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CHART DESCRIPTIONS OF REGULAR BRAIDED SURFACES

SEIICHI KAMADA AND TAKAO MATUMOTO

Abstract. Braided surfaces are surfaces embedded or immersed in the bidisk $D^2 \times D^2$ which are projected onto the second factor of the bidisk as branched covering maps. A simple and embedded braided surface is described by a graph in the 2-disk, called a chart. We generalize the chart description method so that one can consider braided surfaces which are not necessarily simple or embedded. We show that if a braided surface is regular, then it can be described by a chart called regular, and that such a chart is unique up to regular chart move equivalence.

1. Introduction

A graphical method, called the chart description method, was introduced in [5] in order to describe a simple and embedded 2-dimensional braid, which is a special case of braided surfaces. Later, it was proved in [9] that the basic moves (called chart moves) introduced in [5] are sufficient moves. The chart description method was extended to simple and immersed braided surfaces in [10] (cf. Chapter 34 of [11]). It has been used for the study of simple 2-dimensional braids, simple braided surfaces and knotted surfaces in 4-space, cf. [3, 4, 11]. Applying a result of [12], we have the following. (The C-moves are explained later in § 3.)

Theorem 1.1. Any (embedded or immersed) braided surface is described by a chart. Such a chart description is unique up to chart move equivalence. More precisely, let $\Gamma$ and $\Gamma'$ be chart descriptions of braided surfaces $S$ and $S'$. The braided surfaces $S$ and $S'$ are isomorphic (or equivalent, resp.) if and only if $\Gamma$ and $\Gamma'$ are related by a finite sequence of C-moves of type W, C-moves of type B, C-moves of type $\partial$ and isotopies of $D^2_2 \text{ rel } \{y_0\} \cup \Delta_{\Gamma}$ (or isotopies of $D^2_2 \text{ rel } \{y_0\}$, resp.).

A braided surface is called regular if for each singular value there exists exactly one singular point. Since any braided surface is ambiently isotopic to a regular braided surface by an isotopy of $D^2_2 \times D^2_2$ (cf. [11]), it is sufficient to consider regular braided surfaces for study of ambient isotopy classes of braided surfaces and surface-links in 4-space. In order to describe a

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regular braided surface effectively, we introduce the notion of a regular chart. Any regular braided surface is described by a regular chart. Let $\Gamma$ and $\Gamma'$ be regular chart descriptions of regular braided surfaces $S$ and $S'$. The braided surfaces $S$ and $S'$ are isomorphic (or equivalent) if and only if $\Gamma$ and $\Gamma'$ are related by a sequence of moves and isotopies of $D_2^2$ as stated in Theorem 1.1. However, in such a sequence, the regularity condition of a chart is not preserved in general. Some C-moves of type B may change a regular chart to an irregular one.

The following is our main result.

**Theorem 1.2.** Any regular braided surface is described by a regular chart. Such a chart description is unique up to regular chart move equivalence. More precisely, let $\Gamma$ and $\Gamma'$ be regular chart descriptions of regular braided surfaces $S$ and $S'$. The braided surfaces $S$ and $S'$ are isomorphic (or equivalent, resp.) if and only if $\Gamma$ and $\Gamma'$ are related by a finite sequence of C-moves of type W, C-moves of type B keeping the condition of regularity, label shift moves, passing moves, C-moves of type $\partial$, and isotopies of $D_2^2$ rel $\{y_0\}$ (or isotopies of $D_2^2$ rel $\{y_0\}$, resp.).

In § 2 we recall the definition of braided surfaces and monodromy representations. In § 3 the chart description method is explained, and in § 4 the notion of regular charts is introduced. In § 5 regular chart moves, including label shift moves and passing moves, are defined and Theorem 1.2 is proved. In § 6 we recall chart moves for simple (embedded or immersed) 2-dimensional braids introduced in [5, 9, 10] are interpreted in terms of our regular chart moves.

**2. Braided surfaces**

We work in the PL category. An immersed surface $S$ in a 4-manifold $W$ is said to be proper if $S \cap \partial W$ is equal to the boundary of the surface $S$. It is said to be locally flat if every point $x \in S$ has a regular neighborhood $N(x)$ in $W$ such that $(N(x), S \cap N(x), x)$ is homeomorphic to $(D^4, D^2, O)$, $(D^4, D^2_{xy} \cup D^2_{zw}, O)$ or $(D^4_+, D^2_+, O)$, where $D^4$ is the standard 4-disk in Euclidean 4-space coordinated with $x, y, z, w$, $D^2 = D^2_{xy}$ is the 2-disk on the $xy$-plane in $D^4$, $D^2_{zw}$ is the one on the $zw$-plane, $O$ is the origin, and $D^4_+$ and $D^2_+$ are restrictions of $D^4$ and $D^2$ with $x \geq 0$, respectively. In the second case where $(N(x), S \cap N(x), x)$ is homeomorphic to $(D^4, D^2_{xy} \cup D^2_{zw}, O)$, we call $x$ a node of $S$ or a self-intersection of $S$.

Let $D^2_1$ and $D^2_2$ be 2-disks and $pr_i : D^2_1 \times D^2_2 \to D^2_i$ ($i = 1, 2$) the $i$th factor projection. Let $X_m$ be a fixed set of $m$ interior points of $D^2_i$, which are assumed to be on the real line when we regard $D^2_i$ as $\{z \in \mathbb{C}; |z| \leq 1\}$. We identify the braid group $B_m$ with the fundamental group $\pi_1(C_m, X_m)$ of the configuration space $C_m = C_m(\text{Int}D^2_1)$ of $m$ distinct points of the interior, $\text{Int}D^2_1$, of $D^2_1$ (cf. [1, 11]).
We assume that spaces have base points and that the base point of $D_2^2$ is a point of $\partial D_2^2$, which we denote by $y_0$ throughout this paper.

**Definition 2.1** (cf. [13, 14]). A braided surface of degree $m$ is a compact oriented immersed surface $S$ in $D_1^2 \times D_2^2$, which is proper and locally flat, satisfying the following conditions (i) – (iv), where $\iota : S_0 \to D_1^2 \times D_2^2$ is an underlying immersion for $S$ (i.e., $\iota(S_0) = S$) and $p : S_0 \to D_2^2$ is the composition of $\iota$ and the second factor projection $pr_2$.

(i) The map $p : S_0 \to D_2^2$ is a branched covering map of degree $m$.

(ii) For each branch point $x \in S_0$ of $p$, the image $\iota(x)$ is not a node of $S$.

(iii) $\partial S \subset \text{Int}D_1^2 \times \partial D_2^2$.

(iv) $pr_1(S \cap pr_2^{-1}(y_0)) = X_m$.

The boundary $\partial S$ of $S$ is a closed $m$-braid in the solid torus $D_1^2 \times \partial D_2^2$. The $m$-braid obtained from $\partial S$ by cutting off along $pr_2^{-1}(y_0) = D_1^2 \times \{y_0\}$ is called the boundary braid of $S$.

**Definition 2.2** (cf. [5, 10, 16]). A 2-dimensional braid of degree $m$, or a 2-dimensional $m$-braid, is a braided surface of degree $m$ satisfying the following condition (v).

(v) The boundary $\partial S$ of $S$ is the trivial closed $m$-braid $X_m \times \partial D_2^2$ in $D_1^2 \times \partial D_2^2$, i.e., $pr_1(S \cap pr_2^{-1}(y)) = X_m$ for all $y \in \partial D_2^2$.

A branch point of $S$ means the image $\iota(x)$ in $S$ of a branch point $x \in S_0$ of $p$. A singular point of $S$ is a branch point of $S$ or a node of $S$. A singular value of $S$ is a point of $D_2^2$ which is the image under $pr_2$ of a singular point of $S$. The set of singular values of $S$ is denoted by $\Delta_S$.

**Definition 2.3.** A braided surface is **regular** if the following condition (vi) is satisfied.

(vi) For each singular value $y \in D_2^2$, there exists exactly one singular point of $S$.

A braided surface is **simple** if both of (vi) and (vii) are satisfied.

(vii) For each branch point $x \in S$, the local degree of the branched covering at $x$ is 2.

If a braided surface $S$ is an embedded surface (i.e., if there exist no nodes), then we say that $S$ is an **embedded braided surface**.

**Remark 2.4.** We may define a braided surface alternatively as follows: An immersed surface $S$ in $D_1^2 \times D_2^2$ is a braided surface if, for each point $x \in S$, there exists a regular neighborhood $N(x)$ in $D_1^2 \times D_2^2$ satisfying one of the following.

1. $(N(x), S \cap N(x), x)$ is homeomorphic to $(D^4, D^2, O)$, and when we put $E = S \cap N(x)$, the restriction $pr_2|_E : E \to pr_2(E)$ is a homeomorphism. (In this case, we call $x$ a regular point of $S$.)
(2) \((N(x), S \cap N(x), x)\) is homeomorphic to \((D^4, D^2, O)\), and when we put \(E = S \cap N(x)\), the restriction \(pr_2|_E : E \to pr_2(E)\) is a branched covering map with branch point \(x\). (In this case, we call \(x\) a branch point of \(S\).)

(3) \((N(x), S \cap N(x), x)\) is homeomorphic to \((D^4, D^2_{x_0} \cup D^2_{x_w}, O)\), and when we put \(E_1 \cup E_2 = S \cap N(x)\) such that \(E_1\) and \(E_2\) are 2-disks intersecting each other at \(x\) transversely, the restriction \(pr_2|_{E_i} : E_i \to pr_2(E_i)\) is a homeomorphism for \(i = 1, 2\). (In this case, we call \(x\) a node of \(S\).)

(4) \((N(x), S \cap N(x), x)\) is homeomorphic to \((D^4, D^2_0, O)\), and \(x\) is a point of \(\text{Int}D^2_1 \times \partial D^2_2\). (In this case, we call \(x\) a boundary point of \(S\).)

Two braided surfaces \(S\) and \(S'\) of degree \(m\) are said to be equivalent if there exists an isotopy \(\{h_t : D^1_x \times D^2_y \to D^1_x \times D^2_y\}_{t \in [0,1]}\) of the ambient space \(D^1_x \times D^2_y\) satisfying the following (i) – (iii).

(i) \(h_0 = \text{id}\) and \(h_1(S) = S'. \) (\(S\) and \(S'\) are ambiently isotopic by \(\{h_t\}\).)

(ii) For each \(t \in [0,1], \) \(h_t : D^1_x \times D^2_y \to D^1_x \times D^2_y\) is fiber-preserving; namely, there exists an isotopy \(\{h_t : D^2_y \to D^2_y\}_{t \in [0,1]}\) of \(D^2_y\) rel \(\{y_0\}\) with \(h_t \circ pr_2 = pr_2 \circ h_t\).

(iii) For each \(t \in [0,1], \) \(h_t\) fixes the distinguished fiber \(pr_2^{-1}(y_0) = D^2_1 \times \{y_0\}\) over the base point \(y_0\).

Moreover, if

(iv) \(h_t = \text{id} : D^2_y \to D^2_y\) for all \(t\) in the condition (ii),

then we say that \(S\) and \(S'\) are isomorphic.

**Remark 2.5.** It is not difficult to see that two braided surfaces \(S\) and \(S'\) with the same boundary \(\partial S = \partial S'\) are equivalent if and only if there exists an isotopy \(\{h_t\}_{t \in [0,1]}\) of the ambient space \(D^1_x \times D^2_y\) satisfying (i), (ii) and the following (iii)'.

(iii)' For each \(t \in [0,1], \) \(h_t\) fixes the solid torus \(D^2_1 \times \partial D^2_2\) pointwise.

Moreover, \(S\) and \(S'\) with \(\partial S = \partial S'\) are isomorphic if and only if there exists an isotopy \(\{h_t\}_{t \in [0,1]}\) of the ambient space \(D^1_x \times D^2_y\) satisfying (i), (ii), (iii)' and (iv).

Now we recall the monodromy representation of a braided surface.

Let \(S\) be a braided surface and \(\Delta_S\) the singular value set of \(S\). For a path \(c : [0,1] \to D^2_1 \setminus \Delta_S\), we define a path

\[\rho_S(c) : [0,1] \to C_m\]

in the configuration space \(C_m = C_m(\text{Int}D^2_y)\) by

\[\rho_S(c)(t) = pr_1(S \cap pr_2^{-1}(c(t)))\quad \text{for } t \in [0,1].\]

By (iv) of Definition 2.1, for a loop \(c\) in \(D^2_1 \setminus \Delta_S\) with base point \(y_0\), the path \(\rho_S(c)\) is a loop in \(C_m\) with base point \(X_m\). The mapping

\[\rho_S : \pi_1(D^2_1 \setminus \Delta_S, y_0) \to \pi_1(C_m, X_m) = B_m; \quad [c] \mapsto [\rho_S(c)]\]
is a well-defined homomorphism and is called the monodromy representation or the braid monodromy of $S$.

Monodromy representations $\rho_S$ of $S$ and $\rho_{S'}$ of $S'$ are said to be equivalent if there exists a homeomorphism $g : D^2 \to D^2$ satisfying the following (i) and (ii).

(i) $g(\Delta S) = \Delta S'$ and $g|_{\partial D^2} = \text{id}$.

(ii) $\rho_S = \rho_{S'} \circ g_*$, where $g_* : \pi_1(D^2 \setminus \Delta S, y_0) \to \pi_1(D^2 \setminus \Delta S', y_0)$ is the isomorphism induced by $g$.

Proposition 2.6 and 2.7 are proved by the same argument with those in [8] and [11], whose proofs are left to the reader.

**Proposition 2.6** (cf. [8, 10, 11, 13, 14]). Two braided surfaces are isomorphic (or equivalent, resp.) if and only if their monodromy representations are the same (or equivalent, resp).

We define two subsets $G_m$ and $G_m^{\text{reg}}$ of the braid group $B_m$ as follows: An $m$-braid $b$ belongs to $G_m$ if and only if it is conjugate in $B_m$ to a braid $b' = b_1 \Pi b_2 \Pi \cdots \Pi b_c$ which is the split sum of some $m_k$-braids $b_k$ ($k = 1, \ldots, c$) with $\sum_{k=1}^c m_k = m$ satisfying the following.

(i) The closure of each $b_i$ ($i = 1, \ldots, c$) in the 3-sphere $S^3$ is either
   (a) a trivial knot, or
   (b) a Hopf link.

(ii) In case (b) of (i), $b_i$ is a 2-braid.

(iii) $c \neq m$, i.e., $b$ is not the identity element of $B_m$.

An $m$-braid $b$ belongs to $G_m^{\text{reg}}$ if and only if it is conjugate in $B_m$ to a braid $b_1 \Pi b_2 \Pi \cdots \Pi b_c$ satisfying the above (i), (ii) and (iii) and the following:

(iv) $b_1, \ldots, b_c$ are 1-braids except exactly one of them.

When we do not allow the case (b) of (i) of the definition of $G_m$, we have the subset $A_m$ defined in [8, 11].

Let $y$ be an interior point of $D^2$. A sufficiently small simple loop surrounding $y$, which is oriented anticlockwise, is called a meridian around $y$. A meridional loop around $y$ means a loop with base point $y_0$ which is obtained from a meridian loop around $y$ by using a path connecting the meridian loop with $y_0$.

**Proposition 2.7** (cf. [8, 10, 11, 13, 14]). Let $\Delta$ be a finite set of interior points of $D^2$ and let $b_0$ be an $m$-braid. A homomorphism $\rho : \pi_1(D^2 \setminus \Delta, y_0) \to B_m$ is the monodromy representation $\rho_S : \pi_1(D^2 \setminus \Delta S, y_0) \to B_m$ of a braided surface $S$ whose boundary braid is $b_0$ if and only if the following two conditions are satisfied.

(i) If $\eta \in \pi_1(D^2 \setminus \Delta, y_0)$ is represented by a meridional loop, then $\rho(\eta) \in G_m$.

(ii) For the element $[\partial D^2] \in \pi_1(D^2 \setminus \Delta, y_0)$, the image $\rho([\partial D^2]) = b_0$ in $B_m$. 
Moreover this is still valid for embedded braided surfaces $S$ when we replace $G_m$ by $A_m$ in (i). It is also valid for regular braided surfaces $S$ when we replace $G_m$ by $G_m^{reg}$.

3. Chart description

We assume that the $m$ points of $X_m$ are lying on the real line when we regard $D^2_1$ as $\{z \in \mathbb{C}; |z| \leq 1\}$, so that Artin’s standard generators, $\sigma_1, \ldots, \sigma_{m-1}$, of the braid group $B_m = \pi_1(C_m, X_m)$ are defined (cf. [1]).

Let $\Gamma$ be a finite graph in the 2-disk $D^2_2$ such that each edge is oriented and labeled by an integer in $\{1, 2, \ldots, m-1\}$. We say that a path $\eta : [0, 1] \to D^2_2$ is in general position with respect to $\Gamma$ if the following conditions are satisfied.

(i) The image of $\eta$ is disjoint from the vertices of $\Gamma$.

(ii) The preimage $\eta^{-1}(\Gamma)$ is empty or consists of a finite number of interior points of $[0, 1]$, say $t_1, \ldots, t_s$, and assume $t_1 < \cdots < t_s$.

(iii) For each $j$ ($j = 1, \ldots, s$), the path $\eta$ is locally an immersion nearby $t_j$ which intersects an edge of $\Gamma$ transversely.

In this situation, we define the intersection word along $\eta$ with respect to $\Gamma$, denoted by $w_{\eta}(\Gamma)$, to be $\sigma_{i_1}^{-1} \cdots \sigma_{i_s}^{-1}$ where $i_j$ ($j = 1, \ldots, s$) is the label of the edge of $\Gamma$ containing the $j$th intersection $\eta(t_j)$, and $\epsilon_j$ is $+1$ (or $-1$, resp.) if the path $\eta$ at $t = t_j$ intersects the oriented edge of $\Gamma$ from right to left (or left to right, resp.).

Let $\Lambda$ be a set of points of $\partial D^2_2$ (possibly empty) which is disjoint from $y_0$, and each point is labeled by an integer in $\{1, 2, \ldots, m-1\}$ and signed by $+1$ or $-1$.

**Definition 3.1.** An $m$-chart, or simply a chart, in $D^2_2$ with boundary $\Lambda$ is a finite graph $\Gamma$ in the 2-disk $D^2_2$ such that each edge is oriented and labeled by an integer from $\{1, 2, \ldots, m-1\}$, and such that the following conditions (i) and (ii) are satisfied.

(i) $\Gamma \cap \partial D^2_2$ is $\Lambda$ (with respect to labels and signs).

(ii) For each vertex $v$ of $\Gamma$, let $w_v$ be the intersection word along a meridian loop of $v$ with respect to $\Gamma$. Then one of the following occurs.

(a) $w_v \in G_m$, where we regard $w_v$ as an element of the braid group $B_m$.

(b) $w_v = \sigma_i^{-1} \sigma_j^{-1} \sigma_i \sigma_j$ (as a word) for some $i, j$ with $|i - j| > 1$.

(c) $w_v = \sigma_i^{-1} \sigma_j^{-1} \sigma_i^{-1} \sigma_j \sigma_i \sigma_j$ (as a word) for some $i, j$ with $|i - j| = 1$.

A vertex of a chart is called a black vertex, a crossing or a white vertex if the case (a), (b) or (c) occurs, respectively. (See Figure 1.) The set of black vertices of $\Gamma$ is denoted by $\Delta_\Gamma$.

The homomorphism associated with $\Gamma$ is a homomorphism

$$\rho_\Gamma : \pi_1(D^2_2 \setminus \Delta_\Gamma, y_0) \to B_m, \quad [\eta] \mapsto [w_\eta(\eta)].$$
Note that $w_v = 1$ in the braid group $B_m$ for every crossing or white vertex $v$. Thus, we see that the map $\rho_{\Gamma}$ is a well-defined homomorphism.

**Lemma 3.2.** (1) For an $m$-chart $\Gamma$, the homomorphism $\rho_{\Gamma} : \pi_1(D_2^3 \setminus \Delta_\Gamma, y_0) \to B_m$ satisfies the conditions (i) and (ii) of Proposition 2.7, where $\Delta = \Delta_\Gamma$ and $b_0$ is the $m$-braid determined by the intersection word along $\partial D^3_2$ with respect to $\Gamma$.

(2) Let $\Lambda$ be a set of labeled and signed points on $\partial D^3_2$ and let $b_0$ be the $m$-braid determined by the intersection word along $\partial D^3_2$ with respect to $\Lambda$. For a homomorphism $\rho : \pi_1(D_2^3 \setminus \Delta, y_0) \to B_m$ satisfying the conditions (i) and (ii) of Proposition 2.7, there exists an $m$-chart $\Gamma$ with $\rho_{\Gamma} = \rho$ and $\Gamma \cap \partial D^3_2 = \Lambda$.

**Proof.** (1) For each black vertex $v \in \Delta_\Gamma$, the braid $w_v$ belongs to $G_m$. For any meridional element $\eta_v$ for $v$, $\rho_{\Gamma}(\eta_v)$ is a conjugate of $[w_v]$, which belongs to $G_m$. The condition (ii) is obvious by definition. (2) By the same argument with that in [9, 11] (cf. [12]), we can obtain a desired $m$-chart. □

Let $S$ be a braided surface of degree $m$, and $\Gamma$ an $m$-chart with $\rho_S = \rho_{\Gamma}$. Then we call $\Gamma$ a chart description of $S$. (For a given $S$, such a chart exists by Proposition 2.7 and Lemma 3.2, although it is not unique.) Conversely, we call $S$ a braided surface described by $\Gamma$ or a braided surface associated with $\Gamma$. (For a given $\Gamma$, such a braided surface exists and it is unique up to isomorphism by Propositions 2.6, 2.7 and Lemma 3.2.)

Chart moves of type W, of type B, and of type $\partial$ are defined in [12] for chart descriptions of $G$-monodromies in a general setting. In our situation, these moves are described as below.

(1) A **C-move of type W** is a replacement of a chart $\Gamma$ with a chart $\Gamma'$ such that $\Gamma$ and $\Gamma'$ are identical outside a disk region $E$ in $D_2^3$ and such that $\Gamma$ and $\Gamma'$ have no black vertices in $E$. (Some typical C-moves of type W are illustrated in Figure 2. It is known that every C-move of type W is a consequence of the moves depicted in Figure 2, cf. [2, 3, 15].)

(2) A **C-move of type B** is a local replacement nearby a black vertex $v$ of a chart $\Gamma$ illustrated in Figure 3, which changes the word $w_v$ with insertion/deletion of $\sigma_i^{-1}\sigma_i$, or $\sigma_i\sigma_i^{-1}$ for $i = 1, \ldots, m-1$, or $\sigma_i^{-1}\sigma_j^{-1}\sigma_j\sigma_i\sigma_j$ for $i, j$ with $|i-j| > 1$, or $\sigma_i^{-1}\sigma_j^{-1}\sigma_i\sigma_j\sigma_i\sigma_j$ for $i, j$ with $|i-j| = 1$, respectively.

![Figure 1. Vertices](image)
(3) A \textit{C-move of type $\partial$} is a local replacement nearby the boundary $\partial D^2_2$ of a chart $\Gamma$ illustrated in Figure 4, which changes $\Lambda = \Gamma \cap \partial D^2_2$ so that the intersection word along $\partial D^2_2$ with respect to $\Gamma$ changes by insertion/deletion of $\sigma_i^{-1}\sigma_i$, or $\sigma_i\sigma_i^{-1}$ for $i = 1, \ldots, m-1$, or $\sigma_i^{-1}\sigma_i^{-1}\sigma_i\sigma_j$ for $i, j$ with $|i-j| > 1$, or $\sigma_i^{-1}\sigma_i^{-1}\sigma_j\sigma_i\sigma_j$ for $i, j$ with $|i-j| = 1$, respectively. (We assume that C-moves of type $\partial$ are applied away from $y_0 \in \partial D^2_2$.)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{C-moves of type W}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{C-moves of type B}
\end{figure}

\textbf{Definition 3.3.} Two $m$-charts $\Gamma$ and $\Gamma'$ are \textit{chart move isomorphic} (or \textit{chart move equivalent}, resp.) if they are related by a finite sequence of C-moves and isotopies of $D^2_2 \rel \{y_0\} \cup \Delta r$ (or isotopies of $D^2_2 \rel \{y_0\}$, resp.).
Remark 3.4. The C-moves of type B (Figure 3) are equivalent to the moves illustrated in Figure 5 up to C-moves of type W and isotopies of $D_2^2$ rel $\{y_0\} \cup \Delta_r$, and the C-moves of type $\partial$ (Figure 4) are equivalent to the moves in Figure 6. Thus we may add these moves as C-moves in Definition 3.3.

Proof of Theorem 1.1. By Proposition 2.7 and Lemma 3.2, any braided surface has a chart description. Let $S$ and $S'$ be braided surfaces of degree $m$, $\rho_S$ and $\rho_{S'}$ their monodromy representations, $\Gamma$ and $\Gamma'$ their chart descriptions, and $\rho_T$ and $\rho_{T'}$ the associated homomorphisms, respectively. The following conditions (1), (2), (3) and (4) are mutually equivalent.

1. $S$ and $S'$ are isomorphic (or equivalent, resp.).
2. $\rho_S = \rho_{S'}$ (or $\rho_S$ and $\rho_{S'}$ are equivalent, resp.).
3. $\Gamma$ and $\Gamma'$ are chart move isomorphic (or chart move equivalent, resp.).
4. $\rho_T = \rho_{T'}$ (or $\rho_T$ and $\rho_{T'}$ are equivalent, resp.).

The equivalence between (1) and (2) is given in [8] (Lemma 2.6). By definition, (2) and (4) are equivalent. The equivalence between (3) and (4) is proved in [12] (Theorem 12 and § 8 of [12]).
4. Regular charts

Let \( S \) be a braided surface of degree \( m \). For a point \( x \) of \( S \), the local degree of \( x \) means the local degree at \( x \) of the branched covering if \( x \) is not a node. We define the local degree of \( x \) to be 2 if \( x \) is a node. (For a regular point of \( S \), the local degree is 1.) The local degree of \( x \) is denoted by \( \text{deg}_S(x) \).

Let \( \Gamma \) be a chart description of \( S \), and \( y \) a black vertex of \( \Gamma \), which is a singular value of \( S \). The label set of \( y \), denoted by \( \text{Label}_\Gamma(y) \), is the set of labels of the edges which are incident to \( y \). Note that \( \#\text{Label}_\Gamma(y) \geq \tau_S^s(y) \).

(See the proof of the theorem below.)

**Definition 4.1.** In the above situation, \( \Gamma \) is range-reduced at \( y \) if \( \#\text{Label}_\Gamma(y) = \tau_S^s(y) \). A chart is range-reduced if it is range-reduced at every black vertex.

**Lemma 4.2.** Any braided surface has a range-reduced chart description.

**Proof.** Let \( S \) be a braided surface of degree \( m \), and \( y \) a singular value. Let \( \{x_1, \ldots, x_c\} \) be \( S \cap pr_2^{-1}(y) \), where \( c = \#(S \cap pr_2^{-1}(y)) = m - \tau_S^s(y) \). Modifying \( S \) up to isomorphism, we may assume that \( pr_1(x_1), \ldots, pr_1(x_c) \) are on the real line and \( pr_1(x_1) < \cdots < pr_1(x_c) \) where we regard \( D^2_1 \) as \( \{z \in \mathbb{C} : |z| \leq 1\} \). Let \( N_1, \ldots, N_c \) be regular neighborhoods of the points \( pr_1(x_1), \ldots, pr_1(x_c) \) in \( D^2_2 \). Taking a regular neighborhood \( N(y) \) of \( y \) in \( D^2_2 \) sufficiently small, we may assume that the restriction of \( S \) to \( D^2_1 \times N(y) \) is contained in \( \bigcup_{k=1}^c N_k \times N(y) \). The boundary of \( S \cap (D^2_2 \times N(y)) \) is an...
m-braid in the solid torus $D^2 \times \partial N(y)$, say $\ell$, such that $\ell = \ell_1 \Pi \cdots \Pi \ell_c$, with $\ell_k \subset N_k \times \partial N(y)$, and $S \cap (D^2_1 \times N(y))$ is a multiple cone over $\ell = \ell_1 \Pi \cdots \Pi \ell_c$, i.e., $S \cap (D^2_1 \times N(y))$ is the disjoint union $\Pi_{k=1}^c S \cap (N_k \times N(y))$ such that for each $k$, $S \cap (N_k \times N(y))$ is a cone over $\ell_k$ with cone point $x_k$. (Refer to [8] or § 16.4 of [11] for the terminology “multiple cone”.) Each $\ell_k$ is a trivial knot or a Hopf link in the 3-sphere $\partial (N_k \times N(y))$, and the latter case the braid degree of $\ell_k$ is 2. (The former case occurs when the point $x_k$ satisfies (1) or (2) of Remark 2.4, and the latter case occurs when $x_k$ satisfies (3) of Remark 2.4. This is the reason why the local monodromy at $y$ is an element of $G_m$.) Let $m_k$ be the braid degree of $\ell_k$ for $k = 1, \ldots, c$, which is the local degree $\deg_S(x_k)$ of $x_k$. The singular index of $x_k$ of $S$ is $m_k - 1$. By definition $\tau_S^*(y) = m - c = \sum_{k=1}^c (m_k - 1)$.

If $m_k = 1$, then $\ell_k$ is a trivial closed 1-braid. If $m_k > 1$, then $\ell_k$ can be described by a braid word, say $w_k$, in

$$\{\sigma_{m_1+\cdots+m_{k-1}+1}, \sigma_{m_1+\cdots+m_{k-1}+2}, \ldots, \sigma_{m_1+\cdots+m_k-1}\}.$$ 

Since $\ell_k$ is a trivial knot or a closed 2-braid representing a Hopf link, all generators in this set must appear in this word $w_k$. A braid word for $\ell$ is described by the concatenation $w_1 \cdots w_c$. When we construct a chart $\Gamma$ using such a braid word description for $\ell$ in the method used in Theorem 5 of [12], the label set $\text{Label}_\Gamma(y)$ of the black vertex $y$ is

$$\cup_{k=1}^c \{\sigma_{m_1+\cdots+m_{k-1}+1}, \sigma_{m_1+\cdots+m_{k-1}+2}, \ldots, \sigma_{m_1+\cdots+m_k-1}\}.$$ 

Hence $\#\text{Label}_\Gamma(y) = \tau_S^*(y)$. Applying the same argument to each singular value of $S$, we have a range-reduced chart.

**Definition 4.3.** A chart $\Gamma$ is **range-connected** at a black vertex $y$ if $\text{Label}_\Gamma(y)$ consists of consecutive integers.

A black vertex $y$ of a chart $\Gamma$ is called a **nodal black vertex** if it is a singular value of a braided surface $S = S(\Gamma)$ described by $\Gamma$ such that there exists exactly one singular point of $S$ in the fiber over $y$ and the singular point is a node.

If $y$ is a nodal black vertex of a regular chart $\Gamma$ and if $\Gamma$ is range-reduced and range-connected at $y$, then $\text{Label}_\Gamma(y)$ consists of a single integer.

**Definition 4.4.** A nodal black vertex $y$ of a chart $\Gamma$ is **simple** if exactly two edges of $\Gamma$ are incident to $y$ (see Figure 7), otherwise it is called **nonsimple**.

**Definition 4.5.** A chart is **regular** if every black vertex is range-reduced and range-connected and if every nodal black vertex is simple.

![Figure 7. Simple nodal black vertex](image-url)
Lemma 4.6 (Regular chart description). Any regular braided surface has a regular chart description. Conversely, a braided surface described by a regular chart is a regular braided surface.

Proof. Let $S$ be a regular braided surface. For each singular value, there exists exactly one singular point. Thus, a range-reduced chart obtained by the argument of the proof of Lemma 4.2 has range-connected black vertices. When $y$ is a nodal black vertex which is not simple, then more than 2 edges are incident to $y$ and their labels are all the same, say $i$. Since $l_k$ in the proof of Lemma 4.2 is a Hopf link represented by a 2-braid, $w_y$ is equal to $\sigma_i^2$ or $\sigma^{-2}_i$ in the braid group $B_m$. Apply C-moves of type B to make $y$ simple. The converse is obvious. □

5. Regular chart moves

Let $\Gamma$ be a regular chart. Let $y$ be a black vertex of $\Gamma$ and let $\text{Label}_\Gamma(y) = \{s, s+1, \ldots, t\}$ be the label set of $y$. Each move illustrated in Figure 8 shifts the label set by $+1$ or $-1$, where the box means a chart without black vertices. Note that $\{i_1, i_2, \ldots, i_n\} = \{s, s+1, \ldots, t\}$. For (A) of Figure 8, combining fundamental pieces as in Figure 9, we see that the box can be always filled by a chart without black vertices. For example, see Figures 10 and 11. Boxes in the cases (B), (C) and (D) can be filled similarly.

We call the moves illustrated in Figure 8 label shift moves. When it shifts the labels by $+1$, it is called an upper shift, otherwise a lower shift.

A passing move is a move illustrated in Figure 12, where $j$ is an integer with $j < s - 1$ or $j > t + 1$. (Note that $\{i_1, i_2, \ldots, i_n\} = \{s, s+1, \ldots, t\}$.)

The moves in Figure 12 are equivalent to the moves in Figure 13 modulo chart moves of type W.

For the sake of convenience in the proof of Theorem 1.2, we introduce notations for the moves illustrated in Figures 8 and 12. Recall that we are assuming that $\text{Label}_\Gamma(y) = \{s, s+1, \ldots, t\}$. The moves (A), (B), (C) and (D) in Figure 8 are denoted by $L_{s,t}(\sigma_s)$, $L_{s,t}(\sigma_s^{-1})$, $L_{s,t}(\sigma_{s-1})$ and $L_{s,t}(\sigma_{s-1}^{-1})$, respectively. The move (E) in Figure 12 is denoted by $L_{s,t}(\sigma_j)$ if $j < s - 1$ and by $L_{s,t}(\sigma_{j-(t+1-s)})$ if $j > t + 1$. The move (F) in Figure 12 is denoted by $L_{s,t}(\sigma_j^{-1})$ if $j < s - 1$ and by $L_{s,t}(\sigma_{j-(t+1-s)}^{-1})$ if $j > t + 1$. Note that the notation $L_{s,t}[,]$ depends on $\text{Label}_\Gamma(y)$.

Definition 5.1. Two regular charts are regularly chart move isomorphic (or regularly chart move equivalent, resp.) if they are related by a finite sequence of C-moves of type W, C-moves of type B keeping the condition of regularity, label shift moves, passing moves, C-moves of type $\partial$, and isotopies of $D^2_2 \text{ rel } \{y_0\}$ (or isotopies of $D^2_2 \text{ rel } \{y_0\}$, resp.). These moves are called regular chart moves.

Remark 5.2. C-moves of type B keeping the condition of regularity, label shift moves and passing moves are ‘chart moves of transition’ in the sense of [12].
Proof of Theorem 1.2. Any regular braided surface has a regular chart description (Lemma 4.6). Suppose that $\Gamma$ and $\Gamma'$ are regularly chart move
isomorphic (or regularly chart move equivalent, resp.). Label shift moves and passing moves are ‘chart moves of transition’ in the sense of [12], which are consequence of C-moves of type B and type W (Remark 15 of [12]). Therefore, if two regular charts are regularly chart move isomorphic (or regularly chart move equivalent, resp.), then they are chart move isomorphic (or chart move equivalent, resp.). By Theorem 1.1, $S$ and $S'$ are isomorphic (or equivalent, resp.).

We show the converse.

We say that a singular value $y$ of a braided surface $S$ satisfies ‘the condition $(*)$’ if

\[ L_{3,5}[\sigma_3] \]

**Figure 11.** Label shift move $L_{3,5}[\sigma_3]$
\((*)\) If \(\pi_1(S \cap \pi^{-1}_2(y))\) is on the real line where we regard \(D^2_1\) as \(\{z \in C; |z| \leq 1\}\).

(1) First we consider a case where every singular value of \(S\) and \(S'\) satisfies the condition \((*)\) and that \(S'\) is isomorphic to \(S\) keeping the condition \((*)\). Then the label sets \(\text{Label}_\Gamma(y)\) \((y \in \Delta_\Gamma)\) are preserved. By an argument in [12], we see that \(\Gamma\) and \(\Gamma'\) are related by a finite sequence of C-moves of type \(W\), C-moves of type \(B\) preserving the label sets \(\text{Label}_\Gamma(y)\) \((y \in \Delta_\Gamma)\), passing moves, C-moves of type \(\partial\), and isotopies of \(D^2_2\) rel \(\{y_0\} \cup \Delta_\Gamma\). Thus \(\Gamma\) and \(\Gamma'\) are regularly chart move isomorphic.

(2) Next we consider a case where every singular value of \(S\) and \(S'\) satisfies the condition \((*)\) and that \(S'\) is isomorphic to \(S\). Let \(\{S_t\}_{t \in [0,1]}\) be a 1-parameter family of braided surfaces between \(S\) and \(S'\). Let \(y\) be a singular value of \(S\). Without loss of generality we may assume that \(\pi_1(S \cap \pi^{-1}_2(y)) = \pi_1(S' \cap \pi^{-1}_2(y))\). The motion \(\{\pi_1(S_t \cap \pi^{-1}_2(y))\}_{t \in [0,1]}\) is a classical braid of degree \(c = \#(S \cap \pi^{-1}_2(y)) = m - \tau^2_2(y)\). Let \(w = \sigma^2_{i_1} \cdots \sigma^2_{i_\ell}\) be a word description of this braid. We consider a sequence of regular charts, \(\Gamma = \Gamma_0, \Gamma_1, \ldots, \Gamma_\ell\), as follows: Suppose that \(\Gamma_k\) is defined. Let \(\text{Label}_{\Gamma_k}(y) = \{s(k), s(k) + 1, \ldots, t(k)\}\). Apply \(L_{s(k), t(k)}[\sigma^2_{i_k}]\) to the chart \(\Gamma_k\) at \(y\) and let \(\Gamma_{k+1}\) be the result. Let \(\Gamma''\) be the final result \(\Gamma_{\ell}\). By definition \(\Gamma\) is regularly chart move isomorphic to \(\Gamma''\).

On the other hand, let \(S''\) be a braided surface with chart description \(\Gamma''\) satisfying the condition \((*)\). Note that \(S\) and \(S''\) are isomorphic. Let \(\{S'_t\}_{t \in [0,1]}\) be a 1-parameter family of braided surfaces between \(S\) and \(S''\). Without loss of generality we may assume that \(\pi_1(S \cap \pi^{-1}_2(y)) = \pi_1(S'' \cap \pi^{-1}_2(y))\). The motion \(\{\pi_1(S'_t \cap \pi^{-1}_2(y))\}_{t \in [0,1]}\) is a classical braid of degree \(c = \#(S \cap \pi^{-1}_2(y)) = m - \tau^2_2(y)\). This braid is equal to the braid represented by the word \(w\) above. Then \(S'\) and \(S''\) are braided surfaces satisfying the condition \((*)\), and there is a 1-parameter family of braided surfaces between \(S'\) and \(S''\), say \(\{S''_t\}_{t \in [0,1]}\), such that the motion \(\{\pi_1(S''_t \cap \pi^{-1}_2(y))\}_{t \in [0,1]}\) is a classical braid of degree \(c\) such that it is isomorphic to the trivial braid. By replacing \(\{S'_t\}_{t \in [0,1]}\), we may assume that the motion \(\{\pi_1(S''_t \cap \pi^{-1}_2(y))\}_{t \in [0,1]}\) is the trivial braid of degree \(c\). Apply the same argument to each singular value of \(S\). Then by (1), we see that \(\Gamma''\) is regularly chart move isomorphic to \(\Gamma''\). Therefore \(\Gamma\) is regularly chart move isomorphic to \(\Gamma''\).

(3) Let \(S\) and \(S'\) be isomorphic. By definition, \(\Gamma\) (or \(\Gamma'\), resp.) is a regular chart description of some braided surface \(\tilde{S}\) (or \(\tilde{S}'\)) such that \(\tilde{S}\) (or \(\tilde{S}'\)) is isomorphic to \(S\) (or \(S'\)) and satisfies the condition \((*)\). By (2) we see that \(\Gamma\) and \(\Gamma'\) are regularly chart move isomorphic.

(4) Let \(S\) and \(S'\) be equivalent and let \(\Gamma\) and \(\Gamma'\) be their regular chart descriptions. Let \(\{h_t\}\) be an isotopy of \(D^2_1 \times D^2_2\) carrying \(S\) to \(S'\), and let \(\{h_t\}\) be the isotopy of \(D^2_2\) rel \(\{y_0\}\) with \(h_t \circ \pi_2 = \pi_2 \circ h_t\). The chart \(h_1(\Gamma)\) is a regular chart description of \(h_1(S)\). The chart \(\Gamma\) is isotopic to \(h_1(\Gamma)\) by an isotopy of \(D^2_2\) rel \(\{y_0\}\). On the other hand, since \(h_1(S)\) is isomorphic to
$S'$, by (3), $h_1(\Gamma)$ and $\Gamma'$ are regularly chart move isomorphic. Thus $\Gamma$ and $\Gamma'$ are regularly chart move equivalent. □

6. On simple chart description

A chart is simple if it satisfies the conditions of Definition 3.1 such that (a) is replaced with the following condition (a').

(a') $w_i = \sigma_i, \sigma_i^{-1}, \sigma_i^2$ or $\sigma_i^{-2}$ as a word for some $i$.  

In other words, a chart is simple if every black vertex is one of Figure 14. (We call a black vertex is simple if it is as in Figure 14.) By definition, a simple chart is a regular chart.

The following theorem was proved in [10] for simple (immersed) 2-dimensional braids and in [12] for embedded braided surfaces.

**Theorem 6.1** (Simple chart description). Any simple braided surface has a simple chart description. Conversely, a braided surface described by a simple chart is a simple braided surface.

**Proof.** Let $S$ be a simple braided surface. Apply the argument in the proof of Lemma 4.2 and obtain a regular chart description of $S$. If $y$ is a nodal singular value, then $w_y$ is equal to $\sigma_i^2$ or $\sigma_i^{-2}$ as a word for some $i$. If $y$ is a branch value, then $w_y$ is equal to $\sigma_i$ or $\sigma_i^{-1}$ in $B_m$ for some $i$. By C-moves of type B we can change the black vertex $y$ to be simple. The converse is obvious. □

![Figure 14. Simple black vertices and simple nodal black vertices](image)

In [5, 10] (cf. [11]), chart moves (CI-moves, CII-moves, CIII-moves, CIV-moves and CV-moves) for simple charts are introduced. CI-moves are the same with C-moves of type W in this paper. CII-moves, CIII-moves, CIV-moves and CV-moves are illustrated in Figures 15, 16, 17 and 18, respectively. (In the figures, we illustrated an example of possible orientations of the edges. See [5, 9] or [11] for details on the moves.)

![Figure 15. A CII-move](image)

Two simple $m$-charts $\Gamma$ and $\Gamma'$ describe isomorphic (or equivalent, resp.) simple braided surfaces if and only if they are related by a finite sequence
of C-moves and isotopies of $D_2^2 \text{ rel } y_0 \cup \Delta_\Gamma$ (or isotopies of $D_2^2 \text{ rel } y_0$, resp.).

The following theorem is proved in [10] for (immersed) 2-dimensional braids (cf. [11]) and in [12] for embedded braided surfaces.

**Theorem 6.2.** Two simple $m$-charts $\Gamma$ and $\Gamma'$ describe isomorphic (or equivalent, resp.) simple braided surfaces $S$ and $S'$ if and only if they are related by a finite sequence of CI-moves, CII-moves, CIII-moves, CIV-moves, CV-moves, C-moves of type $\partial$ and isotopies of $D_2^2 \text{ rel } y_0 \cup \Delta_\Gamma$ (or isotopies of $D_2^2 \text{ rel } y_0$, resp.).

**Proof.** Suppose that $S$ and $S'$ are isomorphic (or equivalent). In (1) – (3) of the proof of Theorem 1.2, we do not need C-moves of type B. C-moves of type W are CI-moves. Label shift moves are CIII-moves and CV-moves modulo CI-moves, and passing moves are CII-moves and CIV-moves modulo CI-moves. Thus, by the argument of the proof of Theorem 1.2, we see that $\Gamma$ and $\Gamma'$ are related by a finite sequence of CI-moves, CII-moves, CIII-moves, CIV-moves, CV-moves, C-moves of type $\partial$ and isotopies of $D_2^2 \text{ rel } y_0 \cup \Delta_\Gamma$ (or isotopies of $D_2^2 \text{ rel } y_0$, resp.). The converse is obvious.  \qed
Remark 6.3. In Theorems 1.1, 1.2 and 6.2, if $\Gamma$ and $\Gamma'$ have the same boundary then we do not need C-moves of type $\partial$. In particular if $\Gamma$ and $\Gamma'$ be chart descriptions of 2-dimensional braids, then we do not need C-moves of type $\partial$.

REFERENCES