Some general properties of Markov processes

By Takesi Watanabe

(Received Oct. 29, 1958)

The main purpose of this paper is to study the conservative property and the recurrence of Markov processes on a separable locally compact space, following W. Feller's idea of sojourn sets (Feller [2], [3]) and combining them with Green measures.

In §1 we shall give the definition of Markov processes and introduce several notions useful for later considerations. Here we took much from the lecture of Professor H. P. McKean at Kyoto University in 1957-8. Next, we shall derive some results from the hypothesis (H. 1) concerning Green operators which we have introduced at the beginning of §2; we shall impose this hypothesis throughout the subsequent sections. §3 is concerned with the conservative property of Markov processes. Here we shall establish a theorem which characterizes the conservative property using both sojourn sets and Green measures, and we shall derive Feller's theorem concerning purely discontinuous Markov processes (cf. Feller [2; Theorem 7]) as its special case.

In §4 we shall characterize the recurrence either by Green measures or by sojourn sets, putting a new assumption (H. 2) on the continuity of harmonic measures. These results generalize some of the results by J. L. Doob [1] and by W. Feller [3]. In the last section we shall show some applications of the results of §4.

The author wishes to express his hearty thanks to Professor K. Itô for his helpful suggestions.

§1. Definitions of Markov processes and fundamental notions.

Let $E$ be a separable locally compact space. Adding an extra point $\infty$ to $E$ as an isolated one, we shall get a separable locally compact space $\overline{E}=E+\infty$.

We denote a measurable function (sample path) from $[0, +\infty]$ into $\overline{E}$ by $w$ and its position at time $t$ by $w_t$ or $x_t(w)$. Next, let $W$ be the totality of the $w$'s which satisfy the following conditions:

(W. 1) Put
\[
\sigma_w(w) = \inf\{t; x_t(w) = \infty\} \quad \text{if } x_t=\infty \text{ for some } t\geq0,
=\infty \quad \text{otherwise.}
\]

Then $x_t(w) = \infty$ holds for every $t \geq \sigma_w(w)$. Especially we shall define $x_\infty(w) = \infty$. 


The condition (W. 1) shows that every path has no return from \( \infty \) to \( E \).

Now, given an open or closed \( A \) of \( E \), the passage time \( \sigma_A \) for \( A \) is defined by

\[
\sigma_A(w) = \inf \{ t : x_t(w) \in A \} \quad \text{if} \quad x_t(w) \in A \text{ for some } t \geq 0,
\]

\[
= \infty \quad \text{otherwise.}
\]

Further we denote by \( \tilde{\mathcal{B}} \), the smallest Borel field containing the sets \( (w; x_t(w) \in A) \), where \( t \) is an arbitrary fixed time and \( A \) is any Borel set of \( E \). Then we see from (W. 2) that, if \( A \) is open, \( \sigma_A \) is a measurable function from \( [W, \tilde{\mathcal{B}}] \) into \([0, +\infty]\). But we are not sure that, if \( A \) is closed, \( \sigma_A \) is measurable with respect to \( \mathcal{B} \). Hence we need extend \( \tilde{\mathcal{B}} \) to the Borel field \( \mathcal{B} \) generated by \( \tilde{\mathcal{B}} \) and by the sets \( [w; \sigma_F(w) > t] \), where \( F \) is any closed set of \( E \) and \( t \) runs over \([0, +\infty]\). It is clear that every passage time \( \sigma_A \) is a measurable function (random time) from \([W, \mathcal{B}] \) into \([0, +\infty]\). Next, given a random time \( \sigma \), we shall define the stopped path \( w_{\sigma} \) and the shifted path \( w_{\sigma}^t \) as follows:

\[
(w_{\sigma})_t = x_{\min\{\sigma, t\}}(w), \quad (w_{\sigma}^t)_t = x_{\sigma + t}(w).
\]

It is easily shown from the definitions of \( W \) and \( \mathcal{B} \) that \( w_{\sigma} \in W \), \( (w; w_{\sigma} \in B) \in \mathcal{B} \) and consequently \( \mathcal{B}_\sigma \subset \mathcal{B} \), where \( B \) is any set belonging to \( \mathcal{B} \) and \( \mathcal{B}_\sigma \) is the totality of the sets \( (w; w_{\sigma} \in B) \), i.e. the Borel field generated by the sets \( (w; (w_{\sigma})_t \in A) \) and \( (w; \sigma_F(w_{\sigma}) < t) \).

We shall now introduce the notion of a Markov time.

**Definition 1.1.** A random time \( \sigma(w) \) is called a Markov time if \( (w; \sigma(w) \geq t) \in \mathcal{B}_\sigma \) for every \( t \geq 0 \).

We shall mention two lemmas necessary for later considerations without proof.

**Lemma 1.1.** Every Markov time \( \sigma(w) \) is measurable with respect to \( \mathcal{B}_\sigma \),

\[
= \bigcap_{\sigma > 0} \mathcal{B}_\sigma.
\]

**Lemma 1.2.** Every passage time \( \sigma_A(w) \) is a Markov time. Especially, if \( A \) is closed, \( \sigma_A \) is not only measurable with respect to \( \mathcal{B}_{\sigma_A} \), but also to \( \mathcal{B}_{\sigma_A} \).

Next we denote by \( P \) a system \((P_\sigma(\cdot); x \in E)\) of probability measures on \((W, \mathcal{B})\) which satisfy the following conditions:

\begin{itemize}
  \item[(P. 1)] \( P_\sigma(B) \) is measurable as a function of \( x \) for every fixed \( B \in \mathcal{B} \).
  \item[(P. 2)] \( P_\sigma(x_0(w) = x) = 1 \) for every \( x \in E \).
  \item[(P. 3)] \( \mathcal{B} \) coincides with \( \tilde{\mathcal{B}} \) up to \( P_\sigma \)-probability 0 for each \( x \).
\end{itemize}

1) We don't assume the existence of the left limit \( \lim_{t \downarrow \infty} x_t(w) \).
2) c.f. K. Ito and H. P. McKean [6].
3) This means that the completion of \( \tilde{\mathcal{B}} \) with respect to \( P_\sigma \) includes \( \mathcal{B} \). For example, this is true under the hypothesis (A) in Hunt [4].
For every \( x \in \bar{E}, t \geq 0 \) and any bounded Borel function \( f(w) \) on \((W, \mathfrak{B})\),
\[
(1.4) \quad E_\omega(f(w_t))|_{\mathfrak{B}_t} = E_\omega(f(w)) \quad \text{with } P_\omega\text{-probability 1.}
\]
A combination \((W, \mathfrak{B}, P)\) (or simply \( x_t \)) is called a Markov process on \( E \).

Finally, we shall define several notions concerning Markov processes on \( E \).

**Definition 1.2.** We consider two points \( x \) and \( y \) of \( \bar{E} \). Then if \( P_\omega(\sigma_\omega < +\infty) > 0 \)
for every open set \( V \ni y, y \) is said to be accessible from \( x \) and we use the notation \( x \rightarrow y \). If \( x \rightarrow y \) and \( y \rightarrow x \), we say that \( x \) and \( y \) have communication.

**Definition 1.3.** If \( P_\omega(w; \sigma_\omega < +\infty) = 0 \), then we say that the process starting at \( x \) (or briefly \( x \)) is conservative on \( E \).

**Definition 1.4.** \( x \) is called a recurrent point if \( P_\omega(\sigma_\omega < +\infty) < 0 \) holds for any open sets \( U \) and \( V (\bar{V} \subset U) \) containing \( x \).

**Definition 1.5.** \( x \) is called a trap if
\[
P_\omega(x_t(w) = x \text{ for every } t \geq 0) = 1.
\]
According to this terminology, \( \infty \) is a trap.

**Definition 1.6.** An open or closed set \( S \) containing \( x \) is called a sojourn set with the center \( x \), if
\[
P_\omega(\sigma_\omega = \sigma_\omega > 0, 1)
\]
or equivalently, if
\[
P_\omega(\sigma_\omega = \sigma_\omega > 0) > 0.
\]

**§2 Hypothesis (H. 1) and its results.**

First we introduce several notations;
\[
P(t, x, \cdot) = P_\omega(x_t \in \cdot),
\]
\[
P_\omega(t, x, \cdot) = P_\omega(x_t \in \cdot, \sigma_x > t) \quad \text{for any open or closed set } A,
\]
\[
T_t f(x) = E_\omega(f(x_t)) = \int_{x_t} f(y) P(t, x, dy),
\]
\[
G_\alpha f(x) = E_\omega(\int_0^\infty e^{-\alpha t} f(x_t) dt) = \int_0^\infty e^{-\alpha t} T_t f(x) dt \quad \text{for } \alpha > 0.
\]
Here \( f(x) \) is a Borel function on \( \bar{E} \).

We now denote by \( \mathcal{C} \) the totality of functions which are continuous, bounded on \( E \) and equal to 0 at \( \infty \). In the sequel we always assume that our process \( x_t \) satisfies the next hypothesis:

\[
(H. 1) \quad G_\alpha \text{ maps } \mathcal{C} \text{ into } \mathcal{C} \text{ for every } \alpha > 0.
\]

---

1) The complement of a set is always considered with respect to \( \bar{E} \).
This hypothesis is a little weaker than the continuity condition concerning $T_t$ which is usually assumed:\(^1\):

(H. 1)' $T_t$ maps $\mathbb{C}$ into $\mathbb{C}$ for every $t \geq 0$.

We now prove some theorems under (H. 1).

THEOREM 2.1. \textbf{(Strong Markov Property)} If $x_t$ satisfies (H. 1), then for every Markov time $\sigma$ and for any bounded Borel function $f(w)$ on $(W, \mathcal{B})$,

(2.1) $E_x(f(w)) \mid \mathcal{B}_\sigma = E_{x_\sigma}(f(w))$

holds with $P_x$-probability 1.\(^2\)

\textit{Proof.} Since $\infty$ is a trap according to (P. 3), it suffices to show that for every $f \in \mathcal{C}$ and any $B \in \mathcal{B}_\sigma$,

(2.2) $E_x(f(x_{\sigma+})) \mid B = E_x(E_{x_\sigma}(f(x_\sigma)) \mid B).$\(^3\)

Making use of (H. 1) and performing the same calculation as K. Itô \cite[p. 15]{Ito}, we see that for any $B \in \mathcal{B}_\sigma$,

(2.3) $E_x\left(\int_0^\infty e^{-at}f(x_{\sigma+t})dt \mid B\right) = E_x\left(E_{x_\sigma}\left(\int_0^\infty e^{-at}f(x_\sigma)dt\right) \mid B\right)$.

Hence putting

(2.4) $g(t) = E_x(f(x_{\sigma+t}) \mid B), \quad h(t) = E_x(E_{x_\sigma}(f(x_\sigma)) \mid B),$

we have

$\int_0^\infty e^{-at}g(t)dt = \int_0^\infty e^{-at}h(t)dt$.

According to the right continuity of the path, $g(t)$ and $h(t)$ are right continuous, so that it follows from the uniqueness of Laplace transform that for every $t \geq 0$

$g(t) = h(t),$

which is what we wanted to show.

THEOREM 2.2. \textbf{The accessible relation is transitive.} Strictly speaking, if $x \rightarrow y$ and $y \rightarrow z$, then $x \rightarrow z$.

To prove this we shall first show

\textbf{Lemma 2.1.} \textbf{Suppose that} $x \rightarrow y$. Then given any open set $V \subset y$, there exist a $k > 0$ and an open set $U \ni x$ such that for every $\xi \in \bar{U}$

(2.5) $E_\xi(e^{-a\xi}) \geq k$.

---

1) G. Hunt \cite[p. 360]{Hunt} has proved that (H. 1)' is equivalent to (H. 1) under some conditions.

2) K. Ito \cite{Ito} proved this fact under (H. 1)'.

3) $E_x(f(w)) \mid B = \int_B f(w) P_x(dw)$ for $B \in \mathcal{B}$ and a Borel function $f(w)$ on $(W, \mathcal{B})$.

4) $\bar{U}$ means the closure of $U$, i.e. the smallest closed set containing $U$. 
**Proof.** We choose an open set $V'$ whose closure is contained in $V$. Since $x \rightarrow y$, we have

\begin{equation}
E_t(e^{-v'}) > 0.
\end{equation}

According to Urysohn's Lemma there exists a continuous function $f \geq 0$ which is equal to 1 on $\overline{V}'$ and to 0 on $V'$. Now we shall show that

\begin{equation}
E_x(f(x_t)) > 0 \text{ for some } t_0 \geq 0.
\end{equation}

For this purpose we assume that $E_x(f(x_t)) = 0$ for every $t \geq 0$. Then, since $f$ is non-negative and equals to 1 on $\overline{V}'$,

$$P_x(x_t \in V') = 0 \text{ for every } t \geq 0.$$ 

Hence it follows from the right continuity of the path that

$$P_x(x_t \in \overline{V}' \text{ for some } t \geq 0) = 0,$$

which implies $P_x(\sigma' < +\infty) = 0$. This contradicts with (2.6). Thus (2.7) has been proved. Hence again using the right continuity of the path, we have

\begin{equation}
E_x(f(x_t)) > 0 \text{ for every } t \epsilon (t_0, t_0 + \epsilon),
\end{equation}

where $\epsilon$ is a certain positive constant. From this we get

\begin{equation}
E_x\left(\int_0^\infty e^{-t}f(x_t)dt\right) \geq k.
\end{equation}

Therefore making use of (H.1), there exist a $k > 0$ and an open set $U \ni x$ such that for every $\xi \in \overline{U}$

\begin{equation}
E_\xi\left(\int_0^\infty e^{-t}f(x_t)dt\right) \geq k.
\end{equation}

On the other hand since $f = 0$ on $V'$, we get for any $\xi \in E$

\begin{equation}
E_\xi\left(\int_0^\infty e^{-t}f(x_t)dt\right) \leq E_\xi\left(\int_0^\infty e^{-t}dt\right) = E_\xi(e^{-\tau}).
\end{equation}

Thus the lemma was completely proved.

**Proof of Theorem 2.2.** We take an arbitrary open set $V \ni z$. Since $y \rightarrow z$, replacing $x$ by $y$ and $y$ by $z$ in the previous lemma, we have

\begin{equation}
E_\eta(e^{-v'}) \geq k \text{ for every } \eta \in \overline{U},
\end{equation}

where $U$ is an open set containing $y$.

We now put $\sigma'(w) = \sigma_U(w) + \sigma_U(w_{\tau'})$. Since it is evident that $\sigma \leq \sigma'$, we have

\begin{equation}
E_x(e^{-v'}) \geq E_x(e^{-\tau}).
\end{equation}
Here we have used the fact that $x_{\sigma_v} \in U'$ and that $E_a(e^{-\sigma_v}) > 0$.

Finally, we shall give a lemma which is useful for §4.

**Lemma 2.2.** If $x$ is not a trap, there exist a $k > 0$ and an open set $U \ni x$ such that

$$E_{\xi}(\sigma_v) \leq k \quad \text{for every} \quad \xi \in U'.$$

This lemma is essentially due to E. B. Dynkin. The proof of K. Itô [5] under the assumption $(H. 1)'$ is also available to our case under $(H. 1)$.

§ 3. Conservative property.

In this section we assume also $(H. 1)$.

**Theorem 3.1.** The following three conditions are equivalent to each other.

1. The process starting from $x \in E$ is conservative, that is, $P(t, x, E) = 1$ for every $t \geq 0$.
2. For every sojourn set $S$ with the center $x$,

$$E_a(\sigma_S) = \int_0^\infty P(t, x, S) dt = \infty.$$

3. For every open sojourn set $S$ with the center $x$,

$$E_a\left(1_{\tau_S}(x_S) dt\right) = \int_0^\infty P(t, x, S) dt = \infty,$$

where $1_{\tau_S}$ is the indicator function of $S$.

**Proof.** First, it is easily shown that the condition (1) implies the condition (2). In fact, if (1) holds, then $P_a(\sigma_v = \infty) = 1$. Hence by the definition of sojourn sets, $P_a(\sigma_v = \infty) > 0$. From this we get

$$E_a(\sigma_v) = \infty.$$

Next it is evident that the second condition implies the third condition. Finally, we shall show that the third condition implies the first condition. For this purpose, suppose that the condition (1) does not hold, namely that $E_a(e^{-\sigma_v}) > k$ for some $k > 0$.

Let $1_{E}$ be the indicator function of $E$. Then we have

$$G(x, \cdot) = \int_0^\infty P(t, x, \cdot) dt$$

is called Green measure, which may take $+\infty$.\"
Some general properties of Markov processes

\[ E_x \left( \int_0^\omega e^{-\xi_t} (x_t) \, dt \right) = E_x \left( \int_0^\omega e^{-t} \, dt \right) = 1 - E_x(e^{-\sigma_\omega}). \]

But, since \( \xi_t \) belongs to \( \mathcal{G} \), \( E_x(e^{-\sigma_\omega}) \) is continuous in \( x \). Hence we see that \( U = \{ y \in E, E_y(e^{-\sigma_\omega}) > k \} \) is an open set containing \( x \). Thus it is enough to show that (i) \( U \) is a sojourn set with the center \( x \) and that (ii)

(3.3) \[ \int_0^\omega P(t, x, U) \, dt < +\infty. \]

(i) Suppose that \( U \) is not a sojourn set with the center \( x \), that is, \( P_x(\sigma_\omega = \sigma_{E-U}) = 0 \). Then we have

\[ P_x(\sigma_\omega = \sigma_{E-U} < +\infty) = 1, \]

so that noting that, \( U \) is open and that every sample path is right continuous, it is easily shown that

\[ x_{\sigma_\omega} = x_{E-U} e^{E-U} \text{ with } P_x \text{-probability } 1. \]

Hence from the definition of \( U \) we get

\[ E_x(e^{-\omega \sigma_\omega}) \leq k \text{ with } P_x \text{-probability } 1. \]

We now calculate \( E_x(e^{-\sigma_\omega}) \).

\[ k \leq E_x(e^{-\sigma_\omega}) = E_x(e^{-\sigma_{E-U}(w)} - e_{\omega E-U}(w)) \leq E_x(e^{-\sigma_{E-U}(w)} e_{\sigma_{E-U}(w)}) \leq k. \]

This is a contradiction.

(ii) We have for any \( y \) of \( U \)

\[ k \leq E_y(e^{-\omega \sigma_\omega}) \leq P_y(\sigma_\omega \leq t) + e^{-t}, \]

so that, for a large \( t_0 \)

\[ 0 < k' = k - e^{-t_0} < P_y(\sigma_\omega \leq t_0) \]

holds whenever \( y \in U \). Therefore we get for every \( y \in U \)

\[ P(t_0, y, E) = 1 - P_y(\sigma_\omega \leq t_0) < 1 - k'. \]

Hence it is evident that a Borel set \( A = \{ y \in E, P(t_0, y, E) < 1 - k' \} \) contains \( U \). Thus it is enough to show that

(3.4) \[ \int_0^\omega P(t, x, A) \, dt < +\infty. \]

To do this, we shall prove that for any \( n \geq 0 \)
holds independently of \( t \geq 0 \). Noting that \( \infty \) is a trap, we have by the definition of \( A \)

\[
P((n+1)t_0+t, x, E) = \int_A P(nt_0+t, x, dy) P(t_0, y, E) + \int_{E-A} P(nt_0+t, x, dy) P(t_0, y, E) 
\leq (1-k') P(nt_0+t, x, A) + P(nt_0+t, x, E-A).
\]

Consequently,

\[
P((n+1)t_0+t, x, E) + P(nt_0+t, x, A) 
\leq (1-k') P(nt_0+t, x, A) + P(nt_0+t, x, E).
\]

Repeating the same calculation as above, we get

\[
S_{n+1}(t) \leq P((n+1)t_0+t, x, E) + \sum_{i=1}^n P(it_0+t, x, A) 
\leq (1-k') \sum_{i=0}^n P(it_0+t, x, A) + P(t_0, x, E) 
\leq (1-k') S_n(t) + 1.
\]

Since \( S_n(t) \leq 1 \), (3.5) is obtained by induction. Now making use of (3.5) to calculate the left side of (3.4), we have

\[
\int_0^\infty P(t, x, A) dt = \int_0^{t_0} \sum_{i=0}^n P(it_0+t, x, A) dt 
\leq \frac{t_0}{k'} < +\infty.
\]

This completes the proof of Theorem 3.1.

Theorem 3.2. If \( x \) is conservative on \( E \) and if \( x \rightarrow y \), then \( y \) is also conservative on \( E \).

Proof. Suppose that \( y \) is not conservative on \( E \). Then since \( E_y(e^{-\sigma_y}) \) is continuous with respect to \( \eta \), it is greater than some positive \( k \) on the closure of an open set \( V \) containing \( y \).

We now put \( \sigma'_v = \sigma_v + \sigma_{w_1} \). Noting that \( \sigma'_v \geq \sigma_v \), we have

\[
E_w(e^{-\sigma'_v}) \geq E_w(e^{-\sigma_v}) 
= E_w(e^{-\sigma_v} E_{y\mid \{x \rightarrow y\}}(e^{-\sigma_y})) 
\geq k \cdot E_{y\mid \{x \rightarrow y\}}(e^{-\sigma_y}) > 0.
\]

To get the last inequality, we have used the relation \( x \rightarrow y \). Therefore it turns out that \( x \) is not conservative on \( E \), contrary to the assumption.

Next we apply Theorem 3.1 to the Markov processes whose sample path has
only jumps with probability 1. Then we can obtain Feller's theorem concerning so called purely discontinuous Markov processes. But here, for short, we consider the case when \( E \) is a denumerable space, i.e. \( E = \{1, 2, 3, \ldots\} \). In this case our processes defined in §1 automatically have the following properties.

1. Every path \( u \) is a right continuous step function for \( t < \sigma_u \).

2. Since the hypothesis (H.1) is trivially satisfied, the strong Markov property holds for every Markov time (c.f. Theorem 2.1).

3. Define the first jumping time by \( \sigma_i(w) = \sup \{t: x_\nu(w) = x_i(w) \text{ for every } s \leq t\} \). Since \( \sigma_i(w) \) is a Markov time, it follows from 2 that

\[
P(x, \sigma_1(w) \geq t) = e^{-p(x)t},
\]

where \( p(x) \) is a non-negative number which cannot take \( +\infty \) (from the right continuity of path and (P. 2)). If \( p(x) = 0 \), \( x \) is a trap.

4. If \( x \) is not a trap, put

\[
\Pi(x, y) = \Pi(x, \sigma_1(y)),
\]

If \( x \) is a trap, put

\[
\Pi(x, y) = \delta_{xy}^1
\]

Then

\[
\Pi = \left(\begin{array}{cccc}
\Pi(x, y), & x \downarrow 1, 2, \ldots, \infty \\
y \rightarrow 1, 2, \ldots, \infty
\end{array}\right)
\]
is a strictly stochastic matrix on \( \overline{E} \).

Further we denote by \( \Pi^n(x, y) \) the element of the matrix \( \Pi^n = \Pi \cdot \Pi \cdots \Pi \) \((n \geq 1)\). \( \Pi^0 \) is, by definition, the identity matrix on \( \overline{E} \).

Finally we denote by \( \Pi_A \) the restriction of \( \Pi \) to a set \( A \) which is defined by

\[
\Pi_A(x, y) = \begin{cases} 
\Pi(x, y) & \text{if } x, y \in A, \\
0 & \text{otherwise}.
\end{cases}
\]

We understand \( \Pi_A(x, y), \Pi_A^n \) in the same way as for \( \Pi \).

We shall now characterize the conservative property by means of \( p \) and \( \Pi \). These quantities, however, are determined by the generator of the process, so that we can say that the conservative property is characterized by the generator.

**Theorem 3.3.** (W. Feller) The following three conditions are equivalent to each other.

1. The process starting at \( x \) is conservative, that is,

\[
P(t, x, E) = 1 \text{ for every } t \geq 0.
\]

1) \( \delta_{x,y} = \begin{cases} 
1 & \text{if } x = y, \\
0 & \text{otherwise}.
\end{cases} \)
\[(3.10) \quad \sum_{x \in E} P^n(x,y) = 1 \quad (n=0,1,2,\ldots),\]
and
\[(3.11) \quad \sum_{n=0}^{\infty} \sum_{x \in E} P^n(x,y) \frac{1}{p(y)} = \infty,\]
for \(A\) satisfying \((3.12)\).

To prove this we shall prepare two lemmas.

**Lemma 3.1.** Suppose that \((3.10)\) holds. Then \(A\) is a sojourn set with the center \(x\) if and only if \((3.12)\) holds.

**Proof.** It is enough to show that
\[\lim_{n \to \infty} \sum_{x \in E} P^n(x,y) = P_x(\sigma_x = \sigma_\infty).\]
For this we define jumping times:
\[
\sigma_0(w) \equiv 0 \quad \text{(0-th jumping time),}
\]
\[
\sigma_1(w) = \text{first jumping time},
\]
\[
\sigma_2(w) = \sigma_1(w) + \sigma_1(w^+_1) \quad \text{(second jumping time),}
\]
\[
\vdots
\]
\[
\sigma_n(w) = \sigma_{n-1}(w) + \sigma_i(w^+_n) \quad \text{\((n\)-th jumping time).}
\]
Here if \(\sigma_k = \infty\), we put \(\sigma_{k+1} = \sigma_{k+2} = \cdots = \infty\).

First we shall prove that the condition \((3.10)\) implies
\[(3.14) \quad P_x(\sigma_n < \infty, x_{\sigma_n} = \infty) = 0.\]
In fact, if \((3.14)\) is not true, we have
\[
0 < P_x(\sigma_n < \infty, x_{\sigma_n} = \infty)
\]
\[
= P_x(x_{\sigma_1} \in E, x_{\sigma_2} \in E, \ldots, x_{\sigma_{n-1}} \in E, x_{\sigma_n} = \infty)
\]
\[
= \sum_{x \in E} P(x,y) P_x(x_{\sigma_1} \in E, \ldots, x_{\sigma_{n-1}} \in E, x_{\sigma_{n-1}} = \infty)
\]
\[
= P^n(x, \infty).
\]
This contradicts \((3.10)\).
Next it follows from (3.14) that if \( \sigma_{n-1} < \infty \) and if \( \sigma_n = \infty \), then \( x_{\sigma_{n-1}} \) are traps with \( P_x \)-probability 1. Hence we have

\[
P_x(\sigma_d = \sigma_n) = \lim_{n \to \infty} \left\{ \sum_{k=1}^{n} P_x(x_{\sigma} \in A, i < k; \sigma_{k-1} < \infty, \sigma_k = \infty) + P_x(x_{\sigma} \in A, i \leq n; \sigma_n < \infty) \right\}.
\]

To calculate the right side of the above equality we shall denote by \( T \) the totality of traps contained in \( A \). Then we have

\[
P_x(x_{\sigma} \in A, i < k; \sigma_{n-1} < \infty, \sigma_n = \infty)
= \sum_{x \in T} \sum_{y \in A-T} P_{x-y}^{k-1}(x,y) P(y, z)
\]

\[
= \sum_{x \in T} \left( \sum_{y \in \Delta} P_{x-y}^{k-1}(x,y) P(y, z) - \sum_{y \in \Delta} P_{x-y}^{k-2}(x,y) P(y, z) \right)
\]

\[
= \sum_{x \in T} \left( P_{x-z}^{k-1}(x,z) - P_{x-z}^{k-2}(x,z) \right),
\]

\[
P_x(x_{\sigma} \in A, i \leq n; \sigma_n < \infty) = \sum_{y \in A-T} P_{x-y}^{n-1}(x,y).
\]

Therefore we have

\[
\sum_{k=1}^{n} P_x(x_{\sigma} \in A, i < k; \sigma_{n-1} < \infty, \sigma_n = \infty) + P_x(x_{\sigma} \in A, i \leq n; \sigma_n < \infty)
= \sum_{k=1}^{n} \sum_{x \in T} \left( P_{x-y}^{k-1}(x,z) - P_{x-y}^{k-2}(x,z) \right) + \sum_{y \in A-T} P_{x-y}^{n-1}(x,y)
\]

\[
= \sum_{x \in T} P_{x-z}^{n-1}(x,y) + \sum_{y \in A-T} P_{x-y}^{n-1}(x,y)
\]

\[
= \sum_{x \in T} P_{x-z}^{n-1}(x,y).
\]

Thus Lemma 3.1 was proved.

Next we shall introduce several notations:

\[
P = \begin{pmatrix} p(1) & 0 \\ p(2) & 0 \\ \vdots & \vdots \end{pmatrix}, \quad I = \text{identity matrix},
\]

\[
P_d(t) = \begin{pmatrix} p(1) & 0 \\ p(2) & 0 \\ \vdots & \vdots \end{pmatrix} \begin{pmatrix} x \downarrow 1, 2, 3, \cdots, \infty \\ y \downarrow 1, 2, 3, \cdots, \infty \end{pmatrix},
\]

\[
G_d(\alpha) = \begin{pmatrix} p(1) & 0 \\ p(2) & 0 \\ \vdots & \vdots \end{pmatrix} \begin{pmatrix} x \downarrow 1, 2, 3, \cdots, \infty \\ y \downarrow 1, 2, 3, \cdots, \infty \end{pmatrix},
\]

where

\[
G_d(\alpha, x, y) = \int_0^\infty e^{-\alpha t} P_d(t, x, y) dt.
\]

**Lemma 3.2** For any set \( A \) and \( \alpha \geq 0 \),

\[
G_d(\alpha) = \sum_{n=0}^{\infty} ((\alpha I + P)^{-1} P \Delta)^n \cdot (\alpha I + P)^{-1}.
\]

If \( \alpha = 0 \), the both sides of (3.16) may take \( +\infty \).

**Proof.** Define
then we can see that it is equal to
\[ P^*_n(t, x, y) = P^*_0(x_t = y, \sigma_n \leq t < \sigma_{n+1},) \]

Hence we have
\[ G_\alpha(x, y) = \int_0^t e^{-at}P^*_n(t, x, y)dt = C(\alpha + 1)P(x, y). \]

Noting that
\[ P_\alpha(t, x, y) = \sum_{n=0}^\infty P^*_n(t, x, y), \]
(3.16) is obtained immediately.

**Proof of Theorem 3.3.** Summing up Theorem 3.1, Lemma 3.1 and Lemma 3.2, it is evident that the condition (3) implies the condition (1). Hence it suffices to show that the condition (1) implies (3.10). But this is directly derived by the definition of II.

**§4. Recurrence**

In this section we shall assume the following hypothesis (H. 2) besides (H. 1):

(H. 2) Given any closed set \( F \) for every \( f \in \mathfrak{F} \),

\[ E_\alpha(f(x_{\sigma_T})) = \int f(y)h_\alpha(x, dy) \]

belongs to \( \mathfrak{F} \).

Here \( h_\alpha(x, \cdot) \) is the distribution of \( x_{\sigma_T} \) which we call the harmonic measure over \( F \) induced by the process \( x_t \), since this measure is exactly the function-theoretical harmonic measure in case \( x_t \) is the two-dimensional Brownian motion.

Using this notation, we have
\[ P_\alpha(\sigma_T < +\infty) = h_\alpha(x, F). \]

**Theorem 4.1.** \( x \) is a recurrent point if and only if

\[ \int_0^\infty P(t, x, U)dt = \infty \]

for every open set \( U \) containing \( x \).

**Proof.** (i) First we shall show that if \( x \) is a recurrent point, then (4.3) will hold. Suppose in the contrary that for some open set \( U \ni x \)

\[ \int_0^\infty P(t, x, U)dt < \infty. \]
The condition (4.4) implies $E_a(\sigma_{t \varepsilon}) < \infty$, because

\[ \int_0^\infty P(t, x, U) dt = E_a \left( \int_0^\infty \chi_U(x_t) dt \right) \leq E_a \left( \int_0^\infty \chi_U(x_t) dt \right) = E_a (\sigma_{t \varepsilon}), \]

where $\chi_U$ is the indicator function of $U$.

Let $f$ be a continuous function which equals 1 at $x$, 0 on $U^c$ and lies between 0 and 1 elsewhere. Then we have

\[ \lim_{\alpha \to 0} G_\alpha f(x) = G_{0 \alpha} f(x) \leq \int_0^\infty P(t, x, U) dt < +\infty. \]

Therefore, given any small $\varepsilon > 0$, there exists an $\alpha_0$ such that

\[ G_{0 \alpha} f(x) > G_{\alpha} f(x) - \varepsilon. \]

Hence using (H.1), we get

\[ G_{0 \alpha} f(\xi) > G_{\alpha} f(x) - \varepsilon, \]

whenever $\xi$ runs over a certain open set $V$ containing $x$ such that $\overline{V} \subset U$. But since $G_{\alpha} f(\xi)$ is monotone non-increasing as a function of $\alpha$ for any fixed $\xi$, we have

\[ G_{\alpha} f(\xi) \geq G_{0 \alpha} f(x) - \varepsilon \quad \text{for every } \xi \in \overline{V} \text{ and for every } 0 \leq \alpha \leq \alpha_0. \]

On the other hand it follows from the definition of recurrence and from $E_a(\sigma_{t \varepsilon}) < \infty$ that

\[ P_a(\sigma_{t \varepsilon} + \sigma_{(t \varepsilon)^+}) < +\infty) = 1. \]

For short we denote $\sigma_{t \varepsilon}$ by $\sigma_1$ and $\sigma_{t \varepsilon} + \sigma_{(t \varepsilon)^+}$ by $\sigma_2$, respectively. Then according to (4.6),

\[ \lim_{\alpha \to 0} E_a(e^{-\alpha \xi}) = P_\alpha(\sigma_2 < +\infty) = 1. \]

We now calculate $G_{\alpha} f(x)$ for $0 < \alpha \leq \alpha_3$.

\[ G_{\alpha} f(x) \geq G_{\alpha} f(x) = E_a \left( \int_0^\infty e^{-\alpha t} f(x_t) dt \right) \]

\[ \geq E_a \left( \int_0^{\alpha_1} e^{-\alpha t} f(x_t) dt \right) + E_a(e^{-\alpha \xi}; G_{\alpha} f(x_{\alpha_2})) \]

\[ \geq E_a \left( \int_0^{\alpha_1} e^{-\alpha t} f(x_t) dt \right) + (G_{1 \alpha} f(x) - \varepsilon) E_a(e^{-\alpha \xi_1}). \]
Hence we have

\[ G_{0}f(x) \geq \frac{\int_{0}^{\sigma_{1}} e^{-\alpha t} f(x_{t}) dt - \varepsilon \cdot E_{\alpha}(e^{-\alpha \sigma_{2}})}{1 - E_{\alpha}(e^{-\alpha \sigma_{2}})}. \]  

(4.8)

As \( \alpha \) tends to 0, the right side of (4.8) goes to \( \infty \). This is a contradiction.

(ii) Suppose that \( x \) is not a recurrent point. According to Lemma 2.2 and the definition of recurrence, we can choose some constant \( k > 0 \) and two open sets \( U, U' \) containing \( x \) such that

\[ E_{\xi}(\sigma_{0}^{<}) \leq k \quad \text{for every} \quad \xi \in U', \]

(4.9)

and that

\[ P_{\xi}(\sigma_{0}^{<} + \sigma_{U}^{+}(w_{U}^{+}) < + \infty) = \int_{U'} h_{U}^{+}(x, d \theta) h_{U'}(y, U') < 1. \]

(4.10)

Using (H.2), it is easily shown that the right side of (4.10) is continuous in \( x \), so that, for a certain \( \varepsilon > 0 \) and for some open neighbourhood \( V(\bar{V} \subset U') \) of \( x \) we have

\[ P_{\xi}(\sigma_{0}^{<} + \sigma_{U}^{+}(w_{U}^{+}) < + \infty) \leq 1 - \varepsilon, \]

(4.11)

whenever \( \xi \) runs over \( \bar{V} \). Hence noting that

\[ \sigma_{0}^{+} + \sigma_{U}^{+}(w_{U}^{+}) \leq \sigma_{0}^{+} + \sigma_{U}^{+}(w_{U}^{+}), \]

we get from (4.11)

\[ P_{\xi}(\sigma_{0}^{+} + \sigma_{U}^{+}(w_{U}^{+}) < + \infty) \leq 1 - \varepsilon \quad \text{for every} \quad \xi \in \bar{V}. \]

(4.12)

We now define

\[ \sigma_{0}(w) = 0, \quad \sigma_{1}(w) = \sigma_{0}(w), \quad \sigma_{2}(w) = \sigma_{1}(w) + \sigma_{U}^{+}(w_{U}^{+}), \]

(4.13)

\[ \sigma_{2n-1}(w) = \sigma_{2n-2}(w) + \sigma_{U}^{+}(w_{U}^{+}), \quad \sigma_{2n}(w) = \sigma_{2n-1}(w) + \sigma_{U}^{+}(w_{U}^{+}) \]

if \( \sigma_{i} = \infty \), then we put \( \sigma_{i+1} = \sigma_{i+2} = \ldots = \infty. \)

According to (4.12) we get

\[ P_{\xi}(\sigma_{2n} < + \infty) \leq (1 - \varepsilon)^{n}. \]

Using this and (4.9), we shall calculate \( G_{\alpha} \chi_{U'}(x) \).

\[ G_{\alpha} \chi_{U'}(x) = E_{\alpha}(\int_{0}^{\sigma_{1}} e^{-\alpha t} \chi_{U'}(x_{t}) dt) = \sum_{n \geq 2} E_{\alpha}(\int_{\sigma_{2n}}^{\sigma_{2n+1}} e^{-\alpha t} \chi_{U'}(x_{t}) dt). \]

1) The first term of the numerator is positive, decreasing as a function of \( \alpha \) and independent of the \( \varepsilon \). Hence we may assume that it exceeds \( \varepsilon \) for every \( \alpha \leq \alpha_{0} \).

2) Added in proof: Here we have used the fact that \( \sigma_{n} \neq \infty \) with \( P_{\alpha} \) probability 1, which we shall show in the proof of Lemma 4.2.
Some general properties of Markov processes

$$E_x\left(\int_{\sigma_{2n}^a} e^{-\alpha t} \chi(x_t) dt\right) \leq E_x(e^{-\alpha t_{2n}} E_{\sigma_{2n}^a} \left(\int_0^{\sigma_1} e^{-\alpha t} \chi(x_t) dt\right) \leq k(1-\varepsilon)^n.$$

Hence we get

$$G_\alpha \chi(x) \leq k \sum_{n=0}^{\infty} (1-\varepsilon)^n = \frac{k}{\varepsilon} < +\infty.$$

But since the right side of (4.14) is independent of $\alpha$,

$$\int_0^\infty P(t, x, V) dt = G_\alpha \chi(x) = \lim_{\alpha \to 0} G_\alpha \chi(x) \leq \frac{k}{\varepsilon} < +\infty,$$

which is what we wanted to show.

**Corollary.** Every recurrent point is conservative.

Combining Theorem 3.1 and Theorem 4.1, our statement is evident.

Next we shall give two lemmas useful for the subsequent theorems.

**Lemma 4.1.** If $x$ and $y$ have communication with probability 1, both $x$ and $y$ are recurrent.

**Proof.** It is enough to prove that $x$ is recurrent. For this purpose, given any pair of open sets $U$ and $U'$ containing $x$ such that $U' \subset U$ and that $U \ni y$, we shall show

$$P_x(\sigma_{U'} < +\infty) = 1.$$

Using the assumption $P_y(\sigma_{U'} < +\infty) = 1$ and (H.2), given an arbitrary small $\varepsilon > 0$, there exists an open set $V$ containing $y$ such that

$$P_y(\sigma_U < +\infty) \geq 1 - \varepsilon$$

for every $y \in V$.

With no loss of generality we may assume $\bar{U} \cap V = \emptyset$. But since $P_x(\sigma_U < +\infty) = 1$ by the assumption, we have

$$P_x(\sigma_U < +\infty) = E_x(P_{\sigma_U}(\sigma_U < +\infty)) \geq 1 - \varepsilon.$$

Hence noting that $\sigma_U + \sigma_{U'}(w_{U,V}) \leq \sigma_U + \sigma_{U'}(w_{U,V})$, we get

$$P_x(\sigma_U + \sigma_{U'}(w_{U,V}) < +\infty) \geq 1 - \varepsilon,$$

which shows (4.15). Thus Lemma 4.1 was completely proved.

**Lemma 4.2.** Suppose that $x$ is a recurrent point which is not a trap and that $U$ is an open set containing $x$ whose closure is compact and $E_x(\sigma_U) < +\infty$.

Then given any open set $V \subset U$, $\sigma_n$ defined by (4.13) is finite with $P_x$-probability 1 for every $n$, and as $n$ goes to $\infty$, $\sigma_n \uparrow \infty$ with $P_x$-probability 1.

1) Strictly speaking, for any open $U \ni x$ and for any open $V \ni y$

$$P_x(\sigma_U < +\infty) = P_y(\sigma_U < +\infty) = 1.$$
Proof. (1) To prove the first statement it is enough to show that $\sigma_{2n} < +\infty$. From the definition of recurrence, $\sigma_2 < +\infty$. Hence according to (H.2), given an arbitrary small $\varepsilon > 0$, $V' = (\xi; \xi \in V$, $P_\xi(\sigma_2 < +\infty) > 1 - \varepsilon$) is an open set containing $\xi$. Putting $\sigma_i = \sigma_{i-2} + \sigma_i(w_{i-2})$, $\sigma_i = \sigma_i + \sigma_i(w_{i-2})$ and noting that $a_i \geq a_i$ and that $P_\xi(\sigma_i < +\infty) = 1$, we have

$$P_\xi(\sigma_i < +\infty) \geq P_\xi(\sigma_i < +\infty) = E_\xi(P_{\sigma_{i-2}}(\sigma_2 < +\infty)) \geq 1 - \varepsilon,$$

which shows $P_\xi(\sigma_2 < +\infty) = 1$. By the same argument, we have $P_\xi(\sigma_{2n} < +\infty) = 1$.

(2) Since $\sigma_n(w)$ is an increasing sequence of $n$ for any fixed $w$, $\lim_{n \to \infty} \sigma_n(w) = \sigma(w)$ is well defined for every $w$. To prove the second statement it is enough to show that $P_\xi(\sigma = +\infty) = 1$. In the contrary, if we assume that $P_\xi(\sigma < +\infty) > 0$, noting that $P_\xi(\sigma = +\infty) = 1$ (from Corollary to Theorem 4.1) we have $P_\xi(\sigma < +\infty) > 0$. Therefore, the set $\{w; \sigma(w) < \sigma_{\omega}(w)\}$ is not null. If we $(w; \sigma(w) < \sigma_{\omega}(w))$, according to (W.2), $x_{\sigma(w)} = \lim_{n \to \infty} x_{\sigma_n(w)}$ exists. But from the definition of $\sigma_{2n}$ and $\sigma_{2n+1}$, $x_{\sigma_n(w)}$ has to belong both to $V$ and to $U'$. This is impossible.\(^1\)

Theorem 4.2. If $x$ is recurrent and if $x \to y$, $y$ is also recurrent.

Proof. It is enough to show that the condition of Lemma 4.1 is satisfied.

(1) $y$ is accessible from $x$ with $P_\xi$-probability 1.

By the assumption, given any open set $V$ containing $y$, $k = P_\xi(\sigma < +\infty) > 0$. It remains only to show that $k = 1$. Now we take $U$ as in Lemma 4.2 and given any small $\varepsilon > 0$, choose an open set $U' = \{U \subseteq U\}$ containing $x$ which satisfies

$$P_\xi(\sigma < +\infty, \sigma < \sigma_{\omega}) = 1 - \varepsilon \quad \text{for every } \xi \in U'.$$

Then by the second statement of Lemma 4.2,\(^2\) there exists a certain constant $\gamma$ independent of $\varepsilon$ such that for sufficiently large $n > n_0$

$$(4.17) \quad P_\xi(\sigma < +\infty, \sigma < \sigma_{2n}) = 1 - \gamma > 0.$$  

Therefore, using the first statement of Lemma 4.2, we have

$$k = P_\xi(\sigma < +\infty) = P_\xi(\sigma < +\infty, \sigma < \sigma_{2n}) + P_\xi(\sigma < +\infty, \sigma \geq \sigma_{2n})$$

$$\geq \gamma + E_\xi(P_{\sigma_{2n}}(\sigma < +\infty); \sigma_{2n} \leq \sigma)$$

$$\geq \gamma + (1 - \gamma)(k - \varepsilon).$$

Noting that $\gamma \geq \gamma > 0$, we obtain

$$k \geq 1 - \frac{1 - \gamma}{\gamma} \cdot \varepsilon \geq 1 - \frac{1 - \gamma}{\gamma} \cdot \varepsilon.$$  

This shows that $k = 1$.

\(^1\) This part of the proof was suggested by K. Itô.

\(^2\) Lemma 4.2 is applied, replacing $V$ by $U'$. 

Takesi Watanabe
Suppose that for some open set \( U' \exists x \)
\[
P_\sigma(\sigma_\gamma < +\infty) < \alpha < 1.
\]
Here we may further assume that \( \bar{U}' \) is contained in a certain open set \( U \) for which \( P_\sigma(\sigma_\gamma < +\infty) = 1 \). Using (H. 2), we can take an open set \( V \exists y \) such that \( \forall \gamma, \bar{U} = \phi \) and
\[
P_\sigma(\sigma_\gamma < +\infty) \leq \alpha \quad \text{for every } \eta \in \bar{V}.
\]
Next we can see that for a small open \( U''(\subset \bar{U}') \) containing \( x \)
\[
(4.18) \quad P_\sigma(\sigma_\gamma < \sigma_\eta + \sigma_\gamma \omega (w_{\sigma_\gamma})') = \beta > 0. \tag{1}
\]
On the other hand, noting that \( \forall \gamma, \bar{U} = \phi \) and \( \bar{U}'(\subset \bar{U}) \), we have
\[
(4.19) \quad P_\sigma(\sigma_\gamma < +\infty) \leq \alpha \quad \text{for every } \eta \in \bar{V}.
\]
Now putting \( \sigma_2 = \sigma_\gamma + \sigma_\eta \omega (w_{\sigma_\gamma})' \), we shall calculate the left side of (4.18).
\[
\beta = P_\sigma(\sigma_\gamma < \sigma_2) = P_\sigma(\sigma_2 = \sigma_\gamma + \sigma_\eta \omega (w_{\sigma_\gamma})') = E_{\eta}(P_\sigma(\sigma_\eta < +\infty); \sigma_\eta < \sigma_2) \leq \alpha \cdot \beta < \beta.
\]
This is a contradiction. Thus Theorem 4.2 was completely proved.

Next we shall characterize recurrence by means of sojourn sets. For this purpose we shall define a special sojourn set.

**Definition 4.1.** A sojourn set \( S \) is said to be minimal if \( S \) contains no proper sojourn sets. Here we say that \( S' \) is a proper sojourn set of \( S \) if \( S' \) is a sojourn set and if \( S' \subset S \subset S' \).

We now put \( A_x = \{y; x \rightarrow y\} \). Then it is almost evident that \( A_x \) is a closed sojourn set and that \( P_\sigma(\sigma_\alpha = \sigma_\omega) = 1 \).

**Theorem 4.3.** \( x \) is recurrent if and only if \( x \) is conservative and if \( A_x \) is minimal.

**Proof.** (1) Supposing that \( x \) is recurrent, we shall show that \( P_\sigma(\sigma_\omega = \infty) = 1 \) and that \( A_x \) is minimal. First, if \( x \) is a trap, our statement is trivial. Next, if \( x \) is a recurrent point which is not a trap, according to Corollary of Theorem 4.1, \( x \) is conservative. Further as was shown in the proof of Theorem 4.2, any two points which belong to \( A_x \) have communication probability \( 1 \). This shows that \( A_x \) has no proper sojourn sets.

---

1) Define \( \sigma_{\omega n} \) as in (4.13), for a pair of \( U \) and \( U' \). Then if \( \beta = 0 \) for every open \( U''(\subset \bar{U}') \), by the consideration analogous to the first statement of Lemme 4.2, we have
\[
P_\sigma(\sigma_\gamma < \sigma_{\omega n}) = 0 \quad \text{for every } n > 0.
\]
This contradicts the facts that \( \sigma_{\omega n} \rightarrow \infty \) and that \( P_\sigma(\sigma_\gamma < +\infty) > 0 \).
(2) If $x$ is conservative, it follows from Theorem 3.2 that all the points belonging to $A_x$ are conservative. Hence if $A_x$ has no proper sojourn sets, $x$ and any fixed $y \in A$ have communication with probability 1. Therefore, according to Lemma 4.1, $x$ is recurrent.

**Theorem 4.4.** If a compact set $K$ contains no recurrent points, then for every $x \in E$

\[ \int_0^\infty P(t, x, K) < +\infty. \]  

**Proof.** Let $y$ be not recurrent. Then it follows from the proof of Theorem 4.1 that there exists an open neighbourhood $U(y)$ of $y$ such that

\[ \int_0^\infty P(t, y, U(y)) < +\infty \]

holds independently of $y \in U(y)$.

We now take such open sets for every $y \in K$. By compactness of $K$, it is possible to cover $K$ with a finite number of $U(y)$. We shall denote these sets by $U_1, U_2, \ldots, U_n$ and calculate \[ \int_0^\infty P(t, x, U_i) < +\infty \]

using the passage time $\sigma_i$ to $U_i$.

\[ \int_0^\infty P(t, x, U_i) \sigma_i \]

so that we have

\[ \int_0^\infty P(t, x, K) \leq \sum_{i=1}^n \int_0^\infty P(t, x, U_i) \sigma_i \leq +\infty \]

This completes the proof.

**Corollary.** Suppose that $E$ is compact, that at least one point belonging to $E$ is conservative and that any two points of $E$ have communication. Then every point of $E$ is recurrent.

§5. Applications of §4.

In this section we shall apply the results of the preceding section to special cases.

I. Case of $E = \{1, 2, 3, \ldots\}$.

In this case all the processes defined in §1 trivially satisfy the hypotheses (H.1) and (H.2). Therefore the criterion of recurrence by Green measures
(Theorem 4.1) is always true. Furthermore, since the topology in $E$ is discrete, $x$ is recurrent if and only if

$$\int_0^\infty P(t, x, x)dt = \infty.$$  \hspace{1cm} (5.1)

First we shall consider the case in which $x$ is not a trap. Then, putting $A=E$ and $\alpha=0$ in Lemma 3.2, we have

$$\int_0^\infty P(t, x, x')dt = \left( \sum_{n=0}^\infty \Pi^n(x, x') \right) \frac{1}{p(x)}.$$  \hspace{1cm} (5.2)

But $p(x)>0$, as $x$ is not a trap. Therefore the condition (5.1) is equivalent to

$$\sum_{n=0}^\infty \Pi^n(x, x) = \infty.$$  \hspace{1cm} (5.3)

Next, even if $x$ is a trap, (5.3) holds by the definition of $\Pi$. Thus we have

**Theorem 5.1.** Let $x_t$ be a Markov process on a denumerable state space. Then the point $x$ is recurrent if and only if (5.3) holds.

This theorem shows that the recurrence property does not depend on $p(\cdot)$ which describes the speed of our process. Analogously, using only $\Pi$, we can easily reformulate the criterion of recurrence by sojourn sets (Theorem 4.3) in the more concrete form, though the detail is omitted.

**II. Diffusion processes with Brownian hitting probabilities.**

Let $E$ be $n$-dimensional Euclidean space. The process $x_t$ on $E$ defined in §1 is called a *diffusion process with Brownian hitting probabilities* if the harmonic measures of $x_t$ coincide with those of $n$-dimensional Brownian motion, i.e. with the $n$-dimensional classical harmonic measures. In this case it is easily shown that sample paths are continuous with probability 1. Further the continuity of classical harmonic functions implies that the following condition which is stronger than (H. 2) is satisfied:

$$(H. 2)' \quad h_x(x, \cdot) \text{ maps any bounded Borel function } f \text{ into } C.$$  

Next, it follows from the definition that any two points on $E$ have communication and that the recurrence property of $x_t$ is the same as that of $n$-dimensional Brownian motion. But since $n$-dimensional Brownian motion is recurrent\(^1\) for $n \leq 2$ and not recurrent for $n>2$, assuming that our process $x_t$ satisfies (H. 1) we have

**Theorem 5.2.** Let $x_t$ be a diffusion process with Brownian hitting probabilities. Then,

\(^1\) This terminology is due to K. Itô and H. P. McKean.

\(^2\) A Markov process $x_t$ is said to be *recurrent* if every point of $E$ is recurrent with respect to $x_t$. Analogously, the non-recurrence of a process is defined by that of every point in $E$.  

\hspace{1cm}
(1) if \( n \leq 2 \), every point of \( E \) is recurrent and hence we have for every \( x \) and for every open set \( U \) (not necessarily containing \( x \))

\[
P(t, x, U)dt = \infty;
\]

(5.4)

(2) if \( n > 2 \), every point of \( E \) is not recurrent and hence we have for every \( x \) and for every compact set \( K \)

\[
P(t, x, K)dt < +\infty.
\]

(5.5)

III. Green measures of killed processes.

Let \( x_t \) be a Markov process which satisfies (H. 1) and (H. 2). Furthermore we assume that any two points of \( E \) have communication. Given an open set \( A \), the killed process \( x^*_t \) on \( A \) is defined as follows:

\[
x^*_t = x_t \quad \text{if} \quad t < \sigma_A, \\
= \infty \quad \text{if} \quad t \geq \sigma_A.
\]

(5.6)

Now we denote the transition probabilities and the Green measures of \( x^*_t \) by \( P^0(t, x, \cdot) \) and \( G^0(x, \cdot) \), respectively; namely

\[
P^0(t, x, \cdot) = P_x(x_t, \sigma_d > t),
\]

\[
G^0(x, \cdot) = \int_0^\infty P^0(t, x, \cdot)dt.
\]

Since it results from Corollary to Theorem 4.1 and from our assumption that every point of \( E - A \) is not recurrent with respect to \( x^*_t \), it is expected that for every compact set \( K \subset E - A \)

\[
G^0(x, K) < +\infty^1.
\]

(5.7)

In fact, if \( x_t \) is not recurrent, our statement is clear by \( G^0(x, K) \leq G(x, K) \) and Theorem 4.4. On the other hand, if \( x_t \) is recurrent, using the same technique as in Theorem 4.1 and Lemma 4.2, we see that, for every \( x \in E - A \) and for some open set \( U \) containing \( x \), \( G^0(\xi, U) \) is uniformly bounded whenever \( \xi \) runs over \( U \). From this we can easily derive (5.7) in the same way as in Theorem 4.4. Thus we have

**Theorem 5.3** The Green measures of the killed process are finite for every compact sets.

---

1) We cannot derive this fact directly from Theorem 4.4, because we are not sure that the killed process \( x^*_t \) satisfies (H, 1), though (H, 2) is always satisfied.
References


