself.

A vector $v \in N_{\mathbb{R}}$ will be called *generic* if v does not lie on any linear subspace spanned by a cone in $C(\Sigma)$ of dimension less than n. For a generic vector v we set $d_v = \sum_{v \in C(I)} w(I)$, where the sum is understood to be zero if there are no such I.

DEFINITION. A simplicial multi-fan $\Delta = (\Sigma, C, w)$ is called *pre-complete* if the integer d_v is independent of generic vectors v. In this case this integer will be called the *degree* of Δ and will be denoted by $\deg(\Delta)$. It is also called the *Todd genus* of Δ and is denoted by $\mathrm{Td}[\Delta]$. A pre-complete multi-fan Δ is said to be *complete* if the projected multi-fan Δ_K is pre-complete for every $K \in \Sigma$.

A multi-fan is complete if and only if the projected multi-fan Δ_J is pre-complete for every $J \in \Sigma^{(n-1)}$.

Like a toric variety gives rise to a fan, a torus orbifold gives rise to a complete simplicial multi-fan, though this correspondence is not one to one. A torus orbifold is a *closed oriented* orbifold with an effective action (in the sense of orbifold action) of a compact torus of half the dimension of the orbifold with non-empty fixed point set and with some additional conditions on the orientations of certain type of suborbifolds (precise statement will be given later. See [13] for terminologies concerning orbifolds, and [7] for those of torus orbifolds). Cobordism invariants of torus orbifolds are encoded in the associated multi-fans.

Let X be a torus orbifold. A connected component of the fix point set of a subcircle of the torus T is a suborbifold. A suborbifold of this type which has codimension two and contains at least one fixed point of the action of T is called characteristic suborbifold. By the orientation convention included in the definition of torus orbifold, a characteristic suborbifold is equipped with a fixed orientation.

In the following, characteristic suborbifolds will be denoted byX_i. In the multi-fan $\Delta(X) = (\Sigma(X), C(X), w(X))$ the simplicial complex $\Sigma(X)$ is given by

$$\Sigma^{(k)}(X) = \left\{ I \;\middle|\; \#I = k+1, \left(\bigcap_{i \in I} X_i \right)^T \neq \emptyset \right\}.$$

Let S_i be the circle that fixes the points of X_i. Take a point x in X_i. Take an orbifold chart (U_x, V_x, H_x, p_x) around x in which U_x is invariant under the action of S_i and V_x is an Euclidean ball on which H_x acts linearly and the projection $p_x \colon V_x \to U_x$ identifies V_x/H_x with U_x. Then there exist a covering group $\tilde{S}_i$ of S_i and a lifting of the action of S_i to the action of $\tilde{S}_i$ on V_x (exactly its tangent space). Hereafter we shall always take the minimal covering with the above property.

If x is a fixed point of the action of T, U_x can be taken invariant under the action of T and such that $p_x^{-1}(x)$ is a single point. Furthermore if x is in a characteristic suborbifold X_i, then the vector space V_x decomposes into a direct sum $V_x = V_i + V_i^{\perp}$ where $V_i^{\perp}$ is tangent to $p_x^{-1}(U_x \cap X_i)$ and V_i is normal to the tangent space of

$p_x^{-1}(U_x \cap X_i)$ at $p_x^{-1}(x)$ and is endowed with an invariant complex 1-dimensional vector space structure as follows from the definition of torus orbifolds. Then there is a unique isomorphism $\varphi_i\colon S^1 \to \tilde{S}_i$ such that $\varphi_i(z)$ acts by the complex multiplication of $z \in S^1 \subset \mathbb{C}$ on V_i. φ_i depends only on X_i, not on particular choice of x. Let $\pi\colon \tilde{S}_i \to S_i$ denote the covering projection. The homomorphism $\rho_i = \pi \circ \varphi_i\colon S^1 \to S_i \subset T$ defines an element $v_i \in \operatorname{Hom}(S^1, T) = H_2(BT, \mathbb{Z})$. Then $C(X)(I)$ is the cone in $N = H_2(BT, \mathbb{Z})$ with apex at 0 and spanned by $\{v_i \mid i \in I\}$.

Let $\Delta = (\Sigma, C, w)$ be a simplicial multi-fan in a lattice N. The Stanley–Reisner ring or the face ring of the simplicial set Σ is denoted by $H_T^*(\Delta)$. It is the quotient ring of the polynomial ring $\mathbb{Z}[x_i \mid i \in \Sigma^{(1)}]$ by the ideal generated by

$$\left\{ x_K = \prod_{i \in K} x_i \;\middle|\; K \subset \Sigma^{(1)},\ K \notin \Sigma \right\}.$$

When Δ is the fan Δ_X associated to a torus orbifold X, $H_T^*(\Delta_X)_{\mathbb{Q}}$ can be identified with a subring of the equivariant cohomology ring $H_T^*(X)_{\mathbb{Q}}$ of X with respect to the action of compact torus T acting on X (see [10]). (Hereafter we shall use notation $H_T^*(\ )_{\mathbb{Q}}$ to mean $H_T^*(\ ) \otimes \mathbb{Q}$.)

In the sequel we shall often consider a set $\mathcal{V}$ consisting of non-zero edge vectors v_i for each $i \in \Sigma^{(1)}$ such that $v_i \in N \cap C(i)$. We do not require v_i to be primitive. This has meaning for torus orbifolds. For any $K \in \Sigma$ put $\mathcal{V}_K = \{v_i\}_{i \in K}$. Let $N_{K,\mathcal{V}}$ be the sublattice of N_K generated by $\mathcal{V}_K$. The quotient group $N_K/N_{K,\mathcal{V}}$ is denoted by $H_{K,\mathcal{V}}$.

Let $\mathcal{V} = \{v_i\}_{i \in \Sigma^{(1)}}$ be a set of prescribed edge vectors as before. We define a homomorphism $M = N^* = H_T^2(pt) \to H_T^2(\Delta)$ by the formula

$$u = \sum_{i \in \Sigma^{(1)}} \langle u, v_i \rangle x_i. \tag{6}$$

This extends to a homomorphism $H_T^*(pt) \to H_T^*(\Delta)$ and makes $H_T^*(\Delta)$ a ring over $H_T^*(pt)$ (regarded as embedded in $H_T^*(\Delta)$).

Since this definition depends on the set $\mathcal{V}$, the $H_T^*(pt)$-module structure of $H_T^*(\Delta)$ also depends on $\mathcal{V}$. To emphasize this fact we shall use the notation $H_T^*(\Delta, \mathcal{V})$. When all the v_i are taken primitive, the notation $H_T^*(\Delta)$ is used.

Fix $I \in \Sigma^{(n)}$ and let $\{u_i^I\}_{i \in I}$ be the basis of $N^* = H^2(pt)$ dual to $\{v_i\}_{i \in I}$. Define $\iota_I^*\colon H_T^2(\Delta)_{\mathbb{Q}} \to M_{\mathbb{Q}} = H_T^2(pt)_{\mathbb{Q}}$ by

$$\iota_I^*\left(\sum_{i \in \Sigma^{(1)}} d_i x_i \right) = \sum_{i \in I} d_i u_i^I. \tag{7}$$

ι_I^* extends to $H_T^*(\Delta)_{\mathbb{Q}} \to H_T^*(pt)_{\mathbb{Q}}$. It is an $H_T^*(pt)_{\mathbb{Q}}$-module map, since

$$\iota_I^*(u) = u \quad \text{for} \quad u \in H_T^*(pt)_{\mathbb{Q}}.$$

Let S be the multiplicative set in $H_T^*(pt)_{\mathbb{Q}}$ generated by non-zero elements in $H_T^2(pt)_{\mathbb{Q}}$. The push-forward $\pi_*\colon H_T^*(\Delta)_{\mathbb{Q}} \to S^{-1}H_T^*(pt)_{\mathbb{Q}}$ is defined by

$$\pi_*(x) = \sum_{I\in\Sigma^{(n)}} \frac{\iota_I^*(x)}{|H_I|\prod_{i\in I} u_i^I}. \tag{8}$$

It is an $H_T^*(pt)_{\mathbb{Q}}$-module map, and lowers the degrees by $2n$. It is known [7] that, if Δ is a complete simplicial multi-fan, then the image of π_* lies in $H_T^*(pt)_{\mathbb{Q}}$.

Assume that Δ is complete. Let $p_*\colon H_T^*(\Delta)_{\mathbb{Q}} \to \mathbb{Q}$ be the composition of $\pi_*\colon H_T^*(\Delta)_{\mathbb{Q}} \to H_T^*(pt)_{\mathbb{Q}}$ and $H_T^*(pt)_{\mathbb{Q}} \to H_T^0(pt)_{\mathbb{Q}} = \mathbb{Q}$. Note that p_* induces $\int_\Delta\colon H^*(\Delta)_{\mathbb{Q}} \to \mathbb{Q}$ as noted in [7] where $H^*(\Delta)_{\mathbb{Q}}$ is the quotient of $H_T^*(\Delta)_{\mathbb{Q}}$ by the ideal generated by $H_T^+(pt)_{\mathbb{Q}}$. Note that $H^*(\Delta)_{\mathbb{Q}}$ is defined independently of $\mathcal{V}$. If $\bar{x}$ denotes the image of $x \in H_T^*(\Delta)_{\mathbb{Q}}$ in $H^*(\Delta)_{\mathbb{Q}}$, then $\int_\Delta \bar{x} = p_*(x)$.

If X a torus orbifold such that $\Delta_X = \Delta$ then $H_T^*(\Delta)_{\mathbb{Q}}$ is a subring of $H_T^*(X)_{\mathbb{Q}}$. From this it follows that p_* on $H_T^*(\Delta)_{\mathbb{Q}}$ is the restriction of $p_*\colon H_T^*(X)_{\mathbb{Q}} \to \mathbb{Q}$ and $\int_\Delta$ is the ordinary integral $\int_X$ (see [7]).

Let $K \in \Sigma^{(k)}$ and let $\Delta_K = (\Sigma_K, C_K, w_K)$ be the projected multi-fan. The link $\mathrm{Lk}K$ of K in Σ is a simplicial complex consisting of simplices J such that $K\cup J \in \Sigma$ and $K \cap J = \emptyset$. It will be denoted by Σ'_K in the sequel. There is an isomorphism from Σ'_K to Σ_K sending $J \in \Sigma'_K$ to $J \cup K$. Let $\mathcal{V} = \{v_i\}_{i\in\Sigma^{(1)}}$ be a set of prescribed edge vectors as before. Let $\{u_i^K\}_{i\in K}$ be the basis of $N^*_{K,\mathcal{V}}$ dual to $\mathcal{V}_K$. We consider the polynomial ring R_K generated by $\{x_i \mid i \in K \cup \Sigma'^{(1)}_K\}$ and the ideal $\mathcal{I}_K$ generated by monomials $x_J = \prod_{i\in J} x_i$ such that $J \notin \Sigma(K) * \Sigma'_K$ where $\Sigma(K) * \Sigma'_K$ is the join of $\Sigma(K)$ and Σ'_K. We define the equivariant cohomology $H_T^*(\Delta_K)$ of Δ_K with respect to the torus T as the quotient ring $R_K/\mathcal{I}_K$.

If $\mathcal{V}$ is a set of prescribed edge vectors, $H_T^2(pt)$ is regarded as a submodule of $H_T^2(\Delta_K)$ by a formula similar to (6). This defines an $H_T^*(pt)$-module structure on $H_T^*(\Delta_K)$ which will be denoted by $H_T^*(\Delta_K, \mathcal{V})$ to specify the dependence on $\mathcal{V}$. The projection $H_T^*(\Delta, \mathcal{V}) \to H_T^*(\Delta_K, \mathcal{V})$ is defined by sending x_i to x_i for $i \in K \cup \Sigma'^{(1)}_K$ and putting $x_i = 0$ for $i \notin K \cup \Sigma'^{(1)}_K$. The restriction homomorphism $\iota_I^*\colon H_T^*(\Delta_K, \mathcal{V})_{\mathbb{Q}} \to H_T^*(pt)_{\mathbb{Q}}$ for $I \in \Sigma_K^{(n-k)}$ and the push-forward $\pi_*\colon H_T^*(\Delta_K, \mathcal{V})_{\mathbb{Q}} \to S^{-1}H_T^*(pt)_{\mathbb{Q}}$ are also defined in a similar way as before.

Given $\xi = \sum_{i\in K\cup\Sigma'^{(1)}_K} d_i x_i \in H_T^2(\Delta_K, \mathcal{V})_{\mathbb{R}}$, $d_i \in \mathbb{R}$, let A_K^* be the affine subspace in the space $M_{\mathbb{R}}$ defined by $\langle u, v_i\rangle = d_i$ for $i \in K$. Then we introduce a collection $\mathcal{F}_K = \{F_i \mid i \in \Sigma'^{(1)}_K\}$ of affine hyperplanes in A_K^* by setting

$$F_i = \{u \mid u \in A_K^*,\ \langle u, v_i\rangle = d_i\}.$$

The pair $\mathcal{P}_K(\xi) = (\Delta_K, \mathcal{F}_K)$ will be called a *multi-polytope* associated with ξ; see [8]. In case $K = o \in \Sigma^{(0)}$, $\mathcal{P}_K(\xi)$ is simply denoted by $\mathcal{P}(\xi)$.

For $\xi = \sum_{i\in\Sigma^{(1)}} d_i x_i$ and $K \in \Sigma^{(k)}$ put $\xi_K = \sum_{i\in K\cup\Sigma'^{(1)}_K} d_i x_i$ and $\mathcal{P}(\xi)_K = \mathcal{P}_K(\xi_K)$. It will be called the *face* of $\mathcal{P}(\xi)$ corresponding to K.

For $I \in \Sigma_K^{(n-k)}$, i.e. $I \in \Sigma^{(n)}$ with $I \supset K$, we put $u_I = \bigcap_{i \in I} F_i = \bigcap_{i \in I \setminus K} F_i \cap A_K^* \in A_K^*$. Note that u_I is equal to $\iota_I^*(\xi)$. The dual vector space $(N_{\mathbb{R}}^K)^*$ of $N_{\mathbb{R}}^K$ is canonically identified with the subspace $M_{K\mathbb{R}}$ of $M_{\mathbb{R}} = H_T^2(pt)_{\mathbb{R}}$. It is parallel to A_K^*, and u_i^I lies in $M_{K\mathbb{R}}$ for $I \in \Sigma_K^{(n-k)}$ and $i \in I \setminus K$. A vector $v \in N_{\mathbb{R}}^K$ is called generic if $\langle u_i^I, v \rangle \neq 0$ for any $I \in \Sigma_K^{(n-k)}$ and $i \in I \setminus K$. The image in $N_{\mathbb{R}}^K$ of a generic vector in $N_{\mathbb{R}}$ is generic. Take a generic vector $v \in N_{\mathbb{R}}^K$, and define

$$(-1)^I := (-1)^{\#\{j \in I \setminus K \mid \langle u_j^I, v \rangle > 0\}}$$

and

$$(u_i^I)^+ := \begin{cases} u_i^I & \text{if} \quad \langle u_i^I, v \rangle > 0, \\ -u_i^I & \text{if} \quad \langle u_i^I, v \rangle < 0, \end{cases}$$

for $I \in \Sigma_K^{(n-k)}$ and $i \in I \setminus K$. We denote by $C_K^*(I)^+$ the cone in A_K^* spanned by the $(u_i^I)^+$, $i \in I \setminus K$, with apex at u_I, and by ϕ_I its characteristic function. With these understood, we define a function $\mathrm{DH}_{\mathcal{P}_K(\xi)}$ on $A_K^* \setminus \bigcup_i F_i$ by

$$\mathrm{DH}_{\mathcal{P}_K(\xi)} = \sum_{I \in \Sigma_K^{(n-k)}} (-1)^I w(I) \phi_I.$$

As in [8] we call this function the *Duistermaat–Heckman function* associated with $\mathcal{P}_K(\xi)$. When $K = o$, $\mathrm{DH}_{\mathcal{P}(\xi)}$ is defined on $M_{\mathbb{R}} \setminus \bigcup_i F_i$.

The following theorem is fundamental in the sequel, cf. [8] Theorem 2.3 and [7] Corollary 7.4.

Theorem 2.1. *Let Δ be a complete simplicial multi-fan. Let $\xi = \sum_{i \in K \cup \Sigma_K'^{(1)}} d_i x_i \in H_T^2(\Delta_K, \mathcal{V})$ be as above with all d_i integers and put $\xi_+ = \sum_i (d_i + \epsilon) x_i$ with $0 < \epsilon < 1$. Then*

$$\sum_{u \in A_K^* \cap M} \mathrm{DH}_{\mathcal{P}_K(\xi_+)}(u) t^u = \sum_{I \in \Sigma_K^{(n-k)}} \frac{w(I)}{|H_{I,\mathcal{V}}|} \sum_{h \in H_{I,\mathcal{V}}} \frac{\chi_I(\iota_I^*(\xi), h) t^{\iota_I^*(\xi)}}{\prod_{i \in I \setminus K} (1 - \chi_I(u_i^I, h)^{-1} t^{-u_i^I})}, \tag{9}$$

where $\chi_I(u, h) = e^{2\pi \sqrt{-1} \langle u, v(h) \rangle}$ for $u \in N_{I,\mathcal{V}}^*$ and $v(h)$ is a lift of $h \in H_{I,\mathcal{V}}$ to $N_{I,\mathcal{V}}$.

NOTE. The left hand side of (9) is considered as an element in the group ring of M over $\mathbb{R}$ or the character ring $R(T) \otimes \mathbb{R}$ considered as the Laurent polynomial ring in $t = (t_1, \dots, t_n)$. The equality shows that the right hand side, which is a rational function of t, belongs to $R(T) \otimes \mathbb{R}$.

$\xi = \sum_i d_i x_i \in H_T^2(\Delta, \mathcal{V})$ is called T-Cartier if $\iota_I^*(\xi) \in M$ for all $I \in \Sigma^{(n)}$. This condition is equivalent to $u_I \in M$ for all $I \in \Sigma^{(n)}$. In this case $\mathcal{P}(\xi)$ is said lattice multi-polytope. If ξ is T-Cartier, then $\chi_I(\iota_I^*(\xi), h) \equiv 1$. Hence the above formula (9) for $\mathrm{DH}_{\mathcal{P}_K(\xi_{K+})}$ reduces in this case to

$$\sum_{u\in A_K^*\cap M} \mathrm{DH}_{\mathcal{P}_K(\xi_{K+})}(u)t^u = \sum_{I\in\Sigma_K^{(n-k)}} \frac{w(I)}{|H_{I,\mathcal{V}}|} \sum_{h\in H_{I,\mathcal{V}}} \frac{t^{\iota_I^*(\xi_K)}}{\prod_{i\in I\setminus K}(1-\chi_I(u_i^I, h)^{-1}t^{-u_i^I})}. \tag{10}$$

Let $H_T^{**}(\ )$ denote the completed equivariant cohomology ring. The Chern character ch sends $R(T)\otimes\mathbb{R}$ to $H_T^{**}(pt)_\mathbb{R}$ by $\mathrm{ch}(t^u) = e^u$. The image of (10) by ch is given by

$$\sum_{u\in A_K^*\cap M} \mathrm{DH}_{\mathcal{P}_K(\xi_{K+})}(u)e^u = \sum_{I\in\Sigma_K^{(n-k)}} \frac{w(I)}{|H_{I,\mathcal{V}}|} \sum_{h\in H_{I,\mathcal{V}}} \frac{e^{\iota_I^*(\xi_K)}}{\prod_{i\in I\setminus K}(1-\chi_I(u_i^I, h)^{-1}e^{-u_i^I})}. \tag{11}$$

Assume that $\xi = \sum_i d_i x_i \in H_T^2(\Delta, \mathcal{V})$ is T-Cartier. The number $\#(\mathcal{P}(\xi)_K)$ is defined by

$$\#(\mathcal{P}(\xi)_K) = \sum_{u\in A_K^*\cap M} \mathrm{DH}_{\mathcal{P}_K(\xi_{K+})}(u).$$

It is obtained from (11) by setting $u = 0$, that is, it is equal to the image of (11) by $H_T^{**}(pt)_\mathbb{Q} \to H_T^0(pt)_\mathbb{Q}$.

The equivariant Todd class $\mathcal{T}_T(\Delta, \mathcal{V})$ is defined in such a way that

$$\pi_*(e^\xi \mathcal{T}_T(\Delta, \mathcal{V})) = \sum_{u\in M} \mathrm{DH}_{\mathcal{P}(\xi_+)}(u)e^u$$

for ξ T-Cartier. In order to give the definition we need some notations.

For simplicity identify the set $\Sigma^{(1)}$ with $\{1,2,\dots,m\}$ and consider a homomorphism η: $\mathbb{R}^m = \mathbb{R}^{\Sigma^{(1)}} \to N_\mathbb{R}$ sending $\mathbf{a} = (a_1,a_2,\dots,a_m)$ to $\sum_{i\in\Sigma^{(1)}} a_i v_i$. For $K\in\Sigma^{(k)}$ we define

$$\tilde{G}_{K,\mathcal{V}} = \{\mathbf{a} \mid \eta(\mathbf{a}) \in N \text{ and } a_j = 0 \text{ for } j\notin K\}$$

and define $G_{K,\mathcal{V}}$ to be the image of $\tilde{G}_{K,\mathcal{V}}$ in $\tilde{T} = \mathbb{R}^m/\mathbb{Z}^m$. It will be written G_K for simplicity. The homomorphism η restricted on $\tilde{G}_{K,\mathcal{V}}$ induces an isomorphism

$$\eta_K\colon G_K \cong H_{K,\mathcal{V}} \subset T = N_\mathbb{R}/N.$$

Put

$$G_\Delta = \bigcup_{I\in\Sigma^{(n)}} G_I \subset \tilde{T}$$

and

$$DG_\Delta = \bigcup_{I\in\Sigma^{(n)}} G_I\times G_I \subset G_\Delta\times G_\Delta.$$

Let $v(g) = \mathbf{a} = (a_1, a_2, \dots, a_m) \in \mathbb{R}^m$ be a representative of $g \in \tilde{T}$. The factor a_i will be denoted by $v_i(g)$. It is determined modulo integers. If $g \in G_I$, then $v_i(g)$ is necessarily a rational number. Define a homomorphism $\chi_i\colon \tilde{T} \to \mathbb{C}^*$ by

$$\chi_i(g) = e^{2\pi\sqrt{-1}v_i(g)}.$$

Let $g \in G_I$ and $h = \eta_I(g) \in H_{I,\mathcal{V}}$. Then $\eta(v(g)) \in N_I$ is a representative of h in N_I which will be denoted by $v(h)$. Then, for $g \in G_I$ and $i \in I$,

$$v_i(g) \equiv \langle u_i^I, v(h)\rangle \mod \mathbb{Z},$$

and

$$\chi_i(g) = e^{2\pi\sqrt{-1}\langle u_i^I, v(h)\rangle} = \chi_I(u_i^I, h).$$

Let Δ be a complete simplicial multi-fan. Define

$$\mathcal{T}_T(\Delta, \mathcal{V}) = \sum_{g\in G_\Delta}\prod_{i\in\Sigma^{(1)}} \frac{x_i}{1-\chi_i(g)e^{-x_i}} \in H_T^{**}(\Delta, \mathcal{V})_{\mathbb{Q}}.$$

Proposition 2.2. *Let Δ be a complete simplicial multi-fan. Assume that $\xi \in H_T^2(\Delta, \mathcal{V})$ is T-Cartier. Then*

$$\pi_*(e^\xi \mathcal{T}_T(\Delta, \mathcal{V})) = \sum_{u\in M} \mathrm{DH}_{\mathcal{P}(\xi_+)}(u)e^u.$$

Consequently

$$p_*(e^\xi \mathcal{T}_T(\Delta, \mathcal{V})) = \#(\mathcal{P}(\xi)).$$

Proof (cf. [7] Section 8). Let $g \in G_\Delta$ and $I \in \Sigma^{(n)}$. If $g \notin G_I$, then there is an element $i \notin I$ such that $\chi_i(g) \neq 1$; so

$$\frac{x_i}{1-\chi_i(g)e^{-x_i}} = (1-\chi_i(g))^{-1}x_i + \text{higher degree terms}$$

for such i. Hence $i_I^*(x_i/(1-\chi_i(g)e^{-x_i})) = 0$. Therefore, only elements g in G_I contribute to $\iota_I^*(\mathcal{T}_T(\Delta, \mathcal{V}))$. Now suppose $g \in G_I$. Then $\chi_i(g) = 1$ for $i \notin I$, so $\iota_I^*(x_i/(1-\chi_i(g)e^{-x_i})) = 1$ for such i. Finally, since $\iota_I^*(x_i) = u_i^I$ for $i \in I$, we have

$$\iota_I^*(\mathcal{T}_T(\Delta, \mathcal{V})) = \sum_{g\in G_I}\prod_{i\in I}\frac{u_i^I}{1-\chi_i(g)e^{-u_i^I}}.$$

This together with (11) shows that

$$\begin{aligned}\pi_*(e^{\xi}\mathcal{T}_T(\Delta,\mathcal{V})) &= \pi_*\left(e^{\xi}\sum_{g\in G_\Delta}\prod_{i=1}^{m}\frac{x_i}{1-\chi_i(g)e^{-x_i}}\right)\\ &= \sum_{I\in\Sigma^{(n)}}\frac{w(I)e^{\iota_I^*(\xi)}}{|H_{I,\mathcal{V}}|}\sum_{g\in G_I}\frac{1}{\prod_{i\in I}(1-\chi_i(g)e^{-u_i^I})}\\ &= \sum_{u\in M}\mathrm{DH}_{\mathcal{P}(\xi_+)}(u)e^u. \qquad \square\end{aligned}$$

More generally, for $K\in\Sigma^{(k)}$, define $\mathcal{T}_T(\Delta,\mathcal{V})_K$ by

$$\mathcal{T}_T(\Delta,\mathcal{V})_K=\sum_{g\in G_{\Delta_K}}\prod_{i\in\Sigma_K'^{(1)}}\frac{x_i}{1-\chi_i(g)e^{-x_i}}\in H_T^{**}(\Delta,\mathcal{V})_{\mathbb{Q}}.$$

Then the same proof as for Proposition 2.2 yields

Proposition 2.3. *Let Δ be a complete simplicial multi-fan. Assume that $\xi\in H_T^2(\Delta,\mathcal{V})$ is T-Cartier. Then*

$$\pi_*(e^{\xi}x_K\mathcal{T}_T(\Delta,\mathcal{V})_K)=\sum_{u\in A_K^*\cap M}\mathrm{DH}_{\mathcal{P}_K(\xi_{K+})}(u)e^u.$$

for $K\in\Sigma^{(k)}$, where $x_K=\prod_{i\in K}x_i$. Consequently

$$p_*(e^{\xi}x_K\mathcal{T}_T(\Delta,\mathcal{V})_K)=\#(\mathcal{P}(\xi)_K).$$

The lattice $M\cap A_K^*$ defines a volume element dV_K on A_K^*. For $\xi=\sum_{i\in K\cup\Sigma_K'^{(1)}}d_ix_i\in H_T^2(\Delta_K,\mathcal{V})_{\mathbb{R}}$, the volume $\operatorname{vol}\mathcal{P}_K(\xi)$ of $\mathcal{P}_K(\xi)$ is defined by

$$\operatorname{vol}\mathcal{P}_K(\xi)=\int_{A_K^*}\mathrm{DH}_{\mathcal{P}_K(\xi)}\,dV_K^*.$$

Proposition 2.4. *For $\xi=\sum_{i\in\Sigma^{(1)}}d_ix_i\in H_T^2(\Delta,\mathcal{V})_{\mathbb{R}}$*

$$\frac{1}{|H_{K,\mathcal{V}}|}\operatorname{vol}\mathcal{P}(\xi)_K=p_*(e^{\xi}x_K).$$

Proof. We shall give a proof only for the case where ξ is T-Cartier. The general case can be reduced to this case, cf. [7], Lemma 8.6. By Proposition 2.3

$$\#(\mathcal{P}(\xi)_K)=p_*(e^{\xi}x_K\mathcal{T}_T(\Delta,\mathcal{V})_K).$$

The highest degree term with respect to $\{d_i\}$ in the right hand side is nothing but $\operatorname{vol}\mathcal{P}(\xi)_K$ and is equal to

$$p_*\left(\frac{\xi^{n-k}}{(n-k)!}x_K\right)\sum_{g\in G_{\Delta_K}}\left(\prod_{i\in\Sigma_K'^{(1)}}\frac{x_i}{1-\chi_i(g)e^{-x_i}}\right)_0,$$

where the suffix 0 means taking 0-th degree term. But

$$\left(\prod_{i\in\Sigma_K'^{(1)}}\frac{x_i}{1-\chi_i(g)e^{-x_i}}\right)_0=\begin{cases}1 & \text{if} \quad g\in G_K,\\ 0 & \text{if} \quad g\notin G_K.\end{cases}$$

Hence

$$\operatorname{vol}\mathcal{P}(\xi)_K=|G_K|p_*\left(\frac{\xi^{n-k}}{(n-k)!}x_K\right)=|H_{K,\mathcal{V}}|p_*(e^{\xi}x_K). \qquad \square$$

3. Statement of main results

Assume that $1\le k$. For $J\in\Sigma^{(k)}$ let M_J be the annihilator of N_J and put $\omega_J=u_1\wedge\ldots\wedge u_{n-k}\in\bigwedge^{n-k}M\subset\bigwedge^{n-k}M_{\mathbb{Q}}$ where $\{u_1,\ldots,u_{n-k}\}$ is an oriented basis of M_J. Define $f^J(x_i)\in\bigwedge^{n-k+1}M_{\mathbb{Q}}$ by

$$f^J(x_i)=\iota_I^*(x_i)\wedge\omega_J \quad \text{with} \quad J\subset I\in\Sigma^{(n)}.$$

$f^J(x_i)$ is well-defined independently of I containing J. Let $S^*\left(\bigwedge^{n-k+1}M_{\mathbb{Q}}\right)$ be the symmetric algebra over $\bigwedge^{n-k+1}M_{\mathbb{Q}}$. $f^J\colon H_T^2(\Delta)_{\mathbb{Q}}\to\bigwedge^{n-k+1}M_{\mathbb{Q}}$ extends to $f^J\colon H_T^*(\Delta)_{\mathbb{Q}}\to S^*\left(\bigwedge^{n-k+1}M_{\mathbb{Q}}\right)$. For $x=\prod_i x_i^{\alpha_i}\in H_T^{2k}(\Delta)_{\mathbb{Q}}$ we put

$$f^J(x)=(f^J(x_i))^{\alpha_i}.$$

The definition of f^J depends on the orientations chosen, but $f^J(x)/f^J(x_J)$ does not. It belongs to the fraction field of the symmetric algebra $S^*\left(\bigwedge^{n-k+1}M_{\mathbb{Q}}\right)$ and has degree 0. Hence it can be considered as an element of $Rat\left(\mathbb{P}\left(\bigwedge^{n-k+1}N_{\mathbb{Q}}\right)\right)_0$, the field of rational functions of degree 0 on $\mathbb{P}\left(\bigwedge^{n-k+1}N_{\mathbb{Q}}\right)$. Let $\nu^*\colon Rat(\mathbb{P}\left(\bigwedge^{n-k+1}N_{\mathbb{Q}}\right))_0\to Rat(G_{n-k+1}(N_{\mathbb{Q}}))_0$ be the induced homomorphism of the Plücker embedding $\nu\colon G_{n-k+1}(N_{\mathbb{Q}})\to\mathbb{P}\left(\bigwedge^{n-k+1}N_{\mathbb{Q}}\right)$. The image $\nu^*(f^J(x)/f^J(x_J))$ will be denoted by $\mu(x,J)$.

Our first main result is stated in the following

Theorem 3.1. *Let Δ be a complete simplicial multi-fan and $x\in H_T^{2k}(\Delta,\mathcal{V})_{\mathbb{Q}}$. For any $\xi\in H_T^2(\Delta)_{\mathbb{Q}}$ we have*

$$p_*(e^{\xi}x)=\sum_{J\in\Sigma^{(k)}}\mu(x,J)p_*(e^{\xi}x_J) \quad \textit{in} \quad Rat(G_{n-k+1}(N_{\mathbb{Q}}))_0.$$

Corollary 3.2. *Let Δ be a complete simplicial multi-fan in a lattice of rank n. Assume that $\xi \in H_T^2(\Delta, \mathcal{V})$ is T-Cartier. Set*

$$\#(\mathcal{P}(\nu\xi)) = \sum_{k=0}^{n} a_k(\xi)\nu^{n-k}.$$

Then we have

$$a_k(\xi) = \sum_{J \in \Sigma^{(k)}} \mu_k(J) \operatorname{vol} \mathcal{P}(\xi)_J$$

with

$$\mu_k(J) = \frac{1}{|H_{J,\mathcal{V}}|} \nu^* \left(\sum_{h \in H_{J,\mathcal{V}}} \prod_{j \in J} \frac{1}{1 - \chi(u_j^J, h) e^{-f^J(x_j)}} \right)_0$$

in $Rat(G_{n-k+1}(N_{\mathbb{C}}))_0$.

NOTE. It can be proved without difficulty that $\mu_k(J)$ does not depend on the choice of $\mathcal{V}$. Hence one has only to consider the case where all the v_i are primitive.

For the following corollary we need to put an additional condition on the multi-fan Δ.

Corollary 3.3. *Let Δ be a multi-fan. Assume that there is a torus orbifold X such that Δ is isomorphic to Δ_X and $H^*(X)_{\mathbb{Q}}$ is generated by $H^2(X)_{\mathbb{Q}}$. Then for $x \in H_T^{2k}(\Delta)_{\mathbb{Q}}$ the following equality holds.*

$$\bar{x} = \sum_{J \in \Sigma^{(k)}} \mu(x, J)\bar{x}_J \quad in \quad Rat(G_{n-k+1}(N_{\mathbb{Q}}))_0 \otimes_{\mathbb{Q}} H^{2k}(\Delta)_{\mathbb{Q}},$$

where $\bar{x}$ is the image of $x \in H_T^*(\Delta)_{\mathbb{Q}}$ in $H^*(\Delta)_{\mathbb{Q}}$.

REMARK 3.1. If $H^*(X)_{\mathbb{Q}}$ is generated by $H^2(X)_{\mathbb{Q}}$, then $H_T^*(\Delta_X)_{\mathbb{Q}} = H_T^*(X)_{\mathbb{Q}}$. cf. [10], [11].

REMARK 3.2. When Δ is the fan associated to a convex lattice polytope P and $\xi = D$, the Cartier divisor associated to P, we know (see, e.g. [5]) that

$$\mu_0(o) = 1, \quad a_0(\xi) = \operatorname{vol} \mathcal{P}(\xi), \quad \mu_1(i) = \frac{1}{2}, \quad a_1(\xi) = \frac{1}{2} \sum_{i \in \Sigma^{(1)}} \operatorname{vol} \mathcal{P}(\xi)_i.$$

This is also true for simplicial multi-fans and T-Cartier ξ.

As to a_n we have

$$a_n(\xi) = \mathrm{Td}[\Delta].$$

In fact $a_n(\xi) = p_*(\mathcal{T}_T(\Delta, \mathcal{V})) = (\pi_*(\mathcal{T}_T(\Delta, \mathcal{V})))_0$. Thus the above equality follows from the following rigidity property:

Theorem 3.4. *Let Δ be a complete simplicial multi-fan. Then*

$$\pi_*(\mathcal{T}_T(\Delta, \mathcal{V})) = (\pi_*(\mathcal{T}_T(\Delta, \mathcal{V})))_0 = \mathrm{Td}[\Delta].$$

See [7] Theorem 7.2 and its proof. Note that $\mathrm{Td}[\Delta] = 1$ for any complete simplicial fan Δ.

The explicit formula for $\pi_*(\mathcal{T}_T(\Delta, \mathcal{V}))$ is given by

$$\pi_*(\mathcal{T}_T(\Delta, \mathcal{V})) = \sum_{I \in \Sigma^{(n)}} \frac{w(I)}{|H_{I,\mathcal{V}}|} \sum_{h \in H_{I,\mathcal{V}}} \prod_{i \in I} \frac{1}{1 - \chi_I(u_i^I, h) e^{-u_i^I}}.$$

This does not depend on the choice of $\mathcal{V}$ and is in fact equal to $\mathrm{Td}[\Delta]$.

Let Δ be a (not necessarily complete) simplicial fan in a lattice of rank n. Set

$$\mathrm{Td}_T(\Delta) = \sum_{I \in \Sigma^{(n)}} \frac{1}{|H_I|} \sum_{h \in H_I} \prod_{i \in I} \frac{1}{1 - \chi_I(u_i^I, h) e^{-u_i^I}} \in S^{-1} H_T^{**}(pt)_{\mathbb{Q}}.$$

For a simplex I let $\Sigma(I)$ be the simplicial complex consisting of all faces of I. For a fan $\Delta(I) = (\Sigma(I), C)$, $\mathrm{Td}_T(\Delta(I))$ is denoted by $\mathrm{Td}_T(I)$.

Theorem 3.5. $\mathrm{Td}_T(I)$ *is additive with respect to simplicial subdivisions of the cone $C(I)$. Namely, if Δ is the fan determined by a simplicial subdivison of $C(I)$, then the following equality holds*

$$\mathrm{Td}_T(\Delta) = \mathrm{Td}_T(I).$$

For the proof it is sufficient to assume that $\Delta(I)$ and Δ are non-singular. In such a form a proof is given in [12]. The following corollary ensures that $\mu_k(J)$ can be defined for general polyhedral cones as pointed out by Morelli in [12].

Corollary 3.6. *Let $\Delta(J) = (\Sigma(J), C)$ be a fan in a lattice N of rank n where J is a simplex of dimension $k-1$. Then $\mu_k(J) \in Rat(G_{n-k+1}(N_{\mathbb{Q}}))_0$ is additive with respect to simplicial subdivisions of $C(J)$.*

4. Proof of Theorem 3.1 and Corollary 3.3

Proof of Theorem 3.1.

For a primitive sublattice E of N of rank $n-k+1$ let $w_E \in \bigwedge^{n-k+1} N$ be a representative of $\nu(E) \in \mathbb{P}\left(\bigwedge^{n-k+1} N_{\mathbb{Q}}\right)$. The equality in Theorem 3.1 is equivalent to the condition that

$$p_*(e^{\xi}x) = \sum_{J \in \Sigma^{(k)}} \frac{f^J(x)}{f^J(x_J)}(w_E)p_*(e^{\xi}x_J) \quad \text{holds for every generic} \quad E.$$

Let E be a generic primitive sublattice in N of rank $n-k+1$. The intersection $E \cap N_J$ has rank one for each $J \in \Sigma^{(k)}$. Take a non-zero vector $v_{E,J}$ in $E \cap N_J$. (One can choose $v_{E,J}$ to be the unique primitive vector contained in $E \cap C(J)$. But any non-zero vector will suffice for the later use.) For $x \in H_T^{2k}(\Delta)$ and $J \in \Sigma^{(k)}$ the value of $\iota_I^*(x)$ evaluated on $v_{E,J}$ for $I \in \Sigma^{(n)}$ containing J depends only on $\iota_J^*(x)$ so that it will be denoted by $\iota_J^*(x)(v_{E,J})$. Similarly we shall simply write $\langle u_j^J, v_{E,J}\rangle$ instead of $\langle u_j^I, v_{E,J}\rangle$.

Lemma 4.1. *Put* $f_j^J = u_j^J \wedge \omega_J$. *Then*

$$a\langle f_j^J, w_E\rangle = \langle u_j^J, v_{E,J}\rangle,$$

where a is a non-zero constant depending only on $v_{E,J}$.

Proof. Take an oriented basis $u_1,\dots,u_{n-k}$ of M_J. Take also a basis $w_1,\dots,w_{n-k+1}$ of E and write $v_{E,J} = \sum_l c_l w_l$. Then, since $\langle u_i, v_{E,J}\rangle = 0$,

$$\sum_{l=1}^{n-k+1} c_l\langle u_i, w_l\rangle = 0, \quad \text{for} \quad i = 1, \dots, n-k.$$

The matrix $(a_{il}) = (\langle u_i, w_l\rangle)$ has rank $n-k$ and we get

$$(c_1, \dots, c_{n-k+1}) = a(A_1, \dots, A_{n-k+1}), \quad a \neq 0,$$

where

$$A_l = (-1)^{l-1} \det\begin{pmatrix} a_{11} & \cdots & \widehat{a_{1\,l}} & \cdots & a_{1\,n-k+1} \\ \vdots & & \vdots & & \vdots \\ a_{n-k\,1} & \cdots & \widehat{a_{n-k\,l}} & \cdots & a_{n-k\,n-k+1} \end{pmatrix}.$$

Then

$$\begin{aligned}\langle u_j^J, v_{E,J}\rangle &= \sum_{l=1}^{n-k+1} c_l \langle u_j^J, w_l\rangle \\ &= a \sum_{l=1}^{n-k+1} \langle u_j^J, w_l\rangle A_l \\ &= a \det \begin{pmatrix} \langle u_j^J, w_1\rangle & \cdots & \langle u_j^J, w_{n-k+1}\rangle \\ \langle u_1, w_1\rangle & \cdots & \langle u_1, w_{n-k+1}\rangle \\ \vdots & & \vdots \\ \langle u_{n-k}, w_1\rangle & \cdots & \langle u_{n-k}, w_{n-k+1}\rangle \end{pmatrix} \\ &= a\langle f_j^J, w_E\rangle\end{aligned}$$

where $f_j^J = u_j^J \wedge u_1 \wedge \cdots \wedge u_{n-k}$ and $w_E = w_1 \wedge \cdots \wedge w_{n-k+1}$. □

REMARK 4.1. Let X be a torus orbifold of dimension $2n$ and Δ the associated multifan. Let $T = T^n$ be the compact torus acting on X. $E \cap N_J$ determines a subcircle $T_{E,J}^1$ of T. Then $T_{E,J}^1$ pointwise fixes an invariant complex suborbifold X_J. Some of its covering acts on the normal vector space of an Euclidean covering of an invariant neighborhood at each generic point in X_J. Then the numbers $\langle u_j^J, v_{E,J}\rangle$ are weights of this action.

Lemma 4.1 implies that

$$\frac{f^J(x)}{f^J(x_J)}(w_E) = \frac{\iota_J^*(x)}{\prod_{j\in J} u_j^J}(v_{E,J}).$$

Then the equality in Theorem 3.1 holds if and only if

$$p_*(e^\xi x) = \sum_{J\in\Sigma^{(k)}} a \frac{\iota_J^*(x)}{\prod_{j\in J} u_j^J}(v_{E,J}) p_*(e^\xi x_J) \tag{12}$$

holds for every generic E.

The following lemma is easy to prove, cf. e.g. [7] Lemma 8.1.

Lemma 4.2. *The vector space $H_T^{2k}(\Delta)_{\mathbb{Q}}$ is spanned by elements of the form*

$$u_1 \cdots u_{k_1} x_{J_{k_1}}, \quad J_{k_1} \in \Sigma^{(k-k_1)}, \quad u_i \in M_{\mathbb{Q}},$$

with $0 \le k_1 \le k-1$.

NOTE. For $x = u_1 \cdots u_{k_1} x_{J_{k_1}}$, $J_{k_1} \in \Sigma^{(k-k_1)}$, with $k_1 \geq 1$,

$$p_*(e^{\xi} x) = 0.$$

In view of this lemma we proceed by induction on k_1 for $x = u_1 \cdots u_{k_1} x_{J_{k_1}}$.

For $x = x_{J_0}$ with $J_0 \in \Sigma^{(k)}$, the left hand side of (12) is equal to $p_*(e^{\xi} x_{J_0})$. Since $i_J^*(x) = 0$ unless $J = J_0$ and $i_J^*(x)/\prod_{j \in J} u_j^J = 1$ for $J = J_0$, the right hand side is also equal to $p_*(e^{\xi} x_{J_0})$. Hence (12) holds with x of the form $x = x_{J_0}$ for $J_0 \in \Sigma^{(k)}$.

Assuming that (12) holds for x of the form $u_1 \cdots u_{k_1} x_{J_{k_1}}$ with $J_{k_1} \in \Sigma^{(k-k_1)}$, we shall prove that it also holds for $x = u_1 \cdots u_{k_1} u_{k_1+1} x_{J_{k_1+1}}$ with $J_{k_1+1} \in \Sigma^{(k-(k_1+1))}$. Put $K = J_{k_1+1}$.

CASE a). u_{k_1+1} belongs to $M_{K\mathbb{Q}}$, that is, $\langle u_{k_1+1}, v_i \rangle = 0$ for all $i \in K$. In this case

$$u_{k_1+1} = \sum_{i \in \Sigma^{(1)} \setminus K} \langle u_{k_1+1}, v_i \rangle x_i$$

since $\langle u_{k_1+1}, v_i \rangle = 0$ for all $i \in K$. For $i \notin K$, $x_i x_{J_{k_1+1}}$ is either of the form x_{J^i} with $J^i \in \Sigma^{(k-k_1)}$ or equal to 0. Thus, for $x = u_1 \cdots u_{k_1} x_i x_{J_{k_1+1}}$ with $i \notin K$, the equality (12) holds by induction assumption, and it also holds for $x = u_1 \cdots u_{k_1} u_{k_1+1} x_{J_{k_1+1}}$ by linearity.

CASE b). General case. We need the following

Lemma 4.3. *For $K \in \Sigma^{(k-k_1)}$ with $k_1 \geq 1$, the composition homomorphism $M_{K\mathbb{Q}} \subset M_{\mathbb{Q}} \to E_{\mathbb{Q}}^*$ is surjective.*

The proof will be given later. By this lemma, there exists an element $u \in M_{K\mathbb{Q}}$ such that

$$\langle u_{k_1+1}, v_{E,J} \rangle = \langle u, v_{E,J} \rangle \quad \text{for all} \quad J \in \Sigma^{(k)}.$$

Note that $\langle \iota_J^*(u), v_{E,J} \rangle = \langle u, v_{E,J} \rangle$ for any $u \in M_{\mathbb{Q}}$. Then, in (12) for $x = u_1 \cdots u_{k_1} u_{k_1+1} x_{J_{k_1+1}}$ with $J_{k_1+1} \in \Sigma^{(k-(k_1+1))}$, we have

$$\begin{aligned} \iota_J^*(x)(v_{E,J}) &= \left(\prod_{i=1}^{k_1} \langle u_i, v_{E,J} \rangle \right) \langle u_{k_1+1}, v_{E,J} \rangle \\ &= \left(\prod_{i=1}^{k_1} \langle u_i, v_{E,J} \rangle \right) \langle u, v_{E,J} \rangle. \end{aligned}$$

Hence if we put $x' = u_1 \cdots u_{k_1} u x_{J_{k_1+1}}$, the right hand side of (12) is equal to

$$\sum_{J \in \Sigma^{(k)}} \frac{\iota_J^*(x')}{\prod_{j \in J} u_j^J} (v_{E,J}) p_*(e^{\xi} x_J).$$

This last expression is equal to $p_*(e^{\xi}x')$ since x' belongs to Case a). Furthermore $p_*(e^{\xi}x')=0$ and $p_*(e^{\xi}x)=0$ by Note after Lemma 4.2. Thus both side of (12) for $x=u_1\cdots u_{k_1}u_{k_1+1}x_{J_{k_1+1}}$ are equal to 0. This completes the proof of Theorem except for the proof of Lemma 4.3.

Proof of Lemma 4.3. Take a simplex $I\in\Sigma^{(n)}$ which contains K and a simplex $K'\in\Sigma^{(k-1)}$ such that $K\subset K'\subset I$. Such a K' exists since $k-k_1\leq k-1$. Then there are exactly $n-k+1$ simplices $J^1,\dots,J^{n-k+1}\in\Sigma^{(k)}$ such that $K'\subset J^i\subset I$. It is easy to see that the vectors $v_{E,J^1},\dots,v_{E,J^{n-k+1}}$ are linearly independent so that they span $E_{\mathbb{Q}}$. Moreover $M_{K'\mathbb{Q}}$ detects these vectors, that is, $M_{K'\mathbb{Q}}\to M_{\mathbb{Q}}\to E^*_{\mathbb{Q}}$ is surjective. Since $M'_K\subset M_K\subset M$, $M_{K\mathbb{Q}}\to E^*_{\mathbb{Q}}$ is surjective. □

Proof of Corollay 3.2. By Proposition 2.3

$$\#(\mathcal{P}(\nu\xi))=p_*(e^{\nu\xi}\mathcal{T}_T(\Delta,\mathcal{V}))=\sum_{k=0}^{n}a_k(\xi)\nu^{n-k}.$$

Put $x=(\mathcal{T}_T(\Delta,\mathcal{V}))_k\in H_T^{2k}(\Delta,\mathcal{V})_{\mathbb{Q}}$. By Theorem 3.1 and Proposition 2.4

$$a_k(\xi)=\sum_{J\in\Sigma^{(k)}}\nu^*\left(\frac{f^J(x)}{f^J(x_J)}\right)\frac{\operatorname{vol}\mathcal{P}(\xi)_J}{|H_{J,\mathcal{V}}|}.$$

Thus it suffices to show that

$$\frac{f^J(x)}{f^J(x_J)}=\left(\sum_{h\in H_{J,\mathcal{V}}}\prod_{j\in J}\frac{1}{1-\chi(u_j^J,h)e^{-f^J(x_j)}}\right)_0,$$

or

$$f^J(x)=\left(\sum_{h\in H_{J,\mathcal{V}}}\prod_{j\in J}\frac{f^J(x_j)}{1-\chi(u_j^J,h)e^{-f^J(x_j)}}\right)_k.$$

Let $g\in G_\Delta$. If $g\notin G_J$, then there is an element $i\notin J$ such that $\chi_i(g)\neq 1$, and, for such i,

$$f^J\left(\frac{x_i}{1-\chi_i(g)e^{-x_i}}\right)=f^J((1-\chi_i(g))^{-1}x_i+\text{higher degree terms})=0,$$

since $f^J(x_i)=0$. Thus

$$f^J\left(\prod_{i\in\Sigma^{(1)}}\frac{x_i}{1-\chi_i(g)e^{-x_i}}\right)=0$$

for $g \notin G_J$.

If $g \in G_J$, then $\chi_i(g) = 1$ for $i \notin J$. Thus

$$f^J\left(\frac{x_i}{1-\chi_i(g)e^{-x_i}}\right) = f^J\left(1+\frac{1}{2}x_i + \text{higher degree terms}\right) = 1$$

for $g \in G_J$, $i \notin J$. It follows that

$$f^J\left(\sum_{g\in G_\Delta}\prod_{i\in\Sigma^{(1)}}\frac{x_i}{1-\chi_i(g)e^{-x_i}}\right) = \sum_{g\in G_J}\prod_{i\in J}\frac{f^J(x_i)}{1-\chi_i(g)e^{-f^J(x_i)}}.$$

This implies

$$f^J(\mathcal{T}_T(\Delta,\mathcal{V})_k) = \left(\sum_{h\in H_{J,\mathcal{V}}}\prod_{j\in J}\frac{f^J(x_j)}{1-\chi_J(u_j^J,h)e^{-f^J(x_j)}}\right)_k.$$

This finishes the proof of Corollary 3.2. ☐

Proof of Corollary 3.3. Put $x' = \sum_{J\in\Sigma^{(k)}}\mu(x,J)x_J$. Then

$$p_*(e^\xi x') = \sum_{J\in\Sigma^{(k)}}\mu(x,J)p_*(e^\xi x_J) = p_*(e^\xi x)$$

by Theorem 3.1. It follows that $p_*(e^\xi(x'-x)) = 0$. Thus, in order to prove Corollary 3.3, it suffices to show that $p_*(e^\xi y) = 0$, $\forall\xi \in H_T^2(\Delta)_{\mathbb{Q}}$, implies that $p^*(y) = 0$. By the assumption Δ is isomorphic to Δ_X where X is a torus orbifold such that $H^*(X)_{\mathbb{Q}}$ is generated by $H^2(X)_{\mathbb{Q}}$. For such X we know that $H_T^*(\Delta)_{\mathbb{Q}} = H_T^*(X)_{\mathbb{Q}}$ and $H^*(\Delta)_{\mathbb{Q}} = H^*(X)_{\mathbb{Q}}$ by Remark 3.1. In particular $H^*(\Delta)_{\mathbb{Q}}$ satisfies the Poincaré duality. It follows that $p_*(e^\xi y) = 0$ for all ξ implies that $p_*(y) = 0$. ☐

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