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Ken Abe

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ON THE LARGE TIME L^∞-ESTIMATES OF THE STOKES SEMIGROUP IN TWO-DIMENSIONAL EXTERIOR DOMAINS

KEN ABE

Abstract. We prove that the Stokes semigroup is a bounded analytic semigroup on L^∞_σ of angle $\pi/2$ for two-dimensional exterior domains. This result is an end point case of the L^p-boundedness of the semigroup for $p \in (1,\infty)$, established by Borchers and Varnhorn (1993). The proof is based on the non-existence result of bounded steady flows (the Stokes paradox) and some asymptotic formula for the net force of the Stokes resolvent.

1. Introduction

We consider the Stokes equations:

$$\begin{aligned} \partial_t v - \Delta v + \nabla q = 0, \quad \operatorname{div} v &= 0 && \text{in } \Omega \times (0,\infty), \\ v &= 0 && \text{on } \partial\Omega \times (0,\infty), \\ v &= v_0 && \text{on } \Omega \times \{t=0\}, \end{aligned} \tag{1.1}$$

for exterior domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$. It is well known that the solution operator (called the Stokes semigroup)

$$S(t) : v_0 \longmapsto v(\cdot, t),$$

forms an analytic semigroup on L^p_σ for $p \in (1,\infty)$, of angle $\pi/2$ [41], [21], i.e. $S(t)v_0$ is a holomorphic function in the half plane $\{\mathrm{Re}\, t > 0\}$ on L^p_σ. Here, L^p_σ denotes the L^p-closure of $C^\infty_{c,\sigma}$, the space of all smooth solenoidal vector fields with compact support in Ω. The Stokes semigroup $S(t)$ is defined by the Dunford integral of the resolvent of the Stokes operator $A = \mathbb{P}\Delta$ for the Helmholtz projection operator $\mathbb{P} : L^p \longrightarrow L^p_\sigma$ [16], [35], [40]. See, e.g. [29] for analytic semigroups.

We say that an analytic semigroup on a Banach space is a *bounded* analytic semigroup of angle $\pi/2$ if the semigroup is bounded in the sector $\Sigma_\theta = \{t \in \mathbb{C}\backslash\{0\} \mid |\arg t| < \theta\}$ for each $\theta \in (0, \pi/2)$. See, e.g. [6, Definition 3.7.3]. The boundedness in the sector implies the bounds on the positive real line

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$$\|S(t)\| \leq C, \quad \|AS(t)\| \leq \frac{C}{t}, \quad t > 0, \tag{1.2}$$

where $\|\cdot\|$ denotes an operator norm on a Banach space and A is a generator. The estimates (1.2) are important to study large time behavior of solutions to (1.1). In terms of the resolvent, the boundedness of $S(t)$ of angle $\pi/2$ is equivalent to the estimate

$$\|(\lambda - A)^{-1}\| \leq \frac{C}{|\lambda|}, \quad \lambda \in \Sigma_{\theta+\pi/2}. \tag{1.3}$$

When Ω is bounded, the point $\lambda = 0$ belongs to the resolvent set of $A = \mathbb{P}\Delta$ and the Stokes semigroup is a bounded analytic semigroup on L^p_σ of angle $\pi/2$ for $p \in (1,\infty)$. For a half space, the boundedness of the semigroup follows from explicit solution formulas [34], [44], [8].

The boundedness of the Stokes semigroup on L^p_σ for $p \in (1,\infty)$ have been established for exterior domains in $\mathbb{R}^n$ for $n \geq 2$. For $n \geq 3$, the boundedness of $S(t)$ on L^p_σ is proved in [10] based on the resolvent estimate

$$|\lambda|\|v\|_{L^p} + |\lambda|^{1/2}\|\nabla v\|_{L^p} + \|\nabla^2 v\|_{L^p} \leq C\|f\|_{L^p}, \quad 1 < p < \frac{n}{2}, \tag{1.4}$$

for $v = (\lambda - A)^{-1} f$ and $\lambda \in \Sigma_{\theta+\pi/2} \cup \{0\}$. The estimate (1.4) implies (1.3) for $p \in (1, n/2)$ and the case $p \in [n/2, \infty)$ follows from a duality. Due to the restriction on p, the two-dimensional case is more involved. Indeed, the estimate $\|\nabla^2 v\|_{L^p} \leq C\|Av\|_{L^p}$ for $p \in [n/2,\infty)$ does not hold [9]. For $n = 2$, the boundedness of the Stokes semigroup on L^p_σ is proved in [11] based on layer potentials for the Stokes resolvent.

Recently, the case $p = \infty$ has been developed. When Ω is a half space, $S(t)$ forms a bounded analytic semigroup on L^∞_σ of angle $\pi/2$ [14], [42]. For a half space and domains with compact boundaries, we define L^∞_σ by

$$L^\infty_\sigma(\Omega) = \left\{ f \in L^\infty(\Omega) \;\middle|\; \operatorname{div} f = 0 \text{ in } \Omega,\ f \cdot N = 0 \text{ on } \partial\Omega \right\}.$$

Here, N denotes the unit outward normal vector field on $\partial\Omega$. Since $S(t)$ is bounded on L^∞_σ, the associated generator $A = A_\infty$ is also defined for $p = \infty$. For bounded domains [3] and exterior domains [4], analyticity of the semigroup on L^∞_σ follows from the a priori estimate

$$\|v\|_{L^\infty} + t^{1/2}\|\nabla v\|_{L^\infty} + t\|\nabla^2 v\|_{L^\infty} + t\|\partial_t v\|_{L^\infty} + t\|\nabla q\|_{L^\infty} \leq C\|v_0\|_{L^\infty}, \tag{1.5}$$

for $v = S(t)v_0$ and $t \leq T$. The estimate (1.5) is proved by a blow-up argument and implies that $S(t)$ is analytic on L^∞_σ. Moreover, by the resolvent estimates on L^∞_σ [5], $S(t)$ is analytic on L^∞_σ of angle $\pi/2$. When Ω is bounded, $S(t)$ is a bounded analytic semigroup on L^∞_σ of angle $\pi/2$.

In this paper, we consider the boundedness of the Stokes semigroup on L^∞_σ for exterior domains in $\mathbb{R}^n$ for $n \geq 2$. For the Laplace operator or uniformly elliptic operators, a standard approach to prove large time L^∞-estimates of a semigroup is to use a Gaussian upper bound for a complex time heat kernel. See [13, Chapter 3]. However, a kernel of the Stokes semigroup does not satisfy a Gaussian bound since $S(t)$ is unbounded on L^1. See [14], [37] for a half space. Even for exterior domains, $S(t)$ is not bounded on L^1 unless the net force vanishes [28], [22]. It seems no general method to estimate the L^∞-norm of a semigroup for all time without a Gaussian bound.

There is a work by Maremonti [31] who proved the estimate

$$\|S(t)v_0\|_{L^\infty} \leq C\|v_0\|_{L^\infty}, \quad t > 0, \tag{1.6}$$

for exterior domains and $n \geq 3$ based on the finite time estimate in [3]. Subsequently, Hieber and Maremonti [23] proved the estimate $t\|AS(t)v_0\|_{L^\infty} \leq C\|v_0\|_{L^\infty}$ for $t > 0$ and the results are extended in [7] for complex time $t \in \Sigma_\theta$ and $\theta \in (0, \pi/2)$ based on the approach in [31]. The method in [31] seems a perturbation from the heat equation in $\mathbb{R}^n$ and excludes the case $n = 2$.

In the previous work [2], the author studied large time L^∞-estimates of the Stokes semigroup for $n \geq 2$ based on a Liouville theorem for the Stokes equations introduced by Jia, Seregin and Šverák [24], [25]. Liouville theorems are important to study regularity of solutions. See [27], [39] for Liouville theorems of the Navier-Stokes equations. They are also related with large time behavior. Following [24], [25], we say that $v \in L^1_{\mathrm{loc}}(\overline{\Omega} \times (-\infty, 0])$ is an ancient solution to the Stokes equations (1.1) if div $v = 0$ in $\Omega \times (-\infty, 0)$, $v \cdot N = 0$ on $\partial\Omega \times (-\infty, 0)$ and

$$\int_{-\infty}^{0} \int_{\Omega} v \cdot (\partial_t \varphi + \Delta\varphi) \mathrm{d}x\mathrm{d}t = 0,$$

for all $\varphi \in C_c^{2,1}(\overline{\Omega}\times(-\infty, 0])$ satisfying div $\varphi = 0$ in $\Omega\times(-\infty, 0)$ and $\varphi = 0$ on $\partial\Omega\times(-\infty, 0) \cup \Omega \times \{t = 0\}$. The conditions div $v = 0$ and $v \cdot N = 0$ are understood in the sense that

$$\int_{\Omega} v \cdot \nabla\Phi \mathrm{d}x = 0, \quad \text{a.e. } t \in (-\infty, 0),$$

for all $\Phi \in C_c^1(\overline{\Omega})$. Liouville theorems for the Stokes equations has been established in [24] for $\mathbb{R}^n$, $\mathbb{R}^n_+$ and bounded domains. Among others, it is proved in [24] for exterior domains in $\mathbb{R}^n$ for $n \geq 3$ that bounded ancient solutions $v \in L^\infty(\Omega \times (-\infty, 0))$ must satisfy

$$v(x,t) - v_\infty(t) = O(|x|^{-n+2}) \quad \text{as } |x| \to \infty,$$

for some constant $v_\infty(t)$. Since bounded steady flows exist for $n \geq 3$ [9], bounded ancient solutions are non-trivial. If in addition some spatial decay condition is assumed, we can exclude such solutions.

Theorem 1.1 (Liouville theorem on L^p [2])**.** *Let Ω be an exterior domain with C^3-boundary in $\mathbb{R}^n$, $n \geq 2$. Let v be an ancient solution to the Stokes equations (1.1). Assume that*

$$v \in L^\infty(-\infty, 0; L^p) \quad \textit{for } p \in (1,\infty).$$

Then, $v \equiv 0$.

This Liouville property is based on the fact that $S(t)$ is a bounded analytic semigroup on L^p_σ. Since ancient solutions are written as $v(\cdot,t) = S(t+T)v(\cdot,-T)$ for $t \geq -T$ and $T > 0$, the estimate (1.2) and sending $T \to \infty$ reduce the proof to the non-existence of steady flows $\mathrm{Ker}\, A = \{0\}$ on L^p_σ. This approach is available for linear autonomous systems. We note that for the non-linear problem Liouville properties are studied via the large time behavior to a non-autonomous system [38]. See [25] for a Liouville theorem of the Stokes flow on L^∞ based on a duality argument.

Theorem 1.1 is used to prove the large time L^∞-estimate (1.6). By the representation formula for $v = S(t)v_0$ [36], we have

$$v(x,t) = \int_\Omega \Gamma(x-y,t)v_0(y)\mathrm{d}y + \int_0^t \int_{\partial\Omega} V(x-y,t-s)(TN)(y,s)\mathrm{d}H(y)\mathrm{d}s. \tag{1.7}$$

Here, $T = \nabla v + {}^t\nabla v - qI$ is the stress tensor with the identity matrix I and $V = (V_{ij})$ is the Oseen tensor

$$V_{ij}(x,t) = \delta_{ij}\Gamma(x,t) + \partial_i\partial_j \int_{\mathbb{R}^n} E(x-y)\Gamma(y,t)\mathrm{d}y,$$

defined by the heat kernel $\Gamma(x,t) = (4\pi t)^{-n/2}e^{-|x|^2/4t}$ and the fundamental solutions of the Laplace equation E, i.e.

$$E(x) = \begin{cases} \dfrac{1}{n(n-2)\alpha(n)}\dfrac{1}{|x|^{n-2}}, & n \geq 3, \\ -\dfrac{1}{2\pi}\log|x|, & n = 2, \end{cases}$$

where $\alpha(n)$ denotes the volume of the unit ball in $\mathbb{R}^n$.

For $n \geq 3$, the formula (1.7) describes the asymptotic behavior of bounded Stokes flows as $|x| \to \infty$ and $t \to \infty$. Since the Oseen tensor satisfies

$$|V(x,t)| \leq \frac{C}{(|x| + t^{1/2})^n}, \quad x \in \mathbb{R}^n,\ t > 0,$$

we have

$$\left| v(x,t) - \int_\Omega \Gamma(x-y,t)v_0(y)\mathrm{d}y \right| \leq \frac{C}{|x|^{n-2}} \sup_{0<s\leq t} \|T\|_{L^\infty(\partial\Omega)}(s), \quad |x| \geq R,\ t > 0, \tag{1.8}$$

for some constant $R > 0$. The right-hand side is decaying as $|x| \to \infty$ uniformly for all $t > 0$.

The large time estimate (1.6) for $n \geq 3$ is deduced in [2] by using the asymptotic formula (1.8) and the Liouville theorem (Theorem 1.1) by a contradiction argument. Indeed, if (1.6) were false, a sequence of solutions generates a non-trivial ancient solution satisfying $|v(x,t)| \leq C|x|^{-n+2}$ for $|x| \geq R$, $t \in (-\infty, 0]$ and the Liouville theorem yields a contradiction. The boundedness of $S(t)v_0$ in the sector Σ_θ follows the same argument on the half line $\{\arg\ t = \theta\}$.

For $n = 2$, there is a restriction on the net force since the right-hand side of (1.8) might diverge. Indeed, we have

$$\begin{aligned} &\left| v(x,t) - \int_\Omega \Gamma(x-y,t)v_0(y)\mathrm{d}y - \int_0^t V(x,t-s)F(s)\mathrm{d}s \right| \\ &\leq \frac{C}{|x|} \sup_{0<s\leq t} \|T\|_{L^\infty(\partial\Omega)}(s), \quad |x| \geq R,\ t>0, \end{aligned} \tag{1.9}$$

with *the net force*

$$F(s) = \int_{\partial\Omega} TN(y,s)\mathrm{d}H(y).$$

Since $|\int_0^t V(x,s)\mathrm{d}s| \lesssim \log(1+t/|x|^2)$, the decay as $|x| \to \infty$ of the third term in (1.9) is not uniform for $t > 0$ in contrast to (1.8) for $n \geq 3$. If the net force vanishes, the situation is the same as $n = 3$ and we are able to prove (1.6) for $t \in \Sigma_\theta$. For example, when Ω^c is a disk and initial data has some discrete symmetry (called C_m-covariance), the net force vanishes [22], i.e. $F(s) \equiv 0$. The following result includes the case $n = 2$.

Theorem 1.2 (Boundedness on L^∞ for $n \geq 3$ and $n = 2$ with zero net force [2])**.** *(i) For $n \geq 3$, the Stokes semigroup is a bounded analytic semigroup on L^∞_σ of angle $\pi/2$.*
(ii) For n=2, the estimate (1.6) holds for $t \in \Sigma_\theta$ and $v_0 \in L^\infty_\sigma$ for which the net force vanishes (e.g. C_m-covariant vector fields when Ω^c is a disk.)

In this paper, we prove that the assertion (ii) of Theorem 1.2 holds for any bounded initial data $v_0 \in L^\infty_\sigma$. Perhaps the most important vector fields with non-vanishing net force are asymptotically constant solutions of the steady Navier-Stokes flows as $|x| \to \infty$ such as D-solutions or PR-solutions. See [17]. They are bounded and with finite Dirichlet integral. The situation is subtle even for bounded initial data with finite Dirichlet integral for which the fractional power estimate

$$\|\nabla v\|_{L^2} = \|(-A)^{1/2}v\|_{L^2},$$

is available. This estimate holds only for $n = 2$, i.e. the estimate $\|\nabla v\|_{L^p} \leq C\|(-A)^{1/2}v\|_{L^p}$ for $p \in [n, \infty)$ and $n \geq 3$ does not hold [9]. The fractional power estimate implies a uniform bound in the homogeneous L^2-Sobolev space $\dot{H}^1$ and $S(t)v_0$ is merely bounded in BMO even if v_0 is with finite Dirichlet integral, i.e.

$$[S(t)v_0]_{\mathrm{BMO}} \leq C\|v_0\|_{L^\infty\cap \dot{H}^1}, \quad t>0.$$

To prove the large time L^∞-estimate (1.6) for $n = 2$ and any bounded initial data $v_0 \in L^\infty_\sigma$, we analyze the corresponding Stokes resolvent problem:

$$\begin{aligned} \lambda v - \Delta v + \nabla q = f, \quad \operatorname{div} v = 0 \quad &\text{in } \Omega, \\ v = 0 \quad &\text{on } \partial\Omega. \end{aligned} \tag{1.10}$$

Existence and uniqueness of the problem (1.10) for $f \in L^\infty_\sigma$ have been studied in [5]. In particular, the solution operator

$$R(\lambda) : f \longmapsto v(\cdot, \lambda),$$

is a bounded operator on L^∞_σ for $\lambda \in \Sigma_{\theta+\pi/2}$ and for each $\delta > 0$, the estimate $\|R(\lambda)\| \leq C_\delta |\lambda|^{-1}$ holds for $|\lambda| \geq \delta$ with the operator norm $\|\cdot\|$ on L^∞_σ. The operator $R(\lambda)$ is resolvent of some closed operator $A = A_\infty$ on L^∞_σ, i.e. $R(\lambda) = (\lambda - A)^{-1}$. The behavior of $R(\lambda)$ as $\lambda \to 0$ corresponds to the behavior of $S(t)$ as $t \to \infty$. Instead of proving the boundedness of $S(t)$ in Σ_θ, we shall prove the equivalent estimate (1.3) with the operator norm on L^∞_σ. The main result of this paper is the following:

Theorem 1.3 (Boundedness on L^∞ for $n = 2$)**.** *Let Ω be an exterior domain with C^3-boundary in $\mathbb{R}^2$.*
(i) For $\theta \in (0, \pi/2)$, there exists a constant C such that

$$\|R(\lambda)f\|_{L^\infty} \leq \frac{C}{|\lambda|}\|f\|_{L^\infty}, \quad \lambda \in \Sigma_{\theta+\pi/2},\ f \in L^\infty_\sigma. \tag{1.11}$$

(ii) The Stokes semigroup is a bounded analytic semigroup on L^∞_σ of angle $\pi/2$.

There is a difference on the large time behavior for $n = 2$ and $n \geq 3$. By Theorems 1.2 and 1.3, we obtain

$$\|S(t)v_0\|_{L^\infty} + t\|AS(t)v_0\|_{L^\infty} \leq C\|v_0\|_{L^\infty}, \quad t > 0,\ v_0 \in L^\infty_\sigma, \tag{1.12}$$

for exterior domains in $\mathbb{R}^n$ for $n \geq 2$. The estimate (1.12) implies that $S(t)v_0$ is uniformly bounded and approaches a steady flow as $t \to \infty$. For $n = 2$, any bounded solutions of

$$\begin{aligned} -\Delta v + \nabla q = 0, \quad \operatorname{div} v = 0 \quad &\text{in } \Omega, \\ v = 0 \quad &\text{on } \partial\Omega, \end{aligned} \tag{1.13}$$

must be trivial (the Stokes paradox) [12] and therefore $S(t)v_0$ converges to zero locally uniformly in $\overline{\Omega}$ as $t \to \infty$. On the other hand, for $n \geq 3$, bounded steady flows of (1.13) exist and must be asymptotically constant as $|x| \to \infty$. Hence the solution $S(t)v_0$ converges to such a stationary solution as $t \to \infty$. See Remarks 3.3 for rigorous proofs.

If initial data v_0 is decaying as $|x| \to \infty$, $S(t)v_0$ vanishes as $t \to \infty$ for all dimensions $n \geq 2$, i.e. for $v_0 \in C_{0,\sigma}$, $S(t)v_0$ uniformly converges to zero in $\overline{\Omega}$ as $t \to \infty$. Here, $C_{0,\sigma}$ is the L^{∞}-closure of $C^{\infty}_{c,\sigma}$, characterized by

$$C_{0,\sigma}(\Omega) = \left\{ f \in C(\overline{\Omega}) \;\middle|\; \operatorname{div} f = 0 \text{ in } \Omega,\ f = 0 \text{ on } \partial\Omega,\ \lim_{|x|\to\infty} f(x) = 0 \right\}.$$

See [4]. Since $S(t)v_0$ vanishes as $t \to \infty$ for $v_0 \in C^{\infty}_{c,\sigma}$, this property follows from the density in $C_{0,\sigma}$.

There is some issue on the large time behavior of Navier-Stokes flows. By a perturbation argument from the Stokes flow, we are able to construct a unique global-in-time solution of the two-dimensional Navier-Stokes equations for bounded initial data with finite Dirichlet integral [1] satisfying the integral form

$$u(t) = S(t)u_0 - \int_0^t S(t-s)\mathbb{P}u \cdot \nabla u(s)\mathrm{d}s. \tag{1.14}$$

This solution is asymptotically constant if u_0 is, cf. [32]. The large time behavior of this solution is an interesting question since the space $L^{\infty} \cap \dot{H}^1$ includes steady Navier-Stokes flows. See [30] for stability of PR-solutions. It is a question whether solutions of (1.14) remain bounded for all time. The estimate (1.12) implies that the Stokes flow remains bounded for all time and converges to zero locally uniformly in $\overline{\Omega}$ as $t \to \infty$ for any bounded initial data.

The question is non-trivial even for the Cauchy problem for which solutions remain bounded in $\dot{H}^1$ by an a priori estimate of vorticity. This solution is merely bounded in BMO. But a uniform L^{∞}-bound seems unknown. The problem have been studied for merely bounded initial data $u_0 \in L^{\infty}_{\sigma}$ and a polynomial growth bound on the L^{∞}-norm is derived in [45]. It is known that global-in-time solutions satisfy the upper bound $\|u\|_{L^{\infty}} = O(t)$ as $t \to \infty$ [19]. See also [20].

We sketch the proof of Theorem 1.3. Our proof is based on the representation formula for the Stokes resolvent $v = R(\lambda)f$:

$$v(x) = \int_{\Omega} E^{\lambda}(x-y)f(y)\mathrm{d}y + \int_{\partial\Omega} V^{\lambda}(x-y)TN(y)\mathrm{d}H(y), \tag{1.15}$$

for $T = \nabla v + {}^t\nabla v - qI$. Here,

$$E^{\lambda}(x) = \frac{1}{2\pi}K_0(\sqrt{\lambda}|x|) \tag{1.16}$$

is the kernel of the resolvent $(\lambda - \Delta)^{-1}$ and $K_m(\kappa)$ is the modified Bessel function of the second kind of order m. For $\lambda \in \Sigma_{\theta+\pi/2}$, $\sqrt{\lambda}$ denotes the square-root of λ with positive real part, i.e. Re $\sqrt{\lambda} > 0$. The tensor $V^\lambda = (V^\lambda_{ij})$ is the kernel of $\lambda(\lambda - \Delta)^{-1}\mathbb{P}$ for the Helmholtz projection operator $\mathbb{P} = I + \nabla(-\Delta)^{-1}\mathrm{div}$. This tensor has the explicit form [11, p.281],

$$V^\lambda_{ij}(x) = \frac{1}{2\pi}\left(\delta_{ij}e_1\left(\sqrt{\lambda}|x|\right) + \frac{x_ix_j}{|x|^2}e_2\left(\sqrt{\lambda}|x|\right)\right), \tag{1.17}$$

where

$$\begin{aligned} e_1(\kappa) &= K_0(\kappa) + \kappa^{-1}K_1(\kappa) - \kappa^{-2}, \\ e_2(\kappa) &= -K_0(\kappa) - 2\kappa^{-1}K_1(\kappa) + 2\kappa^{-2}, \quad \kappa > 0. \end{aligned}$$

The function $e_1(\kappa)$ has a logarithmic singularity as $\kappa \to 0$ and decaying as $\kappa \to \infty$. The function $e_2(\kappa)$ is bounded for $\kappa > 0$, i.e.

$$\begin{aligned} \left|e_1(\kappa) + \frac{1}{2}\log\kappa\right| + |e_2(\kappa)| &\le C, \qquad && 0 < \kappa \le d, \\ |e_1(\kappa)| + |e_2(\kappa)| &\le C\kappa^{-2}, && \kappa \ge d, \end{aligned} \tag{1.18}$$

for any $d > 0$ with some constant C. Hence

$$V^\lambda(x) = -\frac{1}{4\pi}\left(\log\sqrt{\lambda} + \log|x|\right)I + \tilde{V}^\lambda(x), \tag{1.19}$$

with a bounded function $\tilde{V}^\lambda$ for $|\lambda|^{1/2}|x| \le d$. For $|\lambda|^{1/2}|x| \ge d$, V^λ is bounded.

We shall suppose that λv is uniformly bounded on L^∞ and observe the asymptotic behavior of $|\lambda|\ \|v\|_{L^\infty}$ as $\lambda \to 0$. We take a point $x_\lambda \in \Omega$ such that

$$\|v\|_{L^\infty} \approx |v(x_\lambda)|.$$

The behavior of λv as $\lambda \to 0$ is related with the behavior of f as $|x| \to \infty$. For simplicity of the explanation, we shall consider positive $\lambda > 0$ and asymptotically constant vector fields $f \to f_\infty$ as $|x| \to \infty$ for which $\lambda(\lambda - \Delta)^{-1}f \to f_\infty$ as $\lambda \to 0$.

We first observe that λv converges to zero locally uniformly in $\overline{\Omega}$ as $\lambda \to 0$. Indeed, since $u = \lambda v$ is uniformly bounded on L^∞ and satisfies

$$\begin{aligned} \lambda u - \Delta u + \nabla p = \lambda f, \quad \mathrm{div}\, u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.20}$$

for $p = \lambda q$, by elliptic regularity, u converges to a limit locally uniformly in $\overline{\Omega}$ together with ∇u and p. This pressure p is unique up to constant. Since any bounded solutions of (1.13) must be trivial by the Stokes paradox, it turns out that u, ∇u and p converge to zero locally uniformly in $\overline{\Omega}$. This in particular implies that the stress tensor $T = \nabla u + {}^t\nabla u - pI$ vanishes on $\partial\Omega$ as $\lambda \to 0$.

The behavior of $|\lambda|\,\|v\|_{L^\infty} = |u(x_\lambda)|$ depends on that of the points $\{x_\lambda\}$. If the points $\{x_\lambda\}$ remain bounded, $u(x_\lambda)$ converges to zero as $\lambda \to 0$, i.e. $\lim_{\lambda\to 0} |u(x_\lambda)| = 0$. If the points $\{x_\lambda\}$ diverge, according to the logarithmic singularity of $e_1(\kappa)$ as $\kappa \to 0$, we consider two cases whether $\liminf_{\lambda\to 0} |\lambda|^{1/2}|x_\lambda| > 0$ or $\liminf_{\lambda\to 0} |\lambda|^{1/2}|x_\lambda| = 0$. If $\liminf_{\lambda\to 0} |\lambda|^{1/2}|x_\lambda| > 0$, the kernel $V^\lambda(x_\lambda)$ remains bounded by (1.18). Substituting $x = x_\lambda$ into

$$u(x) = \lambda(\lambda - \Delta)^{-1} f + \int_{\partial\Omega} V^\lambda(x-y) TN(y) \mathrm{d}H(y), \tag{1.21}$$

and sending $\lambda \to 0$ implies $\limsup_{\lambda\to 0} |u(x_\lambda)| \leq \|f\|_{L^\infty}$.

If $\liminf_{\lambda\to 0} |\lambda|^{1/2}|x_\lambda| = 0$, the kernel $V^\lambda(x_\lambda)$ can be singular as $\lambda \to 0$. By (1.19),

$$\begin{aligned} u(x) &= \lambda(\lambda - \Delta)^{-1} f - \frac{1}{4\pi} \log \sqrt{\lambda} \int_{\partial\Omega} TN(y)\mathrm{d}H(y) \\ &\quad - \frac{1}{4\pi} \int_{\partial\Omega} \log|x-y| TN(y)\mathrm{d}H(y) + \int_{\partial\Omega} \tilde{V}^\lambda(x-y) TN(y)\mathrm{d}H(y). \end{aligned} \tag{1.22}$$

For fixed $x \in \Omega$, sending $\lambda \to 0$ implies *the asymptotic formula for the net force*:

$$0 = f_\infty - \frac{1}{4\pi} \lim_{\lambda\to 0} \log \sqrt{\lambda} \int_{\partial\Omega} TN(y)\mathrm{d}H(y). \tag{1.23}$$

The formula (1.23) has been derived for the Oseen approximation by Finn and Smith [15]. It implies that the net force is asymptotically pure drag, i.e. the direction of the net force is asymptotically same as the uniform flow f_∞ as $\lambda \to 0$. By choosing a subsequence, we may assume that $|\lambda|^{1/2}|x_\lambda| \to 0$. We substitute $x = x_\lambda$ into (1.22) and send $\lambda \to 0$. Since $|x_\lambda| \leq |\lambda|^{-1/2}$ for small $\lambda > 0$, we have

$$\frac{1}{4\pi}\left|\int_{\partial\Omega} \log|x_\lambda - y| TN(y)\mathrm{d}H(y)\right| \leq -\frac{1}{4\pi}\log|\lambda|^{1/2}\left|\int_{\partial\Omega} TN(y)\mathrm{d}H(y)\right| + o(1) \quad \text{as } \lambda \to 0.$$

By (1.23), $\limsup_{\lambda\to 0} |u(x_\lambda)| \leq \|f\|_{L^\infty}$. Hence in all cases, the sup-norm of $\lambda v = u$ is controlled by that of f.

Based on this observation, we apply a contradiction argument to obtain the desired estimate (1.11). We suppose that (1.11) were false and obtain sequences $\{f_m\}$ and $\{\lambda_m\} \subset \Sigma_{\theta+\pi/2}$ such that

$$\begin{aligned} &\sup_{\lambda\in\Sigma_{\theta+\pi/2}} |\lambda|\,\|R(\lambda)f_m\|_{L^\infty} = 1, \quad \|f_m\|_{L^\infty} < \frac{1}{m}, \\ &|\lambda_m|\,\|R(\lambda_m)f_m\|_{L^\infty} \geq \frac{1}{2}, \quad \lambda_m \to 0. \end{aligned}$$

We set $u_m = \lambda_m R(\lambda_m) f_m$ and take a point $x_m \in \Omega$ such that $|u_m(x_m)| \geq 1/4$. Since u_m satisfies the Stokes resolvent equations (1.20) for λ_m with the associated pressure p_m, u_m converges to zero locally uniformly in $\overline{\Omega}$ together with ∇u_m and p_m. Then, there are two cases whether

$\liminf_{m\to\infty} |\lambda_m|^{1/2}|x_m| > 0$ or $\liminf_{m\to\infty} |\lambda_m|^{1/2}|x_m| = 0$. Since $\|f_m\|_{L^\infty} \to 0$, in all cases we will see that $1/4 \leq |u_m(x_m)| \to 0$ as $m \to \infty$. This is a contradiction.

This paper is organized as follows. In Section 2, we prove the representation formula (1.15) for solutions of (1.10) for bounded data $f \in L^\infty_\sigma$ and non-existence of bounded solutions of (1.13). In Section 3, we prove Theorem 1.3. After the proof of Theorem 1.3, we note large time behavior of $S(t)v_0$ for $v_0 \in L^\infty_\sigma$.

2. Stokes resolvent on L^∞_σ

We recall some existence and uniqueness result for the Stokes resolvent equations (1.10) for bounded data $f \in L^\infty_\sigma$. To state a result, let $L^p_{\mathrm{ul}}(\overline{\Omega})$ denote the uniformly local L^p-space in $\overline{\Omega}$ for $p \in (1,\infty)$ and $W^{2,p}_{\mathrm{ul}}(\overline{\Omega})$ denote the space of all uniformly local L^p-functions up to second orders. Let $L^\infty_d(\Omega)$ denote the space of all functions $f \in L^1_{\mathrm{loc}}(\overline{\Omega})$ such that $df \in L^\infty(\Omega)$ with the distance function $d(x) = \inf_{y\in\partial\Omega} |x-y|$.

Lemma 2.1 (Resolvent estimates for large λ)**.** *(i) For $p > 2$, $\delta > 0$ and $\theta \in (0,\pi/2)$, there exists a constant C such that for $f \in L^\infty_\sigma$ and $\lambda \in \Sigma_{\theta+\pi/2}$ satisfying $|\lambda| \geq \delta$, there exists a unique solution $(v, \nabla q) \in W^{2,p}_{ul}(\overline{\Omega}) \times (L^p_{ul}(\overline{\Omega}) \cap L^\infty_d(\Omega))$ of (1.10) satisfying*

$$(2.1)\qquad |\lambda|\|v\|_{L^\infty} + |\lambda|^{1/2}\|\nabla v\|_{L^\infty} + |\lambda|^{1/p} \sup_{x\in\Omega}\left\{\|\nabla^2 v\|_{L^p(\Omega_{x,|\lambda|^{-1/2}})} + \|\nabla q\|_{L^p(\Omega_{x,|\lambda|^{-1/2}})}\right\} \leq C\|f\|_{L^\infty},$$

for $\Omega_{x,r} = \Omega \cap B(x,r)$, where $B(x,r)$ denotes an open ball centered at x with radius r.
(ii) The solution operator $R(\lambda) : f \longmapsto v$ is a bounded operator on L^∞_σ and satisfies

$$(2.2)\qquad \|R(\lambda)\| \leq \frac{C}{|\lambda|}, \quad \lambda \in \Sigma_{\theta+\pi/2},\ |\lambda| \geq \delta,$$

with the constant C depending on δ, where $\|\cdot\|$ denotes the operator norm on L^∞_σ.

Proof. See [5, Theorems 1.1 and 1.3]. □

The a priori estimate (2.1) is obtained by applying the localization technique of Masuda [33] and Stewart [43] by using the L^∞-estimate of the pressure. See (2.5) below. The uniqueness follows the same argument. The existence is based on the following approximation lemma for $f \in L^\infty_\sigma$.

Lemma 2.2 (Approximation)**.** *(i) There exists a constant C such that for $f \in L^\infty_\sigma$ there exists a sequence $\{f_m\} \subset C^\infty_{c,\sigma}$ such that*

$$\begin{aligned} &\|f_m\|_{L^\infty} \le C\|f\|_{L^\infty} \\ &f_m \to f \quad \textit{a.e. in } \Omega \quad \textit{as } m \to \infty. \end{aligned} \tag{2.3}$$

(ii) The resolvent $R(\lambda)f_m$ *converges to* $R(\lambda)f$ *locally uniformly in* $\overline{\Omega}$ *as* $m \to \infty$ *for each* $\lambda \in \Sigma_{\theta+\pi/2}$.

Proof. The assertion (i) is proved in [4, Lemma 5.1] by using the Bogovskiĭ operator. Since $R(\lambda)f_m$ is resolvent of the Stokes operator on L^p_σ, i.e. $R(\lambda)f_m = (\lambda - A)^{-1}f_m$ for $A = \mathbb{P}\Delta$, the assertion (ii) follows by applying the a priori estimate (2.1) and uniqueness of (1.10) [5]. □

Remarks 2.3. (i) The associated pressure q of the problem (1.10) is a solution of the Neumann problem

$$\begin{aligned} -\Delta q &= 0 \quad \text{in } \Omega, \\ N \cdot \nabla q &= -N \cdot \nabla^\perp \omega \quad \text{on } \partial\Omega. \end{aligned} \tag{2.4}$$

for $\omega = \partial_1 v^2 - \partial_2 v^1$, $v = {}^t(v^1, v^2)$ and $\nabla^\perp = {}^t(\partial_2, -\partial_1)$. Since $-\Delta v = \nabla^\perp \omega$, this boundary condition follows by taking the normal trace of (1.10). The problem (2.4) has a unique solution satisfying

$$\sup_{x\in\Omega} d(x)|\nabla q(x)| \le C\|\omega\|_{L^\infty(\partial\Omega)}, \tag{2.5}$$

[3], [4], [26] and by using the solution operator $\mathbb{K} : \omega \longmapsto \nabla q$, the associated pressure gradient is represented by $\nabla q = \mathbb{K}\omega$ for $v = R(\lambda)f$ and $f \in L^\infty_\sigma$.
(ii) The operator $R(\lambda)$ is pseudo-resolvent on L^∞_σ with the trivial kernel, i.e. Ker $R(\lambda) = \{0\}$. Indeed, if $v = R(\lambda)f = 0$, we have $\nabla q = \mathbb{K}\omega = 0$ and $f = 0$. Hence by the open mapping theorem, there exits a closed operator A such that $R(\lambda) = (\lambda - A)^{-1}$. We call A the Stokes operator on L^∞_σ.

We shall prove the representation formula (1.15) for solutions of (1.10) with the kernels (1.16) and (1.17).

Lemma 2.4 (Representation formula)**.** *The solution* $v = R(\lambda)f$ *and* $\nabla q = \mathbb{K}\omega$ *for* $\lambda \in \Sigma_{\theta+\pi/2}$ *and* $f \in L^\infty_\sigma$ *is represented by*

$$v(x) = \int_\Omega E^\lambda(x-y)f(y)dy + \int_{\partial\Omega} V^\lambda(x-y)TN(y)dH(y), \quad x \in \Omega, \tag{2.6}$$

for $T = \nabla v + {}^t\nabla v - qI$.

Proof. We denote by $\overline{f}$ the zero extension of f to $\mathbb{R}^2\backslash\overline{\Omega}$. Observe that $(\overline{v},\overline{q})$ is a weak solution of the problem

$$\lambda\overline{v}-\Delta\overline{v}+\nabla\overline{q}=\overline{f}+\mu,\quad \operatorname{div}\overline{v}=0\quad \text{in } \mathbb{R}^2, \tag{2.7}$$

for a measure μ satisfying

$$(\mu,\varphi)=\int_{\partial\Omega}TN(y)\cdot\varphi(y)\mathrm{d}H(y),\quad \varphi\in C_0(\mathbb{R}^2), \tag{2.8}$$

where $C_0(\mathbb{R}^2)$ denotes the space of all continuous functions in $\mathbb{R}^2$ vanishing at space infinity and $(\cdot,\cdot)$ denotes the pairing between $C_0(\mathbb{R}^2)$ and its adjoint space. Indeed, multiplying $\varphi\in C_c^\infty(\mathbb{R}^2)$ by (1.10) and integration by parts imply (2.7) in a weak sense. The formula (2.6) formally follows by multiplying $(\lambda-\Delta)^{-1}\mathbb{P}$ by (2.7). We set $v_1=(\lambda-\Delta)^{-1}\overline{f}$ and $v_2=\overline{v}-v_1$ to see that

$$\lambda v_2-\Delta v_2+\nabla q=\mu,\quad \operatorname{div} v_2=0\quad \text{in } \mathbb{R}^2.$$

By the mollifications $v_{2,\varepsilon}=v_2*\eta_\varepsilon$, $q_\varepsilon=q*\eta_\varepsilon$ and $\mu_\varepsilon=\mu*\eta_\varepsilon$ with the standard mollifier η_ε, $(v_{2,\varepsilon},q_\varepsilon)$ satisfies the above problem for $\mu_\varepsilon\in L^p$ for $p\in[1,\infty]$. By multiplying $(\lambda-\Delta)^{-1}\mathbb{P}$ by the equation, we have

$$v_{2,\varepsilon}(x)=(\lambda-\Delta)^{-1}\mathbb{P}\mu_\varepsilon=\int_{\mathbb{R}^2}V^\lambda(x-y)\mu_\varepsilon(y)\mathrm{d}y=\eta_\varepsilon*\left(\int_{\partial\Omega}V^\lambda(x-y)TN(y)\mathrm{d}H(y)\right).$$

Sending $\varepsilon\to 0$ yields (2.6). This completes the proof. □

The Stokes paradox follows a similar argument using the fundamental tensor of the Stokes equations. The following result is due to Chang and Finn [12, Theorem 3].

Lemma 2.5 (Stokes paradox)**.** *Let $(v,\nabla q)\in W^{2,p}_{loc}(\overline{\Omega})\times L^p_{loc}(\overline{\Omega})$, $p\in(1,\infty)$, satisfy (1.13). Assume that*

$$v(x)=o(\log|x|)\quad \text{as } |x|\to\infty. \tag{2.9}$$

Then, $v\equiv 0$ and $\nabla q\equiv 0$.

Proof. We give a proof for completeness. Observe that the zero extension $(\overline{v},\overline{q})$ is a solution of the problem

$$-\Delta\overline{v}+\nabla\overline{q}=\mu,\quad \operatorname{div}\overline{v}=0\quad \text{in } \mathbb{R}^2, \tag{2.10}$$

for a measure μ defined by (2.8). By the fundamental tensor of the Stokes equations $V = (V_{ij})$ and $Q = (Q_j)$ [18, p.239],

$$V_{ij}(x) = \frac{1}{4\pi}\left(-\delta_{ij}\log|x| + \frac{x_i x_j}{|x|^2}\right), \quad Q_j(x) = \frac{1}{2\pi}\frac{x_j}{|x|^2},$$

we set $(\tilde{v}, \tilde{q})$ by

$$\tilde{v}(x) = \int_{\partial\Omega} V(x-y)TN(y)\mathrm{d}H(y), \quad \tilde{q}(x) = \int_{\partial\Omega} Q(x-y)\cdot TN(y)\mathrm{d}H(y).$$

The functions $\tilde{v}$ and $\tilde{q}$ are locally integrable in $\mathbb{R}^2$ and $\tilde{v} = O(\log|x|)$, $\nabla\tilde{v} = O(|x|^{-1})$ as $|x| \to \infty$. Observe that $u = \overline{v} - \tilde{v}$ and $p = \overline{q} - \tilde{q}$ is a weak solution of

$$-\Delta u + \nabla p = 0, \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^2. \tag{2.11}$$

Since u and p are locally integrable in $\mathbb{R}^2$, by mollification we may assume that they are smooth in $\mathbb{R}^2$. Since $\omega = \partial_1 u^2 - \partial_2 u^1$ is bounded in $\mathbb{R}^2$ and satisfies $-\Delta\omega = 0$ in $\mathbb{R}^2$, ω is constant by the Liouville theorem. By $-\Delta u = 0$ in $\mathbb{R}^2$ and $u = O(\log|x|)$ as $|x| \to \infty$, u and p are constants. Hence by shifting the pressure up to constant

$$v(x) = v_\infty + \int_{\partial\Omega} V(x-y)TN(y)\mathrm{d}H(y), \quad q(x) = \int_{\partial\Omega} Q(x-y)\cdot TN(y)\mathrm{d}H(y), \tag{2.12}$$

for some constant v_∞. This implies

$$\begin{aligned} v(x) &= v_\infty + V(x)\int_{\partial\Omega} TN(y)\mathrm{d}H(y) + O(|x|^{-1}), \\ q(x) &= Q(x)\cdot\int_{\partial\Omega} TN(y)\mathrm{d}H(y) + O(|x|^{-2}) \quad \text{as } |x| \to \infty. \end{aligned}$$

Since $v = o(\log|x|)$ as $|x| \to \infty$, by dividing v by $\log|x|$ and sending $|x| \to \infty$,

$$\int_{\partial\Omega} TN(y)\mathrm{d}H(y) = 0.$$

Hence $v - v_\infty = O(|x|^{-1})$ and $\nabla v,\ q = O(|x|^{-2})$ as $|x| \to \infty$. By multiplying $v - v_\infty$ by (1.13) and integration by parts in $\Omega \cap B(0,R)$,

$$\int_{\Omega\cap B(0,R)} |\nabla v|^2\mathrm{d}x = \int_{\partial B(0,R)} (TN)\cdot(v - v_\infty)\mathrm{d}H(x) \to 0 \quad \text{as } R \to \infty.$$

By $v = 0$ on $\partial\Omega$, $v \equiv 0$ and $\nabla q \equiv 0$ follow. This completes the proof. $\square$

Remark 2.6. For $n \geq 3$, the fundamental tensor of the Stokes equations (2.11) is $V = (V_{ij})$, $Q = (Q_j)$ for

$$V_{ij}(x) = \frac{1}{2n(n-2)\alpha(n)}\left(\frac{\delta_{ij}}{|x|^{n-2}} + (n-2)\frac{x_i x_j}{|x|^n}\right), \quad Q_j(x) = \frac{1}{n\alpha(n)}\frac{x_j}{|x|^n}.$$

In the same way as the proof of Lemma 2.5, we see that any bounded solutions v of (1.13) is of the form (2.12) for some constant v_∞.

3. The resolvent estimate

We prove the estimate (1.11). By the approximation for $f \in L^\infty_\sigma$ (Lemma 2.2), it suffices to show (1.11) for $f \in C^\infty_{c,\sigma}$.

Proposition 3.1.

$$\sup_{\lambda \in \Sigma_{\theta+\pi/2}} |\lambda| \, \|R(\lambda)f\|_{L^\infty} < \infty, \quad f \in C^\infty_{c,\sigma}. \tag{3.1}$$

Proof. Since $C^\infty_{c,\sigma} \subset L^p_\sigma$ for $p \in (1,\infty)$, $R(\lambda)f = (\lambda - A)^{-1}f$ for $A = \mathbb{P}\Delta$ and the Helmholtz projection operator $\mathbb{P}$. The domain $D(A) = W^{2,p} \cap W^{1,p}_0 \cap L^p_\sigma$ is equipped with the graph-norm and $D(A) \subset W^{2,p}$ with continuous injection [21]. Here, $W^{2,p}$ denotes the Sobolev space and $W^{1,p}_0$ denotes the space of all trace zero functions in $W^{1,p}$. By the L^p-resolvent estimate $|\lambda| \, \|R(\lambda)f\|_{L^p} \leq C\|f\|_{L^p}$ [11] and the Sobolev embedding for $p \in (2,\infty)$,

$$\|R(\lambda)f\|_{L^\infty} \leq C\|R(\lambda)f\|_{W^{2,p}} \leq C' \left(\|R(\lambda)f\|_{L^p} + \|AR(\lambda)f\|_{L^p}\right) \leq C''\left(\frac{1}{|\lambda|} + 1\right)\|f\|_{L^p}.$$

Hence $|\lambda| \, \|R(\lambda)f\|_{L^\infty}$ is bounded for $|\lambda| \leq 1$. Since $|\lambda| \, \|R(\lambda)f\|_{L^\infty} \leq C\|f\|_{L^\infty}$ for $|\lambda| \geq 1$ by (2,2), (3.1) follows. □

Lemma 3.2. *There exists a constant C such that (1.11) holds for $f \in C^\infty_{c,\sigma}$ and $\lambda \in \Sigma_{\theta+\pi/2}$.*

Proof. We argue by contradiction. Suppose that (1.11) were false. Then, for $m \geq 1$ there exists $\tilde{f}_m \in C^\infty_{c,\sigma}$ such that

$$M_m = \sup_{\lambda \in \Sigma_{\theta+\pi/2}} |\lambda| \, \|R(\lambda)\tilde{f}_m\|_{L^\infty}(\lambda) > m\|\tilde{f}_m\|_{L^\infty}.$$

By setting $f_m = \tilde{f}_m / M_m$,

$$\sup_{\lambda\in\Sigma_{\theta+\pi/2}} |\lambda|\, \|R(\lambda)f_m\|_{L^\infty}(\lambda) = 1, \quad \|f_m\|_{L^\infty} < \frac{1}{m}.$$

We set $v_m = R(\lambda)f_m$ and take a point $\lambda_m \in \Sigma_{\theta+\pi/2}$ such that

$$|\lambda_m|\, \|v_m\|_{L^\infty} \geq \frac{1}{2}.$$

We may assume that $\lambda_m \to 0$ by (2.2). Observe that $u_m = \lambda_m v_m$ satisfies

$$\begin{aligned} \lambda_m u_m - \Delta u_m + \nabla p_m = \lambda_m f_m, \quad \operatorname{div} u_m = 0 \quad \text{in } \Omega, \\ u_m = 0 \quad \text{on } \partial\Omega, \end{aligned}$$

with some associated pressure p_m. We take a point $x_m \in \Omega$ such that

$$|u_m(x_m)| \geq \frac{1}{4}.$$

We normalize the pressure p_m so that $\int_{\partial\Omega} p_m \mathrm{d}H(y) = 0$. Since $u_m - \Delta u_m + \nabla p_m = \lambda_m(f_m - u_m) + u_m$, applying the resolvent estimates (2.1) for $p > 2$ implies

$$\begin{aligned} \|u_m\|_{W^{1,\infty}} + \sup_{x\in\Omega}\left\{\|\nabla^2 u_m\|_{L^p(\Omega\cap B(x,1))} + \|\nabla p_m\|_{L^p(\Omega\cap B(x,1))}\right\} &\leq C(\|\lambda_m(f_m - u_m)\|_{L^\infty} + \|u_m\|_{L^\infty}) \\ &\leq C', \quad \text{for all } m \geq 1. \end{aligned}$$

Hence $\{u_m\}$ is equi-continuous in $\overline{\Omega}$. By choosing a subsequence (still denoted by $\{u_m\}$), u_m converges to a limit u locally uniformly in $\overline{\Omega}$ together with ∇u_m and p_m. Then the limit u is a bounded solutions of

$$\begin{aligned} -\Delta u + \nabla p = 0, \quad \operatorname{div} u = 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{aligned}$$

with the associated pressure p. Applying Lemma 2.5 implies that $u \equiv 0$ and $\nabla p \equiv 0$. Since $\int_{\partial\Omega} p \mathrm{d}H(y) = 0$, $p \equiv 0$. Hence we have

$$u_m \to 0 \quad \text{locally uniformly in } \overline{\Omega}, \tag{3.2}$$

together with ∇u_m and p_m. In particular, $T_m = \nabla u_m + {}^t\nabla u_m - p_m I \to 0$ uniformly on $\partial\Omega$ as $m \to \infty$.

Suppose that $\limsup_{m\to\infty} |x_m| < \infty$. By choosing a subsequence, we may assume that $\{x_m\}$ converges to some point in $\overline{\Omega}$. This implies that $1/4 \leq |u_m(x_m)| \to 0$, a contradiction. We may assume that $\limsup_{m\to\infty} |x_m| = \infty$. By choosing a subsequence, we may assume that $\lim_{m\to\infty}|x_m| = \infty$. We consider two cases depending on whether $|\lambda_m|^{1/2}|x_m|$ vanishes or not.

Case 1. $\liminf_{m\to\infty}|\lambda_m|^{1/2}|x_m| > 0$.

We may assume that $|\lambda_m|^{1/2}|x_m| \geq d$ for some constant $d > 0$ by choosing a subsequence. By the representation formula (2.6),

$$u_m(x) = (\lambda_m - \Delta)^{-1}\lambda_m f_m + \int_{\partial\Omega} V^{\lambda_m}(x-y)T_m N(y)\mathrm{d}H(y).$$

By $|\lambda_m|^{1/2}|x_m| \geq d$ and (1.18),

$$\sup_{y\in\partial\Omega}|V^{\lambda_m}(x_m - y)| \leq C, \quad \text{for all } m \geq 1.$$

By the L^∞-estimate $|\lambda_m|\,\|(\lambda_m-\Delta)^{-1}f_m\|_{L^\infty} \leq C\|f_m\|_{L^\infty}$,

$$\frac{1}{4} \leq |u_m(x_m)| \leq C\left(\frac{1}{m} + \int_{\partial\Omega}|T_m N(y)|\mathrm{d}H(y)\right) \to 0 \quad \text{as } m\to\infty.$$

Thus Case 1 does not occur.

Case 2. $\liminf_{m\to\infty}|\lambda_m|^{1/2}|x_m| = 0$.

We may assume that $\lim_{m\to\infty}|\lambda_m|^{1/2}|x_m| = 0$. By the representation formula (2.6) and (1.19),

$$\begin{aligned} u_m(x) &= (\lambda_m - \Delta)^{-1}\lambda_m f_m - \frac{1}{4\pi}\log\sqrt{\lambda_m}\int_{\partial\Omega}T_m N(y)\mathrm{d}H(y) \\ &\quad - \frac{1}{4\pi}\int_{\partial\Omega}\log|x-y|T_m N(y)\mathrm{d}H(y) + \int_{\partial\Omega}\tilde{V}^{\lambda_m}(x-y)T_m N(y)\mathrm{d}H(y). \end{aligned} \tag{3.3}$$

By (3.2), sending $m\to\infty$ for fixed $x\in\Omega$ implies

$$0 = \lim_{m\to\infty}\log|\lambda_m|^{1/2}\left|\int_{\partial\Omega}T_m N(y)\mathrm{d}H(y)\right|. \tag{3.4}$$

We substitute $x = x_m$ into (3.3). By (1.18),

$$\sup_{y\in\partial\Omega}|\tilde{V}^{\lambda_m}(x_m - y)| \leq C, \quad \text{for all } m \geq 1.$$

Since $|\lambda_m|^{1/2}|x_m| \leq 1$ for sufficiently large m, $\log|x_m| \leq -\log|\lambda_m|^{1/2}$ and

$$\left|\int_{\partial\Omega}\log|x_m-y|TN(y)\mathrm{d}H(y)\right|\leq\int_{\partial\Omega}\left|\log\left|\frac{x_m}{|x_m|}-\frac{y}{|x_m|}\right|\right||T_mN(y)|\mathrm{d}H(y)+\log|x_m|\left|\int_{\partial\Omega}T_mN(y)\mathrm{d}H(y)\right|$$
$$\leq C\int_{\partial\Omega}|T_mN(y)|\mathrm{d}H(y)-\log|\lambda_m|^{1/2}\left|\int_{\partial\Omega}T_mN(y)\mathrm{d}H(y)\right|.$$

By (3.4) and the dominated convergence theorem,

$$\frac{1}{4}\leq|u_m(x_m)|\leq\frac{C}{m}-\frac{1}{2\pi}\log|\lambda_m|^{1/2}\left|\int_{\partial\Omega}T_mN(y)\mathrm{d}H(y)\right|+C\int_{\partial\Omega}|T_mN(y)|\,\mathrm{d}H(y)$$
$$\to 0\quad\text{as } m\to\infty.$$

We obtained a contradiction. Thus Case 2 does not occur.

We conclude that both Case 1 and Case 2 do not occur. The proof is now complete. □

Proof of Theorem 1.3. For $f\in L^\infty_\sigma$, we take a sequence $\{f_m\}\subset C^\infty_{c,\sigma}$ satisfying (2.3) by Lemma 2.2 (i). Since $|\lambda|\,\|R(\lambda)f_m\|_{L^\infty}\leq C\|f\|_{L^\infty}$ for all $m\geq 1$ and $R(\lambda)f_m$ converges to $R(\lambda)f$ locally uniformly in $\overline{\Omega}$ by Lemma 2.2 (ii), the limit satisfies the desired estimate. Hence the assertion (i) holds. The assertion (ii) follows from the Dunford integral of the resolvent by using (1.11). □

Remarks 3.3. (i) Besides the estimate (1.12), we obtain estimates for spatial derivatives,

$$\|\nabla S(t)v_0\|_{L^\infty}+\|\nabla^2S(t)v_0\|_{L^\infty}\leq C\|v_0\|_{L^\infty},\quad t\geq 1,\quad v_0\in L^\infty_\sigma. \tag{3.5}$$

This follows from (1.12) and the finite time estimate $t^{1/2}\|\nabla S(t)v_0\|_{L^\infty}+t\|\nabla^2S(t)v_0\|_{L^\infty}\leq C\|v_0\|_{L^\infty}$ for $0<t\leq T$ [4].
(ii) For $n\geq 2$ and $v_0\in L^\infty_\sigma$, Lemma 2.5 implies that

$$S(t)v_0\to 0\quad\text{locally uniformly in } \overline{\Omega} \text{ as } t\to\infty. \tag{3.6}$$

In fact, suppose that (3.6) were false. Then, there exists a sequence $\{t_m\}$ such that $t_m\to\infty$ and (3.6) does not hold. By (1.12), (3.5) and choosing a subsequence (still denoted by $\{t_m\}$) $v_m(t)=S(t+t_m)v_0$ converges to a limit v locally uniformly in $\overline{\Omega}\times[0,\infty)$. Since the limit v is bounded and independent of t, $v\equiv 0$ by Lemma 2.5 and $S(t_m)v_0\to 0$ locally uniformly in $\overline{\Omega}$. This is a contradiction.
(iii) For $n\geq 3$ and $v_0\in L^\infty_\sigma$,

$$S(t)v_0\to v\quad\text{locally uniformly in } \overline{\Omega} \text{ as } t\to\infty, \tag{3.7}$$

for some solution v of the stationary Stokes equations (1.13). Since any bounded solutions of (1.13) for $n \geq 3$ must be asymptotically constant as $|x| \to \infty$ by Remark 2.6, $S(t)v_0$ is asymptotically constant as $t \to \infty$ and $|x| \to \infty$ for any bounded initial data $v_0 \in L^\infty_\sigma$.

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(K. Abe) Department of Mathematics, Graduate School of Science, Osaka City University, 3-3-138 Sugimoto, Sumiyoshi-ku Osaka, 558-8585, Japan
E-mail address: kabe@osaka-cu.ac.jp