INERTIA GROUPS AND SMOOTH STRUCTURES
OF \((n-1)\)-CONNECTED \(2n\)-MANIFOLDS

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Abstract

Let \(M^{2n}\) denote a closed \((n-1)\)-connected smoothable topological \(2n\)-manifold. We show that the group \(C(M^{2n})\) of concordance classes of smoothings of \(M^{2n}\) is isomorphic to the group of smooth homotopy spheres \(\tilde{\Theta}_{2n}\) for \(n = 4\) or 5, the concordance inertia group \(I_c(M^{2n}) = 0\) for \(n = 3, 4, 5\) or 11 and the homotopy inertia group \(I_h(M^{2n}) = 0\) for \(n = 4\). On the way, following Wall’s approach [16] we present a new proof of the main result in [9], namely, for \(n = 4, 8\) and \(H^n(M^{2n}; \mathbb{Z}) \cong \mathbb{Z}\), the inertia group \(I(M^{2n}) \cong \mathbb{Z}_2\). We also show that, up to orientation-preserving diffeomorphism, \(M^8\) has at most two distinct smooth structures; \(M^{10}\) has exactly six distinct smooth structures and then show that if \(M^{14}\) is a \(\pi\)-manifold, \(M^{14}\) has exactly two distinct smooth structures.

1. Introduction

We work in the categories of closed, oriented, simply-connected \(Cat\)-manifolds \(M\) and \(N\) and orientation preserving maps, where \(Cat = Diff\) for smooth manifolds or \(Cat = Top\) for topological manifolds. Let \(\tilde{\Theta}_m\) be the group of smooth homotopy spheres defined by M. Kervaire and J. Milnor in [6]. Recall that the collection of homotopy spheres \(\Sigma\) which admit a diffeomorphism \(M \to M \# \Sigma\) form a subgroup \(I(M)\) of \(\tilde{\Theta}_m\), called the inertia group of \(M\), where we regard the connected sum \(M \# \Sigma^m\) as a smooth manifold with the same underlying topological space as \(M\) and with smooth structure differing from that of \(M\) only on an \(m\)-disc. The homotopy inertia group \(I_h(M)\) of \(M^m\) is a subset of the inertia group consisting of homotopy spheres \(\Sigma\) for which the identity map \(M \to M \# \Sigma^m\) is homotopic to a diffeomorphism. Similarly, the concordance inertia group of \(M^m\), \(I_c(M^m) \subseteq \tilde{\Theta}_m\), consists of those homotopy spheres \(\Sigma^m\) such that \(M\) and \(M \# \Sigma^m\) are concordant.

The paper is organized as following. Let \(M^{2n}\) denote a closed \((n-1)\)-connected smoothable topological \(2n\)-manifold. In Section 2, we show that the group \(C(M^{2n})\) of concordance classes of smoothings of \(M^{2n}\) is isomorphic to the group of smooth homotopy spheres \(\tilde{\Theta}_{2n}\) for \(n = 4\) or 5, the concordance inertia group \(I_c(M^{2n}) = 0\) for \(n = 3, 4, 5\) or 11 and the homotopy inertia group \(I_h(M^{2n}) = 0\) for \(n = 4\).

In Section 3, we present a new proof of the following result in [9].

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Theorem 1.1. Let $M^{2n}$ be an $(n - 1)$-connected closed smooth manifold of dimension $2n \neq 4$ such that $H^n(M; \mathbb{Z}) \cong \mathbb{Z}$. Then the inertia group $I(M^{2n}) \cong \mathbb{Z}_2$.

In Section 4, we show that, up to orientation-preserving diffeomorphism, $M^8$ has at most two distinct smooth structures; $M^{10}$ has exactly six distinct smooth structures and if $M^{14}$ is a $\pi$-manifold, then $M^{14}$ has exactly two distinct smooth structures.

2. Concordance inertia groups of $(n - 1)$-connected $2n$-manifolds

We recall some terminology from [6]:

**Definition 2.1.** (a) A homotopy $m$-sphere $\Sigma^m$ is a closed oriented smooth manifold homotopy equivalent to the standard unit sphere $S^m$ in $\mathbb{R}^{m+1}$.

(b) A homotopy $m$-sphere $\Sigma^m$ is said to be exotic if it is not diffeomorphic to $S^m$.

**Definition 2.2.** Define the $m$-th group of smooth homotopy spheres $\Theta_m$ as follows. Elements are oriented $h$-cobordism classes $[\Sigma]$ of homotopy $m$-spheres $\Sigma$, where $\Sigma$ and $\Sigma'$ are called (oriented) $h$-cobordant if there is an oriented $h$-cobordism $(W, \partial_0 W, \partial_1 W)$ together with orientation preserving diffeomorphisms $\Sigma \to \partial_0 W$ and $(\Sigma')^- \to \partial_1 W$. The addition is given by the connected sum. The zero element is represented by $S^m$. The inverse of $[\Sigma]$ is given by $[\Sigma^-]$, where $\Sigma^-$ is obtained from $\Sigma$ by reversing the orientation. M. Kervaire and J. Milnor [6] showed that each $\Theta_m$ is a finite abelian group ($m \geq 1$).

**Definition 2.3.** Two homotopy $m$-spheres $\Sigma_1^m$ and $\Sigma_2^m$ are said to be equivalent if there exists an orientation preserving diffeomorphism $f : \Sigma_1^m \to \Sigma_2^m$.

The set of equivalence classes of homotopy $m$-spheres is denoted by $\bar{\Theta}_m$. The Kervaire–Milnor [6] paper worked rather with the group $\Theta_m$ of smooth homotopy spheres up to $h$-cobordism. This makes a difference only for $m = 4$, since it is known, using the $h$-cobordism theorem of Smale [12], that $\Theta_m \cong \bar{\Theta}_m$ for $m \neq 4$. However the difference is important in the four dimensional case, since $\Theta_4$ is trivial, while the structure of $\bar{\Theta}_4$ is a great unsolved problem.

**Definition 2.4.** Let $M$ be a closed topological manifold. Let $(N, f)$ be a pair consisting of a smooth manifold $N$ together with a homeomorphism $f : N \to M$. Two such pairs $(N_1, f_1)$ and $(N_2, f_2)$ are concordant provided there exists a diffeomorphism $g : N_1 \to N_2$ such that the composition $f_2 \circ g$ is topologically concordant to $f_1$, i.e., there exists a homeomorphism $F : N_1 \times [0, 1] \to M \times [0, 1]$ such that $F|_{N_1 \times 0} = f_1$ and $F|_{N_1 \times 1} = f_2 \circ g$. The set of all such concordance classes is denoted by $C(M)$.

We will denote the class in $C(M)$ of $(M^m \# \Sigma^m, \text{id})$ by $[M^m \# \Sigma^m]$. (Note that $[M^m \# S^m]$ is the class of $(M^m, \text{id})$.)
Definition 2.5. Let $M^m$ be a closed smooth $m$-dimensional manifold. The inertia group $I(M) \subset \tilde{\Theta}_m$ is defined as the set of $\Sigma \in \tilde{\Theta}_m$ for which there exists a diffeomorphism $\phi : M \to M \# \Sigma$.

Define the homotopy inertia group $I_h(M)$ to be the set of all $\Sigma \in I(M)$ such that there exists a diffeomorphism $M \to M \# \Sigma$ which is homotopic to id; $M \to M \# \Sigma$.

Define the concordance inertia group $I_c(M)$ to be the set of all $\Sigma \in I_h(M)$ such that $M \# \Sigma$ is concordant to $M$.

Remark 2.6. (1) Clearly, $I_c(M) \subseteq I_h(M) \subseteq I(M)$.
(2) For $M = S^m$, $I_c(M) = I_h(M) = I(M) = 0$.

Now we have the following:

Theorem 2.7. Let $M^{2n}$ be a closed smooth $(n-1)$-connected $2n$-manifold with $n \geq 3$.
(i) If $n$ is any integer such that $\Theta_{n+1}$ is trivial, then $I_c(M^{2n}) = 0$.
(ii) If $n$ is any integer greater than 3 such that $\Theta_n$ and $\Theta_{n+1}$ are trivial, then
\[ C(M^{2n}) = \{[M^{2n} \# \Sigma] \mid \Sigma \in \tilde{\Theta}_{2n}\} \cong \tilde{\Theta}_{2n}. \]
(iii) If $n = 8$ and $H^n(M; \mathbb{Z}) \cong \mathbb{Z}$, then $M^{2n} \# \Sigma^{2n}$ is not concordant to $M^{2n}$, where $\Sigma^{2n} \in \tilde{\Theta}_{2n}$ is the exotic sphere. In particular, $C(M^{2n})$ has at least two elements.
(iv) If $n$ is any even integer such that $\Theta_n$ and $\Theta_{n+1}$ are trivial, then $I_h(M) = 0$.

Proof. Let $Cat = Top$ or $G$, where $Top$ and $G$ are the stable spaces of self homeomorphisms of $\mathbb{R}^n$ and self homotopy equivalences of $S^{n-1}$ respectively. For any degree one map $f_M : M \to S^{2n}$, we have a homomorphism
\[ f_M^* : [S^{2n}, Cat/O] \to [M, Cat/O]. \]

By Wall [15], $M$ has the homotopy type of $X = \left( \bigvee_{i=1}^k S^n_i \right) \cup_{g} D^{2n}$, where $k$ is the $n$-th Betti number of $M$, $\bigvee_{i=1}^k S^n_i$ is the wedge sum of $n$-spheres which is the $n$-skeleton of $M$ and $g : S^{2n-1} \to \bigvee_{i=1}^k S^n_i$ is the attaching map of $D^{2n}$. Let $\phi : M \to X$ be a homotopy equivalence of degree one and $q : X \to S^{2n}$ be the collapsing map obtained by identifying $S^{2n}$ with $X/\bigvee_{i=1}^k S^n_i$ in an orientation preserving way. Let $f_M = q \circ \phi : M \to S^{2n}$ be the degree one map.

Consider the following Puppe’s exact sequence for the inclusion $i : \bigvee_{i=1}^k S^n_i \hookrightarrow X$ along $Cat/O$:

\[ \cdots \to \left[ \bigvee_{i=1}^k S^n_i, Cat/O \right] \xrightarrow{(S(g))^*} [S^{2n}, Cat/O] \xrightarrow{q^*} [X, Cat/O] \xrightarrow{i^*} \left[ \bigvee_{i=1}^k S^n_i, Cat/O \right]. \]
where $S(g)$ is the suspension of the map $g: S^{2n-1} \to \bigvee_{i=1}^k S_i^n$.

Using the fact that
\[
\bigvee_{i=1}^k S_i^n, \text{Cat/O} \cong \prod_{i=1}^k [S_i^{n+1}, \text{Cat/O}]
\]
and
\[
\bigvee_{i=1}^k S_i^n, \text{Cat/O} \cong \prod_{i=1}^k [S_i^n, \text{Cat/O}],
\]
the above exact sequence (2.1) becomes
\[
\cdots \to \prod_{i=1}^k [S_i^{n+1}, \text{Cat/O}] \xrightarrow{(S(g))^*} [S^{2n}, \text{Cat/O}] \xrightarrow{q^*} [X, \text{Cat/O}] \to \prod_{i=1}^k [S_i^n, \text{Cat/O}].
\]

(i): If $n$ is any integer such that $\Theta_{n+1}$ is trivial and $\text{Cat} = \text{Top}$ in the above exact sequence (2.1), by using the fact that
\[
[S^m, \text{Top/O}] = \tilde{\Theta}_m \quad (m \neq 3, 4)
\]
and $[S^4, \text{Top/O}] = 0$ ([10, pp. 200–201]), we have $q^*: [S^{2n}, \text{Top/O}] \to [X, \text{Top/O}]$ is injective. Hence $f_M^* = \phi^* \circ q^*: \tilde{\Theta}_{2n} \to [M, \text{Top/O}]$ is injective. By using the identifications $\mathcal{C}(M^{2n}) = [M, \text{Top/O}]$ given by [10, pp. 194–196], $f_M^*: \tilde{\Theta}_{2n} \to \mathcal{C}(M^{2n})$ becomes $[\Sigma^{2n}] \to [M \# \Sigma^{2n}]$. $I_c(M)$ is exactly the kernel of $f_M^*$, and so $I_c(M) = 0$. This proves (i).

(ii): If $n > 3$, $\Theta_n$ and $\Theta_{n+1}$ are trivial, and $\text{Cat} = \text{Top}$ then, from the above exact sequence (2.1) we have $q^*: [S^{2n}, \text{Top/O}] \to [X, \text{Top/O}]$ is an isomorphism. This shows that $f_M^* = \phi^* \circ q^*: \tilde{\Theta}_{2n} \to \mathcal{C}(M^{2n})$ is an isomorphism and hence
\[
\mathcal{C}(M^{2n}) = \{[M^{2n} \# \Sigma] | \Sigma \in \tilde{\Theta}_{2n}\}.
\]
This proves (ii).

(iii): If $n = 8$ and $H^n(M; \mathbb{Z}) \cong \mathbb{Z}$, then $M^{2n}$ has the homotopy type of $X = S^n \cup_g D^{2n}$, where $g: S^{2n-1} \to S^n$ is the attaching map. In order to prove $M^{2n} \# \Sigma^{2n}$ is not concordant to $M^{2n}$, by the above exact sequence (2.1) for $\text{Cat} = \text{Top}$, it suffices to prove $q^*: [S^{16}, \text{Top/O}] \to [X, \text{Top/O}]$ is monic, which is equivalent to saying that $(S(g))^*: [SS^8, \text{Top/O}] \to [S^{16}, \text{Top/O}]$ is the zero homomorphism. For the case $g = p$, where $p: S^{15} \to S^8$ is the Hopf map, $(S(g))^*$ is the zero homomorphism, which was proved in the course of the proof of Lemma 1 in [2, pp. 58–59]. This proof works verbatim for any map $g: S^{2n-1} \to S^n$ as well. This proves (iii).

(iv): If $n$ is any even integer such that $\Theta_n$ and $\Theta_{n+1}$ are trivial, then $\pi_{n+1}(G/O) = 0$. This shows that from the above exact sequence (2.1) for $\text{Cat} = G$, $q^*: [S^{2n}, G/O] \to [X, G/O]$ is injective. Then $f_M^* = \phi^* \circ q^*: [S^{2n}, G/O] \to [M, G/O]$ is injective. From
the surgery exact sequences of $M$ and $S^{2n}$, we get the following commutative diagram ([3, Lemma 3.4]):

$\begin{align*}
L_{2n+1}(e) &\longrightarrow \Theta_{2n} \quad \eta_{2n} \quad \pi_{2n}(G/O) \quad \longrightarrow \quad L_{2n}(e) \\
\downarrow f_M &\quad \downarrow f_M \quad \downarrow f_M \\
L_{2n+1}(e) &\longrightarrow S^{\text{Diff}}(M) \quad \eta_M \quad [M, G/O] \quad \longrightarrow \quad L_{2n}(e)
\end{align*}$

(2.2)

By using the facts that $L_{2n+1}(e) = 0$, injectivity of $\eta_{\Theta^{2n}}$ and $\eta_M$ follow from the diagram, and combine with the injectivity of $f_M^*$ to show that $f_M^*: \Theta_{2n} \to S^{\text{Diff}}(M)$ is injective. $I_h(M)$ is exactly the kernel of $f_M^*$, and so $I_h(M) = 0$. This proves (iv).

REMARK 2.8. (i) By M. Kervaire and J. Milnor [6], $\Theta_m = 0$ for $m = 1, 2, 3, 4, 5, 6$ or 12. If $M^{2n}$ is a closed smooth $(n-1)$-connected $2n$-manifold, by Theorem 2.7 (i) and (ii), $I_c(M^{2n}) = 0$ for $n = 3, 4, 5$ or 11 and $C(M^{2n}) \cong \hat{\Theta}_{2n}$ for $n = 4$ or 5.

(ii) If $M$ has the homotopy type of $\bigoplus \mathbb{P}^2$, by Theorem 1.1 and Theorem 2.7 (iii), we have $I_c(M) = 0 \not= I(M)$.

(iii) By Theorem 2.7 (iv), if $M$ has the homotopy type of $\mathbb{HP}^2$, then $I_h(M) = 0$.

DEFINITION 2.9. Let $M$ and $N$ are smooth manifolds. A smooth map $f: M \to N$ is called tangential if for some integers $k$, $l$, $f^*(T(N)) \oplus \epsilon_k^l \cong T(M) \oplus \epsilon_M^I$.

DEFINITION 2.10. Let $M$ be a topological manifold. Let $(N, f)$ be a pair consisting of a smooth manifold $N$ together with a tangential homotopy equivalence of degree one $f: N \to M$. Two such pairs $(N_1, f_1)$ and $(N_2, f_2)$ are equivalent provided there exists a diffeomorphism $g: N_1 \to N_2$ such that $f_2 \circ g$ is homotopic to $f_1$. The set of all such equivalence classes is denoted by $\theta(M)$.

For $M = \mathbb{HP}^2$, [5, Theorem 4] shows $\theta(\mathbb{HP}^2)$ contains at most two elements. Now by Remark 2.8 (iii), we have the following:

Corollary 2.11. $\theta(\mathbb{HP}^2)$ contains exactly two elements, with representatives given by $(\mathbb{HP}^2, \text{id})$ and $(\mathbb{HP}^2 \# \Sigma^8, \text{id})$, where $\Sigma^8$ is the exotic 8-sphere.

3. Inertia groups of projective plane-like manifolds

In [15], C.T.C. Wall assigned to each closed oriented $(n-1)$-connected $2n$-dimensional smooth manifold $M^{2n}$ with $n \geq 3$, a system of invariants as follows:

1. $H = H^n(M; \mathbb{Z}) \cong \text{Hom}(H_n(M; \mathbb{Z}), \mathbb{Z}) \cong \bigoplus_{j=1}^k \mathbb{Z}$, the cohomology group of $M$, with $k$ the $n$-th Betti number of $M$.

2. $I: H \times H \to \mathbb{Z}$, the intersection form of $M$ which is unimodular and $n$-symmetric, defined by

$$I(x, y) = \langle x \cup y, [M] \rangle,$$
where the homology class \([M]\) is the orientation class of \(M\).

(3) A map \(\alpha: H^n(M; \mathbb{Z}) \to \pi_{n-1}(SO_n)\) that assigns each element \(x \in H^n(M; \mathbb{Z})\) to the characteristic map \(\alpha(x)\) for the normal bundle of the embedded \(n\)-sphere \(S^n\) representing \(x\).

Denote by \(\chi = S \circ \alpha : H^n(M; \mathbb{Z}) \to \pi_{n-1}(SO_{n+1}) \cong \text{Hom}(H^n(M; \mathbb{Z}); \mathbb{K})\), where \(S: \pi_{n-1}(SO_n) \to \pi_{n-1}(SO_{n+1})\) is the suspension map. Then

\[
\chi = S \circ \alpha \in H^n(M; \mathbb{K}) = \text{Hom}(H^n(M; \mathbb{Z}); \mathbb{K})
\]

can be viewed as an \(n\)-dimensional cohomology class of \(M\), with coefficients in \(\mathbb{K}\).

The obstruction to triviality of the tangent bundle over the \(n\)-sphere consists of homotopy \(n\)-connected, for \(n \geq 3\). The characteristic map \(x\) is the orientation class of \(S^n\) to its cobordism class. Using surgery, we see \(\Omega_n(1)\) is the usual oriented cobordism group. So \(\Theta_n = \Omega_n(1)\). Similarly, \(\Omega_{n}(2) \cong \Omega_{n}^{\text{Spin}}(n \geq 7)\); since \(B\text{Spin}\) is, in fact, \(3\)-connected, for \(n \geq 8\), \(\Omega_{n}(2) \cong \Omega_{n}(3)\) and \(\Theta_{n}(2) = \Theta_{n}(3) = b\text{Spin}_n\). Here \(b\text{Spin}_n\) consists of homotopy \(n\)-sphere which bound spin manifolds.

In [16], C.T.C. Wall defined the Grothendieck group \(G_{n+1}^{2n}\), a homomorphism \(\vartheta: G_{n+1}^{2n} \to \hat{\Theta}_{2n}\) such that \(\vartheta(G_{n+1}^{2n}) = \Theta_{2n}(n-1)\) and proved the following theorem:

**Theorem 3.1** (Wall). Let \(M^{2n}\) be a closed smooth \((n-1)\)-connected \(2n\)-manifold and \(\Sigma^{2n}\) be a homotopy sphere in \(\hat{\Theta}_{2n}\). Then \(M \# \Sigma^{2n}\) is an orientation-preserving diffeomorphic to \(M\) if and only if

(i) \(\Sigma^{2n} = 0\) in \(\hat{\Theta}_{2n}\) or

(ii) \(\chi \not\equiv 0 \pmod{2}\) and \(\Sigma^{2n} \in \vartheta(G_{n+1}^{2n}) = \Theta_{2n}(n-1)\)

We also need the following result from [1]:

**Theorem 3.2** (Anderson, Brown, Peterson). Let \(\eta_{n}: \hat{\Theta}_{n} \to \Omega_{n}^{\text{Spin}}\) be the homomorphism such that \(\eta_{n}\) sends \(\Sigma^{n}\) to its spin cobordism class. Then \(\eta_{n} \neq 0\) if and only if \(n = 8k + 1\) or \(8k + 2\).
Proof of Theorem 1.1. Let $\xi$ be a generator of $H^n(M^{2n}; \mathbb{Z})$. Consider the case $n = 4$. Then by Itiro Tamura [14] and (3.1), the Pontrjagin class of $M^{2n}$ is given by

$$p_1(M^{2n}) = 2(2h + 1)\xi = \pm 2\chi,$$

where $h \in \mathbb{Z}$. This implies that

$$\chi = \pm (2h + 1)\xi.$$

Likewise, for $n = 8$, we have

$$p_2(M^{2n}) = 6(2k + 1)\xi = \pm 6\chi,$$

where $k \in \mathbb{Z}$. This implies that

$$\chi = \pm (2k + 1)\xi.$$

Therefore in either case, $\chi \not\equiv 0 \mod 2$. Now by Theorem 3.1, it follows that

$$I(M^{2n}) = \Theta_{2n}(n - 1).$$

Since $\Theta_{2n}(n - 1)$ is the kernel of the natural map $i_{n-1}: \tilde{\Theta}_{2n} \to \Omega_{2n}(n - 1)$, where $\Omega_{2n}(n - 1) \cong \Omega_{8}^{\text{Spin}}$ for $n = 4$ and $\Omega_{2n}(n - 1) \cong \Omega_{16}^{\text{String}} \cong \mathbb{Z} \oplus \mathbb{Z}$ for $n = 8$ [4]. Now by Theorem 3.2 and using the fact that $\tilde{\Theta}_{16} \cong \mathbb{Z}_2$ [6], we have $i_{n-1} = 0$ for $n = 4$ and 8. This shows that $\Theta_{2n}(n - 1) = \tilde{\Theta}_{2n}$. This implies that

$$I(M^{2n}) \cong \mathbb{Z}_2.$$

This completes the proof of Theorem 1.1.

4. Smooth structures of $(n - 1)$-connected $2n$-manifolds

Definition 4.1 ($\text{Cat} = \text{Diff}$ or $\text{Top}$-structure sets, [3]). Let $M$ be a closed $\text{Cat}$-manifold. We define the $\text{Cat}$-structure set $S^\text{Cat}(M)$ to be the set of equivalence classes of pairs $(N, f)$ where $N$ is a closed $\text{Cat}$-manifold and $f: N \to M$ is a homotopy equivalence. And the equivalence relation is defined as follows:

$$(N_1, f_1) \sim (N_2, f_2)$$

if there is a $\text{Cat}$-isomorphism $\phi: N_1 \to N_2$ such that $f_2 \circ h$ is homotopic to $f_1$.

We will denote the class in $S^\text{Cat}(M)$ of $(N, f)$ by $[(N, f)]$. The base point of $S^\text{Cat}(M)$ is the equivalence class $[(M, \text{id})]$ of id: $M \to M$.

The forgetful maps $F^\text{Diff}: S^\text{Diff}(M) \to S^\text{Top}(M)$ and $F^\text{Con}: C(M) \to S^\text{Diff}(M)$ fit into a short exact sequence of pointed sets [3]:

$$C(M) \xrightarrow{F^\text{Con}} S^\text{Diff}(M) \xrightarrow{F^\text{Diff}} S^\text{Top}(M).$$
Theorem 4.2. Let $n$ be any integer greater than 3 such that $\Theta_n$ and $\Theta_{n+1}$ are trivial and $M^{2n}$ be a closed smooth $(n-1)$-connected $2n$-manifold. Let $f : N \to M$ be a homeomorphism where $N$ is a closed smooth manifold. Then

(i) there exists a diffeomorphism $\phi : N \to M \# \Sigma^{2n}$, where $\Sigma^{2n} \in \tilde{\Theta}_{2n}$ such that the following diagram commutes up to homotopy:

$$
\begin{array}{ccc}
N & \xrightarrow{\phi} & M \# \Sigma^{2n} \\
\downarrow{f} & & \downarrow{\text{id}} \\
M & & M
\end{array}
$$

(ii) If $I_0(M) = \tilde{\Theta}_{2n}$, then $f : N \to M$ is homotopic to a diffeomorphism.

Proof. Consider the short exact sequence of pointed sets

$$
\mathcal{C}(M) \xrightarrow{F_{Con}} S^{Diff}(M) \xrightarrow{F_{Diff}} S^{Top}(M).
$$

By Theorem 2.7 (ii), we have

$$
\mathcal{C}(M) = \{[M \# \Sigma] \mid \Sigma \in \tilde{\Theta}_{2n} \} \cong \tilde{\Theta}_{2n}.
$$

Since $[(N, f)] \in F^{-1}_{Diff}([(M, \text{id})])$, we obtain

$$
[(N, f)] \in \text{Im}(F_{\text{Con}}) = \{[M \# \Sigma] \mid \Sigma \in \tilde{\Theta}_{2n} \}.
$$

This implies that there exists a homotopy sphere $\Sigma^{2n} \in \tilde{\Theta}_{2n}$ such that $(N, f) \sim (M \# \Sigma^{2n}, \text{id})$ in $S^{Diff}(M)$. This implies that there exists a diffeomorphism $\phi : N \to M \# \Sigma^{2n}$ such that $f$ is homotopic to $\text{id} \circ \phi$. This proves (i).

If $I_0(M) = \tilde{\Theta}_{2n}$, then $\text{Im}(F_{\text{Con}}) = \{([M, \text{id})] \}$ and hence $(N, f) \sim (M, \text{id})$ in $S^{Diff}(M)$. This shows that $f : N \to M$ is homotopic to a diffeomorphism $N \to M$. This proves (ii).

Theorem 4.3. Let $n$ be any integer greater than 3 such that $\Theta_n$ and $\Theta_{n+1}$ are trivial and $M^{2n}$ be a closed smooth $(n-1)$-connected $2n$-manifold. Then the number of distinct smooth structures on $M^{2n}$ up to diffeomorphism is less than or equal to the cardinality of $\tilde{\Theta}_{2n}$. In particular, the set of diffeomorphism classes of smooth structures on $M^{2n}$ is $\{[M \# \Sigma] \mid \Sigma \in \tilde{\Theta}_{2n} \}$.

Proof. By Theorem 4.2 (i), if $N$ is a closed smooth manifold homeomorphic to $M$, then $N$ is diffeomorphic to $M \# \Sigma^{2n}$ for some homotopy $2n$-sphere $\Sigma^{2n}$. This implies that the set of diffeomorphism classes of smooth structures on $M^{2n}$ is $\{[M \# \Sigma] \mid \Sigma \in \tilde{\Theta}_{2n} \}$. This shows that the number of distinct smooth structures on $M^{2n}$ up to diffeomorphism is less than or equal to the cardinality of $\tilde{\Theta}_{2n}$.
Remark 4.4. (1) By Theorem 4.3, every closed smooth 3-connected 8-manifold has at most two distinct smooth structures up to diffeomorphism.
(2) If $M^8$ is a closed smooth 3-connected 8-manifold such that $H^4(M; \mathbb{Z}) \cong \mathbb{Z}$, then by Theorem 1.1, $I(M) \cong \mathbb{Z}_2$. Now by Theorem 4.3, $M$ has a unique smooth structure up to diffeomorphism.
(3) If $M = S^4 \times S^4$, then by Theorem 4.3, $S^4 \times S^4$ has at most two distinct smooth structures up to diffeomorphism, namely, $\{([S^4 \times S^4], [S^4 \times S^4 \# \Sigma])\}$, where $\Sigma$ is the exotic 8-sphere. However, by [11, Theorem A], $I(S^4 \times S^4) = 0$. This implies that $S^4 \times S^4$ has exactly two distinct smooth structures.

Theorem 4.5. Let $M$ be a closed smooth 3-connected 8-manifold with stable tangential invariant $\chi = S \circ \alpha; H_4(M; \mathbb{Z}) \rightarrow \pi_3(SO) = \mathbb{Z}$. Then $M$ has exactly two distinct smooth structures up to diffeomorphism if and only if $\text{Im}(S \circ \alpha) \subseteq 2\mathbb{Z}$.

Proof. Suppose $M$ has exactly two distinct smooth structures up to diffeomorphism. Then by Theorem 4.3, $M$ and $M \not\# \Sigma$ are not diffeomorphic, where $\Sigma$ is the exotic 8-sphere. Since $\delta_8 = \Theta_8(3)$, by Theorem 3.1, the stable tangential invariant $\chi$ is zero (mod 2) and hence $\text{Im}(S \circ \alpha) \subseteq 2\mathbb{Z}$. Conversely, suppose $\text{Im}(S \circ \alpha) \subseteq 2\mathbb{Z}$. Now by Theorem 3.1, $M$ can not be diffeomorphic to $M \# \Sigma$, where $\Sigma$ is the exotic 8-sphere. Now by Theorem 4.3, $M$ has exactly two distinct smooth structures up to diffeomorphism. \hfill \Box

Remark 4.6. If $n = 2, 3, 5, 6, 7$ (mod 8) or the stable tangential invariant $\chi$ of $M^{2n}$ is zero (mod 2), then by [16, Corollary, p. 289] and Theorem 3.1, we have $I(M^{2n}) = 0$. So, by Theorem 4.3, we have the following:

Theorem 4.7. Let $n$ be any integer greater than 3 such that $\Theta_n$ and $\Theta_{n+1}$ are trivial and $M^{2n}$ be a closed smooth $(n-1)$-connected 2n-manifold. If $n = 2, 3, 5, 6, 7$ (mod 8) or the stable tangential invariant $\chi$ of $M^{2n}$ is zero (mod 2), then the set of diffeomorphism classes of smooth structures on $M^{2n}$ is in one-to-one correspondence with group $\delta_{2n}$.

Remark 4.8. (1) By Theorem 4.7, every closed smooth 4-connected 10-manifold has exactly six distinct smooth structures, namely, $\{[M \# \Sigma] \mid \Sigma \in \delta_{10} \cong \mathbb{Z}_6\}$.
(2) If $M^{2n}$ is $n$-parallelisable, almost parallelisable or $\pi$-manifold, then the stable tangential invariant $\chi$ of $M$ is zero [15]. Then by Theorem 4.7, we have the following:

Corollary 4.9. Let $n$ be any integer greater than 3 such that $\Theta_n$ and $\Theta_{n+1}$ are trivial and $M^{2n}$ be a closed smooth $(n-1)$-connected 2n-manifold. If $M^{2n}$ is $n$-parallelisable, almost parallelisable or $\pi$-manifold, then the set of diffeomorphism classes of smooth structures on $M^{2n}$ is in one-to-one correspondence with group $\delta_{2n}$.

Definition 4.10 ([8]). The normal $k$-type of a closed smooth manifold $M$ is the fibre homotopy type of a fibration $p: B \rightarrow BO$ such that the fibre of the map $p$...
is connected and its homotopy groups vanish in dimension $\geq k + 1$, admitting a lift of the normal Gauss map $\nu_M : M \rightarrow BO$ to a map $\tilde{\nu}_M : M \rightarrow B$ such that $\tilde{\nu}_M : M \rightarrow B$ is a $(k + 1)$-equivalence, i.e., the induced homomorphism $\tilde{\nu}_M : \pi_i(M) \rightarrow \pi_i(B)$ is an isomorphism for $i \leq k$ and surjective for $i = k + 1$. We call such a lift a normal $k$-smoothing.

**Theorem 4.11.** Let $n = 5, 7$ and let $M_0$ and $M_1$ be closed smooth $(n - 1)$-connected $2n$-manifolds with the same Euler characteristic. Then

(i) There is a homotopy sphere $\Sigma^{2n} \in \tilde{\Theta}_{2n}$ such that $M_0$ and $M_1 \# \Sigma^{2n}$ are diffeomorphic.

(ii) Let $M^{2n}$ be a closed smooth $(n - 1)$-connected $2n$-manifold such that $[M] = 0 \in \Omega_{2n}^{\text{String}}$ and let $\Sigma$ be any exotic $2n$-sphere in $\tilde{\Theta}_{2n}$. Then $M$ and $M \# \Sigma$ are not diffeomorphic.

Proof. (i): $M_0$ and $M_1$ are $(n - 1)$-connected, and $n$ is 5 or 7; therefore, $p_1/2$ and the Stiefel–Whitney classes $\omega_2$ vanish. So, $M_0$ and $M_1$ are $B\text{String}$-manifolds. Let $\tilde{\nu}_M : M_j \rightarrow B\text{String}$ be a lift of the normal Gauss map $\nu_{M_j} : M_j \rightarrow BO$ in the fibration $p : B\text{String} = BO(8) \rightarrow BO$, where $j = 0$ and 1. Since $B\text{String}$ is 7-connected, $p_8 : \pi_i(B\text{String}) \rightarrow \pi_i(BO)$ is an isomorphism for all $i \geq 8$. This shows that $\tilde{\nu}_M : M_j \rightarrow B\text{String}$ is an $n$-equivalence and hence the normal $(n - 1)$-type of $M_0$ and $M_1$ is $p : B\text{String} \rightarrow BO$. We know that $\Omega_{2n}^{\text{String}} \cong \tilde{\Theta}_{2n}$, where the group structure is given by connected sum [4]. This implies that there always exists $\Sigma^{2n} \in \tilde{\Theta}_{2n}$ such that $M_0$ and $M_1 \# \Sigma^{2n}$ are $B\text{String}$-bordant. Since $M_0$ and $M_1 \# \Sigma^{2n}$ have the same Euler characteristic, by [8, Corollary 4], $M_0$ and $M_1 \# \Sigma^{2n}$ are diffeomorphic.

(ii): Since the image of the standard sphere under the isomorphism $\tilde{\Theta}_{2n} \cong \Omega_{2n}^{\text{String}}$ represents the trivial element in $\Omega_{2n}^{\text{String}}$, we have $[M^{2n}] \neq [M \# \Sigma]$ in $\Omega_{2n}^{\text{String}}$. This implies that $M$ and $M \# \Sigma$ are not $B\text{String}$-bordant. By obstruction theory, $M^{2n}$ has a unique string structure. This implies that $M$ and $M \# \Sigma$ are not diffeomorphic.

**Theorem 4.12.** Let $M$ be a closed smooth 6-connected 14-dimensional $\pi$-manifold and $\Sigma$ is the exotic 14-sphere. Then $M \# \Sigma$ is not diffeomorphic to $M$. Thus, $I(M) = 0$. Moreover, if $N$ is a closed smooth manifold homeomorphic to $M$, then $N$ is diffeomorphic to either $M$ or $M \# \Sigma$.

Proof. It follows from results of Anderson, Brown and Peterson on spin cobordism [1] that the image of the natural homomorphism $\Omega_{14}^{\text{framed}} \rightarrow \Omega_{14}^{\text{Spin}}$ is 0 and $\Omega_{14}^{\text{Spin}} \cong \Omega_{14}^{\text{String}} \cong \mathbb{Z}_2$ [4]. This shows that $[M] = 0 \in \Omega_{14}^{\text{String}}$. Now by Theorem 4.11 (ii), $M \# \Sigma$ is not diffeomorphic to $M$. If $N$ is a closed smooth manifold homeomorphic to $M$, then $N$ and $M$ have the same Euler characteristic. Then by Theorem 4.11 (i), $N$ is diffeomorphic to either $M$ or $M \# \Sigma$. □
Remark 4.13. By the above Theorem 4.12, the set of diffeomorphism classes of smooth structures on a closed smooth 6-connected 14-dimensional $\pi$-manifold $M$ is

$$\{[M], [M \# \Sigma]\} \cong \mathbb{Z}_2,$$

where $\Sigma$ is the exotic 14-sphere. So, the number of distinct smooth structures on $M$ is 2.

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References