Chart descriptions of regular braided surfaces

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CHART DESCRIPTIONS OF REGULAR BRAIDED SURFACES

SEIICHI KAMADA AND TAKAO MATUMOTO

Abstract. Braided surfaces are surfaces embedded or immersed in the bidisk $D^2 \times D^2$ which are projected onto the second factor of the bidisk as branched covering maps. A simple and embedded braided surface is described by a graph in the 2-disk, called a chart. We generalize the chart description method so that one can consider braided surfaces which are not necessarily simple or embedded. We show that if a braided surface is regular, then it can be described by a chart called regular, and that such a chart is unique up to regular chart move equivalence.

1. Introduction

A graphical method, called the chart description method, was introduced in [5] in order to describe a simple and embedded 2-dimensional braid, which is a special case of braided surfaces. Later, it was proved in [9] that the basic moves (called chart moves) introduced in [5] are sufficient moves. The chart description method was extended to simple and immersed braided surfaces in [10] (cf. Chapter 34 of [11]). It has been used for the study of simple 2-dimensional braids, simple braided surfaces and knotted surfaces in 4-space, cf. [3, 4, 11]. Applying a result of [12], we have the following. (The C-moves are explained later in § 3.)

Theorem 1.1. *Any (embedded or immersed) braided surface is described by a chart. Such a chart description is unique up to chart move equivalence. More precisely, let Γ and Γ' be chart descriptions of braided surfaces S and S'. The braided surfaces S and S' are isomorphic (or equivalent, resp.) if and only if Γ and Γ' are related by a finite sequence of C-moves of type W, C-moves of type B, C-moves of type ∂ and isotopies of D_2^2 rel $\{y_0\} \cup \Delta_\Gamma$ (or isotopies of D_2^2 rel $\{y_0\}$, resp.).*

A braided surface is called *regular* if for each singular value there exists exactly one singular point. Since any braided surface is ambiently isotopic to a regular braided surface by an isotopy of $D_1^2 \times D_2^2$ (cf. [11]), it is sufficient to consider regular braided surfaces for study of ambient isotopy classes of braided surfaces and surface-links in 4-space. In order to describe a

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regular braided surface effectively, we introduce the notion of a regular chart. Any regular braided surface is described by a regular chart. Let Γ and Γ' be regular chart descriptions of regular braided surfaces S and S'. The braided surfaces S and S' are isomorphic (or equivalent) if and only if Γ and Γ' are related by a sequence of moves and isotopies of D^2_2 as stated in Theorem 1.1. However, in such a sequence, the regularity condition of a chart is not preserved in general. Some C-moves of type B may change a regular chart to an irregular one.

The following is our main result.

Theorem 1.2. *Any regular braided surface is described by a regular chart. Such a chart description is unique up to regular chart move equivalence. More precisely, let Γ and Γ' be regular chart descriptions of regular braided surfaces S and S'. The braided surfaces S and S' are isomorphic (or equivalent, resp.) if and only if Γ and Γ' are related by a finite sequence of C-moves of type W, C-moves of type B keeping the condition of regularity, label shift moves, passing moves, C-moves of type ∂, and isotopies of D^2_2 rel $\{y_0\} \cup \Delta_\Gamma$ (or isotopies of D^2_2 rel $\{y_0\}$, resp.).*

In § 2 we recall the definition of braided surfaces and monodromy representations. In § 3 the chart description method is explained, and in § 4 the notion of regular charts is introduced. In § 5 regular chart moves, including label shift moves and passing moves, are defined and Theorem 1.2 is proved. In § 6 we recall chart moves for simple (embedded or immersed) 2-dimensional braids introduced in [5, 9, 10] are interpreted in terms of our regular chart moves.

2. Braided surfaces

We work in the PL category. An immersed surface S in a 4-manifold W is said to be *proper* if $S \cap \partial W$ is equal to the boundary of the surface S. It is said to be *locally flat* if every point $x \in S$ has a regular neighborhood $N(x)$ in W such that $(N(x), S \cap N(x), x)$ is homeomorphic to (D^4, D^2, O), $(D^4, D^2_{xy} \cup D^2_{zw}, O)$ or (D^4_+, D^2_+, O), where D^4 is the standard 4-disk in Euclidean 4-space coordinated with x, y, z, w, $D^2 = D^2_{xy}$ is the 2-disk on the xy-plane in D^4, D^2_{zw} is the one on the zw-plane, O is the origin, and D^4_+ and D^2_+ are restrictions of D^4 and D^2 with $x \geq 0$, respectively. In the second case where $(N(x), S \cap N(x), x)$ is homeomorphic to $(D^4, D^2_{xy} \cup D^2_{zw}, O)$, we call x a *node* of S or a *self-intersection* of S.

Let D^2_1 and D^2_2 be 2-disks and $pr_i : D^2_1 \times D^2_2 \to D^2_i$ $(i = 1, 2)$ the ith factor projection. Let X_m be a fixed set of m interior points of D^2_1, which are assumed to be on the real line when we regard D^2_1 as $\{z \in \mathbb{C}; |z| \leq 1\}$. We identify the braid group B_m with the fundamental group $\pi_1(C_m, X_m)$ of the configuration space $C_m = C_m(\mathrm{Int}D^2_1)$ of m distinct points of the interior, $\mathrm{Int}D^2_1$, of D^2_1 (cf. [1, 11]).

We assume that spaces have base points and that the base point of D_2^2 is a point of ∂D_2^2, which we denote by y_0 throughout this paper.

Definition 2.1 (cf. [13, 14])**.** A *braided surface of degree* m is a compact oriented immersed surface S in $D_1^2 \times D_2^2$, which is proper and locally flat, satisfying the following conditions (i) – (iv), where $\iota : S_0 \to D_1^2 \times D_2^2$ is an underlying immersion for S (i.e., $\iota(S_0) = S$) and $p : S_0 \to D_2^2$ is the composition of ι and the second factor projection pr_2.

(i) The map $p : S_0 \to D_2^2$ is a branched covering map of degree m.
(ii) For each branch point $x \in S_0$ of p, the image $\iota(x)$ is not a node of S.
(iii) $\partial S \subset \mathrm{Int} D_1^2 \times \partial D_2^2$.
(iv) $pr_1(S \cap pr_2^{-1}(y_0)) = X_m$.

The boundary ∂S of S is a closed m-braid in the solid torus $D_1^2 \times \partial D_2^2$. The m-braid obtained from ∂S by cutting off along $pr_2^{-1}(y_0) = D_1^2 \times \{y_0\}$ is called the *boundary braid of* S.

Definition 2.2 (cf. [5, 10, 16])**.** A 2*-dimensional braid of degree* m, or a 2*-dimensional* m*-braid*, is a braided surface of degree m satisfying the following condition (v).

(v) The boundary ∂S of S is the trivial closed m-braid $X_m \times \partial D_2^2$ in $D_1^2 \times \partial D_2^2$, i.e., $pr_1(S \cap pr_2^{-1}(y)) = X_m$ for all $y \in \partial D_2^2$.

A *branch point of* S means the image $\iota(x)$ in S of a branch point $x \in S_0$ of p. A *singular point* of S is a branch point of S or a node of S. A *singular value* of S is a point of D_2^2 which is the image under pr_2 of a singular point of S. The set of singular values of S is denoted by Δ_S.

Definition 2.3. A braided surface is *regular* if the following condition (vi) is satisfied.

(vi) For each singular value $y \in D_2^2$, there exists exactly one singular point of S.

A braided surface is *simple* if both of (vi) and (vii) are satisfied.

(vii) For each branch point $x \in S$, the local degree of the branched covering at x is 2.

If a braided surface S is an embedded surface (i.e., if there exist no nodes), then we say that S is an *embedded braided surface*.

Remark 2.4. We may define a braided surface alternatively as follows: An immersed surface S in $D_1^2 \times D_2^2$ is a *braided surface* if, for each point $x \in S$, there exists a regular neighborhood $N(x)$ in $D_1^2 \times D_2^2$ satisfying one of the following.

(1) $(N(x), S \cap N(x), x)$ is homeomorphic to (D^4, D^2, O), and when we put $E = S \cap N(x)$, the restriction $pr_2|_E : E \to pr_2(E)$ is a homeomorhism. (In this case, we call x a *regular point of* S.)

(2) $(N(x), S \cap N(x), x)$ is homeomorphic to (D^4, D^2, O), and when we put $E = S \cap N(x)$, the restriction $pr_2|_E : E \to pr_2(E)$ is a branched covering map with branch point x. (In this case, we call x a *branch point of* S.)

(3) $(N(x), S \cap N(x), x)$ is homeomorphic to $(D^4, D^2_{xy} \cup D^2_{zw}, O)$, and when we put $E_1 \cup E_2 = S \cap N(x)$ such that E_1 and E_2 are 2-disks intersecting each other at x transversely, the restriction $pr_2|_{E_i} : E_i \to pr_2(E_i)$ is a homeomorphism for $i = 1, 2$. (In this case, we call x a *node of* S.)

(4) $(N(x), S \cap N(x), x)$ is homeomorphic to (D^4_+, D^2_+, O), and x is a point of $\mathrm{Int}D^2_1 \times \partial D^2_2$. (In this case, we call x a *boundary point of* S.)

Two braided surfaces S and S' of degree m are said to be *equivalent* if there exists an isotopy $\{h_t : D^2_1 \times D^2_2 \to D^2_1 \times D^2_2\}_{t\in[0,1]}$ of the ambient space $D^2_1 \times D^2_2$ satisfying the following (i) – (iii).

(i) $h_0 = \mathrm{id}$ and $h_1(S) = S'$. (S and S' are ambiently isotopic by $\{h_t\}$.)

(ii) For each $t \in [0, 1]$, $h_t : D^2_1 \times D^2_2 \to D^2_1 \times D^2_2$ is fiber-preserving; namely, there exists an isotopy $\{\underline{h}_t : D^2_2 \to D^2_2\}_{t\in[0,1]}$ of D^2_2 rel $\{y_0\}$ with $\underline{h}_t \circ pr_2 = pr_2 \circ h_t$.

(iii) For each $t \in [0, 1]$, h_t fixes the distinguished fiber $pr_2^{-1}(y_0) = D^2_1 \times \{y_0\}$ over the base point y_0.

Moreover, if

(iv) $\underline{h}_t = \mathrm{id} : D^2_2 \to D^2_2$ for all t in the condition (ii),

then we say that S and S' are *isomorphic*.

Remark 2.5. It is not difficult to see that two braided surfaces S and S' with the same boundary $\partial S = \partial S'$ are equivalent if and only if there exists an isotopy $\{h_t\}_{t\in[0,1]}$ of the ambient space $D^2_1 \times D^2_2$ satisfying (i), (ii) and the following (iii)$'$.

(iii)$'$ For each $t \in [0, 1]$, h_t fixes the solid torus $D^2_1 \times \partial D^2_2$ pointwise.

Moreover, S and S' with $\partial S = \partial S'$ are isomorphic if and only if there exists an isotopy $\{h_t\}_{t\in[0,1]}$ of the ambient space $D^2_1 \times D^2_2$ satisfying (i), (ii), (iii)$'$ and (iv).

Now we recall the monodromy representation of a braided surface.

Let S be a braided surface and Δ_S the singular value set of S. For a path $c : [0, 1] \to D^2_2 \setminus \Delta_S$, we define a path

$$\rho_S(c) : [0, 1] \to C_m$$

in the configuration space $C_m = C_m(\mathrm{Int}D^2_1)$ by

$$\rho_S(c)(t) = pr_1(S \cap pr_2^{-1}(c(t))) \quad \text{for } t \in [0, 1].$$

By (iv) of Definition 2.1, for a loop c in $D^2_2 \setminus \Delta_S$ with base point y_0, the path $\rho_S(c)$ is a loop in C_m with base point X_m. The mapping

$$\rho_S : \pi_1(D^2_2 \setminus \Delta_S, y_0) \to \pi_1(C_m, X_m) = B_m; \quad [c] \mapsto [\rho_S(c)]$$

is a well-defined homomorphism and is called the *monodromy representation* or the *braid monodromy* of S.

Monodromy representations ρ_S of S and $\rho_{S'}$ of S' are said to be *equivalent* if there exists a homeomorphism $g : D_2^2 \to D_2^2$ satisfying the following (i) and (ii).

(i) $g(\Delta_S) = \Delta_{S'}$ and $g|_{\partial D_2^2} = \mathrm{id}$.
(ii) $\rho_S = \rho_{S'} \circ g_*$, where $g_* : \pi_1(D_2^2 \setminus \Delta_S, y_0) \to \pi_1(D_2^2 \setminus \Delta_{S'}, y_0)$ is the isomorphism induced by g.

Propositions 2.6 and 2.7 are proved by the same argument with those in [8] and [11], whose proofs are left to the reader.

Proposition 2.6 (cf. [8, 10, 11, 13, 14])**.** *Two braided surfaces are isomorphic (or equivalent, resp.) if and only if their monodromy representations are the same (or equivalent, resp).*

We define two subsets G_m and G_m^{reg} of the braid group B_m as follows: An m-braid b belongs to G_m if and only if it is conjugate in B_m to a braid $b' = b_1 \amalg b_2 \amalg \cdots \amalg b_c$ which is the split sum of some m_k-braids b_k ($k = 1, \ldots, c$) with $\sum_{k=1}^{c} m_k = m$ satisfying the following.

(i) The closure of each b_i ($i = 1, \ldots, c$) in the 3-sphere S^3 is either
  (a) a trivial knot, or
  (b) a Hopf link.
(ii) In case (b) of (i), b_i is a 2-braid.
(iii) $c \neq m$, i.e., b is not the identity element of B_m.

An m-braid b belongs to G_m^{reg} if and only if it is conjugate in B_m to a braid $b_1 \amalg b_2 \amalg \cdots \amalg b_c$ satisfying the above (i), (ii) and (iii) and the following;

(iv) $b_1, \ldots, b_c$ are 1-braids except exactly one of them.

When we do not allow the case (b) of (i) of the definition of G_m, we have the subset A_m defined in [8, 11].

Let y be an interior point of D_2^2. A sufficiently small simple loop surrounding y, which is oriented anticlockwise, is called a *meridian loop around* y. A *meridional loop around* y means a loop with base point y_0 which is obtained from a meridian loop around y by using a path connecting the meridian loop with y_0.

Proposition 2.7 (cf. [8, 10, 11, 13, 14])**.** *Let Δ be a finite set of interior points of D_2^2 and let b_0 be an m-braid. A homomorphism $\rho : \pi_1(D_2^2 \setminus \Delta, y_0) \to B_m$ is the monodromy representation $\rho_S : \pi_1(D_2^2 \setminus \Delta_S, y_0) \to B_m$ of a braided surface S whose boundary braid is b_0 if and only if the following two conditions are satisfied.*

(i) *If $\eta \in \pi_1(D_2^2 \setminus \Delta, y_0)$ is represented by a meridional loop, then $\rho(\eta) \in G_m$.*
(ii) *For the element $[\partial D_2^2] \in \pi_1(D_2^2 \setminus \Delta, y_0)$, the image $\rho([\partial D_2^2]) = b_0$ in B_m.*

Moreover this is still valid for embedded braided surfaces S when we replace G_m by A_m in (i). *It is also valid for regular braided surfaces S when we replace G_m by $G_m^{\rm reg}$.*

3. Chart description

We assume that the m points of X_m are lying on the real line when we regard D_1^2 as $\{z \in \mathbb{C}; |z| \leq 1\}$, so that Artin's standard generators, $\sigma_1, \ldots, \sigma_{m-1}$, of the braid group $B_m = \pi_1(C_m, X_m)$ are defined (cf. [1]).

Let Γ be a finite graph in the 2-disk D_2^2 such that each edge is oriented and labeled by an integer in $\{1, 2, \ldots, m-1\}$. We say that a path $\eta : [0,1] \to D_2^2$ is *in general position with respect to* Γ if the following conditions are satisfied.

(i) The image of η is disjoint from the vertices of Γ.
(ii) The preimage $\eta^{-1}(\Gamma)$ is empty or consists of a finite number of interior points of $[0,1]$, say $t_1, \ldots, t_s$, and assume $t_1 < \cdots < t_s$.
(iii) For each j $(j = 1, \ldots, s)$, the path η is locally an immersion nearby t_j which intersects an edge of Γ transversely.

In this situation, we define the *intersection word along η with respect to* Γ, denoted by $w_\Gamma(\eta)$, to be $\sigma_{i_1}^{\epsilon_1} \cdots \sigma_{i_s}^{\epsilon_s}$ where i_j $(j = 1, \ldots, s)$ is the label of the edge of Γ containing the jth intersection $\eta(t_j)$, and ϵ_j is $+1$ (or -1, resp.) if the path η at $t = t_j$ intersects the oriented edge of Γ from right to left (or left to right, resp.).

Let Λ be a set of points of ∂D_2^2 (possibly empty) which is disjoint from y_0, and each point is labeled by an integer in $\{1, 2, \ldots, m-1\}$ and signed by $+1$ or -1.

Definition 3.1. An *m-chart*, or simply a *chart*, in D_2^2 with boundary Λ is a finite graph Γ in the 2-disk D_2^2 such that each edge is oriented and labeled by an integer from $\{1, 2, \ldots, m-1\}$, and such that the following conditions (i) and (ii) are satisfied.

(i) $\Gamma \cap \partial D_2^2$ is Λ (with respect to labels and signs).
(ii) For each vertex v of Γ, let w_v be the intersection word along a meridian loop of v with respect to Γ. Then one of the following occurs.
  (a) $w_v \in G_m$, where we regard w_v as an element of the braid group B_m.
  (b) $w_v = \sigma_i^{-1}\sigma_j^{-1}\sigma_i\sigma_j$ (as a word) for some i, j with $|i-j| > 1$.
  (c) $w_v = \sigma_i^{-1}\sigma_j^{-1}\sigma_i^{-1}\sigma_j\sigma_i\sigma_j$ (as a word) for some i, j with $|i-j| = 1$.

A vertex of a chart is called a *black vertex*, a *crossing* or a *white vertex* if the case (a), (b) or (c) occurs, respectively. (See Figure 1.) The set of black vertices of Γ is denoted by Δ_Γ.

The *homomorphism associated with* Γ is a homomorphism

$$\rho_\Gamma : \pi_1(D_2^2 \setminus \Delta_\Gamma, y_0) \to B_m, \quad [\eta] \mapsto [w_\Gamma(\eta)].$$

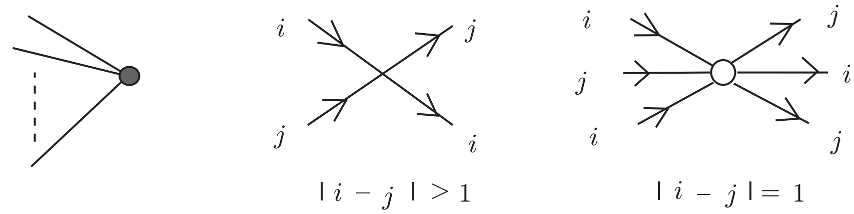

Figure 1. Vertices

Note that $w_v = 1$ in the braid group B_m for every crossing or white vertex v. Thus, we see that the map ρ_Γ is a well-defined homomorphism.

Lemma 3.2. (1) *For an m-chart Γ, the homomorphism $\rho_\Gamma : \pi_1(D_2^2 \setminus \Delta_\Gamma, y_0) \to B_m$ satisfies the conditions* (i) *and* (ii) *of Proposition 2.7, where $\Delta = \Delta_\Gamma$ and b_0 is the m-braid determined by the intersection word along ∂D_2^2 with respect to Γ.*

(2) *Let Λ be a set of labeled and signed points on ∂D_2^2 and let b_0 be the m-braid determined by the intersection word along ∂D_2^2 with respect to Λ. For a homomorphism $\rho : \pi_1(D_2^2 \setminus \Delta, y_0) \to B_m$ satisfying the conditions* (i) *and* (ii) *of Proposition 2.7, there exists an m-chart Γ with $\rho_\Gamma = \rho$ and $\Gamma \cap \partial D_2^2 = \Lambda$.*

Proof. (1) For each black vertex $v \in \Delta_\Gamma$, the braid w_v belongs to G_m. For any meridional element η_v for v, $\rho_\Gamma(\eta_v)$ is a conjugate of $[w_v]$, which belongs to G_m. The condition (ii) is obvious by definition. (2) By the same argument with that in [9, 11] (cf. [12]), we can obtain a desired m-chart. □

Let S be a braided surface of degree m, and Γ an m-chart with $\rho_S = \rho_\Gamma$. Then we call Γ a *chart description of* S. (For a given S, such a chart exists by Proposition 2.7 and Lemma 3.2, although it is not unique.) Conversely, we call S a *braided surface described by* Γ or a *braided surface associated with* Γ. (For a given Γ, such a braided surface exists and it is unique up to isomorphism by Propositions 2.6, 2.7 and Lemma 3.2.)

Chart moves of type W, of type B, and of type ∂ are defined in [12] for chart descriptions of G-monodromies in a general setting. In our situation, these moves are described as below.

(1) A *C-move of type W* is a replacement of a chart Γ with a chart Γ' such that Γ and Γ' are identical outside a disk region E in D_2^2 and such that Γ and Γ' have no black vertices in E. (Some typical C-moves of type W are illustrated in Figure 2. It is known that every C-move of type W is a consequence of the moves depicted in Figure 2, cf. [2, 3, 15].)

(2) A *C-move of type B* is a local replacement nearby a black vertex v of a chart Γ illustrated in Figure 3, which changes the word w_v with insertion/deletion of $\sigma_i^{-1}\sigma_i$, or $\sigma_i\sigma_i^{-1}$ for $i = 1, \ldots, m-1$, or $\sigma_i^{-1}\sigma_j^{-1}\sigma_i\sigma_j$ for i, j with $|i-j| > 1$, or $\sigma_i^{-1}\sigma_j^{-1}\sigma_i^{-1}\sigma_j\sigma_i\sigma_j$ for i, j with $|i-j| = 1$, respectively.

(3) A *C-move of type* ∂ is a local replacement nearby the boundary ∂D_2^2 of a chart Γ illustrated in Figure 4, which changes $\Lambda = \Gamma \cap \partial D_2^2$ so that the intersection word along ∂D_2^2 with respect to Γ changes by insertion/deletion of $\sigma_i^{-1}\sigma_i$, or $\sigma_i\sigma_i^{-1}$ for $i = 1, \dots, m-1$, or $\sigma_i^{-1}\sigma_j^{-1}\sigma_i\sigma_j$ for i, j with $|i-j| > 1$, or $\sigma_i^{-1}\sigma_j^{-1}\sigma_i^{-1}\sigma_j\sigma_i\sigma_j$ for i, j with $|i-j| = 1$, respectively. (We assume that C-moves of type ∂ are applied away from $y_0 \in \partial D_2^2$.)

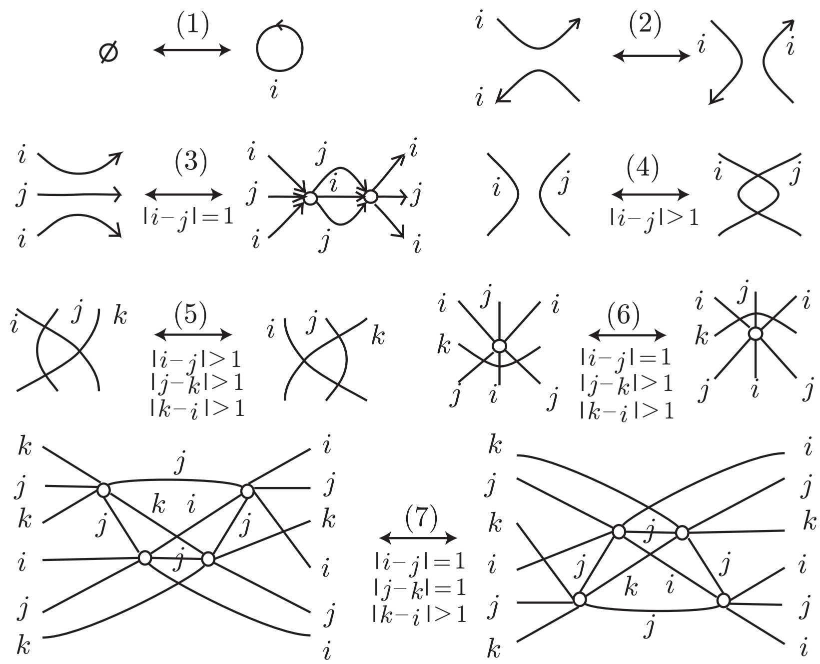

FIGURE 2. C-moves of type W

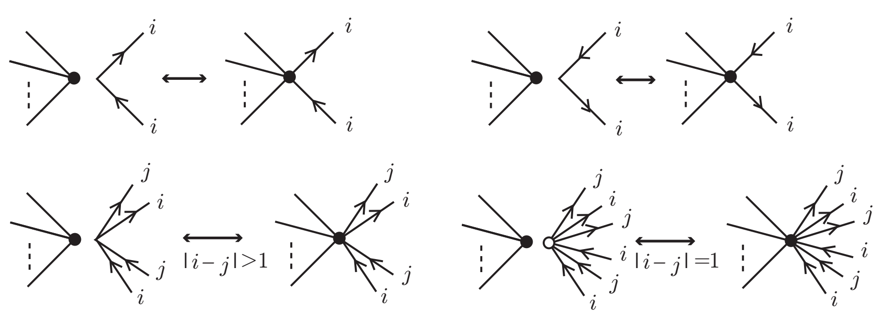

FIGURE 3. C-moves of type B

Definition 3.3. Two m-charts Γ and Γ' are *chart move isomorphic* (or *chart move equivalent*, resp.) if they are related by a finite sequence of C-moves and isotopies of D_2^2 rel $\{y_0\} \cup \Delta_\Gamma$ (or isotopies of D_2^2 rel $\{y_0\}$, resp.).

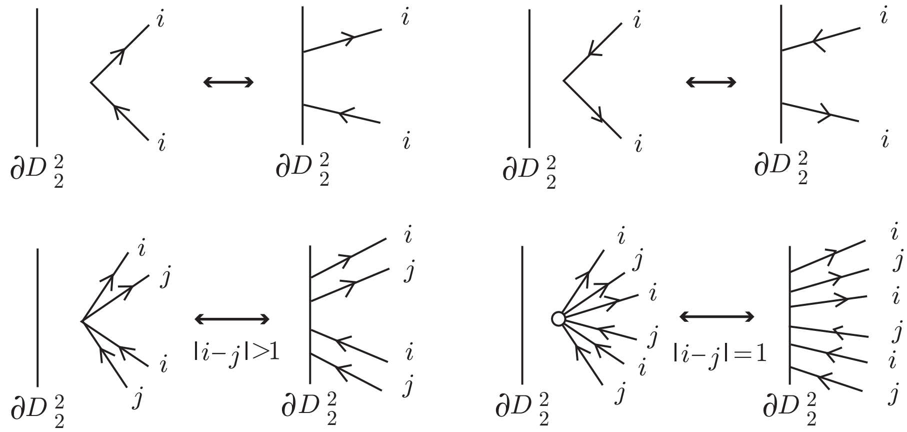

FIGURE 4. C-moves of type ∂

Remark 3.4. The C-moves of type B (Figure 3) are equivalent to the moves illustrated in Figure 5 up to C-moves of type W and isotpies of D^2_2 rel $\{y_0\} \cup \Delta_\Gamma$, and the C-moves of type ∂ (Figure 4) are equivalent to the moves in Figure 6. Thus we may add these moves as C-moves in Definition 3.3.

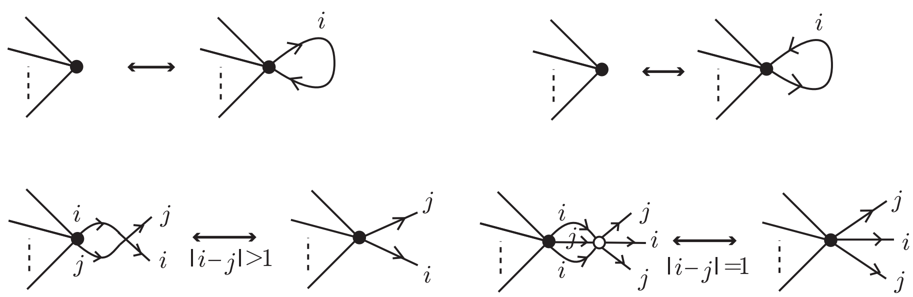

FIGURE 5. C-moves of type B

Proof of Theorem 1.1. By Proposition 2.7 and Lemma 3.2, any braided surface has a chart description. Let S and S' be braided surfaces of degree m, ρ_S and $\rho_{S'}$ their monodromy representations, Γ and Γ' their chart descriptions, and ρ_Γ and $\rho_{\Gamma'}$ the associated homomorphisms, respectively. The following conditions (1), (2), (3) and (4) are mutually equivalent.

(1) S and S' are isomorphic (or equivalent, resp.).
(2) $\rho_S = \rho_{S'}$ (or ρ_S and $\rho_{S'}$ are equivalent, resp.).
(3) Γ and Γ' are chart move isomorphic (or chart move equivalent, resp.).
(4) $\rho_\Gamma = \rho_{\Gamma'}$ (or ρ_Γ and $\rho_{\Gamma'}$ are equivalent, resp.).

The equivalence between (1) and (2) is given in [8] (Lemma 2.6). By definition, (2) and (4) are equivalent. The equivalence between (3) and (4) is proved in [12] (Theorem 12 and § 8 of [12]). □

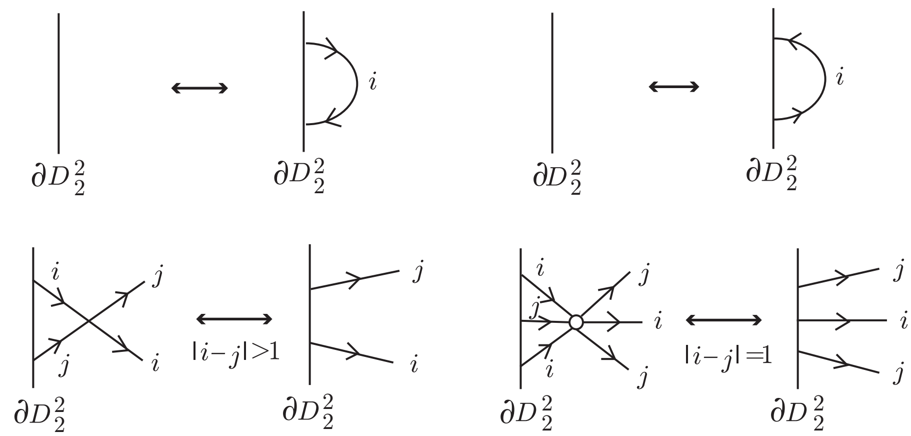

FIGURE 6. C-moves of type ∂

4. Regular charts

Let S be a braided surface of degree m. For a point x of S, the *local degree* of x means the local degree at x of the branched covering if x is not a node. We define the *local degree* of x to be 2 if x is a node. (For a regular point of S, the local degree is 1.) The local degree of x is denoted by $\deg_S(x)$. For a point x of S, we define the *singular index*, $\tau_S(x)$, to be $\deg_S(x) - 1$. Then x is a singular point of S if and only if $\tau_S(x) > 0$. For a point $y \in D_2^2$, we define the *singular index*, $\tau_S^*(y)$, of y to be $m - \#(S \cap pr_2^{-1}(y))$. Then y is a singular value of S if and only if $\tau_S^*(y) > 0$. Note that, for any point $y \in D_2^2$,

$$\tau_S^*(y) = \sum_{x \in S \cap pr_2^{-1}(y)} \tau_S(x).$$

Let Γ be a chart description of S, and y a black vertex of Γ, which is a singular value of S. The *label set* of y, denoted by $\mathrm{Label}_\Gamma(y)$ is the set of labels of the edges which are incident to y. Note that $\#\mathrm{Label}_\Gamma(y) \geq \tau_S^*(y)$. (See the proof of the theorem below.)

Definition 4.1. In the above situation, Γ is *range-reduced at* y if $\#\mathrm{Label}_\Gamma(y) = \tau_S^*(y)$. A chart is *range-reduced* if it is range-reduced at every black vertex.

Lemma 4.2. Any braided surface has a range-reduced chart description.

Proof. Let S be a braided surface of degree m, and y a singular value. Let $\{x_1, \ldots, x_c\}$ be $S \cap pr_2^{-1}(y)$, where $c = \#(S \cap pr_2^{-1}(y)) = m - \tau_S^*(y)$. Modifying S up to isomorphism, we may assume that $pr_1(x_1), \ldots, pr_1(x_c)$ are on the real line and $pr_1(x_1) < \cdots < pr_1(x_c)$ where we regard D_1^2 as $\{z \in \mathbb{C}; |z| \leq 1\}$. Let $N_1, \ldots, N_c$ be regular neighborhoods of the points $pr_1(x_1), \ldots, pr_1(x_c)$ in D_1^2. Taking a regular neighborhood $N(y)$ of y in D_2^2 sufficiently small, we may assume that the restriction of S to $D_1^2 \times N(y)$ is contained in $\cup_{k=1}^c N_k \times N(y)$. The boundary of $S \cap (D_1^2 \times N(y))$ is an

m-braid in the solid torus $D_1^2 \times \partial N(y)$, say ℓ, such that $\ell = \ell_1 \amalg \cdots \amalg \ell_c$, with $\ell_k \subset N_k \times \partial N(y)$, and $S \cap (D_1^2 \times N(y))$ is a multiple cone over $\ell = \ell_1 \amalg \cdots \amalg \ell_c$, i.e., $S \cap (D_1^2 \times N(y))$ is the disjoint union $\amalg_{k=1}^c S \cap (N_k \times N(y))$ such that for each k, $S \cap (N_k \times N(y))$ is a cone over ℓ_k with cone point x_k. (Refer to [8] or § 16.4 of [11] for the terminology "multiple cone".) Each ℓ_k is a trivial knot or a Hopf link in the 3-sphere $\partial(N_k \times N(y))$, and the latter case the braid degree of ℓ_k is 2. (The former case occurs when the point x_k satisfies (1) or (2) of Remark 2.4, and the latter case occurs when x_k satisfies (3) of Remark 2.4. This is the reason why the local monodromy at y is an element of G_m.) Let m_k be the braid degree of ℓ_k for $k = 1, \ldots, c$, which is the local degree $\deg_S(x_k)$ of x_k. The singular index of x_k of S is $m_k - 1$. By definition $\tau_S^*(y) = m - c = \sum_{k=1}^c (m_k - 1)$.

If $m_k = 1$, then ℓ_k is a trivial closed 1-braid. If $m_k > 1$, then ℓ_k can be described by a braid word, say w_k, in

$$\{\sigma_{m_1+\cdots+m_{k-1}+1}, \sigma_{m_1+\cdots+m_{k-1}+2}, \ldots, \sigma_{m_1+\cdots+m_k-1}\}.$$

Since ℓ_k is a trivial knot or a closed 2-braid representing a Hopf link, all generators in this set must appear in this word w_k. A braid word for ℓ is described by the concatenation $w_1 \cdots w_c$. When we construct a chart Γ using such a braid word description for ℓ in the method used in Theorem 5 of [12], the label set $\mathrm{Label}_\Gamma(y)$ of the black vertex y is

$$\cup_{k=1}^c \{\sigma_{m_1+\cdots+m_{k-1}+1}, \sigma_{m_1+\cdots+m_{k-1}+2}, \ldots, \sigma_{m_1+\cdots+m_k-1}\}.$$

Hence $\#\mathrm{Label}_\Gamma(y) = \tau_S^*(y)$. Applying the same argument to each singular value of S, we have a range-reduced chart. □

Definition 4.3. A chart Γ is *range-connected* at a black vertex y if $\mathrm{Label}_\Gamma(y)$ consists of consecutive integers.

A black vertex y of a chart Γ is called a *nodal black vertex* if it is a singular value of a braided surface $S = S(\Gamma)$ described by Γ such that there exists exactly one singular point of S in the fiber over y and the singular point is a node.

If y is a nodal black vertex of a regular chart Γ and if Γ is range-reduced and range-connected at y, then $\mathrm{Label}_\Gamma(y)$ consists of a single integer.

Definition 4.4. A nodal black vertex y of a chart Γ is *simple* if exactly two edges of Γ are incident to y (see Figure 7), otherwise it is called *nonsimple.*

Definition 4.5. A chart is *regular* if every black vertex is range-reduced and range-connected and if every nodal black vertex is simple.

FIGURE 7. Simple nodal black vertex

Lemma 4.6 (Regular chart description)**.** *Any regular braided surface has a regular chart description. Conversely, a braided surface described by a regular chart is a regular braided surface.*

Proof. Let S be a regular braided surface. For each singular value, there exists exactly one singular point. Thus, a range-reduced chart obtained by the argument of the proof of Lemma 4.2 has range-connected black vertices. When y is a nodal black vertex which is not simple, then more than 2 edges are incident to y and their labels are all the same, say i. Since ℓ_k in the proof of Lemma 4.2 is a Hopf link represented by a 2-braid, w_y is equal to σ_i^2 or σ_i^{-2} in the braid group B_m. Apply C-moves of type B to make y simple. The coverse is obvious. □

5. Regular chart moves

Let Γ be a regular chart. Let y be a black vertex of Γ and let $\mathrm{Label}_\Gamma(y) = \{s, s+1, \dots, t\}$ be the label set of y. Each move illustared in Figure 8 shifts the label set by $+1$ or -1, where the box means a chart without black vertices. Note that $\{i_1, i_2, \dots, i_n\} = \{s, s+1, \dots, t\}$. For (A) of Figure 8, combining fundamental pieces as in Figure 9, we see that the box can be always filled by a chart without black vertices. For example, see Figures 10 and 11. Boxes in the cases (B), (C) and (D) can be filled similarly.

We call the moves illustrated in Figure 8 *label shift moves*. When it shifts the labels by $+1$, it is called an *upper shift*, otherwise a *lower shift*.

A *passing move* is a move illustrated in Figure 12, where j is an integer with $j < s-1$ or $j > t+1$. (Note that $\{i_1, i_2, \dots, i_n\} = \{s, s+1, \dots, t\}$.)

The moves in Figure 12 are equivalent to the moves in Figure 13 modulo chart moves of type W.

For the sake of convenience in the proof of Theorem 1.2, we introduce notations for the moves illustrated in Figures 8 and 12. Recall that we are assuming that $\mathrm{Label}_\Gamma(y) = \{s, s+1, \dots, t\}$. The moves (A), (B), (C) and (D) in Figure 8 are denoted by $L_{s,t}[\sigma_s]$, $L_{s,t}[\sigma_s^{-1}]$, $L_{s,t}[\sigma_{s-1}]$ and $L_{s,t}[\sigma_{s-1}^{-1}]$, respectively. The move (E) in Figure 12 is denoted by $L_{s,t}[\sigma_j]$ if $j < s-1$ and by $L_{s,t}[\sigma_{j-(t+1-s)}]$ if $j > t+1$. The move (F) in Figure 12 is denoted by $L_{s,t}[\sigma_j^{-1}]$ if $j < s-1$ and by $L_{s,t}[\sigma_{j-(t+1-s)}^{-1}]$ if $j > t+1$. Note that the notation $L_{s,t}[\cdot]$ depends on $\mathrm{Label}_\Gamma(y)$.

Definition 5.1. Two regular charts are *regularly chart move isomorphic* (or *regularly chart move equivalent*, resp.) if they are related by a finite sequence of C-moves of type W, C-moves of type B keeping the condition of regularity, label shift moves, passing moves, C-moves of type ∂, and isotopies of D_2^2 rel $\{y_0\} \cup \Delta_\Gamma$ (or isotopies of D_2^2 rel $\{y_0\}$, resp.). These moves are called *regular chart moves*.

Remark 5.2. C-moves of type B keeping the condition of regularity, label shift moves and passing moves are 'chart moves of transition' in the sense of [12].

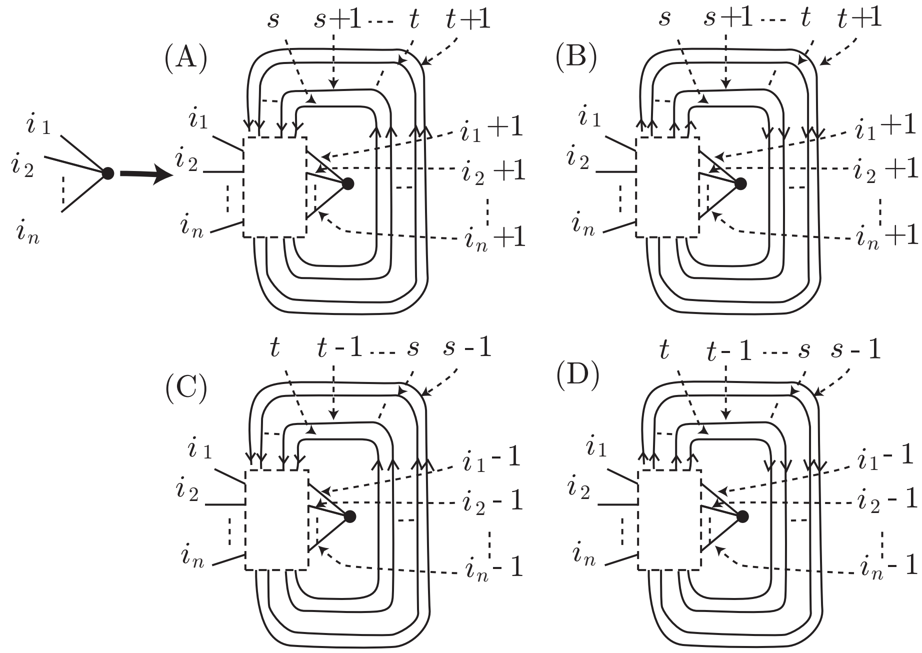

FIGURE 8. Label shift moves $L_{s,t}[\sigma_s]$, $L_{s,t}[\sigma_s^{-1}]$, $L_{s,t}[\sigma_{s-1}]$ and $L_{s,t}[\sigma_{s-1}^{-1}]$

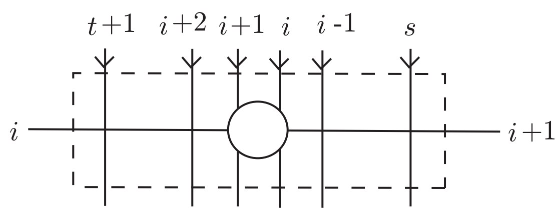

FIGURE 9. A fundamental piece

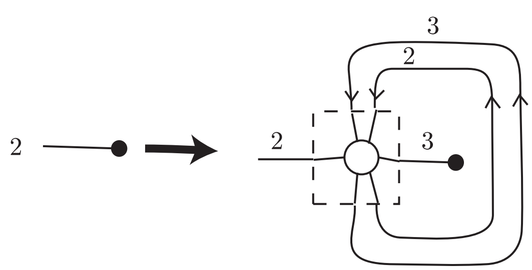

FIGURE 10. Label shift move $L_{2,2}[\sigma_2]$

Proof of Theorem 1.2. Any regular braided surface has a regular chart description (Lemma 4.6). Suppose that Γ and Γ' are regularly chart move

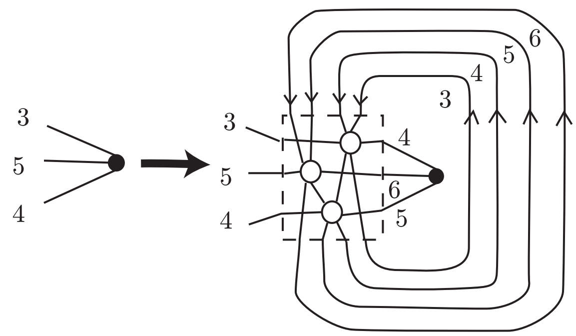

FIGURE 11. Label shift move $L_{3,5}[\sigma_3]$

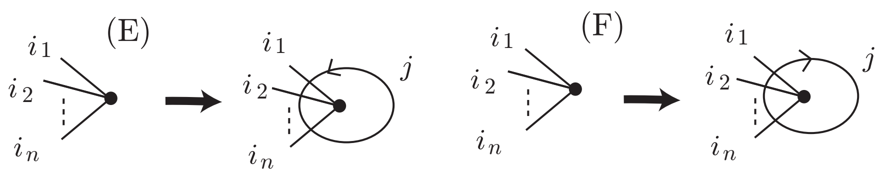

FIGURE 12. (E) passing move $L_{s,t}[\sigma_j]$ for $j < s-1$ or $L_{s,t}[\sigma_{j-(t+1-s)}]$ for $j > t+1$, and (F) passing move $L_{s,t}[\sigma_j^{-1}]$ for $j < s-1$ or $L_{s,t}[\sigma_{j-(t+1-s)}^{-1}]$ for $j > t+1$

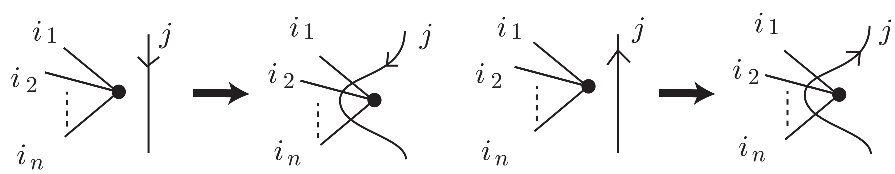

FIGURE 13. Passing moves

isomorphic (or regularly chart move equivalent, resp.). Label shift moves and passing moves are 'chart moves of transition' in the sense of [12], which are consequence of C-moves of type B and type W (Remark 15 of [12]). Therefore, if two regular charts are regularly chart move isomorphic (or regularly chart move equivalent, resp.), then they are chart move isomorphic (or chart move equivalent, resp.). By Theorem 1.1, S and S' are isomorphic (or equivalent, resp.).

We show the converse.

We say that a singular value y of a braided surface S satisfies '*the condition* $(*)$' if

($*$) $pr_1(S \cap pr_2^{-1}(y))$ is on the real line where we regard D_1^2 as $\{z \in \mathbb{C}; |z| \leq 1\}$.

(1) First we consider a case where every singular value of S and S' satisfies the condition ($*$) and that S' is isomorphic to S keeping the condition ($*$). Then the label sets $\mathrm{Label}_\Gamma(y)$ ($y \in \Delta_\Gamma$) are preserved. By an argument in [12], we see that Γ and Γ' are related by a finite sequence of C-moves of type W, C-moves of type B preserving the label sets $\mathrm{Label}_\Gamma(y)$ ($y \in \Delta_\Gamma$), passing moves, C-moves of type ∂, and isotopies of D_2^2 rel $\{y_0\} \cup \Delta_\Gamma$. Thus Γ and Γ' are regularly chart move isomorphic.

(2) Next we consider a case where every singular value of S and S' satisfies the condition ($*$) and that S' is isomorphic to S. Let $\{S_t\}_{t\in[0,1]}$ be a 1-parameter family of braided surfaces between S and S'. Let y be a singular value of S. Without loss of generality we may assume that $pr_1(S \cap pr_2^{-1}(y)) = pr_1(S' \cap pr_2^{-1}(y))$. The motion $\{pr_1(S_t \cap pr_2^{-1}(y))\}_{t\in[0,1]}$ is a classical braid of degree $c = \#(S \cap pr_2^{-1}(y)) = m - \tau_S^*(y)$. Let $w = \sigma_{i_1}^{\epsilon_1} \cdots \sigma_{i_\ell}^{\epsilon_\ell}$ be a word description of this braid. We consider a sequence of regular charts, $\Gamma = \Gamma_0, \Gamma_1, \cdots, \Gamma_\ell$, as follows: Suppose that Γ_k is defined. Let $\mathrm{Label}_{\Gamma_k}(y) = \{s(k), s(k)+1, \cdots, t(k)\}$. Apply $L_{s(k),t(k)}[\sigma_{i_k}^{\epsilon_k}]$ to the chart Γ_k at y and let Γ_{k+1} be the result. Let Γ'' be the final result Γ_{i_ℓ}. By definition Γ is regularly chart move isomorphic to Γ''.

On the other hand, let S'' be a braided surface with chart description Γ'' satisfying the condition ($*$). Note that S and S'' are isomorphic. Let $\{S'_t\}_{t\in[0,1]}$ be a 1-parameter family of braided surfaces between S and S''. Without loss of generality we may assume that $pr_1(S \cap pr_2^{-1}(y)) = pr_1(S'' \cap pr_2^{-1}(y))$. The motion $\{pr_1(S'_t \cap pr_2^{-1}(y))\}_{t\in[0,1]}$ is a classical braid of degree $c = \#(S \cap pr_2^{-1}(y)) = m - \tau_S^*(y)$. This braid is eqaul to the braid repesented by the word w above. Then S' and S'' are braided surfaces satisfying the condition ($*$), and there is a 1-parameter family of braided surfaces between S' and S'', say $\{S''_t\}_{t\in[0,1]}$, such that the motion $\{pr_1(S''_t \cap pr_2^{-1}(y))\}_{t\in[0,1]}$ is a classical braid of degree c such that it is isotopic to the trivial braid. By replacing $\{S''_t\}_{t\in[0,1]}$, we may assume that the motion $\{pr_1(S''_t \cap pr_2^{-1}(y))\}_{t\in[0,1]}$ is the trival braid of degree c. Apply the same argument to each singular value of S. Then by (1), we see that Γ' is regularly chart move isomorphic to Γ''. Therefore Γ is regularly chart move isomorphic to Γ'.

(3) Let S and S' be isomorphic. By definition, Γ (or Γ', resp.) is a regular chart description of some braided surface $\tilde{S}$ (or $\tilde{S}'$) such that $\tilde{S}$ (or $\tilde{S}'$) is isomorphic to S (or S') and satisfies the condition ($*$). By (2) we see that Γ and Γ' are regularly chart move isomorphic.

(4) Let S and S' be equivalent and let Γ and Γ' be their regular chart descriptions. Let $\{h_t\}$ be an isotopy of $D_1^2 \times D_2^2$ carrying S to S', and let $\{\underline{h}_t\}$ be the isotopy of D_2^2 rel $\{y_0\}$ with $\underline{h}_t \circ pr_2 = pr_2 \circ h_t$. The chart $\underline{h}_1(\Gamma)$ is a regular chart description of $h_1(S)$. The chart Γ is isotopic to $\underline{h}_1(\Gamma)$ by an isotopy of D_2^2 rel $\{y_0\}$. On the other hand, since $h_1(S)$ is isomorphic to

S', by (3), $\underline{h}_1(\Gamma)$ and Γ' are regularly chart move isomorphic. Thus Γ and Γ' are regularly chart move equivalent. □

6. On simple chart description

A chart is *simple* if it satisfies the conditions of Definition 3.1 such that (a) is replaced with the following condition (a′).

(a′) $w_v = \sigma_i$, σ_i^{-1}, σ_i^2 or σ_i^{-2} as a word for some i.

In other words, a chart is simple if every black vertex is one of Figure 14. (We call a black vertex is *simple* if it is as in Figure 14.) By definition, a simple chart is a regular chart.

The following theorem was proved in [10] for simple (immersed) 2-dimensional braids and in [12] for embedded braided surfaces.

Theorem 6.1 (Simple chart description)**.** *Any simple braided surface has a simple chart description. Conversely, a braided surface described by a simple chart is a simple braided surface.*

Proof . Let S be a simple braided surface. Apply the argument in the proof of Lemma 4.2 and obtain a regular chart description of S. If y is a nodal singular value, then w_y is equal to σ_i^2 or σ_i^{-2} as a word for some i. If y is a branch value, then w_y is equal to σ_i or σ_i^{-1} in B_m for some i. By C-moves of type B we can change the black vertex y to be simple. The coverse is obvious. □

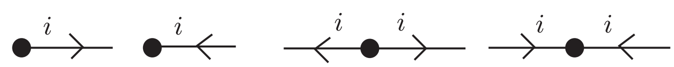

Figure 14. Simple black vertices and simple nodal black vertices

In [5, 10] (cf. [11]), chart moves (CI-moves, CII-moves, CIII-moves, CIV-moves and CV-moves) for simple charts are introduced. CI-moves are the same with C-moves of type W in this paper. CII-moves, CIII-moves, CIV-moves and CV-moves are illustrated in Figures 15, 16, 17 and 18, respectively. (In the figures, we illustrated an example of possible orientations of the edges. See [5, 9] or [11] for details on the moves.)

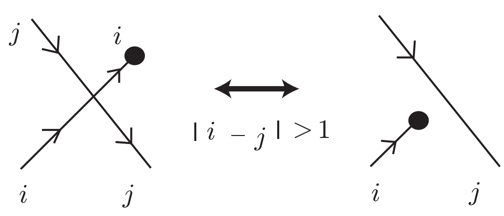

Figure 15. A CII-move

Two simple m-charts Γ and Γ' describe isomorphic (or equivalent, resp.) simple braided surfaces if and only if they are related by a finite sequence

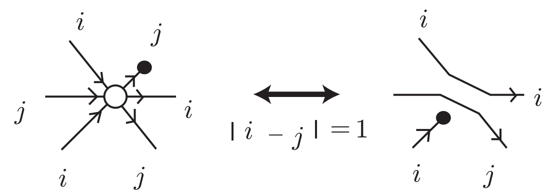

FIGURE 16. A CIII-move

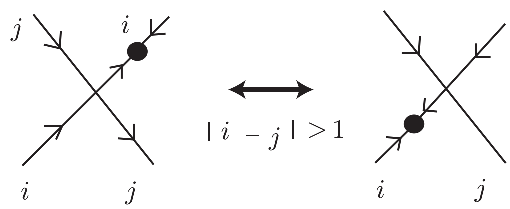

FIGURE 17. A CIV-move

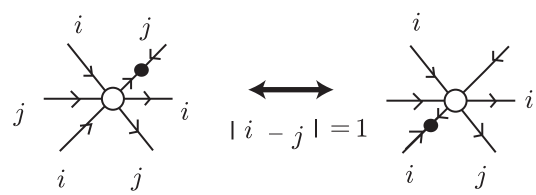

FIGURE 18. A CV-move

of C-moves and isotopies of D_2^2 rel $\{y_0\} \cup \Delta_\Gamma$ (or isotopies of D_2^2 rel $\{y_0\}$, resp.).

The following theorem is proved in [10] for (immersed) 2-dimensional braids (cf. [11]) and in [12] for embedded braided surfaces.

Theorem 6.2. *Two simple m-charts Γ and Γ' describe isomorphic (or equivalent, resp.) simple braided surfaces S and S' if and only if they are related by a finite sequence of CI-moves, CII-moves, CIII-moves, CIV-moves, CV-moves,C-moves of type ∂ and isotopies of D_2^2 rel $\{y_0\} \cup \Delta_\Gamma$ (or isotopies of D_2^2 rel $\{y_0\}$, resp.).*

Proof . Suppose that S and S' are isomorphic (or equivalent). In (1) – (3) of the proof of Theorem 1.2, we do not need C-moves of type B. C-moves of type W are CI-moves. Label shift moves are CIII-moves and CV-moves modulo CI-moves, and passing moves are CII-moves and CIV-moves mudulo CI-moves. Thus, by the argument of the proof of Theorem 1.2, we see that Γ and Γ' are related by a finite sequence of CI-moves, CII-moves, CIII-moves, CIV-moves, CV-moves,C-moves of type ∂ and isotopies of D_2^2 rel $\{y_0\} \cup \Delta_\Gamma$ (or isotopies of D_2^2 rel $\{y_0\}$, resp.). The converse is obvious. $\square$

Remark 6.3. *In Theorems 1.1, 1.2 and 6.2, if* Γ *and* Γ' *have the same boundary then we do not need C-moves of type* ∂. *In particular if* Γ *and* Γ' *be chart descriptions of* 2*-dimensional braids, then we do not need C-moves of type* ∂.

References

[1] J. S. Birman, *Braids, Links, and Mapping Class Groups*, Ann. Math. Studies **82**, Princeton Univ. Press, Princeton, N.J., (1974).

[2] J. S. Carter and M. Saito, *Knotted surfaces, braid moves and beyond*, in "Knots and Quantum Gravity", pp. 191–229, Oxford Science Publishing, (1994).

[3] J. S. Carter and M. Saito, *Knotted Surfaces and Their Diagrams*, Math. Surveys and Monographs **55**, Amer. Math. Soc., (1998).

[4] S. Carter, S. Kamada and M. Saito, *Surfaces in* 4*-Space*, Encyclopaedia of Mathematical Sciences **142**, Springer-Verlag, Berlin Heidelberg New York, (2004).

[5] S. Kamada, *Surfaces in* R^4 *of braid index three are ribbon*, J. Knot Theory Ramifications **1** (1992), 137–160.

[6] S. Kamada, 2*-dimensional braids and chart descriptions*, in "Topics in Knot Theory", NATO ASI Series C, **399**, pp. 277–287, Kluwer Academic Publisher, (1992).

[7] S. Kamada, *A characterization of groups of closed orientable surfaces in* 4*-space*, Topology **33** (1994), 113–122.

[8] S. Kamada, *On braid monodromies of non-simple braided surfaces*, Math. Proc. Camb. Phil. Soc. **120** (1996), 237–245.

[9] S. Kamada, *An observation of surface braids via chart description*, J. Knot Theory Ramifications **4** (1996), 517–529.

[10] S. Kamada, *Surfaces in* 4*-space: A view of normal forms and braidings*, Lectures at Knots 96 (ed. S. Suzuki), pp. 39–71, World Scientific Publishing Co., Singapore, (1997).

[11] S. Kamada, *Braid and Knot Theory in Dimension Four*, Math. Surveys and Monographs **95**, Amer. Math. Soc., (2002).

[12] S. Kamada, *Graphic descriptions of monodromy representations*, Topology Appl. **154** (2007), 1430–1446.

[13] L. Rudolph, *Braided surfaces and Seifert ribbons for closed braids*, Comment. Math. Helv. **58** (1983), 1–37.

[14] L. Rudolph, *Special positions for surfaces bounded by closed braids* Rev. Mat. Iberoamericana **1** (1985), 93–133.

[15] K. Tanaka, *A note on* CI*-moves*, in "Intelligence of Low Dimensional Topology", Series of Knots and Everything Vol. 40, pp. 307–314, World Scientific Publishing Co., Singapore, (2007).

[16] O. Ya. Viro, Lecture given at Osaka City University, September, 1990.

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