Note on the dimension of modules and algebras

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Let $A$ be a ring with unit element. The left dimension (notation: $l \dim AA$), the left injective dimension ($l \text{ inj. dim. } AA$) and the left weak dimension ($w \text{. l. dim } AA$) for left $A$-modules and the left global dimension ($l \text{. gl. dim } A$) and the global weak dimension ($w \text{. gl. dim } A$) of $A$ are those defined in [3].

Let $A$ and $A'$ be rings and $\psi$ a ring homomorphism of $A$ to $A'$. Then each left $A'$-module $A$ may be regarded as a left $A$-module, by setting, for $\lambda \in A' \ a \in A$ $\lambda \cdot a = \psi \lambda \cdot a$.

If $A'$ is $A$-projective in this sense, the following inequalities are shown in [3]; $l \dim AA \leq l \dim rA$, $w \text{. l. dim } AA \leq w \text{. l. dim } rA$ and $l \text{. inj. dim } AA \leq l \text{. inj. dim } rA$ for left $A'$-modules $A$.

M. Auslander [1] has shown that $l \text{. gl. dim } A = \sup l \dim \frac{A}{I}$ where $I$ ranges over all left ideals of $A$ and obtained some relations among $l \text{. gl. dim } A_1$, $l \text{. gl. dim } A_2$ and $l \text{. gl. dim } A_1 \otimes A_2$ in the special cases where $A_1$ and $A_2$ are algebras over a field $K$.

If $\mathfrak{N}$ is a two-sided ideal in $A$, there is in general very little relation between $l \text{. gl. dim } A$ and $l \text{. gl. dim } \frac{A}{\mathfrak{N}}$; it was however proved in Eilenberg-Nagao-Nakayama [6] that if $l \text{. gl. dim } A \leq 1$ and $A$ is semi-primary, then $\text{gl. dim } \frac{A}{\mathfrak{N}} < \infty$.

Now, we show in section 1 of the present note that for each left $A$-module $A$ we have $l \dim AA = l \dim A_a A_a^*$, $w \text{. l. dim } AA = w \text{. l. dim } A_a A_a^*$ and $l \text{. inj. dim } AA = l \text{. inj. dim } A_a A_a^*$ and conversely, for each left $A_a$-module $A$, $l \dim AA = l \dim A_a$ and so on, where $A_a$ is the total matrix ring of order $n$ over $A$. Hence, as the special case of $A_1 \otimes A_2$ we obtain $l \text{. gl. dim } A = l \text{. gl. dim } A_u$ and $w \text{. gl. dim } A = w \text{. gl. dim } A_u$ for any ring $A$ and further if $A$ is an algebra over a commutative ring $K$, we obtain $\dim A = \dim A_u$.

In section 2 we show that the analogous theorem to Auslander’s is valid for $w \text{. gl. dim } A$ and some characterization of ring $A$ with $w \text{. gl. dim } A \leq n$ or $l \text{. gl. dim } A \leq n$ ($n \geq 1$). In section 3, we assume that $\psi$ is a ring homomorphism of $A$ to $A'$ and $l \dim A' = 0$ or $r \dim A' = 0$, then we obtain some relations between the dimensions of $A$ and $A'$, regarding $A'$-modules $A$ as $A$-modules. In particular, if two sided ideal $\mathfrak{N}$ is equal to $ae$ or $ea$ ($e = e^2$), we obtain $l \text{. gl. dim } A \leq l \text{. gl. dim } \frac{A}{\mathfrak{N}}$ and $w \text{. gl. dim } A \leq w \text{. gl. dim } \frac{A}{\mathfrak{N}}$. 

In section 4 we show that \( \text{w. gl. dim } A = 0 \) if and only if \( A \) is regular, hence we obtain an example of the case that \( \text{l. gl. dim } A > \text{w. gl. dim } A \). Finally in section 5 we study some relations between the dimensions of \( A \) and \( eAe \) under some assumptions. The definitions and notions employed in this paper are based on those introduced by H. Cartan and S. Eilenberg [3].

Let \( A \) be a ring with unit element and \( A_n \) be the total matrix ring of order \( n \) over \( A \). We assume that each \( A \)-module is unitary and that each ring homomorphism maps unit upon unit. If two rings \( A \) and \( \Gamma \) and a ring homomorphism \( \psi \) of \( A \) to \( \Gamma \) are given, than each left \( \Gamma \)-module \( A \) may be regarded as a left \( A \)-module, by setting, for \( a \in A \), \( \lambda \in A \)

\[
\lambda a = \psi(\lambda)a.
\]

In particular \( \Gamma \) may be regarded as \( A \)-module.

The following lemma is an immediate consequence of [3; XVI, Exer. 5]

**Lemma 1.** Let \( A, \Gamma \) and \( \psi \) be as above. Then

if \( \text{w. l. dim } \Lambda \Gamma = 0 \)

we have \( \text{w. l. dim } \Lambda A \leq \text{w. l. dim } rA \),

if \( \text{l. dim } \Lambda \Gamma = 0 \)

we have \( \text{l. dim } \Lambda A \leq \text{l. dim } rA \), and

if \( \text{w. r. dim } \Lambda \Gamma = 0 \)

we have \( \text{l. inj. dim } \Lambda A \leq \text{l. inj. dim } rA \), for each left \( \Gamma \)-module \( A \).

Let \( A \) be a left \( A \)-module and \( A^n \) and \( A_n \) be the direct sums of \( n \) and \( n^2 \) \( \Lambda \)'s, respectively. The left operations of \( A_n \) over \( A^n \) and \( A_n \) are defined, by setting, for \( \lambda = (\lambda_{ij}) \in A_n \), \( a = (a_1 \ldots a_n) \in A^n \)

\[
\lambda a = (\sum_j \lambda_{ij} a_j, \ldots, \sum_j \lambda_{nj} a_j)
\]

\( \lambda a = (\sum_k \lambda_{ik} a_{kj}) \).

\( A^n \) and \( A_n \) become left \( A_n \)-modules under these operations. We define a ring homomorphism \( \varphi \) of \( A \) to \( A_n \) as follows,

\[
\varphi(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}
\]

for \( \lambda \in A \).

\( A^n \) and \( A_n \) become left \( A \)-modules by (1), (2) and (3), and these coincide with natural direct sums of \( n \) and \( n^2 \) \( A \)'s as \( A \)-modules respectively.

**Proposition 1.** If a left \( A \)-module \( A \) is projective, then the left \( A_n \)-module \( A^n \) so is.

**Proof.** If \( A \neq A \), we have \( A_n = A^n \oplus \ldots \oplus A^n \) as \( A_n \)-module, hence \( A^n \) is \( A_n \)-projective. Thus by a direct sum argument we have proposition.

**Proposition 2.** Each left \( A_n \)-module \( A \) is \( A_n \)-isomorphic to \((e_{11} A)^n\), where we regard \( e_{11} A \) as a left \( A \)-module.

**Proof.** We have a decomposition of \( A \) as follows,

\[
A = e_{11} A + e_{21} A + \ldots + e_{nn} A
\]

and \( e_{11} A \) is \( A \)-isomorphic to \( e_{11} A \). We obtain a \( A_n \)-isomorphism of \( A \) to \((e_{11} A)^n\) by the following correspondence, for \( a \in A \), \( a' \in (e_{11} A)^n \)

\[
a = e_{11} a_1 + e_{21} a_2 + \ldots + e_{nn} a_n \leftrightarrow a' = e_{11} a_1 + e_{12} a_2 + \ldots + e_{nn} a_n.
\]
Proposition 3. For left \( A \)-modules \( A, B \) and a right \( A \)-module \( C \), we have isomorphism

\[
\text{Hom} \ \lambda(A, B) \approx \text{Hom} \ \lambda_n(A^n, B^n), C \otimes_A^R A \approx C_n \otimes^R A^n.
\]

Proof. We denote an element \((0 \cdots 0 a 0 \cdots 0)\) by \( a^{(i)} \). Any element \( f \) of \( \text{Hom} \ \lambda_n(A^n, B^n) \) is uniquely decided by the image of the first component of \( A \), for \( f(a^{(1)}) = f(e_{11}a^{(1)}) = e_{11}f(a^{(1)}) \). And since \( f(a^{(1)}) = f(e_{11}a^{(1)}) = e_{11}f(a^{(1)}) \), \( f \) is uniquely determined by a element of \( \text{Hom} \ \lambda(A, B) \).

Next we have

\[
(c_1, \ldots, c_n) \otimes (a_1, \ldots, a_n) = \sum_{i,j} c_i \otimes a_{i+1}^{(j)}.
\]

If \( i \neq j \), \( c^i \otimes a_{i+1}^{(j)} = c^i e_{ij} \otimes a_{i+1}^{(j)} = c^i \otimes e_{ij} a_{i+1}^{(j)} = 0 \), and \( c^i \otimes a^{(i)} = c^i e_{11} \otimes a^{(i)} = c^i \otimes e_{11} a^{(i)} = c^i \otimes a^{(i)} \), hence \( (c_1 \ldots c_n) \otimes (a_1 \ldots a_n) = \sum_{i} c_i^{(1)} \otimes a_i^{(1)} \).

We define an epimorphic mapping \( \varphi : C \otimes_A A \rightarrow C^n \otimes_A A^n \) by setting

\[
\varphi (c \otimes a) = c^{(1)} \otimes a^{(1)}.
\]

Conversely define a mapping \( \varphi : C^n \otimes_A A^n \rightarrow C \otimes_A A \) by setting

\[
\varphi (c^{(1)} \otimes a^{(1)}) = c_1 \otimes a_1,
\]

this mapping is defined independent on the choice of representatives.

Then \( \varphi \) is epimorphic and \( \varphi \circ \varphi \) is the identity mapping. Therefore \( \varphi \) is isomorphic.

Proposition 4. Let \( A, B \) and \( C \) be as above, then we have isomorphisms

\[
\text{Ext} \ \lambda(A, B) \approx \text{Ext} \ \lambda_n(A^n, B^n), \text{Tor} \ \lambda(C, A) \approx \text{Tor} \ \lambda_n(C^n, A^n).
\]

Proof. Let

\[
X_m \xrightarrow{d_m} X_{m-1} \xrightarrow{d_{m-1}} \cdots \xrightarrow{d_1} X_1 \xrightarrow{d_0} X_0 \xrightarrow{d_0} A \rightarrow 0
\]

be a projective resolution of \( A \). By the natural manner we can extend this sequence to a \( \lambda_n \)-projective resolution of \( A^n \), using proposition 1, as follows

\[
\cdots \xrightarrow{d_m} X_m \xrightarrow{d_m} X_{m-1} \xrightarrow{d_{m-1}} \cdots \xrightarrow{d_1} X_1 \xrightarrow{d_0} X_0 \xrightarrow{d_0} A^n \rightarrow 0
\]

Passing to homology yields the desired results in virtue of the definitions of Ext and Tor.

Corollary 1. For each left \( A \)-module \( A \) we have l. dim. \( \lambda A = l. \dim \lambda_n A^n, \lambda \) inj. dim \( \lambda A = l. \dim \lambda_n A^n \), and w. l. dim \( \lambda_n A = w. \dim \lambda A^n \).

Proof. We have immediately the conclusion for l. dim \( \lambda \) by lemma 1 and the consideration in the proof of proposition. Let \( B \) be a left \( \lambda_n \)-module, then we have following isomorphisms from propositions 2 and 4.

\[
\text{Ext} \ \lambda(\lambda_e, B, A) \approx \text{Ext} \ \lambda_n((\lambda_e, B)^n, A^n) \approx \text{Ext} \ \lambda_n(B, A^n).
\]

Hence \( \text{l. inj. dim.} \ \lambda_n A^n \geq \text{l. inj. dim.} \ \lambda A \).

The inverse inequality is obtained from lemma 1, noting that \( A^n \) is the direct sum.
of \( n \) \( A \)'s as \( A \) module.

It is similar for \( w. l. \dim \).

**Remark 1.** From corollary 1 and Theorem 18 of Eilenberg-Nakayama [4] we can obtain the well known result that \( A \) is quasi-Frobenius if and only if \( A_n \) so is.

**Corollary 2.** For each left \( A_n \)-module \( A \) we have
\[
\dim AA = \dim A_n A, \quad \text{and} \quad \dim AA = \dim A_n A
\]
and \( w. l. \dim AA = w. l. \dim A_n A \)

**Proof.** Observe that \( A \) is the direct sum of \( n \) \((e_{11} A)\)'s as a \( A \)-module, we have by propositions 1 and 2
\[
\dim AA = \dim A_n (e_{11} A)^n = \dim A_{e11} A = \dim AA.
\]

It is similar for the remainders.

From the above two corollaries we have

**Theorem 1.** \( l. \dim A \leq l. \dim A_n \), \( w. \dim A = w. \dim A_n \).

Now, let \( A \) be an algebra over a commutative ring \( K \). And we have \( A^* = A \otimes A^* \) where \( A^* \) is the inverse algebra. As for two sided \( A \)-modules \( A \), the standard procedure will be to convert them into left modules over \( A' \). Further we observe that \((A_n)^e = A^n \otimes A^n \) is isomorphic to \( A \otimes A^* \otimes K_n = (A')_n \).

Hence from corollary 2 we have \( l. \dim A^* A = l. \dim (A')_n A \) for each two sided \( A_n \)-modul \( A \).

In particular, setting \( A = A_n \) we have

**Theorem 2.** \( \dim A = \dim A_n \)

**Proposition 5.** The following properties are equivalent, respectively:

a) \( A \) is left hereditary,

b) \( A_n \) is left hereditary,

and

a') \( A \) is left semi-hereditary,

b') \( A_n \) is left semi-hereditary,

The first statements are clear from Theorem 1 and [3, VI, 2.8]. For the proof of the second statements we need the following well known result, (cf. [2: 23-15]).

Let \( I \) be left ideal of \( A_n \) and \( m(I) \) be the left \( A \)-module consisting of the first row of elements in \( I \).

Then the correspondence \( I \leftrightarrow m(I) \) gives one to one correspondence between the left ideals of \( A_n \) and the \( A \)-submodule of \( n \)-dimensional vector space \( A^n \) over \( A \). Moreover, \( m(I) \) is finitely generated as a \( A \)-module if and only if \( I \) has finite generators as a left ideal.

Now we assume that \( A \) is left semi-hereditary. If \( I \) is a finitely generated left ideal of \( A_n \), we have from the above remark and corollary 1 of proposition 4

\[
l. \dim A_{nI} = l. \dim A_{n} m(I)^{n} = l. \dim A m(I).
\]

From [3: I, 6.2] \( l. \dim A m(I) = 0 \), hence \( A_n \) is left semihereditary. Conversely, let
An be left semi-hereditary and I be a finitely generated left ideal of A, then we have $l. \dim A I = l. \dim A_n I_n$ and since $I_n$ is finitely generated as a left ideal of $A_n$, $l. \dim A_n I_n = 0$. Therefore A is left semi-hereditary.

2. Now we study here some properties of weak dimensions of rings.

**Lemma 2.** Let A be a left A-module and consider an exact sequence

$$0 \to B \to P \to A \to 0$$

where $w. l. \dim \lambda P = 0$. If $w. l. \dim \lambda A \neq 0$, then $w. l. \dim \lambda B = w. l. \dim \lambda A - 1$, and if $w. l. \dim \lambda A = 0$, then $w. l. \dim \lambda B = 0$.

It is clear (cf. [3; VI, 2.3]).

The following theorem is analogous to Auslander's theorem in the case of left dimensions.

**Theorem 3.**

a) $w. gl. \dim A = \sup w. l. \dim \lambda B$

b) $= \sup w. l. \dim \lambda A / I$

where B ranges over all left A-modules generated by a single element and I ranges over all left ideals of A.

If further $w. gl. \dim A \neq 0$

c) $w. gl. \dim A = 1 + \sup. w. l. \dim \lambda I$

**Proof.** a) $\to$ b) $\to$ c) is clear from lemma 2. Hence we prove here only the statement a) of the theorem. This proof is based on

**Lemma 3.** Let A be a left A-module, I a non empty well ordered set and $(A_i)_{i \in I}$ a family of sumodules of A such that $\cup \ A_i = A$ and if $i \in I$ and $i \ngeq j$, then $A_i \supseteq A_j$.

If $w. l. \dim \lambda (A_i/A'_i) \leq n$ for all $i \in I$ where $A'_i = \cup_{j < i} \ A_j$, $A'_1 = (0)$ (1 is the least element of I), then $w. l. \dim \lambda A \leq n$.

**Proof.** If $n = 0$ the then for all $i \in I$ we have $w. l. \dim \lambda A_i/A'_i = 0$. From the exact sequence

$$0 \to A'_i \to A_i \to A_i/A'_i \to 0$$

we have for each right A-module B and $n \geq 1$

$$0 = Tor^A_{n+1}(B, A_i/A'_i) \to Tor^A_n(B, A'_i) \to Tor^A_n(B, A_i) \to Tor^A_n(B, A_i/A'_i) = 0.$$ 

Hence $Tor^A_n(B, A'_i)$ is isomorphic to $Tor^A_n(B, A_i)$, that is, $w. l. \dim \lambda A'_i = w. l. \dim \lambda A_i$. By our assumption we have $w. l. \dim \lambda (A_i/A'_i) = w. l. \dim \lambda A_1 = 0$. Then we can use the transfinite induction. We assume that all modules $A_j$ such as $j < i$ are those with $w. l. \dim \lambda A_j = 0$. If $i$ is not a limit element, we have $A'_i = A_{i-1}$ and by the above remark $w. l. \dim \lambda A_i = 0$. If $i$ is a limit element, then $A'_i$ is the direct limit of $A_j (j < i)$ and inclusion mappings $\pi_j: (j \leq j' < i)$ (see [5; VIII, Exer. B]). Since Tor commutes with the direct limit, we have $Tor^A_n (B, A'_i) = 0$ for $n > 0$. Hence by the above remark we obtain $w. l. \dim \lambda A_i = 0$.

For $n > 0$ we can use the same method as that of proof of [1; pr. 3]. The proof of a) of theorem is also similar to that of [1; Th. 1].
From lemma 2, Theorem 3 and the analogous properties to them in the case of the left dimensions we have the following corollary which is a generalization of [3; I, 5.4].

**Corollary.** The following properties are equivalent for \( n \geq 1 \), respectively:

a) \( l. \text{gl. dim } A \leq n \)

b) For each \( A \)-submodule \( A \) of a left projective \( A \)-module we have \( l. \dim \Lambda A \leq n - 1 \)

and

a') \( w. \text{gl. dim } A \leq n \)

b') For each \( A \)-submodule \( A \) of a left \( A \)-module \( P \) with \( w. l. \dim \Lambda P = 0 \) we have \( w. l. \dim \Lambda A \leq n - 1 \)

3. we now consider some relations between dimensions of two ring \( A \) and \( \Gamma \) which are connected by a ring homomorphism \( \psi \) of \( A \) to \( \Gamma \).

**Proposition 6.** Let \( A \), \( \Gamma \) and \( \psi \) be as above and we assume that \( l. \dim \Lambda \Gamma = 0 \) and \( l. \dim \Gamma B = 1 \) implies \( l. \dim \Lambda B = 1 \) for left \( \Gamma \)-modules \( B \). Then we have \( l. \dim \Lambda A = l. \dim \Gamma A \) for each left \( \Gamma \)-module \( A \) with \( l. \dim \Gamma A < \infty \).

**Proof.** If \( l. \dim \Gamma A = 0 \), \( l. \dim \Lambda A = 0 \) by lemma 1. Now, we assume that the proposition is proved for left \( \Gamma \)-modules \( A' \) with \( l. \dim \Lambda A' < q \), \( 1 < q < \infty \), and that \( l. \dim \Gamma A = q \). There exists a \( \Gamma \)-exact sequence of \( A \) with \( X \) projective as

\[
\begin{array}{c}
0 \longrightarrow Q \longrightarrow X \longrightarrow A \longrightarrow 0.
\end{array}
\]

Since \( l. \dim \Gamma A > 1 \), we have \( l. \dim \Gamma Q = q - 1 \), hence by the hypothesis of induction \( l. \dim \Lambda Q = q - 1 \) and \( l. \dim \Lambda X = 0 \). Regarding (E) as \( A \)-exact sequence, we have \( l. \dim \Lambda A = q \).

If there are the same assumptions for weak or injective dimensions, it is true for them. In particular if \( \psi \) is epimorphic, the second condition of proposition is satisfied (cf. cor of pr. 9).

**Proposition 7.** Let \( A \), \( \Gamma \) and \( \psi \) be as above and \( \mathfrak{l} \) be a left ideal of \( A \). We set \( \mathfrak{l}^* = \Gamma \psi(\mathfrak{l}) \). If \( w. r. \dim \Lambda \mathfrak{l} = 0 \), then \( l. \dim \Gamma / \mathfrak{l}^* \leq l. \dim \Lambda A / \mathfrak{l} \).

**Proof.** We obtain the following commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\xi_A} & A / \mathfrak{l} \\
\uparrow{\psi} & & \downarrow{\phi} \\
\Gamma & \xrightarrow{\xi_{\Gamma}} & \Gamma / \mathfrak{l}^* 
\end{array}
\]

where \( \xi_A \) is the natural mapping of \( A \) to \( A / \mathfrak{l} \) and \( \xi_{\Gamma} \) is that of \( \Gamma \) to \( \Gamma / \mathfrak{l}^* \) and \( \phi: A / \mathfrak{l} \longrightarrow \Gamma / \mathfrak{l}^* \) is defined as follows,

for \( \lambda \in A / \mathfrak{l} \) (\( \lambda \) is a residue class of \( \lambda \) mod \( \mathfrak{l} \))

\[
\phi(\lambda) = \overline{\psi(\lambda)} \quad (\overline{\psi(\lambda)} \text{ is a residue class of } \psi(\lambda) \text{ mod } \mathfrak{l}^*).
\]
We define a homorphism $g$ of $\Gamma \otimes A/I$ to $\Gamma/I^*$ as follows; for $\gamma \in \Gamma$, $\bar{a} \in A/I$ $g(\Gamma \otimes \bar{a}) = \gamma \varphi(\bar{a}) = \gamma \varphi(\bar{a})$. Observe that $\Gamma \otimes A/I = \Gamma \otimes I$ and the kernel of $g$ is $I^* \otimes I$. For $x \in I^* \otimes I$ we have $x = \sum \gamma_i \varphi(l_i) \otimes 1 = \sum \gamma_i \otimes l_i = 0$ where $\gamma_i \in \Gamma$, $l_i \in I$. Hence $g$ is isomorphic. Since by the assumption $\text{Tor}^A_*(\Gamma, A/I) = 0$ for $n > 0$ we have from the mapping theorem [3; VIII, 3.1]

$$\text{Tor}^A(A/I, C) \approx \text{Tor}^A(\Gamma, A/I^*),$$

$$\text{Ext}^l(A/I, C) \approx \text{Ext}^l(A/I, C)$$

for right $\Gamma$-modules $A$ and left $\Gamma$-modules $C$. This proves the first half. For the second half we have the same theorem as the mapping theorem and we can prove the last statements.

**Corollary.** Let $\varphi$ be epimorphic and $N$ be its kernel. If $w. r. \dim A/N = 0$, then $l. gl. \dim A/N \leq l. gl. \dim A$. And if $w. r. \dim A/N = 0$ or $w. l. \dim A/N = 0$, then $w. gl. \dim A/N \leq w. gl. \dim A$.

**Proposition 8.** Let $A$, $\Gamma$ be semi-primary and a ring homomorphism $\varphi$ of $A$ to $\Gamma$ be given. And let $Na$ be the radical of $A$ and we assume that $Na$ is $\Gamma$-primary and that $r. \dim A = 0$. Then we have for each right $\Gamma$-module $A$ and left $\Gamma$-module $B$

$$r. \dim \varphi A = r. \dim \varphi A, \quad l. \dim \varphi B = l. \dim \varphi B.$$

**Proof.** From the consideration in proposition 7 we obtain the following isomorphism,

$$\text{Tor}^A(A, A/Na) \approx \text{Tor}^A(A, A/Nr).$$

We have from the analogous properties of [1; pr. 7] such equivalent relations as

$$r. \dim \varphi A < n \iff \text{Tor}^A(A, A/Na) = 0 \iff \text{Tor}^A(\Gamma, A/Nr) = 0 \iff r. \dim \varphi A < n.$$

It is similar for left injective dimension.

**Proposition 9.** Let $\mathfrak{N}$ be a two sided ideal of $A$ and we assume that $w. r. \dim A/\mathfrak{N} = 0$ or $w. l. \dim A/\mathfrak{N} = 0$, then we have for each left $A/\mathfrak{N}$-module $B$ and right $A/\mathfrak{N}$-module $C$

$$\text{Tor}^A(C, B) \approx \text{Tor}^A(\mathfrak{N}, B).$$

And if $l. \dim A/\mathfrak{N} = 0$ or $r. \dim A/\mathfrak{N} = 0$, we have $\text{Ext}^A(A/\mathfrak{N}, B)$ for each left $A/\mathfrak{N}$-modules $A$ and $B$.

**Proof.** It is easily seen that Hom $A(A/\mathfrak{N}, B)$ is isomorphic to $B$. We define a homomorphism $\varphi$ of $A/\mathfrak{N} \otimes A$ to $A/\mathfrak{N} A$ by setting, for $1 \otimes a \in A/\mathfrak{N} \otimes A(\bar{1} \otimes \bar{a}$ is a residue class of $1 \mod \mathfrak{N})$

$$\varphi(\bar{1} \otimes \bar{a}) = \bar{a} \quad (\bar{a} \text{ is a residue class of a mod } \mathfrak{N}A).$$

Then it is clear that $\varphi$ is isomorphic. From [3, VI. pr. 4.1.2.3.4] we obtain isomorphisms.

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(1) A ring $A$ is called semi-primary if it contains a nilpotent two-sided ideal $N$ such that the residue ring $A/N$ is semi-simple. It does not coincide with "half primar" of Deuring, Algebren, Ergebn. Math.
COROLLARY. If \( l \dim A/\mathfrak{A} = 0 \) we have for left \( A/\mathfrak{A} \)-modules \( A \) \( l \dim A = l \dim A/\mathfrak{A} \). If \( w. r. \dim A/\mathfrak{A} = 0 \), then \( l. \text{inj. dim} \ A = l. \text{inj. dim} \ A/\mathfrak{A} \).

Proof. For each left \( A/\mathfrak{A} \)-module \( B \) we have an isomorphism: \( \text{Ext} A (A, B) \cong \text{Ext} A/\mathfrak{A} (A, B) \), hence we obtain \( l. \dim A/\mathfrak{A} = l. \dim A \). The inverse inequality is obtained from lemma 1. It is similar for the remainders.

**Theorem 4.** If a two-sided ideal \( \mathfrak{A} \) of \( A \) is generated by an idempotent element \( e \) as a left ideal or a right, then \( l. \text{gl. dim} A \cong l. \text{dim} A/\mathfrak{A} \), \( w. \text{gl. dim} A \cong w. \text{gl. dim} A/\mathfrak{A} \).

**Remark 2.** If \( \Gamma \) is a crossed product over \( A \) with a finite complete outer automorphisms \( \mathfrak{G} \) of \( A \), then all the assumptions of propositions 7 and 8 are satisfied.

If \( \Gamma \) is a commutative semi-primary ring and \( \mathfrak{G} \) is a finite complete automorphisms of \( \Gamma \) and \( A \) is the \( \mathfrak{G} \)-invariant subring of \( \Gamma \), then \( \Gamma \) and \( A \) satisfy all assumptions of propositions 7 and 8.

**Proposition 10.** Let \( \Gamma \) be a crossed product over \( A \) as above, then

\[ \text{gl. dim} A = \text{gl. dim} \Gamma. \]

**Proof.** Let \( A \) be a left \( A \)-module. We define a \( \Gamma \)-module \( \mathfrak{p}(A) \) as follows,

\[ \mathfrak{p}(A) = \sum_{\sigma \in \mathfrak{G}} V_{\sigma} A \quad (\{ V_{\sigma} \text{ is a base of } \mathfrak{p}(A) \}) \]

for \( x \in A, V_{\sigma} a \in V_{\sigma} A \)

\[ x(V_{\sigma} a) = V_{\sigma} x^a, \]

\[ u_{\tau}(V_{\sigma} a) = V_{\tau} a_{\tau, \sigma} a \]

where \( \{ u_{\tau} \} \) is a base of \( \Gamma \) over \( A \) and \( \{ a_{\tau, \sigma} \} \) is a factor set of \( \Gamma \) over \( A \). Since \( u_{1} A \) is a direct summand of \( \mathfrak{p}(A) \) as left \( A \) module we obtain by lemma \( l. \dim r \mathfrak{p}(A) \) \( \geq l. \dim A \). Which proves proposition.

Observing that we can obtain naturally a \( \Gamma \) projective resolution of \( \mathfrak{p}(A) \) from \( A \)-projective one of \( A \), we have \( l. \dim \mathfrak{p}(A) = l. \dim r \mathfrak{p}(A) \).

If \( A \) is semi-primary, from proposition 8 we obtain,

**Corollary 1.** If \( A \) is semi-primary, then \( \text{gl. dim} \ A = \text{gl. dim} \Gamma \).

We obtain a similar result for the second example of remark 2 as follows.

**Corollary 2.** Let \( A \) and \( \Gamma \) be the same as the second example, then

\[ \text{gl. dim} A = \text{gl. dim} \Gamma. \]

4. We now characterize rings \( A \) with \( w. \text{gl. dim} A = 0 \)

**Proposition 11.** Let \( I \) be a left ideal of \( A \). Then

\[ w. l. \dim A/I = 0 \quad \text{if and only if}, \quad \text{for each right} \ A \text{-module} \ A \text{ and each right} \ A \text{-submodule} A' \text{ of } A \text{ } A \cap A/I = A' \text{ holds.} \]

**Proof.** We assume \( w. l. \dim A/I = 0 \) and we obtain an exact sequence as follows

\[ 0 \rightarrow A' \rightarrow A \rightarrow A/A' \rightarrow 0 \]

From our assumption we obtain the exact sequence; \( 0 \rightarrow A \otimes A/I \rightarrow A \otimes A/I \rightarrow A/A' \otimes A/I \rightarrow 0 \). By the isomorphism \( \psi \) in the proof of proposition 9 \( A' \cap A/I = A' \) holds. Conversely if \( A' \cap A/I = A' \) we obtain \( w. l. \dim A/I = 0 \) by the above consideration.

We call an element \( a \) of a ring \( A \) regular if there exists such an element \( x \) as
axa = a and a left ideal \( I \) regular if all elements of \( I \) are regular.

**Proposition 12** If a left ideal \( I \) is regular then

\[
\text{w. l. dim } A/I = 0
\]

**Proof.** For a right \( A \)-module \( A \) and its submodule \( A' \) we prove the equality \( A' \cap A I = A' I \). For \( x \in A' \cap A I \), we have \( x = \sum a_i y_i \), \( a_i \in A \), \( y_i \in I \).

Since \( I \) is regular, the left ideal generated by \( \langle y_i \rangle \) is generated by an idempotent \( e \). Hence \( x \cdot e = \sum a_i y_i \cdot e = \sum a_i x 
\in A' I \)

**Lemma 4** For each left \( A \)-module \( B \) we obtain \( \text{w. l. dim } A B \leq n \) if and only if \( \text{Tor}^{A+1} (xAB) = 0 \), where \( xA \) is a right \( A \)-module generated by a single element \( x \).

**Proof.** The "if part" is trivial. It is sufficient to show \( \text{Tor}^{A+1} (A, B) = 0 \) for each finitely generated right \( A \)-module \( A \), since \( \text{Tor} \) commutes with the direct limits. We assume that it is true for right \( A \)-module \( A' \) generated by \( (n-1) \) elements. Let \( A \) be generated by \( x_1 \ldots x_n \) and \( A' \) by \( x_1 \ldots x_{n-1} \), then we obtain \( 0 \rightarrow A' \rightarrow A \rightarrow A/A' \rightarrow 0 \). Then \( 0 \rightarrow 0 = \text{Tor}^{A+1} (A', B) \rightarrow \text{Tor}^{A+1} (A, B) \rightarrow \text{Tor}^{A+1} (A/A', B) \rightarrow 0 \) is exact, that is, \( \text{Tor}^{A+1} (A, B) = 0 \). We have the lemma by the induction.

**Corollary** We have for each left \( A \)-modul \( B \)

\[
\text{w. l. dim } A B \leq n \quad \text{if and only if } \text{Tor}^{A+1} (A/B, B) = 0
\]

for each right ideal \( \mathfrak{N} \) of \( A \).

**Proposition 13** Let \( I \) be a left ideal of \( A \). Then \( \text{w. l. dim } A I = 0 \) if and only if \( I \cap \mathfrak{N} = \mathfrak{N} I \) holds for each right ideal \( \mathfrak{N} \) of \( A \).

**Proof.** If we replace \( A \) by \( A \) and \( A' \) by \( \mathfrak{N} \) in proposition 11, we obtain the first half. Conversely, we assume \( I \cap \mathfrak{N} = \mathfrak{N} I \). From the exact sequence:

\[
0 \rightarrow \mathfrak{N} \rightarrow A \rightarrow A/\mathfrak{N} \rightarrow 0,
\]

we obtain the following exact one:

\[
0 \rightarrow \text{Tor}^{A} (A/\mathfrak{N}, A/I) \rightarrow \mathfrak{N} \otimes A/\mathfrak{N} \rightarrow A \otimes A/\mathfrak{N} \rightarrow 0.
\]

By our assumption we see that the third arrow is monomor-
phic and \( \text{Tor}^{A} (A/\mathfrak{N}, A/I = 0 \). Hence we obtain the proposition by lemma 4.

**Corollary** If \( \text{w. l. dim } A I = 0 \), then for any element \( x \) of \( I \) \( x \) contains \( x \) and \( I \) is idempotent: \( I^2 = I \). In particular if \( I \) is principal \( (I = Aa) \) then \( \text{w. l. dim } A I = 0 \) if and only if there exists some element \( x \) in \( Aa \) as \( a \cdot x = a \).

From propositions 12 and 13 and theorem 3 we obtain

**Theorem 5** For each ring \( A \), the following conditions are equivalent:

a) \( \text{w. gl. dim } A = 0 \)

b) \( A' \cap A I = A' I \) for each right \( A \) module \( A \), each right \( A \)-sub-module \( A' \) of \( A \) and each left ideal \( I \) of \( A \).

c) \( A \) is regular

From corollary of proposition 7 and proposition 12

**Theorem 6** If \( \mathfrak{N} \) is a regular two-sided ideal of \( A \), then

\[
\text{l. gl. dim } A \geq \text{l. gl. dim } A/\mathfrak{N} \quad \text{and } \text{w. gl. dim } A \geq \text{w. gl. dim } A/\mathfrak{N}.
\]

If \( A \) is regular without minimal conditions, for instance a direct product of infinite number of fiels, the w. gl. dim \( A \) is smaller then gl. dim \( A \). We note that from
theorems 1 and 5 we obtain that \( A \) is regular if and only if \( A \) so is, which was obtained by Neumann [7] and that if \( A \) is regular, then \( A \) is semi-hereditary.

5. We consider now some relations between dimensions of \( A \) and \( eAe(e = e^e) \) under particular assumptions.

Let \( A \) and \( B \) be left \( eAe \)-modules.

Since \( Ae \) is a direct sum of \( eAe \) and \( (1-e)Ae \), we may regard \( B \) as a sub-module of \( Ae \otimes B \). Hence we obtain an isomorphism: \( \text{Hom} \left( (eAe \otimes A, Ae \otimes B) \right) \cong \text{Hom} \left( (eAe \otimes A, Ae \otimes B) \right) \)

\[
\varphi(a) = f(e \otimes a) = e \cdot f(e \otimes a), \quad \varphi g((e \otimes a) = e \cdot g(e \otimes a).
\]

**Proposition 14** If \( \text{Tor}^{eAe}_{n} (A, A) = 0 \) for \( n > 0 \) and a left \( eAe \) module \( A \), then \( \text{Ext}_{eAe}^{n} (A, B) \cong \text{Ext}_{eAe}^{n} (A, B) \) for each left \( eAe \)-module \( B \).

**Proof.** Since \( Ae \) is \( A \)-projective, we obtain the proposition by the same consideration as that of the change of rings in [3, VI].

We can obtain the analogous proposition to the above one for \( \text{Tor} \).

**Proposition 14a** If \( \text{Tor}^{eAe}_{n} (A, eA) = 0 \) or \( \text{Tor}^{eAe}_{n} (eA, B) = 0 \) for \( n > 0 \) and a right \( eAe \) module \( A \) and a left \( eAe \) module \( B \). Then \( \text{Tor}_{n} (A \otimes eA, Ae \otimes B) \cong \text{Tor}_{n} (eA, B) \).

**Proof.** We only note that since \( eA \otimes eA \) is isomorphic to \( eAe \) as a two sided \( eAe \) module by the mapping: \( \lambda \lambda_4 \lambda_5 \lambda_6 \rightarrow \lambda \lambda_4 \lambda_5 \lambda_6 \), we obtain \( (A \otimes eA) \otimes (eAe \otimes B) \cong A \otimes B \).

**Proposition 15** If \( w.r. \dim eAe = 0 \), we obtain

\[
\dim A_{eAe} \otimes A = w \cdot \dim eAe, \quad w.l. \dim eAe \otimes A = w.l. \dim eAe A
\]

for each left \( eAe \) module \( A \).

**Proof.** If \( \dim eAe A \) is infinite, proposition is clear from the above. We prove it by induction with respect to the dimension of \( n \). It is clear for \( n = 0 \). We assume the proposition for each module \( A \) with \( \dim eAe A \leq n-1 \). We take an exact sequence of a left \( eAe \) module \( A \) with \( \dim eAe A = n \): \( 0 \rightarrow Q \rightarrow P \rightarrow A \rightarrow 0 \), where \( P \) is \( eAe \)-projective. By the hypothesis we obtain \( \dim eAe \otimes Q = n-1 \) and \( \dim eAe \otimes A = 0 \). Furthermore we can obtain the exact sequence of \( eAe \): \( 0 \rightarrow eAe \otimes A \rightarrow eAe \otimes A \rightarrow 0 \) from the above one. Hence we have \( \dim eAe \otimes A = n \) for \( \dim eAe \otimes A = 0 \). For the weak dimension we only observe that we can obtain the exact sequence: \( 0 \rightarrow B \otimes eAe \rightarrow C \otimes eAe \) from a \( A \)-exact one: \( 0 \rightarrow B \rightarrow C \) and further if \( w.l. \dim eAe A = 0 \) we have finally the exact one:

\[
0 \rightarrow B \otimes eAe \otimes A \rightarrow C \otimes eAe \otimes A.
\]

From the proposition we can obtain

**Theorem 7** If \( w.r. \dim eAe A = 0 \) then we obtain

\( l.g.l. \dim A \geq l.g.l. \dim eAe \) and \( w.g.l. \dim A \geq w.g.l. \dim eAe \)

In order to obtain an analogous theorem to this we need the following lemma

**Lemma 5** If \( \dim eAe eAe = 0 \), we have for each left \( A \)-module \( A l. \dim A \)
I. dim e A e  eA.

Proof Let \( \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow A \longrightarrow 0 \) be a projective resolution of \( A \). Then \( \longrightarrow eX_2 \longrightarrow eX_1 \longrightarrow eX_0 \longrightarrow eA \longrightarrow 0 \) is clearly an \( eAe \)-projective resolution of \( eA \) from our assumption. This proves proposition.

**Theorem 8** If \( I. \dim eA e = 0 \) then

\[ l. \quad \text{gl. dim } A \cong l. \quad \text{gl. dim } eA. \]

Proof Let \( I' \) be a left ideal of \( eAe \). Then \( I = A' \) is a left ideal of \( A \) contained in \( eAe \) and further \( A/I \) is isomorphic to \( Ae/\{A(1-e)\} \). From lemma 5 we obtain

\[ \dim A/I = \dim Ae/\{Ae/\{e\} \} = \dim eAe/\{e\} = \dim eAe/I'. \]

Next we consider algebras over a commutative ring \( K \).

**Proposition 16** If \( I. \dim eAe A = r. \dim eAe A = 0 \), then

\[ \dim A \cong \dim eA. \]

Proof It is easily seen that \( (eA)^e \) is isomorphic to \( (e \otimes e^*) A^e (e \otimes e^*) \) and \( l. \dim (eAe)^e A^e \) is equal to \( r. \dim eAe A^e \). Hence from lemma 5 and [3, IX, 2.5] we obtain

\[ \dim A^e A \cong \dim (e \otimes e^*) A = \dim (eAe)^e eAe. \]

Remark 3 If we take the total matrix ring of order \( n \) over \( A \) instead of \( A \) and \( e_{11} \) instead of \( e \), then our hypotheses are satisfied and propositions 14 and 14a coincide with proposition 4.

We can easily obtain isomorphisms of propositions 4, 14 and 14a by using the formulas (4) and (4a) of [3, XVI, 4].

**Bibliography**


