The structure of rings whose quotient rings are primitive rings with minimal one sided ideals

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Recently A. W. Goldie [2] has proved that the quotient ring of a prime ring with some ascending chain condition is a simple ring with minimal condition. In this note we shall show that we can obtain the properties of a ring whose quotient ring is a primitive ring with minimal one sided ideals (P.M.I.), which are analogous to those of a prime ring in [2]. The following example shows that there exists such a ring.

Let \( I \) be the ring of rational integers. Let \( R_n \) be a sub-ring of matrix ring with infinite degree over the ring of rational numbers such that

\[
\begin{pmatrix}
(a_{ij})
\end{pmatrix}
\begin{pmatrix}
m_1

m_2

\vdots
\end{pmatrix}
\in I, \quad (a_{ij}) \in I_n.
\]

Let \( R = \bigcup_n R_n \), then if an element \( a \) of \( R \) is not zero divisor, \( a \) is the following form:

\[
a = \begin{pmatrix}
(a_{ij})
\end{pmatrix}
\begin{pmatrix}
m_1

m_2

\vdots
\end{pmatrix}
\mid a_{ij} \neq 0, \quad m_i \neq 0.
\]

Hence the right (and left) quotient ring of \( R \) is \( Q = \bigcup Q_n \):

\[
Q_n = \begin{pmatrix}
(a_{ij})
\end{pmatrix}
\begin{pmatrix}
m_1

m_2

\vdots
\end{pmatrix}
\in Q_n \quad \text{and} \quad m_i \in Q',
\]

where \( Q' \) is the ring of rational numbers, and \( Q \) is P.M.I.

In this note there are many statements which overlap [2], but we shall repeat those for the sake of completeness.

1. Preliminaries.

Let \( R \) be a ring with the right and left quotient ring \( Q \) and we shall call non zero divisor elements regular elements. We shall denote one sided ideals of \( R \) by Roman and ones of \( Q \) by German.

We have the following statements.
(1) If \( c_1, c_2, \ldots, c_n \) are regular elements of \( R \), then there exist regular elements \( d_1, d_2, \ldots, d_n \) and \( c \) such that
\[
c_i^{-1} = d_i c^{-1}.
\]
We can prove this by the induction with respect to \( n \), cf. Asano [1], and [2] Lemma 4.2.

(2) If \( A \) is a right, left and two sided ideal respectively, then \( AQ, QA \) and \( QAQ \) consist of \( ac^{-1} \), \( c^{-1} a \) and \( d^{-1}ac^{-1} \), \( a \in A \) and \( c, d \in R \), respectively.

Cf. [2] Lemma 4.3.

(3) Let \( r \) be a non zero right ideal of \( Q \), then \( \tau \cap R \neq (0) \).

Let \( S \) be a sub-set of \( Q \). We shall define the following annihilators.
\[
S_r = \{ x \in R, \ Sx = (0) \}, \\
S_r^* = \{ x \in Q, \ Sx = (0) \} \quad \text{and} \\
\tilde{S} = \{ a \in R, \text{there exists a regular element } b \text{ in } R \text{ such that} \\
b^{-1}a \in S \} \cup (S \cap R).
\]

(4) Let \( r \) be a right ideal of \( Q \), then
\[
r = (r \cap R)Q.
\]

It is clear \( r \supseteq (r \cap R)Q \). If \( x \in r \) then \( x = ac^{-1} \), \( c \in R \) and \( a = xc \in r \cap R \). Hence \( x \in (r \cap R)Q \).

(5) \[ S_r = S_r^* \cap R \quad \text{and} \quad S_r^* = S_rQ. \]

It is clear that \( S_r^* \supseteq S_rQ \). If \( a \in S_r^* \) and \( a = bc^{-1} \), \( b, c \in R \), then \( (0) = Sa = Sbc^{-1} \) hence \( b \in S_r \).

We have clearly

(6) \[ S_r^* = (\tilde{S})_r^* \quad \text{and} \quad (\tilde{S})_r = S_r^* \cap R. \]

Let \( l \) be a left ideal of \( Q \), then

(7) \[ (l \cap R)_r = l_r^* \cap R. \]

By the definition \( \tilde{l} = l \cap R \) and by (6) we have \( (l \cap R)_r = \tilde{l}_r = l_r^* \cap R \).

(8) Let \( I_r \) be a maximal annihilator in \( R \), then \( I_rQ \) is so in \( Q \).

Let \( I_r^* \) be a maximal annihilator in \( Q \), then \( I_r^* \cap R \) is so in \( R \).

It is clear that \( I_rQ \) is an annihilator. If there exists an annihilator \( I_r^* \) such that \( I_r^* \supseteq I_rQ \), then \( (l \cap R)_r = l_r^* \cap R \supseteq I_rQ \supseteq R \). By (3) \( l \cap R \neq (0) \), and \( (l \cap R)_r = R \), hence \( (l \cap R)_r = I_r \) and \( I_r^* = (l_r^* \cap R)Q = (l \cap R)_rQ = I_rQ \). Conversely let \( r \) be a maximal annihilator, then \( r \cap R \) is an annihilator in \( R \) by (7). If \( I_r \supseteq r \cap R \), by (4) we have
\[
r = (r \cap R)Q \supseteq I_rQ = I_r^*, \quad \text{hence} \quad r = I_r^* \supseteq I_r \quad \text{and} \quad r \cap R = I_r^{r}. 
\]
Let $I \subseteq (0)$ be a right ideal in $R$. We shall call maximal right ideals $J$ with $J \cap I = (0)$ complements of $I$ (denoted by $I^e$, $I^{e'}$, \ldots).

Let $I$ be a right ideal in $R$. For any complement $I^e$ of $I$ in $R$ there exists a complement $(IQ)^{e'}$ of $IQ$ such that

$$(9) \quad I^eQ = (IQ)^{e'},$$

and conversely for any complement $(IQ)^{e'}$ of $IQ$ there exists a complement $I^e$ of $I$ satisfying (9).

If $x \in (IQ \cap I^eQ)$, then $x = ic^{-1} = jd^{-1}$, $i \in I$, $j \in I^e$ and we have by (2) $c^{-1} = af^{-1}$, $d^{-1} = bf^{-1}$, hence $ia = jb \in I \cap I^e = (0)$ and $x = 0$. If there exists a right ideal $I$ such that $I^eQ \subseteq I$ and $i \cap IQ = (0)$, then $i \cap R \cap I \subseteq i \cap IQ = (0)$, hence since $I^e \subseteq i \cap R$, $I^e = i \cap R$ and $I^eQ = (1 \cap R)Q = I$. Therefore $I^eQ$ is a complement of $IQ$. Conversely let $(IQ)^{e'}$ be a complement, then from the fact $(IQ)^{e'} \cap R \cap I = (0)$, $(IQ)^{e'} \cap R \supseteq I^e$ hence $(IQ)^{e'} = (IQ)^{e'} \cap R \subseteq I^eQ$. From the above $I^eQ = (IQ)^{e''}$, hence $(IQ)^{e'} = (IQ)^{e''} = I^eQ$.

Let $i$ be a right ideal in $Q$. For any complement $i^e$ of $i$ in $Q$ there exists a complement $(i \cap R)^{e'}$ of $(i \cap R)$ in $R$ such that

$$(10) \quad i^e \cap R = (i \cap R)^{e'}$$

and conversely for any complement $(i \cap R)^{e'}$ there exists a complement right ideal $i^e$ in $Q$ satisfying (10).

From the fact $(i \cap R)^{e'} \cap R = (0)$ we have $i^e \cap R \subseteq (i \cap R)^{e'}$. $i^e = (i \cap R)Q \subseteq (i \cap R)^{e'}Q = (i \cap R)^{e''} = i^{e''}$ by (9). Hence $i^e = i^{e''}$ and $i^e \cap R = i^{e''} \cap R = (i \cap R)^{e''} \cap R \supseteq (i \cap R)^{e'}$. Conversely $(i \cap R)^{e'} \cap (i \cap R) = (0)$, then $i \cap R \cap R = (i \cap R)^{e'}Q = (0)$. Hence $(i \cap R)^{e'}Q \subseteq i^e$ for some complement $I^e$ of $I$ and $i^e \cap R \supseteq (i \cap R)^{e'}$. By the above $i^e \cap R = (i \cap R)^{e''} \supseteq (i \cap R)^{e'}$, hence $i^e \cap R = (i \cap R)^{e'}$.

2. Uniform right ideals.

We can classify the right ideals in $R$ as follows;

$I \equiv J$ if and only if there exist regular elements $d, d' \in R$ such that for any elements $r \in I$, $r' \in J$, $rd, r'd' \in I$.

It is clear that

$I \equiv J$ if and only if $IQ = JQ$.

We shall denote the class containing $I$ by $[I]$.

**Proposition 1.** The right ideals in $Q$ are lattice isomorphic to $[I]$. 

**Proof.** From the definition and (3) it is clear that this correspondence is onto and that $(I, I_Q) \subseteq [I]$. If $x \in I, a \subseteq I_Q, x = r_1 q_1 = r_2 q_2$, $r_1 \in I$, and by (1) we have $x = r_1 p_1 t^{-1} = r_2 p_2 t^{-1}$, hence $r_1 p_1 = r_2 p_2 \in I \cap I_Q$ and $x \in (I \cap I_Q)Q$. We have clearly $(I \cap I_Q)Q = I_Q \cap I_Q$. 

Thus, the ideal $I \cap I_Q$ is derived from the correspondence $I \rightarrow [I]$.
$[I]Q\cap R$ is the unique maximal right ideal in $[I]$. Since $Q$ is P.M.I. there exist minimal right ideals and we call a right ideal in $R$ which corresponds to a minimal right ideal in $Q$ an uniform right ideal and the unique maximal right ideal in this class basic right ideal.

**Proposition 2.** If $U$ is a uniform right ideal, then for any non zero right ideals $I, J (\subseteq U)$ $I \cap J \not= \{0\}$.

*Proof.* Since $U$ is uniform, $UQ$ is irreducible, hence $IQ = JQ = UQ$. From Proposition 1 $I \cap J \not= \{0\}$.

*Lemma 1.* Let $Q$ be a P.M.I. ring. If a right ideal $r$ is not minimal, then it contains at least two minimal right ideals.

*Proof.* Let $r$ contain only one minimal right ideal $r_0$. Then $r_3 \subseteq r_0 \cap r_3$ and $r_3 = r_5 = eQ$ where $3$ is the socle of $Q$. Hence $r_3 = er_3$. For any elements $r \in r, z \in 3$ we have $rz = erz$ i.e. $(er-r)z = 0$. Therefore $er-r \in 3 = \{0\}$ and $er \not= r$. Hence $er = r = eQ$.

**Proposition 3.** Let $U$ be a right ideal in $R$. If for any non zero right ideals $I, J$ in $U \ I \cap J \not= \{0\}$, then $U$ is uniform.

*Proof.* If $U$ is not uniform, there exist two minimal right ideals $r_1, r_2$ in $UQ$ by Lemma 1. Since $r_1 \cap U \not= \{0\}$, $r_2 \cap U \not= \{0\}$ and $r_1 \cap r_2 \cap U = \{0\}$, it is a contradiction.

**Proposition 4.** Let $I$ be a right ideal in $R$. $I$ is uniform if and only if there exist elements $y_1, y_2$ and regular elements $y_1, y_2$ in $R$ such that for any elements $x, x' \in I$ $xy_1 = x'y_1, x'y_2 = xy_2$.

*Proof.* Let $xq^{-1}$ and $x'q'^{-1}$ be elements in $IQ$. Then by the hypothesis $x'y = xy$ with regular element $y$. Hence $x'q'^{-1} = xyy'^{-1}q'^{-1} = xq^{-1}qyq'y^{-1}q'^{-1} \in xQ$, therefore $IQ$ is irreducible. The converse is similar.

**Proposition 5.** There exist mutually isomorphic uniform right ideals in any two classes which contain basic right ideals.

*Proof.* Let $I_1$ and $I_2$ be basic. Since $Q$ is P.M.I. there exists a $Q$-isomorphism $\lambda$ of $I_1Q$ to $I_2Q$. Let $I_1Q = e_1Q, e_i = r_i x_i^{-1}, r_i \in I_1, x_i \in R$ and $\lambda(e_i) = e_2q_i, q_i \in Q$. Then $\lambda(r_i) = \lambda(e_i x_i) = e_2q_i x_i$. If we put $x_2q_i = yz^{-1}, y, z \in R$, we have $0 \not= \lambda(r_1z) = e_2q_i x_i z = e_2x_2^2 s_2 q_i x_i z = r_2 y$. Since $I_1Q$ and $I_2Q$ are irreducible, $[r_1zR^1]^\sim = [I_1]$ and $[r_2yR^1] = [I_2]$. Hence $\lambda$ sends $r_1zR^1$ isomorphically onto $r_2yR^1$.

If $e$ is a primitive idempotent in $R$, then so is $e$ in $Q$, hence $eR$ is basic. But basic right ideals are not always principal even if $R$ has the unit. For example, let $K$ be a field and $x$ be an independent over $K$ and $R_0$ be the subring of elements in $K[x]$ without constant-term. If we put $R = EK + \cup (R_0)_n$ as in the first ex-

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1) Mr. Kanzaki kindly pointed out to me this proof.
2) $ar^a$ means the right ideal in $R$ generated by $a$.++
ample, then its quotient ring is \( Q = EK \cup K(x)_n \). Let \( r = e_{11}Q \). If \( r \cap R \) is principal: \( r \cap R = \left( \begin{array}{cc} f_1, f_2, \ldots, f_n \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \end{array} \right) \), then there exist \( g_1, \ldots, g_n \) and \( k \neq 0 \in K \) such that 
\( f_i(k + g_1) + \cdots + f_n g_n = x \), hence min. degree of \( f_i = 1 \). On the other hand there exist \( g_1, \ldots, g_n \) and \( k' \neq 0 \in K \) such that 
\( f_i(k'_1 + g_1) + \cdots + f_n g_n = 0 \), hence min. degree of \( f_i(x) \geq 2 \). This is a contradiction. Next example shows that basic right ideals are not always mutually isomorphic. Let \( R = (R_0)_n + e_{33} K + \cdots + e_{nn} K \). If an element \( x \) of \( R \) is not a zero-divisor in \( R \) then \( x \) is regular in \( K(x)_n \), for the adjoint of \( x \) is in \( R \). Let \( (x_{ij}), (y_{ij}) \) be elements of \( R \), and suppose that \( (x_{ij}) \) is non zero-divisor. Then \( (x_{ij})^{-1} (y_{ij}) | x_{ij} | E = \text{adj} (x_{ij}) \cdot (y_{ij}) \) is in \( R \), hence \( (x_{ij}) \cdot (y_{ij}) = (y_{ij}) | x_{ij} | E \) and \( | x_{ij} | E \) is a non zero divisor. Therefore \( R \) has the quotient ring \( Q = K(x)_n \). \( e_{11}Q \cap R \) is basic and not principal, because if \( e_{11} Q \cap R = (e_{11} f_1 + e_{12} f_2 + \cdots + e_{nn} f_n) R, f_i \in R_0 \) then \( x = \sum_{i=1}^n f_i g_i, g_i \in R_0 \) which is a contradiction. On the other hand \( e_{33} Q \cap R = e_{33} R \) is basic and principal. Therefore \( e_{11}Q \cap R \) is not isomorphic to \( e_{33} R \).

**Proposition 6.** Any right ideal \( I \) in \( R \) contains a uniform right ideal in \( R \).

**Proof.** Since \( Q \) is P.M.I., \( IQ \) contains a minimal right ideal \( \tau \) in \( Q \), and further \( (0) = I \cap \tau = I \cap \tau \cap R \) and \( (I \cap \tau \cap R) Q = \tau \), hence \( I \cap \tau \cap R \) is uniform.

**Proposition 7.** Let \( U \) be a uniform right ideal in \( R \). Then

\[
U_I = \{ x \in R \mid x_{rr} U = \{ 0 \} \}.
\]

**Proof.** If \( xu = 0 \) for any \( u \in U \), then since \( UQ \) is irreducible, \( UQ = uQ \), hence \( xuQ = xuQ = (0) \). Therefore \( x \in U_I \).

An element \( u \) in \( R \) is called right uniform if \( uR^1 \) is a uniform right ideal (equivalently if \( uR \) is uniform \( (R_I = R_r = (0)) \)).

We can define similarly left uniform elements. But the left uniform elements coincide with the right uniform elements, because if \( u \) is left uniform, then \( Qu = Qe \) is irreducible where \( e \) is a primitive idempotent, since \( Q \) is P.M.I., \( eQ \) is irreducible, hence \( uQ = ueQ \) is also irreducible. Therefore \( u \) is right uniform, and the converse is similar. Hence we may call right (left) uniform elements simply uniform elements.

**Proposition 8.** Let \( I \) be a right ideal in \( R \). If there exists some uniform element \( u \) such that \( u_{rr} I = (0) \), then \( I \) is uniform. Furthermore if \( R \) is prime, then the converse is true.

**Proof.** If \( u_{rr} I = (0) \), for any element \( aq^{-1} \in u_{rr} \cap IQ \), \( a \in I \) we have \( ua = 0 \), hence \( a \in I \cap u_{rr} = (0) \) and so \( u_{rr} \cap IQ = (0) \). Let \( \theta \) be a mapping: \( q \rightarrow uq \). Since \( \theta^{-1}(0) \cap IQ = (0) \), we have a isomorphism \( IQ \approx uQ \), hence \( I \) is uniform. Let \( R \) be prime and \( I \) be uniform. If \( u_{rr} I = (0) \) for all element \( u \) in \( I \), then \( I^2 = 0 \) by Proposition 7. This is a contradiction.
From the definition \( xU \) is uniform if \( U \) is so, hence the sum \( R_0 \) of all uniform right ideals is two sided ideal and \( R_0 \) is the sum of all uniform elements. Therefore \( R_0 \) coincides with the sum of all left uniform ideals. Furthermore \( R_0Q \) is the socle \( \mathcal{S} \) of \( Q \). \( R_0Q \subseteq \mathcal{S} \) and since \( (\cap \mathcal{S} R)^2 = \mathcal{S} \), for \( x \in \mathcal{S} \), \( x \in \mathcal{S} \) and \( x \in R_0Q \).

**Theorem 1.** The cardinal numbers of the maximal length of direct-sums of basic right ideals are equal. Further if \( Q \) is a sub-P.M.I. ring of \( \mathcal{L}_m(m) \) with \( \alpha \)-dim = \( \alpha \)-dim \( m' \), then the cardinal numbers for basic left ideals coincide with ones for basic right ideals, where \( \mathcal{L}_m(m) \) is the ring of continuous endomorphisms of \( m \), topologized by \( m' \)-topology, and \( \Delta \) is the division ring of \( \mathcal{L}_m(m) \)-endomorphisms of \( m \).

**Proof.** Let \( B = \{ B_a \} \) be the set of basic right ideals. We can order direct-sums \( S_j = \bigoplus_{a \in I} B_a \) of elements \( B_a \) of \( B \) as follows:

\[ S_i \supset S_j \quad \text{if and only if} \quad S_i = S_j \bigoplus \bigoplus_{a \in I \setminus J} B_a. \]

By the Zorn’s Lemma there exists a maximal element \( S_0 \) in this order. Then \( S_0 \) meets all basic right ideals. If \( S_0Q \subseteq \mathcal{S} \) there exists a minimal right ideal \( r_0 \) such that \( r_0 \cap S_0Q = \{0\} \). Hence \( \mathcal{S} = R \cap r_0 \cap S_0Q \cap R \). Since \( R \cap r_0 \) is basic, it is a contradiction. Therefore \( S_0Q = \mathcal{S} \). Since \( Q \) is P.M.I. the right dimension of \( \mathcal{S} \) is constant. It is also true for left basic ideals. Further if \( Q \) is as in Theorem, then the left dimension coincides with the right one.

**Theorem 2.** Let \( U \) be a uniform right ideal in \( R \) and \( e(U) \) be the \( R \)-endomorphism ring of \( U \). Then non zero element of \( e(U) \) is non singular. \( e(U) \) has the right quotient division ring which is the \( Q \)-endomorphism ring of \( Q \)-irreducible module. 

**Proof.** If \( \phi \in e(U) \), then \( \phi \) can be extended to a \( Q \)-endomorphism of \( UQ \). Because if \( uq^{-1} = u'q'^{-1} \in UQ \), then there exist \( p, s, d \) by (1) such that \( q^{-1} = pd^{-1}, q'^{-1} = sd^{-1} \), hence \( \phi(uq^{-1}) = \phi(uq^{-1}) = \phi(u)p d^{-1} = \phi(u')d^{-1} = \phi(u')sd^{-1} = \phi(u')q'^{-1} \). Since \( UQ \) is irreducible, the \( Q \)-endomorphism ring of \( UQ \) is a division ring. Hence if \( \phi \) is not zero, then \( \phi \) is non singular. Let \( \phi \) be any \( Q \)-endomorphism of \( UQ \). Then there exists \( y \) in \( UQ \) such that \( \phi(y) = u \in U \); \( y = u'x^{-1} \), \( u' \in U \) and for any element \( w \) in \( U \) \( \phi_\lambda(w) = \phi(u'w) = \phi(yxw) = uwx = \lambda_x w \) where \( \lambda_a : x \rightarrow ax \), \( x \in R \). Hence \( \psi = \lambda_ux^\lambda_{a^2} \).

3. Complements and annihilators.

**Theorem 3.** Let \( B \) be basic then \( B = B_{tr} \). A right ideal \( B \) in \( R \) is basic if and only if \( B \) is a minimal annihilator. A right ideal \( M \) in \( R \) is a maximal annihilator if and only if \( M = u \), where \( u \) is a uniform element.

**Proof.** Let \( B \) be basic, then \( B = BQ \cap R \) and \( BQ = eQ \), \( e = e' \). By (7) \( B_{tr} = (BQ \cap R)_{tr} = BQ_{tr} \cap R = eQ_{tr} \cap R = eQ \cap R = B \). If \( B \subseteq L_r \) then \( (QL)_r = L_r Q \subseteq BQ \). Since \( BQ \) is irreducible \( BQ = (QL)_r \). Hence \( B = BQ \cap R = (QL)_{tr} \cap R = L_r \). Therefore
B is a minimal annihilator. Let \( L = L_r \) be a minimal annihilator. If \( L_r Q \subseteq L'_r \) for some subset \( L' \) in \( Q \), then \( L_r = L_r, Q \subseteq L'_r \cap R = L'_r \) by (6). Hence \( L_r = L' \) and \( L_r Q = L'_r Q = L'_r \). Therefore \( L_r Q \) is also a minimal annihilator. Let \( r = eQ \) be an irreducible right ideal in \( Q \) contained in \( L_r Q \). Then \( eQ = (Q_{L_r})_r \) and since \( L_r Q \) is a minimal annihilator, \( eQ = L_r Q \), hence \( L_r = L_r Q \cap R \) is basic. From Proposition 7 we have \( B_r = u_r \) for any element \( u \) in \( B \). Conversely if \( u \) is a uniform element, then \( Qu = u_r \) is irreducible, hence \( (Qu)_r = u_r \) is a maximal right ideal. By (8) \( u_r = u_r \cap R \) is a maximal annihilator.

**Theorem 4.** Let \( M \) be a right ideal in \( R \). \( M \) is a maximal complement in \( R \) if and only if \( MQ \) is a maximal one of right ideals \( r \) with \( (r; Q)_r = (0) \) and \( MQ = M \) or if and only if \( M = B^c \) where \( B \) is basic. Let \( M \) be a maximal complement in \( R \). Then (1) for any basic right ideal \( B \subseteq B \) or \( M \cap B = (0) \), (2) \( M \) is minimal irreducible, (3) if \( M \) is of the maximal length of direct-sum of basic right ideals contained in \( M \), then there exists a basic right ideal \( B \) such that \( M \otimes B \) is of the maximal length of direct-sum of basic right ideals in \( R \) and (4) \( M^c \) is basic. Maximal annihilators are maximal complements.

**Proof.** Let \( M \) be a maximal complement in \( R \); \( M = I^c \). By (9) \( MQ = (IQ)^c \). Let \( MQ \subseteq i \). Since \( (i \cap R)^c = i^c \cap R \supseteq M, MQ = (i^c \cap R) Q = i^c \), hence \( MQ \) is a maximal complement in \( Q \), and \( MQ \cap R = M \). Let \( r \) be a right ideal with \( (r; Q)_r = (0) \) and \( r \supseteq MQ \). Then since \( r \supseteq i \) there exists a minimal right ideal \( t_0 \) such that \( r \cap t_0 = (0) \). Hence \( r \) is contained in a maximal complement. Therefore \( r = MQ \). Conversely if \( MQ \) satisfies the property mentioned in Theorem, then \( MQ \supseteq i \) and \( MQ \cap t_0 = (0) \); \( t_0 \) a minimal right ideal, and \( MQ \subseteq t_0 \). Since \( (t_0 ; Q) = (0), MQ = t_0 \). If \( MQ \subseteq \gamma \), then \( (\gamma ; Q)_r = (0) \). By (10) \( M = MQ \cap R = (t_0 ; R) = (t_0 \cap R)^c \). Further if \( M \subseteq \gamma \) then \( MQ \subseteq I^c Q = (IQ)^c \), hence \( MQ = I^c Q \). Therefore \( M = I^c \), and \( M \) is a maximal complement. Let \( M \) be a maximal complement in \( R \), then \( MQ \) is so in \( Q \). Hence there exists a minimal right ideal \( t_0 \) such that \( t_0 \cap MQ = (0) \) and \( MQ = t_0 \). \( M = MQ \cap R = t_0 \cap R = (t_0 \cap R)^c \) by (10) and \( t_0 \cap R \) is basic. Conversely let \( M = B^c \). \( MQ = B^c \) and \( BQ \) is minimal. If \( \gamma = (BQ^c \cap Q) \oplus BQ \cap t_1 \), where \( \gamma \) is the socle of \( Q \), then \( \gamma (\in (BQ^c) \oplus t_1 \cap BQ) = x_1 + x_2, x_1 \in BQ^c, x_2 \in t_1 \), we have \( x_1 = y - x_2 \in (BQ^c \oplus t_1) \cap BQ^c \subseteq \gamma \cap (BQ^c) \). Hence \( (BQ^c) \cap t_1 \cap BQ = (0) \) and \( t_1 = (0) \). If \( MQ \subseteq r \), then \( r^c \supseteq \gamma \) hence \( r^c, BQ = (0) \) and \( r^c = MQ \). Therefore \( MQ \) is a maximal complement in \( Q \) and further \( B^c = M \subseteq MQ \cap R = (BQ^c)_r \cap R = (BQ\cap R)^c = B^c \) and we have \( M = MQ \cap R \). 1) Let \( B \) be basic. Since \( BQ \) is a minimal right ideal, \( BQ \subseteq MQ \) or

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3) From this theorem a right ideal \( I \) is called irreducible if \( I = M \cap N \) implies \( I = M \) or \( I = N \).
BQ ∩ MQ = (0). Hence B ⊆ MQ ∩ R = M or M ∩ B = (0). 2). If M ⊆ N, M ⊆ S, and 
M = N ∩ S, then MQ ⊆ SQ for MQ = SQ implies M = S. Hence SQ ⊆ 3 and NQ ⊆ 3. 
Therefore NQ ∩ SQ ⊆ 3 and this is a contradiction. If M ⊆ M, then M = (M ∪ 
M') ∩ M, hence M is minimal irreducible. From the above argument and the fact 
that MQ ∩ R = M, M is a maximal complement. Let MQ ∩ r = (0) for a minimal 
right ideal r. Since MQ ⊕ r ⊆ MQ, MQ ⊕ r ⊆ 3. We define the right ideal 

\[ I = \{ j ∈ MQ, \text{ there exists an element } z ∈ 3 \text{ such that } z = j + r, r ∈ r \}. \]

Then \[ i \subseteq 3 ∩ MQ \] and \[ z = j + r, j = z ⊕ r, r ∈ r \] are minimal ideals. 
M = MQ ∩ R ⊆ 3 ∩ MQ ∩ R = \Sigma r ∩ R. If (MQ)′ is not minimal, then it contains two minimal 
right ideals, r₁, r₂ by Lemma 1. Hence MQ ∩ (r₁ ∪ r₂) = (0) and (MQ ∪ r₁) ∩ r₂ = (0). 
Therefore since (MQ)′ is minimal and (MQ)′ = M′Q, M′ is uniform and by (1) 
M′ is basic. Let M′ be a maximal annihilator. By (8) M′Q = 1 = e is so in Q. 
If \[ t₀ = Qe \] is a minimal left ideal in I, then \[ I_r*x = 1 = e(1 – e)Q \] and \[ I_r*e Q = (0). \]
Since \[ I_r*e \] is maximal, \[ I_r*e \] is a maximal complement. 

The following example with field Q/3 analogous to the first one in this note 
shows that a maximal complement is not always a maximal annihilator. Let r 
be the right ideal generated by elements \[ e_{11} + e_{21}, e_{12} + e_{22}, \ldots. \] Since \[ (m/\text{mod} : d) = 1, \] 
r is a maximal right ideal contained in \[ 3 \] where \[ m \] is an irreducible Q-module and 
d is its Q-endomorphism ring. If \[ r^k ⊆ r \] then an element \[ x \] of \[ r^k − r \] is of the 
following from 

\[ x = x₁ + αE, \quad α ∈ d \quad \text{and} \quad x₁ ∈ 3. \]

If \[ α = 0 \], then \[ x_e ≡ αe ≡ x ∈ r^k \] for a sufficiently large \( i \). Hence \[ r^k ⊆ 3 \]. If \[ α = 0 \], then \[ x ∈ 3 \]. Therefore \[ r^k ⊆ 3 \]. From Theorem 4 \( R ∩ r \) is a maximal complement but not 
a maximal annihilator since \( r \) is not maximal. Furthermore in this ring \( R \) if a 
right ideal \( M \) is minimal irreducible and \( M = MQ ∩ R \), then \( M \) is maximal comple-
ment. Because if \( M \) is minimal irreducible then \( MQ \) is so in \( Q \). Since \( r \) is 
minimal irreducible \( MQ ⊆ 3 \), hence \( MQ ⊆ r₀ \) for some minimal right ideal \( r₀ \). If 
\( r₀ ⊆ MQ \) then \( MQ = r₀ ∩ (MQ ⊕ r₀) \) is not irreducible. Hence \( MQ = r₀ \) and by the first 
mention in the proof \( MQ \) is a maximal complement. 

**Theorem 5.** If \( Q \) satisfies the minimal conditions, then the complement right 
ideals coincide with the annihilator right ideals. A right ideal \( M \) is a maximal 
complement if and only if \( M \) is minimal irreducible and \( M \) contains no regular 
elements. 

**Proof.** Let \( I = J^c \) be a complement right ideal. \( IQ = J^c Q = (JQ)^c = (eQ)^c \) 
= (1 – e)Q = (Qe), where \( e^2 = e \), \( JQ = eQ \), because \( Q \) is a simple ring with minimal 
conditions. On the other hand if \( IQ ∩ R = I \setminus I \) then \( I^c J = (0) \), hence \( (0) = I^c Q ∩ 
JQ \) which is a contradiction. Therefore \( I = J^c = J^c Q ∩ R = (Qe) r ∩ R = (Qe) r ∩ R \). 
Conversely if \( I = J_r \) then \( J_r = J^c r ∩ R = (eQ)^c r ∩ R \) where \( J_r^c = (1 – e)Q \). By (10)
$J_r = J^* \cap R = (eQ)^c \cap R = (eQ \cap R)^c$. Let $M$ be a minimal irreducible right ideal with $MQ = Q$. Then there exists a maximal right ideal $r$ which contains $M$, $r \cap R \supseteq M$ and since from Theorem 4 $r \cap R$ is minimal irreducible, $M = r \cap R$ is a maximal complement.

Bibliography