REMARKS ON “THE DORFMEISTER–NEHER THEOREM ON ISOPARAMETRIC HYPERSURFACES”

REIKO MIYAOKA

(Received June 12, 2013, revised November 7, 2013)

Abstract
Sections 7 and 8 of “The Dorfmeister–Neher theorem on isoparametric hypersurfaces”, (Osaka J. Math. 46, 695–715) are the heart of the paper, but a lack of clear argument causes some questions, although the statement is true. The purpose of the present paper is to make it clear.

1. Dim $E = 2$ (§7 [2])

We follow the notation and the argument in [2]. First, we correct a typo in the last term of the displayed formula right above (35) of [2]: $(\Lambda_{63}^3)^2$ should be $(\Lambda_{63}^4)^2$.

We call a vector field $v(t)$ along $L_6$ parametrized by $p(t)$ even when $v(t + \pi) = v(t)$, and odd when $v(t + \pi) = -v(t)$. Note that $E$ consists of $\nabla_{e_6}^k e_3(t)$, $k = 0, 1, \ldots$ which are all odd or all even, and $W$ consists of $\nabla_{e_6}^k \nabla_{e_6} e_6(t)$ of which evenness and oddness is the opposite of $E$, since $L(t + \pi) = -L(t)$.

Proposition 7.1 ([2]) $\dim E = 2$ does not occur at any point of $M_+$. 

Proof. $\dim E = 2$ implies $\dim W = 1$, and so $W$ consists of even vectors ($\nabla_{e_6} e_6$ never vanish by Remark 5.3 of [2]). Thus $E$ consists of odd vectors. For $X_1$, $Z_1$, $X_2$, $Z_2$ on p. 709, $X_1$ is parallel to $\nabla_{e_6} e_3$ at $p_0 = p(0)$ and $p(\pi)$, and so has opposite sign at $p(0)$ and $p(\pi)$. Note that $Z_1 \in W$ is a constant unit vector parallel to $\nabla_{e_6} e_6(t)$. Also, $\text{span}\{X_2, Z_2\}$ is parallel since this is the orthogonal complement of $E \oplus W$. Because $D_1(\pi) = D_3(0)$ and $D_2(\pi) = D_4(0)$ etc. hold, four cases occur:

$$(e_1 + e_5)(\pi) = (e_1 + e_5)(0) \quad \text{and} \quad (e_2 + e_4)(\pi) = (e_2 + e_4)(0),$$

$$(e_1 + e_5)(\pi) = (e_1 + e_5)(0) \quad \text{and} \quad (e_2 + e_4)(\pi) = -(e_2 + e_4)(0),$$

$$(e_1 + e_5)(\pi) = -(e_1 + e_5)(0) \quad \text{and} \quad (e_2 + e_4)(\pi) = (e_2 + e_4)(0),$$

$$(e_1 + e_5)(\pi) = -(e_1 + e_5)(0) \quad \text{and} \quad (e_2 + e_4)(\pi) = -(e_2 + e_4)(0).$$

2000 Mathematics Subject Classification. 53C40.
In the first case, \( \alpha(\pi) = -\alpha(0) \) and \( \beta(\pi) = -\beta(0) \) follow. Then \( X_2 \) becomes even and \( Z_2 \) becomes odd, which contradicts that span\{\( X_2, Z_2 \)\} is parallel. In the second case, \( \alpha(\pi) = -\alpha(0) \) and \( \beta(\pi) = \beta(0) \) hold, and so \( X_2 \) is odd, and \( Z_2 \) is even, again a contradiction. Other cases are similar.

2. \( \text{Dim } E = 3 \) (§8 [2])

When \( \text{Dim } E = 3 \), \( e_3(t) \) is an even vector, since \( E \) is parallel along \( L_6 \). Using Proposition 8.1 [2], we extend \( e_1, e_2, e_4, e_5 \) as follows: Taking the double cover \( \tilde{c}(t) \) of \( c(t) \), i.e., \( t \in [0, 4\pi) \), if necessary, we choose a differentiable frame \( e_i(t) \) as follows: First take \( e_1(t), e_2(t) \) continuously for \( t \in [0, 4\pi) \). Then we define \( e_3(t) = e_1(t + \pi) \) and \( e_4(t) = e_2(t + \pi) \) for \( t \in [0, 3\pi) \). Thus we have a differentiable frame \( e_i(t) \) for \( t \in [0, 3\pi) \), though we only need \( t \in [0, 2\pi] \).

With respect to this frame, we can take a differentiable orthonormal frame of \( E \) and \( E^\perp \) by

\[
e_3(t), \quad X_1 = \alpha(t)(e_1 + e_3)(t) + \beta(t)(e_2 + e_4)(t),
\]

\[
X_2(t) = \frac{1}{\sqrt{\sigma(t)}} \left( \frac{\beta(t)}{\sqrt{3}} (e_1 - e_3)(t) - \sqrt{3} \alpha(t)(e_2 - e_4)(t) \right)
\]

and

\[
Z_1(t) = \frac{1}{\sqrt{\sigma(t)}} \left( \sqrt{3} \alpha(t)(e_1 - e_3)(t) + \frac{\beta(t)}{\sqrt{3}} (e_2 - e_4)(t) \right),
\]

\[
Z_2(t) = \beta(t)(e_1 + e_3) - \alpha(t)(e_2 + e_4)(t),
\]

where \( \alpha(t), \beta(t), \sigma(t) \) are differentiable for \( t \in [0, 3\pi] \), satisfying

\[
\alpha^2(t) + \beta^2(t) = \frac{1}{2}, \quad \sigma(t) = 2 \left( 3\alpha^2(t) + \frac{\beta^2(t)}{3} \right).
\]

Note that \( \sigma(t) = \sigma(t + \pi) \) holds, since \( \sigma(t) \) is an eigenvalue of \( T(t) = {}^tRR(t) \) (see (45) [2] and the statement after it).

**Proposition 8.2** ([2]) \( \sigma(t) \) is constant and takes values 1/3 or 3.

**Remark.** We need not distinguish the case \( \sigma = 1 \) in the proof.

Proof of Proposition 8.2 ([2]). From (3), the conclusion follows if we show \( \alpha(t)\beta(t) \equiv 0 \). Suppose \( \alpha(t)\beta(t) \neq 0 \). By definition, we have

\[
e_1(\pi) = e_5(0), \quad e_2(\pi) = e_4(0).
\]
We must be careful for
\[ e_5(\pi) = e_1(2\pi) = \epsilon_1 e_1(0), \quad e_4(\pi) = e_2(2\pi) = \epsilon_2 e_2(0), \]
where \( \epsilon_i = \pm 1 \). However, since \( e_3 \) is even and by (4), we obtain
\[ \epsilon := \epsilon_1 = \epsilon_2. \]

**Case 1** \( \epsilon = 1 \). In this case, we have
\[
X_1(\pi) = \alpha(\pi)(e_1(\pi) + e_5(\pi)) + \beta(\pi)(e_2(\pi) + e_4(\pi)) \\
= \alpha(\pi)(e_5(0) + e_1(0)) + \beta(\pi)(e_4(0) + e_2(0)),
\]
which belongs to \( E \), and is orthogonal to \( e_3(0) \) and \( X_2(0) \). Thus we obtain
\[
X_1(\pi) = \tilde{\epsilon} X_1(0), \quad \text{namely, } \alpha(\pi) = \tilde{\epsilon} \alpha(0), \quad \beta(\pi) = \tilde{\epsilon} \beta(0),
\]
where \( \tilde{\epsilon} = \pm 1 \). On the other hand, we have
\[
X_2(\pi) = \frac{1}{\sqrt{\sigma(\pi)}} \left( \frac{\beta(\pi)}{\sqrt{3}} (e_1(\pi) - e_3(\pi)) - \sqrt{3} \alpha(\pi)(e_2(\pi) - e_4(\pi)) \right) \\
= \frac{1}{\sqrt{\sigma(0)}} \left( \frac{\beta(\pi)}{\sqrt{3}} (e_5(0) - e_1(0)) - \sqrt{3} \alpha(\pi)(e_4(0) - e_2(0)) \right),
\]
where we use \( \sigma(\pi) = \sigma(0) \). Thus from (6), we obtain
\[ X_2(\pi) = -\tilde{\epsilon} X_2(0). \]
However, because \( E \) is parallel, \( X_1 \) and \( X_2 \) should be both even or both odd, a contradiction.

**Case 2** \( \epsilon = -1 \). In this case, we have
\[
X_1(\pi) = \alpha(\pi)(e_1(\pi) + e_5(\pi)) + \beta(\pi)(e_2(\pi) + e_4(\pi)) \\
= \alpha(\pi)(e_5(0) - e_1(0)) + \beta(\pi)(e_4(0) - e_2(0)),
\]
which belongs to \( E \), and is orthogonal to \( e_3(0) \) and \( X_1(0) \). Thus we obtain
\[
X_1(\pi) = \tilde{\epsilon} X_2(0), \quad \text{namely, } \alpha(\pi) = -\tilde{\epsilon} \frac{\beta(0)}{\sqrt{3} \sigma(0)}, \quad \text{and } \beta(\pi) = \tilde{\epsilon} \frac{\sqrt{3} \alpha(0)}{\sqrt{\sigma(0)}},
\]
for \( \tilde{\epsilon} = \pm 1 \). On the other hand, we see that
\[
X_2(\pi) = \frac{1}{\sqrt{\sigma(\pi)}} \left( \frac{\beta(\pi)}{\sqrt{3}} (e_1(\pi) - e_3(\pi)) - \sqrt{3} \alpha(\pi)(e_2(\pi) - e_4(\pi)) \right) \\
= \frac{1}{\sqrt{\sigma(0)}} \left( \frac{\beta(\pi)}{\sqrt{3}} (e_5(0) + e_1(0)) - \sqrt{3} \alpha(\pi)(e_4(0) + e_2(0)) \right)
\]
where we use $\sigma(\pi) = \sigma(0)$. Because it belongs to $E$ and is orthogonal to $e_3(0)$ and $X_2(0)$, and further because $(X_1(0), X_2(0)) \leftrightarrow (X_1(\pi), X_2(\pi))$ should be orientation preserving, we obtain,

\[(11) \, X_2(\pi) = -\tilde{e}_1 X_1(0), \text{ namely, } \frac{\beta(\pi)}{\sqrt{3} \sigma(0)} = -\tilde{e}_1 \alpha(0) \text{ and } \frac{\sqrt{3} \alpha(\pi)}{\sqrt{\sigma(0)}} = -\tilde{e}_1 \beta(0).\]

However, then (9) and (11) have no solution.

These contradictions are caused by the assumption $\alpha(t) \beta(t) \neq 0$. Thus $\alpha(t) \beta(t) \equiv 0$ follows. Now, by the argument in §9 [2], we obtain

**Theorem 2.1** ([1], [2]) *Isoparametric hypersurfaces with $(g, m) = (6, 1)$ are homogeneous.*

ACKNOWLEDGMENT. The author thanks to U. Abresch and A. Siffert for their questions and indication.

References


Mathematical Institute
Graduate School of Sciences
Tohoku University
Sendai, 980-8578
Japan
e-mail: r-miyaok@m.tohoku.ac.jp