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STABILITY OF LAMB DIPOLES

KEN ABE AND KYUDONG CHOI

ABSTRACT. The Lamb dipole is a traveling wave solution to the two-dimensional Euler equations introduced by S. A. Chaplygin (1903) and H. Lamb (1906) at the early 20th century. We prove orbital stability of this solution based on a vorticity method initiated by V. I. Arnold. Our method is a minimization of a penalized energy with multiple constraints that deduces existence and orbital stability for a family of traveling waves. As a typical case, orbital stability of the Lamb dipole is deduced by characterizing a set of minimizers as an orbit of the dipole by a uniqueness theorem in the variational setting.

1. INTRODUCTION

1.1. **Lamb dipoles.** We consider the two-dimensional vorticity equations:

$$
\begin{aligned}
\partial_t \zeta + v \cdot \nabla \zeta = 0, \quad v = k * \zeta \quad &\text{in } \mathbb{R}^2 \times (0,\infty), \\
\zeta = \zeta_0 \quad &\text{on } \mathbb{R}^2 \times \{t = 0\},
\end{aligned}
\tag{1.1}
$$

with the kernel $k(x) = (2\pi)^{-1} x^{\perp} |x|^{-2}$, $x^{\perp} = {}^t(-x_2, x_1)$. The equations (1.1) admit *a vortex pair*, i.e., a solution of the form

$$
\begin{aligned}
v(x,t) &= u(x + u_\infty t) - u_\infty, \\
\zeta(x,t) &= \omega(x + u_\infty t),
\end{aligned}
$$

vanishing at space infinity with a constant velocity $u_\infty \in \mathbb{R}^2$. Vortex pairs are pairs of compactly supported dipoles, symmetrically placed with opposite signs, translating in one direction. They are theoretical models of coherent vortex structures in large-scale geophysical flows. See, e.g., [27], [19] for experimental works. By rotational invariance of (1.1), we take $u_\infty = {}^t(-W, 0)$, $W > 0$, without loss of generality. Substituting (v, ζ) into (1.1) implies the steady Euler equations for (u, ω) in a half plane:

$$
\begin{aligned}
u \cdot \nabla \omega = 0 \quad &\text{in } \mathbb{R}^2_+, \\
u \to u_\infty \quad &\text{as } |x| \to \infty.
\end{aligned}
\tag{1.2}
$$

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In the 3rd edition of the book "Hydrodynamics" published at 1906, H. Lamb [30, p.231] noted an explicit solution to (1.2), generally referred to as *the Lamb dipole* (Chaplygin-Lamb dipole), a solution $\omega_L = \lambda \max\{\Psi_L, 0\}$, $u_L = {}^t(\partial_{x_2}\Psi_L, -\partial_{x_1}\Psi_L)$, $0 < \lambda < \infty$, of the form

$$\Psi_L(x) = \begin{cases} C_L J_1(\lambda^{1/2} r)\sin\theta, & r \le a, \\ -W\left(r - \dfrac{a^2}{r}\right)\sin\theta, & r > a, \end{cases} \tag{1.3}$$

with the constants

$$C_L = -\frac{2W}{\lambda^{1/2} J_0(c_0)}, \quad a = c_0 \lambda^{-1/2},$$

where (r,θ) is the polar coordinate and $J_m(r)$ is the m-th order Bessel function of the first kind. The constant c_0 is the first zero point of J_1, i.e., $J_1(c_0) = 0$, $c_0 = 3.8317\cdots$, $J_0(c_0) < 0$. The parameter $\lambda > 0$ denotes the strength of the vortex and is related with its impulse by

$$\int_{\mathbb{R}^2_+} x_2 \omega_L \mathrm{d}x = \frac{c_0^2 \pi W}{\lambda}.$$

The Lamb dipole (1.3) is the simplest explicit solution to (1.2), symmetric for the x_2-variable, which is a special case of non-symmetric Chaplygin dipoles, independently found by S. A. Chaplygin in 1903 [14], [15]. See [39] for their origins.

The Lamb dipole is considered as a stable vortex structure in a two-dimensional flow. Its stability has been studied by an experimental work [19] and also by a numerical work [24]. On the other hand, despite the explicit form of this classical solution, its mathematical stability had been an open question since the solution was introduced by S. A. Chaplygin and H. Lamb at the early 20th century. For solutions with a single-signed vortex such as a circular vortex [48], [43] or a rectangular vortex [5], stability results have been developed, while no stability result was known for the Lamb dipole which has a multi-signed vortex and forms a traveling wave.

There is an interesting relation with *solitons* in the theory of nonlinear wave equations. One of classical models that describes propagation of a wave may be the KdV equation [29]. More generally for the gKdV equation,

$$\partial_t w + \partial_x^3 w + \partial_x(w^p) = 0, \quad x \in \mathbb{R},\ t > 0,$$

for an integer $p \ge 2$, there exists a soliton solution of the form $w(x,t) = Q_c(x - ct)$ for $c > 0$ and $Q_c(x) = c^{1/(p-1)} Q(c^{1/2} x)$, where

$$Q(x) = \left(\frac{p+1}{2\cosh^2((p-1)x/2)}\right)^{1/(p-1)},$$

is called soliton, which is a unique positive solution of the elliptic problem $\partial_x^2 Q + Q^p = Q$, up to translation. Stability of this soliton is well known when the problem is globally well-posed. Indeed for $2 \leq p < 5$, the gKdV equation is globally well-posed, and if initial data is close to the soliton, the solution remains nearby the soliton for all time by admitting translation of Q [6], [49]. Such stability is termed *orbital stability*. For $p = 5$, this soliton is unstable [36] and a finite time blow-up occurs [40], [37]. The Euler equations may have some aspects of the wave equation. Even for the three-dimensional case, vortex rings form traveling waves. We shall establish the orbital stability theorem for the Lamb dipole which is the most typical traveling wave.

In the sequel, we identify a function ζ_0 in $\mathbb{R}^2_+$ with an odd extension to $\mathbb{R}^2$ for the x_2-variable, i.e., $\zeta_0(x_1, x_2) = -\zeta_0(x_1, -x_2)$. Since a classical solution to (1.1) exists and is symmetric for the x_2-variable for sufficiently smooth initial data [34], a standard approximation argument implies the existence of a symmetric global weak solution $\zeta \in BC([0,\infty); L^2 \cap L^1(\mathbb{R}^2))$ for symmetric initial data $\zeta_0 \in L^2 \cap L^1(\mathbb{R}^2)$ [35]. Here, $BC([0,\infty); X)$ denotes the space of all bounded continuous functions from $[0,\infty)$ into a Banach space X. Among other results, our simplest result is the following:

Theorem 1.1. *Let $0 < \lambda, W < \infty$. The Lamb dipole ω_L is orbitally stable in the sense that for $\nu > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for $\zeta_0 \in L^2 \cap L^1(\mathbb{R}^2_+)$ satisfying $x_2\zeta_0 \in L^1(\mathbb{R}^2_+)$, $\zeta_0 \geq 0$, $\|\zeta_0\|_1 \leq \nu$ and*

$$\inf_{y\in\partial\mathbb{R}^2_+} \left\{ \|\zeta_0 - \omega_L(\cdot + y)\|_2 + \|x_2(\zeta_0 - \omega_L(\cdot + y))\|_1 \right\} \leq \delta,$$

there exists a global weak solution $\zeta(t)$ of (1.1) satisfying

$$\inf_{y\in\partial\mathbb{R}^2_+} \left\{ \|\zeta(t) - \omega_L(\cdot + y)\|_2 + \|x_2(\zeta(t) - \omega_L(\cdot + y))\|_1 \right\} \leq \varepsilon, \quad \textit{for all } t \geq 0.$$

Remark 1.2. As we will see later in Remarks 5.2 (i), the smallness condition in Theorem 1.1 can be replaced with a slightly weaker condition $\inf_{y\in\partial\mathbb{R}^2_+} \|\zeta_0 - \omega_L(\cdot + y)\|_2 + \left|\int x_2\zeta_0 \mathrm{d}x - \mu\right| \leq \delta$ for $\mu = c_0^2\pi W/\lambda$.

Orbital stability of traveling waves to the two-dimensional Euler equations is first studied by Burton, Lopes and Lopes [12] based on a variational principle using a rearrangement and the concentration compactness principle. See Burton [11] for a recent improvement. The works [12], [11] proved orbital stability for a broad class of vortex pairs though stability of Lamb dipole was unknown. We prove Theorem 1.1 by using a simpler variational principle in a restricted class of vortex pairs.

1.2. **Vorticity method.** Theorem 1.1 is a particular case of our general stability theorem. Let us consider the existence problem (1.2). The equation $(1.2)_1$ implies that the vorticity is a first integral of the stream line, i.e. an integral curve of $u = {}^t(\partial_{x_2}\Psi, -\partial_{x_1}\Psi)$ for the stream function Ψ. Therefore ω is locally a function of Ψ. We assume that ω is globally represented by $\omega = \lambda f(\Psi)$ with some function $f(t)$ and $\lambda > 0$. Then solutions of (1.2) can be constructed by the semi-linear elliptic problem for $\gamma \geq 0$:

$$
(1.4) \qquad \begin{aligned} -\Delta\Psi &= \lambda f(\Psi) \quad \text{in } \mathbb{R}^2_+, \\ \Psi &= -\gamma \qquad \text{on } \partial\mathbb{R}^2_+, \\ \partial_{x_1}\Psi \to 0, \quad &\partial_{x_2}\Psi \to -W \quad \text{as } |x| \to \infty. \end{aligned}
$$

The function f is called a vorticity function which is prescribed by a non-negative and non-decreasing function. In this paper, we shall take

$$
f(t) = t_+, \quad t_+ = \max\{t, 0\},
$$

for which the Lamb dipole Ψ_L is a solution to (1.4) for $\gamma = 0$ and spt $\omega_L = \overline{B(0,a) \cap \mathbb{R}^2_+}$, i.e., $\omega_L = \lambda f(\Psi_L)$. Here $B(0,a)$ is an open disk centered at the origin with the radius $a > 0$. The three parameters $W, \gamma \geq 0$ and $\lambda > 0$ are referred to as propagation speed, flux constant and strength parameter. We chose the flux constant γ so that $\Psi = 0$ on the boundary of the vortex core spt $\omega = \overline{\Omega}$. The problem (1.4) is a free-boundary problem since the vortex core Ω is a priori unknown. Once the core is found, one can find Ψ by solving the two problems:

$$
\begin{aligned} &-\Delta\Psi = \lambda\Psi \quad \text{in } \Omega, \quad \Psi = 0 \quad \text{on } \partial\Omega, \\ &-\Delta\Psi = 0 \quad \text{in } \mathbb{R}^2_+\backslash\overline{\Omega}, \ \Psi = -\gamma \quad \text{on } \partial\mathbb{R}^2_+, \ \partial_{x_1}\Psi \to 0, \ \partial_{x_2}\Psi \to -W \quad \text{as } |x| \to \infty. \end{aligned}
$$

On the other hand, the core is characterized as $\Omega = \{x \in \mathbb{R}^2_+ \mid \Psi(x) > 0\}$ by a maximum principle. The function $\Psi = \psi - Wx_2 - \gamma$ is represented by the Green function of the Laplace operator subject to the Dirichlet boundary condition in a half plane

$$
(1.5) \qquad \psi(x) = \int_{\mathbb{R}^2_+} G(x,y)\omega(y)\mathrm{d}y, \quad G(x,y) = \frac{1}{4\pi}\log\left(1 + \frac{4x_2y_2}{|x-y|^2}\right).
$$

To study existence and stability of solutions to (1.4), we consider a variational principle based on vorticity, called a vorticity method, originating from the idea of Kelvin [45], initiated by Arnold [3], [4]. See also Benjamin [7] for vortex rings. For vortex pairs, vorticity methods were developed by Turkington [46] and Burton [8]. See also Norbury [42] and Yang [50] for a stream function method.

Our approach is based on the vorticity method of Friedman-Turkington [23], [22] developed for vortex rings. For $0 < \mu, \nu, \lambda < \infty$, we set a space of admissible functions

$$
K_{\mu,\nu} = \left\{ \omega \in L^2(\mathbb{R}^2_+) \ \middle| \ \omega \geq 0, \ \int_{\mathbb{R}^2_+} x_2\omega \mathrm{d}x = \mu, \ \int_{\mathbb{R}^2_+} \omega \mathrm{d}x \leq \nu \right\}.
$$

We construct solutions of (1.4) by maximizing a penalized energy

$$E_{2,\lambda}[\omega] = E[\omega] - \frac{1}{2\lambda}\int_{\mathbb{R}^2_+} \omega^2 \mathrm{d}x, \quad E[\omega] = \frac{1}{2}\int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2_+} G(x,y)\omega(x)\omega(y)\mathrm{d}x\mathrm{d}y.$$

For a notational convenience, we formulate the maximization problem as a minimization of $-E_{2,\lambda}$ and denote by

$$I_{\mu,\nu,\lambda} = \inf_{\omega\in K_{\mu,\nu}} \{-E_{2,\lambda}[\omega]\}. \tag{1.6}$$

The constants W, $\gamma \geq 0$ are Lagrange multipliers. This formulation is slightly different from that of [23], [22], where admissible functions are restricted to a space of symmetric functions for $x_1 \in \mathbb{R}$. More precisely, the method in [23], [22] applies to prove compactness of a minimizing sequence satisfying

$$\begin{aligned} &\omega(x_1, x_2) = \omega(-x_1, x_2), \\ &\omega(x_1, x_2) \text{ is non-increasing for } x_1 > 0. \end{aligned} \tag{1.7}$$

The condition (1.7) is essential for the method in [23], [22]. In fact, since the energy $-E_{2,\lambda}$ is invariant by translation for the x_1-variable, translation of any minimizer is a minimizing sequence. In this paper, without assuming (1.7), we shall show that any minimizing sequence is relatively compact by translation for the x_1-variable by using the concentration compactness principle of Lions [31]. The following Theorem 1.3 is an improvement of [23], [22] in terms of vortex pairs.

Theorem 1.3. *Let $0 < \mu, \nu, \lambda < \infty$. For any minimizing sequence $\{\omega_n\}$ satisfying $\omega_n \in K_{\mu_n,\nu}$, $\mu_n \to \mu$ and $-E_{2,\lambda}[\omega_n] \to I_{\mu,\nu,\lambda}$, there exists a sequence $\{y_n\} \subset \partial\mathbb{R}^2_+$ such that $\{\omega_n(\cdot + y_n)\}$ and $\{x_2\omega_n(\cdot + y_n)\}$ are relatively compact in $L^2(\mathbb{R}^2_+)$ and $L^1(\mathbb{R}^2_+)$, respectively. In particular, the problem (1.6) has a minimizer in $K_{\mu,\nu}$.*

A novelty in the present paper is the adaptation of the vorticity method of [23], [22], instead of [46] which prescribes that mass is exactly $\nu > 0$ for admissible functions. As proved in [23], [22] for vortex rings, mass becomes strictly less than $\nu > 0$ for small $\lambda > 0$ with fixed μ, ν. Indeed, the variational principle in [46] does not provide solutions of (1.4) for small $\lambda > 0$. Our existence for small $\lambda > 0$ seems a new result although the above formulation is noted in [46]. See also [42].

Removing the restriction on the strength parameter is essential in the present work since solutions of (1.4) approach a Lamb dipole as $\lambda \to 0$. We shall rigorously state this claim in Theorem 1.5 below. For fixed μ, ν, solutions of (1.6) form a one parameter family for $0 < \lambda < \infty$. In particular, solutions approach a Dirac measure as $\lambda \to \infty$ and in contrast a Lamb dipole as $\lambda \to 0$. A variational characterization of the Lamb dipole is studied in [9], [10] for solutions to (1.4) for $\gamma = 0$.

Orbital stability of vortex pairs is a consequence of compactness of a minimizing sequence. We use conservations of L^q-norms, impulse and penalized energy of (1.1):

$$\begin{aligned} \|\zeta\|_q(t) &= \|\zeta_0\|_q, \quad 1 \le q \le 2, \\ \|x_2\zeta\|_1(t) &= \|x_2\zeta_0\|_1, \\ E_{2,\lambda}(\zeta)(t) &= E_{2,\lambda}(\zeta_0), \qquad \text{for all } t \ge 0. \end{aligned} \tag{1.8}$$

Although a global weak solution $\zeta(t)$ of (1.1) obtained by an approximation argument [35] might have weak regularity at $t = 0$, by the renormalization property of DiPerna-Lions [18], the constructed weak solution satisfies the conservations (1.8), i.e., $\zeta(t) \in K_{\mu,\nu}$ for $\zeta_0 \in K_{\mu,\nu}$. In general, $\zeta(t)$ is not symmetric and non-increasing for the x_1-variable even if ζ_0 is. The renormalization property of weak solutions to the two-dimensional Euler equations is due to [33].

The vorticity method not only constructs stationary solutions as lowest energy solutions but also deduces their stability by compactness of a minimizing sequence, cf. [13] for dispersive equations. For the Euler equations, research on orbital stability goes back to Benjamin [7]. See Wan [47] for an early work. For vortex pairs, the first orbital stability result appeared in Burton, Lopes and Lopes [12] for a certain class of solutions to (1.2) by a vorticity method based on a rearrangement for a prescribed function. See [28], [12] for a physical background and an introduction to the problem. The method of [12] yields existence of solutions to (1.4) for small $W > 0$, $\gamma = 0$ with unknown $f(t)$, $\lambda > 0$ and deduces their stability by the L^2-norm $\|\zeta\|_2$ for compactly supported ζ_0 close to the orbit in the stronger norm $\|\zeta\|_2 + |\int x_2\zeta \mathrm{d}x|$. Burton [11] recently proved orbital stability by using one norm $\|\zeta\|_p + \|\zeta\|_1 + |\int x_2\zeta \mathrm{d}x|$, $p > 2$, for both stability of ζ and an initial condition of ζ_0. We prove existence of (1.4) by prescribing $f(t) = t_+$, $\lambda > 0$ and deduce their stability by the norm $\|\zeta\|_2 + \|x_2\zeta\|_1$ without assuming compact support for ζ_0. Let $S_{\mu,\nu,\lambda}$ denote the set of minimizers of (1.6). Theorem 1.3 implies:

Theorem 1.4. *For $0 < \mu, \nu, \lambda < \infty$, $S_{\mu,\nu,\lambda}$ is orbitally stable in the sense that for $\varepsilon > 0$, there exists $\delta > 0$ such that for $\zeta_0 \in L^2 \cap L^1(\mathbb{R}^2_+)$ satisfying $x_2\zeta_0 \in L^1(\mathbb{R}^2_+)$, $\zeta_0 \ge 0$, $\|\zeta_0\|_1 \le \nu$ and*

$$\inf_{\omega \in S_{\mu,\nu,\lambda}} \{\|\zeta_0 - \omega\|_2 + \|x_2(\zeta_0 - \omega)\|_1\} \le \delta, \tag{1.9}$$

there exists a global weak solution $\zeta(t)$ of (1.1) satisfying

$$\inf_{\omega \in S_{\mu,\nu,\lambda}} \{\|\zeta(t) - \omega\|_2 + \|x_2(\zeta(t) - \omega)\|_1\} \le \varepsilon, \quad \text{for all } t \ge 0. \tag{1.10}$$

Theorem 1.4 is a general stability theorem for a family of vortex pairs for $0 < \lambda < \infty$. If the set of minimizers is characterized as an orbit $O(\omega) = \{\omega(\cdot + y) \mid y \in \partial\mathbb{R}^2_+\}$ for some vortex pair, one can deduce orbital stability of the vortex pair itself. Since translation of a minimizer ω of (1.6) is also a minimizer, the orbit $O(\omega)$ is a subset of $S_{\mu,\nu,\lambda}$. The converse inclusion is a uniqueness issue. See [1] for uniqueness of the Hill's spherical vortex rings and [9], [10] of the Lamb dipoles.

In this paper, we prove uniqueness of minimizers of (1.6) for small $\lambda > 0$, i.e., $\mu\nu^{-1}\lambda^{1/2} \le M_1$ for some $M_1 > 0$. As proved later, for small $\lambda > 0$, the flux constant γ vanishes. This implies that ψ/x_2 is a positive solution to the elliptic problem in $\mathbb{R}^4$, i.e., for $y = {}^t(y', y_4) \in \mathbb{R}^4$,

$$-\Delta_y \left(\frac{\psi(y_4, |y'|)}{|y'|} \right) = \lambda f \left(\frac{\psi(y_4, |y'|)}{|y'|} - W \right) \qquad \text{in } \mathbb{R}^4.$$

Since positive solutions $\psi/|y'|$ of the above problem are radially symmetric for some point on $\{y' = 0\}$ [9], minimizers of (1.6) for small $\lambda > 0$ must be translation of a Lamb dipole ω_L for $W > 0$. As a consequence, it turns out that $S_{\mu,\nu,\lambda} = O(\omega_L)$ for $\mu\nu^{-1}\lambda^{1/2} \leq M_1$ and (1.10) is orbital stability of the Lamb dipole itself. By the constraint on the impulse, the speed $W > 0$ is uniquely determined by $W = \mu\lambda/(c_0^2\pi)$.

Theorem 1.5. *Let $0 < \mu, \nu, \lambda < \infty$ satisfy $\mu\nu^{-1}\lambda^{1/2} \leq M_1$ for some absolute constant $M_1 > 0$. Let ω_L be the Lamb dipole for $W = \mu\lambda/(c_0^2\pi)$. Then, minimizers of (1.6) are translation of the Lamb dipole, i.e.,*

$$S_{\mu,\nu,\lambda} = \left\{ \omega_L(\cdot + y) \;\middle|\; y \in \partial\mathbb{R}^2_+ \right\}. \tag{1.11}$$

The characterization (1.11) implies that $S_{\mu,\nu,\lambda}$ is independent of large $\nu > 0$ for fixed μ, λ, i.e., $\mu\nu^{-1}\lambda^{1/2} \leq M_1$. Therefore for given $\lambda, W > 0$, $\nu > 0$ and $\mu = c_0^2\pi W/\lambda$, we take $\tilde{\nu} = \max\{\nu, \mu\lambda^{1/2}M_1^{-1}\}$ so that $S_{\mu,\tilde{\nu},\lambda} = O(\omega_L)$. Theorem 1.1 is then deduced from Theorem 1.4.

There is a possibility that uniqueness holds even for solutions to (1.4) for small $\gamma > 0$. See [41], [2] for uniqueness of vortex rings. If the uniqueness holds, one can characterize $S_{\mu,\nu,\lambda}$ as an orbit of some deformed vortex pair supported away from the boundary $\partial\mathbb{R}^2_+$. Theorem 1.4 may include stability of such solutions.

There are few remarks related with nonlinear wave equations. Orbital stability is concerned with stability about a shape of a wave. Indeed, Theorem 1.1 implies that the shape of ω_L is stable by a perturbation for all $t \geq 0$. A more advanced question is the asymptotic behavior of the perturbation $\zeta(t)$ as $t \to \infty$. One may expect that a perturbation approaches some fixed traveling wave as $t \to \infty$. Such stability is termed *asymptotic stability* in the study of nonlinear wave equations. Another issue is interaction between traveling waves. Stability of two Lamb dipoles or more generally stability of a finite number of the dipoles are open questions. We refer to a survey [44] on stability of solitons.

In this paper, we considered the vorticity function $f(t) = t_+$ to prove the orbital stability of the Lamb dipole. Our method is also applied to prove orbital stability of more general vortex pairs and also vortex rings. For example, we are able to take $f(t) = t_+^{1/(p-1)}$ as a vorticity function to study existence and orbital stability of vortex pairs for $4/3 < p < \infty$ and vortex rings for $6/5 < p < \infty$. The stability norm can be replaced with the L^p-norm with the weighted L^1-norm.

A special case is $p = \infty$ for which the vorticity function becomes an indicator function. The penalized energy can be replaced with the kinetic energy whose minimizers are vortex patches [23], [22]. This class particularly includes the Hill's spherical vortex rings. See [16] for a stability result.

We outline the proof of Theorem 1.3. Applicability of the concentration compactness principle to free boundary problems is noted in the original paper of Lions [32, p.279], though little is known on stability of evolving free boundaries. The first application to stability of traveling waves to the two-dimensional Euler equations is due to Burton, Lopes and Lopes [12] in which stability of a set of minimizers is proved for a large class of vortex pairs based a variational principle with unknown vorticity functions. The main contribution of the present work is the reformulation of the problem by prescribing $f(t) = t_+$ and adjusting a variational principle of vortex rings developed by Friedman-Turkington [23], [22] to vortex pairs so that the Lamb dipole is obtained as a minimizer. This variational principle involves *multiple constraints* and cannot appeal to the subadditivity condition of a minimum found by Lions [31] to obtain compactness of a minimizing sequence. We sketch the key part of the proof below.

In the sequel, we reduce the problem to the case $\nu = \lambda = 1$ by the scaling

$$\hat{\omega}(x) = \frac{1}{\lambda\nu}\omega\left(\frac{x}{\lambda^{1/2}}\right). \tag{1.12}$$

If $\omega \in K_{\mu,\nu}$, $\hat{\omega} \in K_{M,1}$ for $M = \mu\nu^{-1}\lambda^{1/2}$ and $E_{2,1}[\hat{\omega}] = \nu^{-2}E_{2,\lambda}[\omega]$. We abbreviate the notation as $K_\mu = K_{\mu,1}$, $I_\mu = I_{\mu,1,1}$, $E_2[\omega] = E_{2,1}[\omega]$, and $S_\mu = S_{\mu,1,1}$.

To prove compactness of a minimizing sequence of (1.6), we apply a concentration compactness principle and exclude possibilities of dichotomy and vanishing of the sequence. Since I_μ is negative and decreasing for $\mu \in (0,\infty)$, vanishing cannot occur. The problem is to exclude dichotomy of the sequence. Let us consider for simplicity a minimizing sequence $\{\omega_n\} \subset K_\mu$ satisfying $\omega_n = \omega_{1,n} + \omega_{2,n}$, $\omega_{1,n}, \omega_{2,n} \geq 0$, and for $0 < \alpha < \mu$,

$$\alpha = \int_{\mathbb{R}^2_+} x_2\omega_{1,n}\mathrm{d}x, \quad \mu - \alpha = \int_{\mathbb{R}^2_+} x_2\omega_{2,n}\mathrm{d}x, \quad \operatorname{dist}(\operatorname{spt}\omega_{1,n}, \operatorname{spt}\omega_{2,n}) \to \infty.$$

Observe that for example if $\omega_{1,n}$ and $\omega_{2,n}$ are compactly supported and move away for the x_1-direction, the sequence $\{\omega_n\}$ is not compact in L^2. If we have the strict subadditivity of I_μ, i.e., $I_\mu < I_\alpha + I_{\mu-\alpha}$ for $0 < \alpha < \mu$, we immediately conclude that this cannot occur by letting $n \to \infty$ in $E_2[\omega_n] \leq E_2[\omega_{1,n}] + E_2[\omega_{2,n}] + o(1)$.

The main difficulty is the fact that K_μ has the multiple constraints (impulse = μ, mass ≤ 1) which is an obstacle to deduce the strict subadditivity of I_μ from the scaling property of E_2. See [31, Corollary II.1]. We overcome this difficulty by reducing the problem to compactness of a sequence satisfying (1.7) and existence of minimizers of (1.6) by using Steiner symmetrization $\omega^*_{i,n}$, i.e., a rearrangement of $\omega_{i,n}$ satisfying (1.7), $E_2[\omega_{i,n}] \leq E_2[\omega^*_{i,n}]$, conserving L^q-norms, $1 \leq q \leq 2$, and impulse. Since $\omega^*_{i,n}$ is non-increasing for $x_1 > 0$, we are able to show that the weak convergence $\omega^*_{i,n} \rightharpoonup \overline{\omega}_i$ in L^2 implies the convergence of the kinetic energy $E[\omega^*_{i,n}] \to E[\overline{\omega}_i]$. This yields

$$\begin{aligned} &-I_\mu \leq E_2[\overline{\omega}_1] + E_2[\overline{\omega}_2], \\ &\alpha \geq \int_{\mathbb{R}^2_+} x_2\overline{\omega}_1\mathrm{d}x, \quad \mu - \alpha \geq \int_{\mathbb{R}^2_+} x_2\overline{\omega}_2\mathrm{d}x, \quad \|\overline{\omega}_1\|_1 + \|\overline{\omega}_2\|_1 \leq 1. \end{aligned}$$

A contradiction is deduced from the existence of minimizers of (1.6) (satisfying (1.7)).

Indeed, there exists a maximizer ω_1 of E_2 (a minimizer of $-E_2$) under the constraints $\int x_2\overline{\omega}_1 \mathrm{d}x \leq \alpha$ and $\|\overline{\omega}_1\|_1 \leq 1 - \|\overline{\omega}_2\|_1$ for fixed $\overline{\omega}_2$. The maximizer satisfies $\int x_2\omega_1 \mathrm{d}x = \alpha$ with compact support. Therefore we are able to replace $\overline{\omega}_1$ with ω_1 and apply the same for $\overline{\omega}_2$ for fixed ω_1. Since we can assume that spt $\omega_1 \cap$ spt $\omega_2 = \emptyset$ by translation for the x_1-variable,

$$-I_\mu \leq E_2[\omega_1] + E_2[\omega_2] = E_2[\omega_1 + \omega_2] - \int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2_+} G(x,y)\omega_1(x)\omega_2(y)\mathrm{d}x\mathrm{d}y \leq -I_\mu.$$

This implies $\omega_i \equiv 0$ for $i = 1$ or 2, a contradiction to $\mu = \int x_2(\omega_1 + \omega_2)\mathrm{d}x$.

The existence of the minimizer ω_1 follows from the compactness of a minimizing sequence satisfying (1.7). Since we can assume that a minimizing sequence satisfies (1.7) by Steiner symmetrization, the existence of the minimizer ω_1 follows from the convergence of the kinetic energy.

This paper is organized as follows. In Section 2, we prove that I_μ is negative and decreasing for $\mu \in (0,\infty)$ and that minimizers of (1.6) are solutions of (1.4) with compact support. In Section 3, we prove compactness of the kinetic energy for a sequence satisfying (1.7) and existence of minimizers of (1.6). In Section 4, we prove Theorem 1.3 by a concentration compactness principle. In Section 5, we prove existence of symmetric global weak solutions to (1.1) and deduce Theorem 1.4 by a contradiction argument. In Section 5, we prove Theorem 1.5 by applying a symmetry result for positive solutions of the semi-linear elliptic problem [20].

2. A minimization problem

We begin with estimates for the kinetic energy $E[\omega]$. Thanks to the finiteness of the impulse $x_2\omega \in L^1$, the kinetic energy is finite for $\omega \in L^2 \cap L^1$ and agrees with the Dirichlet energy for the stream function. By using energy estimates, we show that I_μ is decreasing for $\mu \in (0,\infty)$ and any minimizing sequence of I_μ is a bounded sequence in L^2. In the subsequent section, we prove properties of minimizers.

2.1. Properties of I_μ.

For the later usage in the proofs of Theorems 1.3 and 1.4, we estimate difference of two energies.

Proposition 2.1. *The estimates*

$$\left|\int_{\mathbb{R}^2_+} G(x,y)\omega(y)dy\right| \le Cx_2^{1/2}\|\omega\|_1^{1/2}\|\omega\|_2^{1/2}, \tag{2.1}$$

$$E[\omega] \le C\|x_2\omega\|_1^{1/2}\|\omega\|_1\|\omega\|_2^{1/2}, \tag{2.2}$$

$$\left|\int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2_+} G(x,y)\omega_1(x)\omega_2(y)dxdy\right| \le C\|\omega_1\|_1^{1/2}\|\omega_1\|_2^{1/2}\|x_2\omega_2\|_1^{1/2}\|\omega_2\|_1^{1/2}, \tag{2.3}$$

$$|E[\omega_1]-E[\omega_2]| \le C\|\omega_1-\omega_2\|_1^{1/2}\|\omega_1-\omega_2\|_2^{1/2}\|x_2(\omega_1+\omega_2)\|_1^{1/2}\|\omega_1+\omega_2\|_1^{1/2}, \tag{2.4}$$

hold for $\omega, \omega_i \in L^2 \cap L^1(\mathbb{R}^2_+)$ satisfying $x_2\omega, x_2\omega_i \in L^1(\mathbb{R}^2_+)$ with some constant C, independent of ω, ω_i, $i = 1, 2$.

Proof. We define ψ_1 in terms of ω_1 by using (1.5). By Hölder's inequality, for $q \in (1,2)$, $1/q = \theta + (1-\theta)/2$,

$$|\psi_1(x)| \le \left(\int_{\mathbb{R}^2_+} G(x,y)^{q'}\mathrm{d}y\right)^{1/q'}\|\omega_1\|_q \le Cx_2^{2/q'}\|\omega_1\|_q \le Cx_2^{1-\theta}\|\omega_1\|_1^{\theta}\|\omega_1\|_2^{1-\theta}.$$

Taking $\theta = 1/2$ implies (2.1) and

$$\begin{aligned}\left|\int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2_+} G(x,y)\omega_1(y)\omega_2(x)\mathrm{d}x\mathrm{d}y\right| = \left|\int_{\mathbb{R}^2_+}\psi_1(x)\omega_2(x)\mathrm{d}x\right| &\le C\|\omega_1\|_1^{1/2}\|\omega_1\|_2^{1/2}\int_{\mathbb{R}^2_+} x_2^{1/2}|\omega_2(x)|\mathrm{d}x \\ &\le C\|\omega_1\|_1^{1/2}\|\omega_1\|_2^{1/2}\|x_2\omega_2\|_1^{1/2}\|\omega_2\|_1^{1/2}.\end{aligned}$$

Thus (2.3) holds. The estimate (2.2) follows from (2.3). We suppress the integral region. Observe that

$$\begin{aligned}2(E[\omega_1]-E[\omega_2]) &= \iint G(x,y)\omega_1(x)\omega_1(y)\mathrm{d}x\mathrm{d}y - \iint G(x,y)\omega_2(x)\omega_2(y)\mathrm{d}x\mathrm{d}y \\ &= \iint G(x,y)\tilde{\omega}(x)\omega_1(y)\mathrm{d}x\mathrm{d}y + \iint G(x,y)\omega_2(x)\tilde{\omega}(y)\mathrm{d}x\mathrm{d}y,\end{aligned}$$

for $\tilde{\omega} = \omega_1 - \omega_2$ and by $G(x,y) = G(y,x)$,

$$\iint G(x,y)\omega_2(x)\tilde{\omega}(y)\mathrm{d}x\mathrm{d}y = \iint G(y,x)\omega_2(y)\tilde{\omega}(x)\mathrm{d}x\mathrm{d}y = \iint G(x,y)\tilde{\omega}(x)\omega_2(y)\mathrm{d}x\mathrm{d}y.$$

We see that

$$2(E[\omega_1]-E[\omega_2]) = \iint G(x,y)\tilde{\omega}(x)\hat{\omega}(y)\mathrm{d}x\mathrm{d}y, \quad \hat{\omega} = \omega_1 + \omega_2.$$

Thus (2.4) follows from (2.3). This completes the proof. □

We show that the Dirichlet integral of the stream function is finite.

Proposition 2.2. *For $\omega \in L^2 \cap L^1(\mathbb{R}^2_+)$ satisfying $x_2\omega \in L^1(\mathbb{R}^2_+)$ and $\omega \geq 0$ ($\omega \not\equiv 0$), the stream function (1.5) satisfies $\psi > 0$ in $\mathbb{R}^2_+$,*

$$\psi(x) \to 0 \quad \text{as } |x| \to \infty, \tag{2.5}$$

$$E[\omega] = \frac{1}{2}\|\nabla\psi\|_2^2. \tag{2.6}$$

Proof. By

$$\psi(x) = \int_{\mathbb{R}^2_+} G(x,y)\omega(y)\mathrm{d}y = \int_{|x-y|\geq x_2/2} G(x,y)\omega(y)\mathrm{d}y + \int_{|x-y|<x_2/2} G(x,y)\omega(y)\mathrm{d}y,$$

and $G(x,y) \leq \pi^{-1}x_2y_2|x-y|^{-2}$,

$$\int_{|x-y|\geq x_2/2} G(x,y)\omega(y)\mathrm{d}y \leq \frac{4}{\pi x_2}\|y_2\omega\|_1.$$

By Hölder's inequality, $1/q + 1/q' = 1$, $1/q = \theta + (1-\theta)/2$,

$$\begin{aligned}\int_{|x-y|<x_2/2} G(x,y)\omega(y)\mathrm{d}y &\leq \left(\int_{|x-y|<x_2/2} G(x,y)^{q'}\mathrm{d}y\right)^{1/q'} \left(\int_{|x-y|<x_2/2} \omega(y)^q\mathrm{d}y\right)^{1/q} \\ &\leq C x_2^{2/q'} \|\omega\|^{\theta}_{L^1(|x-y|<x_2/2)} \|\omega\|^{1-\theta}_{L^2(|x-y|<x_2/2)}.\end{aligned}$$

Since

$$\int_{|x-y|<x_2/2} \omega(y)\mathrm{d}y \leq \frac{2}{x_2}\|y_2\omega\|_1,$$

we have

$$\int_{|x-y|<x_2/2} G(x,y)\omega(y)\mathrm{d}y \leq \frac{C}{x_2^{4/q-3}}\|x_2\omega\|^{\theta}_{L^1}\|\omega\|^{1-\theta}_{L^2\cap L^1}.$$

Hence by (2.1) and for $\delta \in (0,1)$, by taking $q \in (1,2]$ sufficiently small,

$$\psi(x) \leq \frac{C_\delta}{(1+x_2)^{1-\delta}}\left(\|x_2\omega\|_{L^1} + \|\omega\|_{L^2\cap L^1}\right), \quad x \in \mathbb{R}^2_+. \tag{2.7}$$

We take a sequence $\{\omega_n\} \subset C_c^\infty(\mathbb{R}^2_+)$ such that $\omega_n \to \omega$ in $L^2 \cap L^1(\mathbb{R}^2_+)$ and $x_2\omega_n \to x_2\omega$ in $L^1(\mathbb{R}^2_+)$. By (2.7),

$$\psi(x) = \int_{\mathbb{R}^2_+} G(x,y)(\omega(y)-\omega_n(y))\mathrm{d}y + \int_{\mathbb{R}^2_+} G(x,y)\omega_n(y)\mathrm{d}y$$
$$\leq C\left(\|x_2(\omega-\omega_n)\|_{L^1} + \|\omega-\omega_n\|_{L^2\cap L^1}\right) + \frac{x_2}{\pi \inf_{y\in \mathrm{spt}\ \omega_n}|x-y|^2}\|y_2\omega_n\|_{L^1}.$$

Sending $|x|\to\infty$ and then $n\to\infty$ imply (2.5).

We take a non-increasing function $\theta \in C^\infty_c[0,\infty)$ satisfying $\theta = 1$ in $[0,1]$, $\theta = 0$ in $[2,\infty)$ and set the cut-off function by $\theta_R(x) = \theta(|x|/R)$. Since $-\Delta\psi = \omega$ in $\mathbb{R}^2_+$ and $\psi(x_1,0)=0$, by multiplying $\psi\theta_R$ by $-\Delta\psi=\omega$ and integration by parts,

$$\int_{\mathbb{R}^2_+}\left(|\nabla\psi|^2\theta_R - \frac{1}{2}\psi^2\Delta\theta_R\right)\mathrm{d}x = \int_{\mathbb{R}^2_+}\psi\omega\theta_R\mathrm{d}x.$$

Since $\psi\to 0$ as $|x|\to\infty$ by (2.5), the second term vanishes as $R\to\infty$. Hence (2.6) follows from the monotone convergence theorem. □

We prove that the function I_μ is negative and decreasing for $\mu\in(0,\infty)$ by using (2.2).

Lemma 2.3.

(2.8) $$I_0 = 0,$$

(2.9) $$-\infty < I_\mu < 0, \quad 0<\mu<\infty,$$

(2.10) $$I_\mu < I_\alpha, \quad 0<\alpha<\mu.$$

Proof. Since

$$I_\mu = -\sup_{\omega\in K_\mu} E_2[\omega], \qquad E_2[\omega] = E[\omega] - \frac{1}{2}\int_{\mathbb{R}^2_+}\omega^2\mathrm{d}x,$$

we shall show that

(2.11) $$0 < \sup_{\omega\in K_\mu} E_2[\omega] < \infty, \quad 0<\mu<\infty,$$

(2.12) $$\sup_{\omega\in K_\alpha} E_2[\omega] < \sup_{\omega\in K_\mu} E_2[\omega], \quad 0<\alpha<\mu.$$

The property (2.8) is trivial since $K_0 = \{0\}$. By (2.2) and Young's inequality, for arbitrary $\varepsilon>0$ and $\omega\in K_\mu$,

$$E_2[\omega] \leq C\|x_2\omega\|_1^{1/2}\|\omega\|_1\|\omega\|_2^{1/2} - \frac{1}{2}\|\omega\|_2^2$$
$$\leq \frac{3}{4}\left(\frac{C}{\varepsilon^{1/2}}\|x_2\omega\|_1^{1/2}\|\omega\|_1\right)^{4/3} + \left(\frac{\varepsilon^2}{4}-\frac{1}{2}\right)\|\omega\|_2^2.$$

Thus for $\varepsilon \leq \sqrt{2}$,

$$\sup_{\omega \in K_\mu} E_2[\omega] \leq C\mu^{2/3} < \infty.$$

We set $\omega_1 = 1_{B(0,a)\cap\mathbb{R}^2_+}$ for $B(0,a) = \{x \in \mathbb{R}^2 \mid |x| < a\}$ and choose $a > 0$ so that $\int x_2\omega_1 \mathrm{d}x = \mu$. Set $\omega_\sigma(x) = \sigma^3\omega_1(\sigma x)$, $\sigma > 0$, and observe that

$$\begin{aligned}
\int_{\mathbb{R}^2_+} x_2\omega_\sigma \mathrm{d}x &= \int_{\mathbb{R}^2_+} x_2\omega_1 \mathrm{d}x = \mu, \\
\int_{\mathbb{R}^2_+} \omega_\sigma \mathrm{d}x &= \sigma \int_{\mathbb{R}^2_+} \omega_1 \mathrm{d}x, \\
E_2[\omega_\sigma] &= \sigma^2\left(E[\omega_1] - \frac{\sigma^2}{2}\int_{\mathbb{R}^2_+} \omega_1^2 \mathrm{d}x\right).
\end{aligned}$$

Thus for sufficiently small $\sigma > 0$, $\omega_\sigma \in K_\mu$ and

$$\sup_{\omega \in K_\mu} E_2[\omega] \geq E_2[\omega_\sigma] > 0.$$

We proved (2.11).

It remains to show (2.12). For $\omega \in K_\alpha$, $\omega_\tau(x) = \tau^{-2}\omega(\tau^{-1}x)$, $\tau > 1$, satisfies

$$\begin{aligned}
\int_{\mathbb{R}^2_+} x_2\omega_\tau(x) \mathrm{d}x &= \tau \int_{\mathbb{R}^2_+} x_2\omega(x) \mathrm{d}x = \tau\alpha, \\
\int_{\mathbb{R}^2_+} \omega_\tau(x) \mathrm{d}x &= \int_{\mathbb{R}^2_+} \omega(x) \mathrm{d}x \leq 1.
\end{aligned}$$

Hence $\omega_\tau \in K_{\tau\alpha}$ and

$$\sup_{\tilde{\omega} \in K_{\tau\alpha}} E_2[\tilde{\omega}] \geq E_2[\omega_\tau] = E[\omega] - \frac{1}{2\tau^2}\int_{\mathbb{R}^2_+} \omega^2 \mathrm{d}x = E_2[\omega] + \frac{1}{2}\left(1 - \frac{1}{\tau^2}\right)\int_{\mathbb{R}^2_+} \omega^2 \mathrm{d}x > E_2[\omega].$$

By taking a supremum for $\omega \in K_\alpha$,

$$\sup_{\tilde{\omega} \in K_{\tau\alpha}} E_2[\tilde{\omega}] \geq \sup_{\omega \in K_\alpha} E_2[\omega].$$

If $\sup_{\tilde{\omega} \in K_{\tau\alpha}} E_2[\tilde{\omega}] = \sup_{\omega \in K_\alpha} E_2[\omega]$, there exists a maximizing sequence $\{\omega_n\} \subset K_\alpha$ such that $E_2[\omega_n] \to \sup_{\omega \in K_\alpha} E_2[\omega]$ and $\omega_n \to 0$ in L^2. By (2.2), $E_2[\omega_n] \to 0$. This contradicts (2.11). Hence $\sup_{\tilde{\omega} \in K_{\tau\alpha}} E_2[\tilde{\omega}] > \sup_{\omega \in K_\alpha} E_2[\omega]$ and (2.12) holds by taking $\tau = \mu/\alpha$. The proof is complete. $\square$

Remarks 2.4. (i) The strict subadditivity

$$I_\mu < I_\alpha + I_{\mu-\alpha}, \quad 0 < \alpha < \mu,$$

is unknown, cf. Lions [31].
(ii) Any minimizing sequence $\{\omega_n\}$ satisfying $\omega_n \in K_{\mu_n}$, $\mu_n \to \mu$ and $-E_2[\omega_n] \to I_\mu$ is uniformly bounded in L^2. Indeed, by (2.2) and Young's inequality, for arbitrary $\varepsilon > 0$ and $\omega \in K_\mu$,

$$\begin{aligned}\frac{1}{2}\|\omega\|_2^2 + E_2[\omega] = E[\omega] &\leq C\|x_2\omega\|_1^{1/2}\|\omega\|_1\|\omega\|_2^{1/2} \\ &\leq \frac{3}{4}\left(\frac{C}{\varepsilon^{1/2}}\|x_2\omega\|_1^{1/2}\|\omega\|_1\right)^{4/3} + \frac{\varepsilon^2}{4}\|\omega\|_2^2.\end{aligned}$$

By taking $\varepsilon = 1$,

$$\|\omega\|_2^2 \leq C\|x_2\omega\|_1^{2/3}\|\omega\|_1^{4/3} - 4E_2[\omega].$$

Thus by $I_\mu < 0$, the minimizing sequence $\{\omega_n\}$ satisfies $\limsup_{n\to\infty}\|\omega_n\|_2 \leq C\mu^{1/3}$.

2.2. **Properties of minimizers.** We show that minimizers of (1.6) are solutions to (1.4) for some $W > 0$ and $\gamma \geq 0$ with compact support. As noted below in Remarks 2.6 (iii), the flux constant γ vanishes if μ is sufficiently small.

Proposition 2.5. *Each minimizer $\omega \in S_\mu$ satisfies*

$$\begin{aligned}&\omega = f(\psi - Wx_2 - \gamma), \\ &\psi(x) = \int_{\mathbb{R}^2_+} G(x,y)\omega(y)dy,\end{aligned} \tag{2.13}$$

for some constants $W, \gamma \geq 0$, uniquely determined by ω.

Proof. The proof follows from a standard argument, e.g., [23], [22] for vortex rings. We set the space $K = \{\omega \in L^2 \cap L^1(\mathbb{R}^2_+) \mid x_2\omega \in L^1(\mathbb{R}^2_+)\}$ equipped with the norm $\|\omega\|_K = \|\omega\|_{L^2\cap L^1} + \|x_2\omega\|_{L^1}$. By (2.2) and (2.4), the functional $E_2 : K \to \mathbb{R}$ is continuous. For ψ defined in terms of $\omega \in K$ by using $(2.13)_2$, the estimate (2.3) implies that

$$\left|\int_{\mathbb{R}^2_+}(\psi - \omega)\eta \mathrm{d}x\right| \leq C\|\omega\|_K\|\eta\|_K$$

for all $\eta \in K$. The functional E_2 has a Gateaux derivative E_2' at any point $\omega \in K$ and

$$E_2'(\omega)\eta = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}E_2(\omega + \varepsilon\eta)\bigg|_{\varepsilon=0} = \int_{\mathbb{R}^2_+} (\psi - \omega)\,\eta \mathrm{d}x.$$

The functional $E_2'(\omega)$ is a bounded linear operator for each $\omega \in K$ and $E_2' : K \to K^*$ is continuous. Thus the Fréchet derivative E_2' exists and is continuous on K.

We take an arbitrary minimizer $\omega \in S_\mu$. Since $I_\mu < 0$ by (2.9), the minimizer ω is nontrivial. We take a constant $\delta_0 > 0$ such that $|\{x \in \mathbb{R}^2_+ \mid \omega \geq \delta_0\}| > 0$. Here $|E|$ denotes the Lebesgue measure of a set $E \subset \mathbb{R}^2_+$. We take compactly supported $h_1, h_2 \in L^\infty(\mathbb{R}^2_+)$ such that $\operatorname{spt} h_i \subset \{\omega \geq \delta_0\}$, $i = 1, 2$,

$$\int_{\mathbb{R}^2_+} h_1(x)\mathrm{d}x = 1, \quad \int_{\mathbb{R}^2_+} x_2 h_1(x)\mathrm{d}x = 0,$$
$$\int_{\mathbb{R}^2_+} h_2(x)\mathrm{d}x = 0, \quad \int_{\mathbb{R}^2_+} x_2 h_2(x)\mathrm{d}x = 1.$$

We take an arbitrary $\delta \in (0, \delta_0)$ and compactly supported $h \in L^\infty(\mathbb{R}^2_+)$ such that $h \geq 0$ on $\{0 \leq \omega \leq \delta\}$. We set

$$\eta = h - \left(\int_{\mathbb{R}^2_+} h\mathrm{d}x\right)h_1 - \left(\int_{\mathbb{R}^2_+} x_2 h\mathrm{d}x\right)h_2$$

so that $\int \eta \mathrm{d}x = 0$ and $\int x_2\eta \mathrm{d}x = 0$. Observe that $\omega + \varepsilon\eta \geq \delta - \varepsilon\|\eta\|_\infty \geq 0$ on $\{\omega \geq \delta\}$ for small $\varepsilon > 0$. Since $\eta = h \geq 0$ on $\{0 \leq \omega \leq \delta\}$, $\omega + \varepsilon\eta \geq 0$ on $\{0 \leq \omega \leq \delta\}$. Hence $\omega + \varepsilon\eta \in K_\mu$. Since ω is a minimizer of (1.6),

$$E_2'(\omega)\eta \leq 0. \tag{2.14}$$

By the definition of η,

$$E_2'(\omega)\eta = E_2'(\omega)h - E_2'(\omega)h_1\left(\int_{\mathbb{R}^2_+} h\mathrm{d}x\right) - E_2'(\omega)h_2\left(\int_{\mathbb{R}^2_+} x_2 h\mathrm{d}x\right).$$

By setting $\gamma = E_2'(\omega)h_1$ and $W = E_2'(\omega)h_2$,

$$0 \geq E_2'(\omega)h - \gamma\left(\int_{\mathbb{R}^2_+} h\mathrm{d}x\right) - W\left(\int_{\mathbb{R}^2_+} x_2 h\mathrm{d}x\right) = \int_{\mathbb{R}^2_+} (\psi - Wx_2 - \gamma - \omega)\,h\mathrm{d}x = \int_{0\leq\omega\leq\delta} + \int_{\omega>\delta}.$$

We set $\Psi = \psi - Wx_2 - \gamma$. Since h is an arbitrary function satisfying $h \geq 0$ on $\{0 \leq \omega \leq \delta\}$,

$$\begin{aligned} \Psi - \omega &= 0 \quad \text{on } \{\omega > \delta\}, \\ \Psi - \omega &\leq 0 \quad \text{on } \{0 \leq \omega \leq \delta\}. \end{aligned} \tag{2.15}$$

Since $\delta > 0$ is arbitrary, sending $\delta \to 0$ implies

$$\begin{aligned} \Psi - \omega &= 0 \quad \text{on } \{\omega > 0\}, \\ \Psi &\leq 0 \quad \text{on } \{\omega = 0\}. \end{aligned} \tag{2.16}$$

If $\Psi > 0$, $\omega = \Psi$. If $\Psi \leq 0$, $\omega = 0$. Thus $\omega = \Psi_+$ and (2.13) holds.

We take a sequence $\{x_n\}$, $x_n = {}^t(x_{1,n}, x_{2,n})$, such that $\omega(x_n) \to 0$ and $x_{n,1} \to \infty$, $x_{n,2} \to 0$. By (2.16),

$$\limsup_{n\to\infty} (\psi(x_n) - W x_{n,2} - \gamma) \leq 0.$$

Hence $\gamma \geq 0$. By taking an another sequence $\{x_n\}$ such that $\omega(x_n) \to 0$ and $x_{n,1} \to 0$, $x_{n,2} \to \infty$, $W \geq 0$ follows.

We show uniqueness of W, γ. Suppose that ω satisfies (2.13) for $W_*, \gamma_* \geq 0$. Then, $\Psi = \psi - W_* x_2 - \gamma_*$ satisfies (2.15) for $\delta \in (0, \delta_0)$. Hence,

$$\begin{aligned} 0 \geq \int_{\mathbb{R}^2_+} (\Psi - \omega)\, h \mathrm{d}x &= \int_{\mathbb{R}^2_+} (\psi - \omega - \gamma_* - W_* x_2)\, h \mathrm{d}x \\ &= E_2'(\omega)h - \gamma_* \left(\int_{\mathbb{R}^2_+} h \mathrm{d}x \right) - W_* \left(\int_{\mathbb{R}^2_+} x_2 h \mathrm{d}x \right), \end{aligned}$$

for compactly supported $h \in L^\infty(\mathbb{R}^2_+)$ satisfying $h \geq 0$ on $\{0 \leq \omega \leq \delta\}$. By taking $h = \pm h_1, \pm h_2$, $E_2'(\omega)h_1 = \gamma_*$, $E_2'(\omega)h_2 = W_*$ follow. The proof is complete. □

Remarks 2.6. (i) The constant W is positive by the identity [46, p.1062],

$$W = \left(\frac{1}{2\pi} \int_{\mathbb{R}^2_+} \int_{\mathbb{R}^2_+} \frac{x_2 + y_2}{|x - y^*|^2} \omega(x)\omega(y) \mathrm{d}x \mathrm{d}y \right) \left(\int_{\mathbb{R}^2_+} \omega(x) \mathrm{d}x \right)^{-1}, \ y^* = {}^t(y_1, -y_2), \tag{2.17}$$

for minimizers $\omega \in S_\mu$. The identity (2.17) follows by multiplying $\partial_{x_2}\Psi = \partial_{x_2}\psi - W$ by ω and integration by parts.

(ii) Every minimizer $\omega \in S_\mu$ for $\gamma > 0$ satisfies

$$\int_{\mathbb{R}^2_+} \omega \mathrm{d}x = 1.$$

Indeed, suppose that $\int \omega \mathrm{d}x < 1$. We set

$$\eta = h - \left(\int_{\mathbb{R}^2_+} x_2 h \mathrm{d}x \right) h_2,$$

by h and h_2 as in the proof of Proposition 2.5 and observe that $\int x_2\eta \mathrm{d}x = 0$. Then, by $\int \omega \mathrm{d}x < 1$,

$$\int_{\mathbb{R}^2_+} x_2(\omega + \varepsilon\eta)\mathrm{d}x = \int_{\mathbb{R}^2_+} x_2\omega \mathrm{d}x = \mu,$$
$$\int_{\mathbb{R}^2_+} (\omega + \varepsilon\eta)\mathrm{d}x \leq 1,$$

for small $\varepsilon > 0$. Thus $\omega + \varepsilon\eta \in K_\mu$. By minimality of ω, (2.14) holds for ω and η and we have

$$\begin{aligned} \psi - Wx_2 - \omega = 0 \quad &\text{on } \{\omega > 0\}, \\ \psi - Wx_2 \leq 0 \quad &\text{on } \{\omega = 0\}. \end{aligned}$$

This implies $(2.13)_1$ for $\gamma = 0$, a contradiction to $\gamma > 0$.
(iii) There exists a constant $M_1 > 0$ such that if $0 < \mu \leq M_1$, then every minimizer $\omega \in S_\mu$ satisfies

$$\int_{\mathbb{R}^2_+} \omega \mathrm{d}x < 1.$$

In particular, $\gamma = 0$ by (ii). Indeed, suppose that $\int \omega \mathrm{d}x = 1$. By $\mu = \int_{\mathbb{R}^2_+} x_2\omega \mathrm{d}x \geq 2\mu \int_{x_2 \geq 2\mu} \omega \mathrm{d}x$,

$$\int_{0<x_2<2\mu} \omega \mathrm{d}x = 1 - \int_{x_2 \geq 2\mu} \omega \mathrm{d}x \geq \frac{1}{2}.$$

Observe that by $\omega = \Psi_+ \leq \psi$,

$$\begin{aligned} \int_{0<x_2<2\mu} \omega \mathrm{d}x &\leq \int_{0<x_2<2\mu} \mathrm{d}x \int_{\mathbb{R}^2_+} G(x,y)\omega(y)\mathrm{d}y \\ &= \int_{0<y_2<2\mu} \mathrm{d}y \int_{\mathbb{R}^2_+} G(y,x)\omega(x)\mathrm{d}x \\ &= \int_{\mathbb{R}^2_+} \omega(x)\mathrm{d}x \int_{0<y_2<2\mu} G(x,y)\mathrm{d}y = \int_{0<x_2<4\mu}\int_{0<y_2<2\mu} + \int_{x_2 \geq 4\mu}\int_{0<y_2<2\mu}. \end{aligned}$$

For $0 < x_2 < 4\mu$, we have

$$\int_{0<y_2<2\mu} G(x,y)\mathrm{d}y \leq C\mu^2.$$

In fact, by

$$\int_{0<y_2<4\mu} G(x,y)\mathrm{d}y = \int_{\substack{0<y_2<4\mu,\\ |x-y|<x_2/2}} + \int_{\substack{0<y_2<4\mu,\\ |x-y|\geq x_2/2}} .$$

we estimate

$$\int_{\substack{0<y_2<4\mu,\\ |x-y|<x_2/2}} G(x,y)\mathrm{d}y \leq \frac{1}{4\pi}\int_{|x-y|<x_2/2} \log\left(1+\frac{4x_2y_2}{|x-y|^2}\right)\mathrm{d}y = \frac{x_2^2}{4\pi}\int_{|z|<1/2} \log\left(1+\frac{4(1-z_2)}{|z|^2}\right)\mathrm{d}z$$
$$\leq C\mu^2.$$

For $|x-y| \geq x_2/2$, the triangle inequality yields $|x-y^*| \leq 5|x-y|$ for $y^* = {}^t(y_1, -y_2)$. By $G(x,y) \leq \pi^{-1}x_2y_2|x-y|^{-2}$,

$$\int_{\substack{0<y_2<4\mu,\\ |x-y|\geq x_2/2}} G(x,y)\mathrm{d}y \leq \frac{1}{\pi}\int_{\substack{0<y_2<4\mu,\\ |x-y|\geq x_2/2}} \frac{x_2y_2}{|x-y|^2}\mathrm{d}y \leq \frac{25}{\pi}\int_{\substack{0<y_2<4\mu,\\ |x-y|\geq x_2/2}} \frac{x_2y_2}{|x-y^*|^2}\mathrm{d}y \leq C\mu^2.$$

Hence we have the desired estimate.

For $x_2 \geq 4\mu$, by $x_2 - y_2 \geq x_2/2$,

$$\int_{0<y_2<2\mu} G(x,y)\mathrm{d}y \leq \frac{x_2}{\pi}\int_{0<y_2<2\mu} \frac{y_2}{|x-y|^2}\mathrm{d}y \leq C\mu^2.$$

Hence $1/2 \leq \int_{0<x_2<2\mu} \omega \mathrm{d}x \leq C\mu^2 \to 0$ as $\mu \to 0$, a contradiction.

The positivity of $W > 0$ will be shown to imply compactness of support for minimizers. We denote by $BUC(\overline{\mathbb{R}^2_+})$ the space of all bounded uniformly continuous functions in $\overline{\mathbb{R}^2_+}$ and by $C^\alpha(\overline{\mathbb{R}^2_+})$ the space of all Hölder continuous functions of exponent $0 < \alpha < 1$ in $\overline{\mathbb{R}^2_+}$. For an integer $k \geq 0$, $BUC^{k+\alpha}(\overline{\mathbb{R}^2_+})$ denotes the space of all $\psi \in BUC(\overline{\mathbb{R}^2_+})$ such that $\partial_x^l \psi \in BUC(\overline{\mathbb{R}^2_+}) \cap C^\alpha(\overline{\mathbb{R}^2_+})$, for $|l| \leq k$.

Proposition 2.7. *For $\omega \in S_\mu$, the stream function $(2.13)_2$ satisfies $\psi \in BUC^{2+\alpha}(\overline{\mathbb{R}^2_+})$, $0 < \alpha < 1$, $\psi/x_2 \in BUC^{1+\alpha}(\overline{\mathbb{R}^2_+})$ and*

$$\frac{\psi(x)}{x_2} \to 0 \quad \text{as } |x| \to \infty. \tag{2.18}$$

Proof. Since $\omega \in L^1 \cap L^2$, the representation $(2.13)_2$ implies $\nabla^2\psi \in L^q, q \in (1,2)$ and $\nabla\psi \in L^p$, $1/p = 1/q - 1/2$. By $(2.13)_1$ and (2.5), ψ satisfies

$$
\begin{aligned}
-\Delta\psi(x) &= f(\psi - Wx_2 - \gamma) \quad \text{in } \mathbb{R}^2_+, \\
\psi &= 0 \quad \text{on } \partial\mathbb{R}^2_+, \\
\psi &\to 0 \quad \text{as } |x| \to \infty.
\end{aligned}
\tag{2.19}
$$

By the Lipschitz continuity of f, $\partial_x^l\psi \in L^p_{\mathrm{ul}}(\overline{\mathbb{R}^2_+})$, $|l| = 3$. Here, $L^p_{\mathrm{ul}}(\overline{\mathbb{R}^2_+})$ denotes the uniformly local L^p-space in $\overline{\mathbb{R}^2_+}$. Hence $\psi \in BUC^{2+\alpha}(\overline{\mathbb{R}^2_+})$ by Sobolev embedding. Since $\psi(x_1, 0) = 0$ and

$$
\frac{\psi(x_1,x_2)}{x_2} = \int_0^1 (\partial_2\psi)(x_1, x_2 s)\mathrm{d}s,
$$

$\psi/x_2 \in BUC^{1+\alpha}(\overline{\mathbb{R}^2_+})$ follows. By (2.6) and Hardy's inequality [38, 2.7.1],

$$
\left\| \frac{\psi}{x_2} \right\|_2 \leq 2\, \|\nabla\psi\|_2\,,
$$

$\psi/x_2 \in BUC(\overline{\mathbb{R}^2_+}) \cap L^2(\mathbb{R}^2_+)$ and (2.18) follows. □

Lemma 2.8. *The support of $\omega \in S_\mu$ is compact in $\overline{\mathbb{R}^2_+}$.*

Proof. Since $\mathrm{spt}\,\omega = \overline{\{x \in \mathbb{R}^2_+ \mid \psi(x) - Wx_2 - \gamma > 0\}}$ for $W > 0$ and $\gamma \geq 0$ by $(2.13)_1$ and (2.16),

$$
Wx_2 \leq \psi(x), \qquad x \in \mathrm{spt}\,\omega.
$$

Since $\psi/x_2 \to 0$ as $|x| \to \infty$ by (2.18), the assertion follows. □

To prove Theorem 1.5 later in Section 6, we state properties of the associated stream function.

Lemma 2.9. *For $\omega \in S_\mu$, the stream function $\psi \in BUC^{2+\alpha}(\overline{\mathbb{R}^2_+})$, $0 < \alpha < 1$, is a positive solution of (2.19) satisfying $\psi/x_2 \in BUC^{1+\alpha}(\overline{\mathbb{R}^2_+})$, (2.18) and for*

$$
\Omega = \left\{ x \in \mathbb{R}^2_+ \;\middle|\; \psi(x) - Wx_2 - \gamma > 0 \right\},
$$

$\overline{\Omega}$ is compact in $\overline{\mathbb{R}^2_+}$. If $0 < \mu \leq M_1$, then $\gamma = 0$, where M_1 is the constant as in Remarks 2.6 (iii).

Proof. The assertion follows from Propositions 2.2, 2.7, Lemma 2.8 and Remarks 2.6 (iii). □

3. Existence of minimizers

We show that if the minimizing sequence $\{\omega_n\}$ satisfies (1.7), then the kinetic energy $E[\omega_n]$ is concentrated on a bounded domain $Q = \{x \in \mathbb{R}^2_+ \mid |x_1| < AR,\ x_2 < R\}$ and the weak convergence of the sequence $\{\omega_n\}$ in L^2 implies the convergence of the energy $E[\omega_n]$. Once we have the convergence of the energy, the existence of minimizers easily follows.

Proposition 3.1 (Steiner symmetrization)**.** *For $\omega \geq 0$ satisfying $\omega \in L^2 \cap L^1(\mathbb{R}^2_+)$ and $x_2\omega \in L^1(\mathbb{R}^2_+)$, there exists $\omega^* \geq 0$ such that*

$$\begin{aligned} &\omega^*(x_1, x_2) = \omega^*(-x_1, x_2), \\ &\omega^*(x_1, x_2) \text{ is non-increasing for } x_1 > 0. \end{aligned} \tag{3.1}$$

Moreover,

$$\begin{aligned} &\|\omega^*\|_q = \|\omega\|_q \quad 1 \leq q \leq 2, \\ &\|x_2\omega^*\|_1 = \|x_2\omega\|_1, \\ &E(\omega^*) \geq E(\omega). \end{aligned}$$

Proof. See [21, Appendix I], [46, p.1053]. □

For the later usage in the proof of Theorem 1.3, we state a result for general $0 < \mu, \nu < \infty$ with $\lambda = 1$. We first find a minimizer of $-E_2$ in a slightly larger space $\tilde{K}_{\mu,\nu} \supset K_{\mu,\nu}$ and then prove that the impulse of this minimizer is exactly $\mu > 0$. The goal of this section is to prove:

Lemma 3.2. *For $0 < \mu, \nu < \infty$, set*

$$\tilde{K}_{\mu,\nu} = \left\{ \omega \in L^2(\mathbb{R}^2_+) \;\middle|\; \omega \geq 0,\ \int_{\mathbb{R}^2_+} x_2\omega dx \leq \mu,\ \int_{\mathbb{R}^2_+} \omega dx \leq \nu \right\}.$$

(i) There exists $\omega \in \tilde{K}_{\mu,\nu}$ such that

$$E_2[\omega] = \sup_{\tilde{\omega} \in \tilde{K}_{\mu,\nu}} E_2[\tilde{\omega}].$$

(ii) This maximizer $\omega \in \tilde{K}_{\mu,\nu}$ satisfies (1.7),

$$\int_{\mathbb{R}^2_+} x_2\omega dx = \mu,$$

and is with compact support in $\overline{\mathbb{R}^2_+}$.

The proof of Lemma 3.2 is parallel to the case for vortex rings [23], [22] and is given later. We first use the monotonicity $(1.7)_2$ and deduce a decay estimate for the stream function for the x_1-variable.

Proposition 3.3. *Let $A \geq 1$. Let ψ be the stream function (1.5) for $\omega \in L^2 \cap L^1(\mathbb{R}^2_+)$ satisfying $x_2\omega \in L^1(\mathbb{R}^2_+)$ and $\omega \geq 0$. Assume that (1.7) holds for ω. Then,*

$$\psi(x) \leq C\left(\left(\frac{x_2}{A}\right)^{1/2}\|\omega\|_1^{1/2}\|\omega\|_2^{1/2} + \frac{1}{A}\|\omega\|_1 + x_2\left(\frac{A}{x_1}\right)^2\|x_2\omega\|_1\right), \quad x_2 \leq \frac{|x_1|}{A}. \tag{3.2}$$

The constant C is independent of ω and A.

Proof. By replacing A with $A/2$, we prove (3.2) for $x_2 \leq 2|x_1|/A$ and $A \geq 2$. We may assume that $x_1 > 0$. Observe that for a non-increasing function $g(t) \geq 0$ for $t > 0$,

$$\int_{t-t/A}^{t+t/A} g(s)\mathrm{d}s \leq \frac{4}{A}\|g\|_{L^1(0,\infty)} \quad t > 0,\ A \geq 2,$$

by $tg(t) \leq \|g\|_1$, $t > 0$. Applying this to ω implies

$$\int_{|x_1-y_1|<x_1/A} \omega(y)\mathrm{d}y \leq \frac{4}{A}\|\omega\|_1.$$

We set

$$\psi(x) = \int_{|x-y|<x_2/2} G(x,y)\mathrm{d}y + \int_{|x-y|\geq x_2/2} G(x,y)\mathrm{d}y =: \psi_1 + \psi_2.$$

The conditions $x_2 \leq 2x_1/A$ and $|x-y| < x_2/2$ imply $|x_1 - y_1| < x_1/A$. By Hölder's inequality for $1/q = \theta + (1-\theta)/2$, $1/q + 1/q' = 1$,

$$\begin{aligned}\psi_1(x) = \int_{\substack{|x-y|<x_2/2,\\ |x_1-y_1|<x_1/A}} G(x,y)\omega(y)\mathrm{d}y &\leq \left(\int_{\mathbb{R}^2_+} G(x,y)^{q'}\mathrm{d}y\right)^{1/q'}\left(\int_{|x_1-y_1|<x_1/A}\omega^q(y)\mathrm{d}y\right)^{1/q} \\ &\leq Cx_2^{2/q'}\|\omega\|^{\theta}_{L^1(|x_1-y_1|<x_1/A)}\|\omega\|^{1-\theta}_{L^2(|x_1-y_1|<x_1/A)}.\end{aligned}$$

Taking $\theta = 1/2$ yields $\psi_1(x) \leq C(x_2/A)^{1/2}\|\omega\|_1^{1/2}\|\omega\|_2^{1/2}$. We set

$$\psi_2(x) = \int_{\substack{|x-y|\geq x_2/2,\\ |x_1-y_1|<x_1/A}} G(x,y)\mathrm{d}y + \int_{\substack{|x-y|\geq x_2/2,\\ |x_1-y_1|\geq x_1/A}} G(x,y)\mathrm{d}y =: \psi_2^1 + \psi_2^2.$$

By $G(x,y) \leq \pi^{-1} x_2 y_2 |x-y|^{-2}$,

$$\psi_2^1(x) \leq \frac{1}{\pi} \int_{\substack{|x-y|\geq x_2/2,\\ |x_1-y_1|<x_1/A}} \frac{x_2 y_2}{|x-y|^2}\omega(y)\mathrm{d}y \leq \frac{6}{\pi}\int_{|x_1-y_1|<x_1/A} \omega(y)\mathrm{d}y \leq \frac{24}{\pi A}\|\omega\|_1,$$

$$\psi_2^2(x) \leq \frac{1}{\pi} \int_{\substack{|x-y|\geq x_2/2,\\ |x_1-y_1|\geq x_1/A}} \frac{x_2 y_2}{|x-y|^2}\omega(y)\mathrm{d}y \leq \frac{x_2}{\pi}\left(\frac{A}{x_1}\right)^2 \|y_2\omega\|_1.$$

We have obtained (3.2). □

The stream function estimate (3.2) will now be shown to imply that the kinetic energy $E[\omega]$ is concentrated on a bounded domain $Q = \{x \in \mathbb{R}^2_+ \mid |x_1| < AR,\ x_2 < R\}$.

Proposition 3.4. *Under the assumption of Proposition 3.3,*

$$\int_{\mathbb{R}^2_+\setminus Q} \psi(x)\omega(x)dx \leq \frac{C}{\min\{A,R\}^{1/2}}\left(\|\omega\|^2_{L^1\cap L^2} + \|x_2\omega\|^2_{L^1}\right). \tag{3.3}$$

The constant C is independent of ω and $A, R \geq 1$.

Proof. We decompose

$$\int_{\mathbb{R}^2_+\setminus Q} \psi(x)\omega(x)\mathrm{d}x = \int_{x_2\geq R} \psi(x)\omega(x)\mathrm{d}x + \int_{\substack{x_2<R,\\ |x_1|\geq AR}} \psi(x)\omega(x)\mathrm{d}x,$$

and estimate by (2.1)

$$\int_{x_2\geq R} \psi(x)\omega(x)\mathrm{d}x \leq C\|\omega\|_{L^1}^{1/2}\|\omega\|_{L^2}^{1/2} \int_{x_2\geq R} x_2^{1/2}\omega\mathrm{d}x \leq \frac{C}{R^{1/2}}\|\omega\|_{L^1\cap L^2}\|x_2\omega\|_{L^1}.$$

Since $|x_1| \geq AR$ and $x_2 < R$ imply $x_2 \leq x_1/A$, applying (3.2) yields

$$\begin{aligned}\int_{\substack{x_2<R,\\ |x_1|\geq AR}} \psi(x)\omega(x)\mathrm{d}x &\leq C \int_{\substack{x_2<R,\\ |x_1|\geq AR}} \left(\left(\frac{x_2}{A}\right)^{1/2}\|\omega\|_{L^2\cap L^1} + \frac{1}{A}\|\omega\|_{L^1} + x_2\frac{1}{R^2}\|x_2\omega\|_{L^1}\right)\omega(x)\mathrm{d}x \\ &\leq \frac{C}{\min\{A,R\}^{1/2}}\left(\|\omega\|^{3/2}_{L^1\cap L^2}\|x_2\omega\|^{1/2}_{L^1} + \|\omega\|^2_{L^1\cap L^2} + \|x_2\omega\|^2_{L^1}\right).\end{aligned}$$

By Young's inequality, (3.3) follows. ◻

Proposition 3.4 implies that the kinetic energy $E[\omega]$ is continuous by the weak continuity in a certain proper subset of L^2, as we now show.

Lemma 3.5. *Let $\{\omega_n\}$ be a sequence such that*

$$\sup_{n\geq 1}\{\|\omega_n\|_{L^2\cap L^1}+\|x_2\omega_n\|_{L^1}\}<\infty,$$
$$\omega_n \rightharpoonup \omega \quad \text{in } L^2(\mathbb{R}^2_+) \quad \text{as } n\to\infty.$$

Assume that each ω_n satisfies (1.7). Then,

$$E[\omega_n]\to E[\omega] \quad \text{as } n\to\infty.$$

Proof. We decompose the energy into two terms

$$2E[\omega_n]=\int_{\mathbb{R}^2_+}\psi_n(x)\omega_n(x)\mathrm{d}x=\int_Q\psi_n(x)\omega_n(x)\mathrm{d}x+\int_{\mathbb{R}^2_+\backslash Q}\psi_n(x)\omega_n(x)\mathrm{d}x,$$

and observe that

$$\begin{aligned}\int_Q\psi_n(x)\omega_n(x)\mathrm{d}x&=\int_Q\omega_n(x)\mathrm{d}x\int_{\mathbb{R}^2_+}G(x,y)\omega_n(y)\mathrm{d}y\\&=\int_Q\omega_n(x)\mathrm{d}x\int_QG(x,y)\omega_n(y)\mathrm{d}y+\int_Q\omega_n(x)\mathrm{d}x\int_{\mathbb{R}^2_+\backslash Q}G(x,y)\omega_n(y)\mathrm{d}y.\end{aligned}$$

By $G(x,y)=G(y,x)$,

$$\int_Q\omega_n(x)\mathrm{d}x\int_{\mathbb{R}^2_+\backslash Q}G(x,y)\omega_n(y)\mathrm{d}y=\int_Q\omega_n(y)\mathrm{d}y\int_{\mathbb{R}^2_+\backslash Q}G(x,y)\omega_n(x)\mathrm{d}x\leq\int_{\mathbb{R}^2_+\backslash Q}\psi_n(x)\omega_n(x)\mathrm{d}x.$$

Applying (3.3) yields

$$\left|2E[\omega_n]-\int_Q\int_QG(x,y)\omega_n(x)\omega_n(y)\mathrm{d}x\mathrm{d}y\right|\leq 2\int_{\mathbb{R}^2_+\backslash Q}\psi_n(x)\omega_n(x)\mathrm{d}x\leq\frac{C}{\min\{A,R\}^{1/2}}.$$

By estimating $E[\omega]$ in the same way,

$$2\left|E[\omega_n]-E[\omega]\right|\leq\left|\int_Q\int_QG(x,y)\left(\omega(x)\omega(y)-\omega_n(x)\omega_n(y)\right)\mathrm{d}x\mathrm{d}y\right|+\frac{C}{\min\{A,R\}^{1/2}}$$

Since $G(x,y) \in L^2(Q\times Q)$ and $\omega_n(x)\omega_n(y) \rightharpoonup \omega(x)\omega(y)$ in $L^2(Q\times Q)$, sending $n\to\infty$ and $A,R\to\infty$ imply the desired result. □

Proof of Lemma 3.2. By the scaling (1.12), we reduce to the case $0<\mu<\infty$, $\nu=1$ with an abbreviated notation $\tilde{K}_{\mu,1}=\tilde{K}_{\mu}$. Let $\{\omega_n\}\subset\tilde{K}_{\mu}$ be a maximizing sequence of E_2. By Steiner symmetrization, we may assume that ω_n satisfies (1.7). Since $\{\omega_n\}$ is uniformly bounded in L^2 as we proved in Remarks 2.4 (ii), by choosing a subsequence (still denoted by $\{\omega_n\}$), there exists $\omega\in L^2$ such that $\omega_n \rightharpoonup \omega$ in L^2 and $\|\omega\|_2 \leq \liminf_{n\to\infty}\|\omega_n\|_2$. The limit ω belongs to $\tilde{K}_{\mu}$ and satisfies (1.7). Since $\{\omega_n\}$ satisfies the assumption of Lemma 3.5,

$$\sup_{\tilde{\omega}\in\tilde{K}_{\mu}} E_2[\tilde{\omega}] = \lim_{n\to\infty} E_2[\omega_n] = \lim_{n\to\infty} E[\omega_n] - \frac{1}{2}\liminf_{n\to\infty}\|\omega_n\|_2^2 \leq E[\omega] - \frac{1}{2}\|\omega\|_2^2 = E_2[\omega].$$

Thus $\omega\in\tilde{K}_{\mu}$ is a maximizer. We proved (i).

Since $\sup_{\omega\in\tilde{K}_{\mu}} E_2[\omega]>0$ as we proved in (2.9), the maximizer ω is a non-trivial function and satisfies (2.13) for some constants $W,\gamma\geq 0$ as in Proposition 2.5. By the identity (2.17), we have $W>0$. It remains to show

$$\int_{\mathbb{R}^2_+} x_2\omega \mathrm{d}x = \mu.$$

Suppose that $\int x_2\omega \mathrm{d}x<\mu$. We set

$$\eta = h - \left(\int_{\mathbb{R}^2_+} h\mathrm{d}x\right) h_1,$$

by h and h_1 as in the proof of Proposition 2.5 so that $\int \eta \mathrm{d}x=0$. Then by $\int x_2\omega\mathrm{d}x<\mu$,

$$\begin{aligned}\int_{\mathbb{R}^2_+}(\omega+\varepsilon\eta)\mathrm{d}x &= \int_{\mathbb{R}^2_+}\omega\mathrm{d}x \leq 1,\\ \int_{\mathbb{R}^2_+} x_2(\omega+\varepsilon\eta) &\leq \mu,\end{aligned}$$

for small $\varepsilon>0$. Thus $\omega+\varepsilon\eta\in\tilde{K}_{\mu}$. By the maximality of $\omega\in\tilde{K}_{\mu}$, (2.14) holds for ω and η and for $\gamma=E_2'[\omega]h_1$,

$$0\geq E_2'[\omega]\eta = E_2'[\omega]h - E_2'[\omega]h_1\int_{\mathbb{R}^2_+} h\mathrm{d}x = \int_{\mathbb{R}^2_+}(\psi-\gamma-\omega)h\mathrm{d}x.$$

In the same way as in the proof of Proposition 2.5, this implies that

$$\begin{aligned}\psi-\gamma-\omega &= 0, \quad \text{on } \{\omega>0\},\\ \psi-\gamma &\leq 0, \quad \text{on } \{\omega=0\}.\end{aligned}$$

Thus (2.13) holds for $W = 0$. Thanks to the uniqueness of W by Proposition 2.5, this yields a contradiction to $W > 0$. The compactness of spt ω follows from Lemma 2.8. We have proved (ii). □

Remark 3.6. It is observed from the proof of Lemma 3.2 that after taking Steiner symmetrization, $\{\omega_n\}$ satisfies $\lim_{n\to\infty} \|\omega_n\|_2 = \|\omega\|_2$ and hence $\omega_n \to \omega$ in L^2. We will see in the next section that any maximizing sequence is relatively compact in L^2 by translation for the x_1-variable without the condition (1.7).

4. Concentrated compactness

We prove Theorem 1.3. For a minimizing sequence of (1.6) which does not satisfy the symmetric and non-increasing condition (1.7), Lemma 3.5 cannot be directly applied to prove compactness of the sequence. Instead, we apply a concentration compactness principle to get compactness of the minimizing sequence up to translation for the x_1-variable. The main difficulty appears when we need to exclude the possibility of dichotomy of the sequence since the strict subadditivity of I_μ is unknown as in Remarks 2.4 (i). To overcome this difficulty, we use the idea from Steiner symmetrization and reduce the problem to the compactness of a symmetric and non-increasing sequence (Lemma 3.5) and the existence of minimizers of (1.6) (Lemma 3.2).

As used in [31], [13], the concentration-compactness lemma is available even if the mass is not exactly the same. See also [12, Lemma 1].

Lemma 4.1. *Let $0 < \mu < \infty$. Let $\{\rho_n\} \subset L^1(\mathbb{R}^2_+)$ satisfy*

$$\rho_n \geq 0 \quad n \geq 1, \qquad \int_{\mathbb{R}^2_+} \rho_n dx = \mu_n \to \mu \quad \text{as } n \to \infty.$$

There exists a subsequence $\{\rho_{n_k}\}$ satisfying the one of the following:
(i) (Compactness) There exists a sequence $\{y_k\} \subset \overline{\mathbb{R}^2_+}$ such that $\rho_{n_k}(\cdot + y_k)$ is tight, i.e., for arbitrary $\varepsilon > 0$ there exists $R > 0$ such that

$$\liminf_{k\to\infty} \int_{B(y_k,R)\cap\mathbb{R}^2_+} \rho_{n_k} dx \geq \mu - \varepsilon. \tag{4.1}$$

(ii) (Vanishing) For each $R > 0$,

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}^2_+} \int_{B(y,R)\cap\mathbb{R}^2_+} \rho_{n_k} dx = 0. \tag{4.2}$$

(iii) (Dichotomy) There exists $\alpha \in (0, \mu)$ such that for arbitrary $\varepsilon > 0$ there exist $k_0 \geq 1$ and $\{\rho_k^1\}$, $\{\rho_k^2\} \subset L^1(\mathbb{R}^2_+)$ such that spt $\rho_k^1 \cap$ spt $\rho_k^2 = \emptyset$, $0 \leq \rho_k^i \leq \rho_{n_k}$, i=1,2,

$$\begin{aligned}(4.3)\qquad &\limsup_{k\to\infty}\left\{\|\rho_{n_k}-\rho_k^1-\rho_k^2\|_{L^1}+\left|\int_{\mathbb{R}^2_+}\rho_k^1dx-\alpha\right|+\left|\int_{\mathbb{R}^2_+}\rho_k^2dx-(\mu-\alpha)\right|\right\}\leq\varepsilon,\\ &dist\,(spt\,\rho_k^1, spt\,\rho_k^2)\to\infty \quad as\ k\to\infty.\end{aligned}$$

Proof. The assertion is proved in [31, Lemma I.1] for the whole space and the fixed mass $\mu_n = \mu$ by using Lévy's concentration function. The proof also applies to a half space. The case $\mu_n \to \mu$ is reduced to the fixed mass case by setting $\tilde{\rho}_n = \rho_n\mu/\mu_n$. □

Remark 4.2. The case (i) is further divided into two cases: (a) $\limsup_{k\to\infty} y_{2,k} = \infty$ for $y_k = {}^t(y_{1,k}, y_{2,k})$ and (b) $\sup_{k\geq 1} y_{2,k} < \infty$. In the case (b), we may assume that $y_{2,k} = 0$ by replacing R. In fact, $B({}^t(y_{1,k}, 0), R') \supset B(y_k, R)$ for $R' = \sup_{k\geq 1} y_{2,k} + R$. Hence

$$\liminf_{k\to\infty}\int_{B({}^t(y_{1,k},0),R')}\rho_{n_k}\mathrm{d}x \geq \mu-\varepsilon.$$

Proof of Theorem 1.3. Let $\{\omega_n\}$ be a minimizing sequence such that $\omega_n \in K_{\mu_n}$, $\mu_n \to \mu$ and $-E_2[\omega_n] \to I_\mu$ as $n \to \infty$. By Remarks 2.4 (ii), $\{\omega_n\}$ is uniformly bounded in L^2. We set $\rho_n = x_2\omega_n$ and apply Lemma 4.1. Then, for a certain subsequence still denoted by $\{\omega_n\}$, one of the three cases, (iii) Dichotomy, (ii) Vanishing, (i) Compactness, should occur. We shall exclude the first two cases to get compactness of the sequence.

Case 1. *Dichotomy:*
There exists some $\alpha \in (0, \mu)$ such that for arbitrary $\varepsilon > 0$, there exist $k_0 \geq 1$ and $\{\omega_{1,n}\}, \{\omega_{2,n}\} \subset L^1$ such that $\omega_{3,n} = \omega_n - \omega_{1,n} - \omega_{2,n}$ satisfies spt $\omega_{1,n} \cap$ spt $\omega_{2,n} = \emptyset$, $0 \leq \omega_{i,n} \leq \omega_n$, $i = 1, 2, 3$, and

$$\begin{aligned}&\limsup_{n\to\infty}\{\|x_2\omega_{3,n}\|_1+|\alpha_n-\alpha|+|\beta_n-(\mu-\alpha)|\}\leq\varepsilon,\\ &\alpha_n=\int_{\mathbb{R}^2_+}x_2\omega_{1,n}\mathrm{d}x,\quad \beta_n=\int_{\mathbb{R}^2_+}x_2\omega_{2,n}\mathrm{d}x,\\ &d_n=\mathrm{dist}\,(\mathrm{spt}\,\omega_{1,n},\mathrm{spt}\,\omega_{2,n})\to\infty\quad \text{as } n\to\infty.\end{aligned}$$

By choosing a subsequence, we may assume that $\sup_n \|x_2\omega_{3,n}\|_1 \leq 2\varepsilon$, $\alpha_n \to \overline{\alpha}$ and $\beta_n \to \overline{\beta}$. By suppressing the integral region, we see that

$$
\begin{aligned}
2E[\omega_n] &= \iint G(x,y)\omega_n(x)\omega_n(y)\mathrm{d}x\mathrm{d}y \\
&= \iint G(x,y)\omega_{1,n}(x)\omega_{1,n}(y)\mathrm{d}x\mathrm{d}y + \iint G(x,y)\omega_{2,n}(x)\omega_{2,n}(y)\mathrm{d}x\mathrm{d}y \\
&+ 2\iint G(x,y)\omega_{1,n}(x)\omega_{2,n}(y)\mathrm{d}x\mathrm{d}y + \iint G(x,y)(2\omega_n(x)-\omega_{3,n}(x))\omega_{3,n}(y)\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

Applying (2.3) implies

$$
\begin{aligned}
\left|\iint G(x,y)(2\omega_n(x)-\omega_{3,n}(x))\omega_{3,n}(y)\mathrm{d}x\mathrm{d}y\right| &\leq C\|2\omega_n-\omega_{3,n}\|_1^{1/2}\|2\omega_n-\omega_{3,n}\|_2^{1/2}\|x_2\omega_{3,n}\|_1^{1/2}\|\omega_{3,n}\|_1^{1/2} \\
&\leq C\varepsilon^{1/2}.
\end{aligned}
$$

Since $G(x,y)\leq \pi^{-1}x_2y_2|x-y|^{-2}$,

$$
\iint G(x,y)\omega_{1,n}(x)\omega_{2,n}(y)\mathrm{d}x\mathrm{d}y = \iint_{|x-y|\geq d_n} G(x,y)\omega_{1,n}(x)\omega_{2,n}(y)\mathrm{d}x\mathrm{d}y \leq \frac{\mu^2}{\pi d_n^2}.
$$

Hence

$$
E_2[\omega_n] = E[\omega_n] - \frac{1}{2}\int_{\mathbb{R}^2_+}\omega_n^2\mathrm{d}x \leq E_2[\omega_{1,n}] + E_2[\omega_{2,n}] + \frac{\mu^2}{\pi d_n^2} + C\varepsilon^{1/2}.
$$

We take a Steiner symmetrization $\omega_{i,n}^*$ of $\omega_{i,n}$ to see that

$$
\begin{aligned}
&E_2[\omega_n] \leq E_2[\omega_{1,n}^*] + E_2[\omega_{2,n}^*] + \frac{\mu^2}{\pi d_n^2} + C\varepsilon^{1/2}, \\
&\|\omega_{1,n}^*\|_1 + \|\omega_{2,n}^*\|_1 \leq 1, \quad \|\omega_{1,n}^*\|_2 + \|\omega_{2,n}^*\|_2 \leq C, \\
&\alpha_n = \int_{\mathbb{R}^2_+} x_2\omega_{1,n}^*\mathrm{d}x, \quad \beta_n = \int_{\mathbb{R}^2_+} x_2\omega_{2,n}^*\mathrm{d}x.
\end{aligned}
$$

By choosing a subsequence (still denoted by $\{\omega_{i,n}^*\}$), $\omega_{i,n}^* \rightharpoonup \overline{\omega}_i^\varepsilon$ in L^2 and $\|\overline{\omega}_i^\varepsilon\|_2 \leq \liminf_{n\to\infty}\|\omega_{i,n}^*\|_2$. Since $\omega_{i,n}^*$ is symmetric and non-increasing for $x_1>0$, we apply Lemma 3.5 to get the convergence of the kinetic energy

$$
\lim_{n\to\infty} E[\omega_{i,n}^*] = E[\overline{\omega}_i^\varepsilon], \quad i=1,2.
$$

Sending $n\to\infty$ implies that

$$
\begin{aligned}
&-I_\mu \leq E_2[\overline{\omega}_1^\varepsilon]+E_2[\overline{\omega}_2^\varepsilon]+C\varepsilon^{1/2},\\
&\|\overline{\omega}_1^\varepsilon\|_1+\|\overline{\omega}_2^\varepsilon\|_1\leq 1,\quad \|\overline{\omega}_1^\varepsilon\|_2+\|\overline{\omega}_2^\varepsilon\|_2\leq C,\\
&\overline{\alpha}\geq \int_{\mathbb{R}^2_+} x_2\overline{\omega}_1^\varepsilon \mathrm{d}x,\quad \overline{\beta}\geq \int_{\mathbb{R}^2_+} x_2\overline{\omega}_2^\varepsilon \mathrm{d}x.
\end{aligned}
$$

Since $\overline{\omega}_i^\varepsilon$ for $\varepsilon>0$ is also symmetric and non-increasing for $x_1>0$, applying the same argument for $\overline{\omega}_i^\varepsilon$ and sending $\varepsilon\to 0$ implies that $\overline{\omega}_i^\varepsilon \rightharpoonup \overline{\omega}_i$ in $L^2(\mathbb{R}^2_+)$ and

$$
\begin{aligned}
&-I_\mu \leq E_2[\overline{\omega}_1]+E_2[\overline{\omega}_2],\\
&\|\overline{\omega}_1\|_1+\|\overline{\omega}_2\|_1\leq 1,\\
&\alpha\geq \int_{\mathbb{R}^2_+} x_2\overline{\omega}_1 \mathrm{d}x,\quad \mu-\alpha\geq \int_{\mathbb{R}^2_+} x_2\overline{\omega}_2 \mathrm{d}x.
\end{aligned}
$$

If $\overline{\omega}_1\equiv 0$ and $\overline{\omega}_2\equiv 0$, we have $-I_\mu\leq 0$, a contradiction to $I_\mu<0$ by (2.9). We may assume that $\overline{\omega}_1\not\equiv 0$. We set $\nu_1=1-\|\overline{\omega}_2\|_1>0$ and apply Lemma 3.2 to take a maximizer $\omega_1\in\tilde{K}_{\alpha,\nu_1}$ of

$$
E_2[\omega_1]=\sup_{\omega\in\tilde{K}_{\alpha,\nu_1}} E_2[\omega],
$$

such that $\int x_2\omega_1\mathrm{d}x=\alpha$ and spt ω_1 is compact in $\overline{\mathbb{R}^2_+}$. Hence

$$
\begin{aligned}
&-I_\mu \leq E_2[\omega_1]+E_2[\overline{\omega}_2],\\
&\|\omega_1\|_1+\|\overline{\omega}_2\|_1\leq 1,\\
&\alpha= \int_{\mathbb{R}^2_+} x_2\omega_1 \mathrm{d}x,\quad \mu-\alpha\geq \int_{\mathbb{R}^2_+} x_2\overline{\omega}_2 \mathrm{d}x.
\end{aligned}
$$

If $\overline{\omega}_2\equiv 0$, we have $-I_\mu\leq -I_\alpha$, a contradiction to $I_\mu<I_\alpha$ by (2.10). We may assume that $\overline{\omega}_2\not\equiv 0$. By setting $\nu_2=1-\|\omega_1\|_1>0$ and taking a maximizer $\omega_2\in\tilde{K}_{\mu-\alpha,\nu_2}$ with compact support in the same way,

$$
\begin{aligned}
&-I_\mu \leq E_2[\omega_1]+E_2[\omega_2],\\
&\|\omega_1\|_1+\|\omega_2\|_1\leq 1,\\
&\alpha= \int_{\mathbb{R}^2_+} x_2\omega_1 \mathrm{d}x,\quad \mu-\alpha= \int_{\mathbb{R}^2_+} x_2\omega_2 \mathrm{d}x.
\end{aligned}
$$

By translation for the x_1-variable, we may assume that spt $\omega_1\cap$ spt $\omega_2=\emptyset$. Since $\omega_1+\omega_2\in K_\mu$,

$$-I_\mu \le E_2[\omega_1] + E_2[\omega_2] = E_2[\omega_1+\omega_2] - \iint G(x,y)\omega_1(x)\omega_2(y)\mathrm{d}x\mathrm{d}y$$
$$\le -I_\mu - \iint G(x,y)\omega_1(x)\omega_2(y)\mathrm{d}x\mathrm{d}y \le -I_\mu.$$

Hence, $\omega_i \equiv 0$ for $i = 1$ or 2. This contradicts $\mu = \int_{\mathbb{R}^2_+} x_2(\omega_1+\omega_2)\mathrm{d}x$. Thus dichotomy does not occur.

Case 2. *Vanishing:*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^2_+} \int_{B(y,R)\cap\mathbb{R}^2_+} x_2\omega_n \mathrm{d}x = 0, \quad \text{for each } R > 0.$$

We shall show that $\lim_{n\to\infty} E[\omega_n] = 0$. Since $E_2[\omega_n] \le E[\omega_n]$, this implies $I_\mu \ge 0$, a contradiction to $I_\mu < 0$.

We set

$$2E[\omega_n] = \iint G(x,y)\omega_n(x)\omega_n(y)\mathrm{d}x\mathrm{d}y = \iint_{|x-y|\ge R} + \iint_{|x-y|<R}.$$

Since $G(x,y) \le \pi^{-1}x_2y_2|x-y|^{-2}$,

$$\iint_{|x-y|\ge R} G(x,y)\omega_n(x)\omega_n(y)\mathrm{d}x\mathrm{d}y \le \frac{\mu^2}{\pi R^2}.$$

We divide the second term into two terms

$$\iint_{|x-y|<R} G(x,y)\omega_n(x)\omega_n(y)\mathrm{d}x\mathrm{d}y = \iint_{\substack{|x-y|<R,\\ G\ge Rx_2y_2}} + \iint_{\substack{|x-y|<R,\\ G<Rx_2y_2}},$$

and observe that

$$\iint_{\substack{|x-y|<R,\\ G<Rx_2y_2}} G(x,y)\omega_n(x)\omega_n(y)\mathrm{d}x\mathrm{d}y \le R\mu\left(\sup_{y\in\mathbb{R}^2_+}\int_{B(y,R)\cap\mathbb{R}^2_+} x_2\omega_n(x)\mathrm{d}x\right) \to 0 \quad \text{as } n\to\infty.$$

We may assume that $R \ge 1$. The condition $G \ge Rx_2y_2$ implies $|x-y| \le R^{-1/2}$. Since $|x-y^*| \le 2x_2 + R^{-1/2}$, $y^* = {}^t(y_1, -y_2)$,

$$G(x,y) = -\frac{1}{2\pi}\left(\log|x-y| - \log|x-y^*|\right) \le \frac{1}{\pi}\left(\left|\log|x-y|\right| + x_2\right),$$

$$\left(\int_{|x-y|<R^{-1/2}} G(x,y)^2 \mathrm{d}y\right)^{1/2} \leq C(R)(1+x_2),$$

and $C(R) \to 0$ as $R \to \infty$. Hence

$$\begin{aligned}\iint_{\substack{|x-y|<R,\\ G\geq Rx_2y_2}} G(x,y)\omega_n(x)\omega_n(y)\mathrm{d}x\mathrm{d}y &\leq \iint_{|x-y|<R^{-1/2}} G(x,y)\omega_n(x)\omega_n(y)\mathrm{d}x\mathrm{d}y \\ &\leq \|\omega_n\|_2 \int_{\mathbb{R}^2_+} \omega_n(x)\left(\int_{|x-y|<R^{-1/2}} G(x,y)^2\mathrm{d}y\right)^{1/2} \mathrm{d}x \\ &\leq C(R)'.\end{aligned}$$

Sending $n \to \infty$, and then $R \to \infty$ implies $\lim_{n\to\infty} E[\omega_n] = 0$. Thus vanishing does not occur.

Case 3. *Compactness:*
There exists a sequence $\{y_n\} \subset \overline{\mathbb{R}^2_+}$ such that for arbitrary $\varepsilon > 0$, there exists $R > 0$ such that

$$\liminf_{n\to\infty} \int_{B(y_n,R)\cap\mathbb{R}^2_+} x_2\omega_n \mathrm{d}x \geq \mu - \varepsilon.$$

By translation for the x_1-variable, we may assume that $y_n = {}^t(0, y_{2,n})$. Then, there are two cases whether (a) $\limsup_{n\to\infty} y_{2,n} = \infty$ or (b) $\sup_{n\geq 1} y_{2,n} < \infty$. We shall first show that the case (a) does not occur.

(a) $\limsup_{n\to\infty} y_{2,n} = \infty$. We may assume that $\lim_{n\to\infty} y_{2,n} = \infty$ and $\sup_n \|x_2\omega_n\|_{L^1(\mathbb{R}^2_+\backslash B(y_n,R))} \leq 2\varepsilon$ by choosing a subsequence. We shall show that $\lim_{n\to\infty} E[\omega_n] = 0$. This implies $-I_\mu = \lim_{n\to\infty} E_2[\omega_n] \leq \lim_{n\to\infty} E[\omega_n] = 0$, a contradiction to $I_\mu < 0$.

We set

$$2E[\omega_n] = \int_{\mathbb{R}^2_+} \psi_n\omega_n \mathrm{d}x = \int_{B(y_n,R)\cap\mathbb{R}^2_+} + \int_{\mathbb{R}^2_+\backslash B(y_n,R)},$$

for

$$\psi_n(x) = \int_{\mathbb{R}^2_+} G(x,y)\omega_n(y)\mathrm{d}y.$$

By (2.1),

$$\int_{B(y_n,R)\cap\mathbb{R}^2_+} \psi_n\omega_n \mathrm{d}x \leq \left\|\frac{\psi_n}{x_2^{1/2}}\right\|_\infty \int_{B(y_n,R)\cap\mathbb{R}^2_+} x_2^{1/2}\omega_n \mathrm{d}x \leq \frac{C\mu}{(y_{2,n}-R)^{1/2}} \to 0 \quad \text{as } n\to\infty.$$

By Hölder's inequality,

$$\int_{\mathbb{R}^2_+\setminus B(y_n,R)} \psi_n\omega_n \mathrm{d}x \le \left\| \frac{\psi_n}{x_2^{1/2}} \right\|_\infty \left(\int_{\mathbb{R}^2_+\setminus B(y_n,R)} x_2\omega_n \mathrm{d}x \right)^{1/2} \left(\int_{\mathbb{R}^2_+\setminus B(y_n,R)} \omega_n \mathrm{d}x \right)^{1/2} \le C\varepsilon^{1/2}.$$

Thus sending $n \to \infty$, and then $\varepsilon \to 0$ implies $\lim_{n\to\infty} E[\omega_n] = 0$. Thus case (a) does not occur.

(b) $\sup_{n\ge} y_{2,n} < \infty$. We may assume that $y_{2,n} = 0$ by taking sufficiently large $R > 0$ as noted in Remark 4.2, i.e., for $B = B(0, R)$,

$$\liminf_{n\to\infty} \int_{B\cap\mathbb{R}^2_+} x_2\omega_n \mathrm{d}x \ge \mu - \varepsilon.$$

Since $\{\omega_n\}$ is uniformly bounded in L^2, by choosing a subsequence, $\omega_n \rightharpoonup \omega$ in L^2 for some ω. By sending $n \to \infty$,

$$\int_{\mathbb{R}^2_+} x_2\omega \mathrm{d}x = \mu.$$

Hence $\omega \in K_\mu$. We shall show that

$$\lim_{n\to\infty} E[\omega_n] = E[\omega]. \tag{4.4}$$

This implies that

$$-I_\mu = \lim_{n\to\infty} E_2[\omega_n] \le \lim_{n\to\infty} E[\omega_n] - \frac{1}{2}\liminf_{n\to\infty} \|\omega_n\|_2^2 \le E_2[\omega] \le -I_\mu.$$

Hence $\lim_{n\to\infty} \|\omega_n\|_2 = \|\omega\|_2$ and $\omega_n \to \omega$ in L^2 follows. By

$$\int_{\mathbb{R}^2_+} x_2|\omega_n - \omega|\mathrm{d}x = \int_{B\cap\mathbb{R}^2_+} x_2|\omega_n - \omega|\mathrm{d}x + \int_{\mathbb{R}^2_+\setminus B} x_2|\omega_n - \omega|\mathrm{d}x \le C\|\omega_n - \omega\|_2 + C'\varepsilon,$$

sending $n \to \infty$ and then $\varepsilon \to 0$ implies $x_2\omega_n \to x_2\omega$ in L^1. Since $E_2[\omega_n] \to E_2[\omega]$, the limit $\omega \in K_\mu$ is a minimizer of I_μ.

It remains to show (4.4). We decompose

$$2E[\omega_n] = \int_{\mathbb{R}^2_+} \psi_n\omega_n \mathrm{d}x = \int_{B\cap\mathbb{R}^2_+} + \int_{\mathbb{R}^2_+\setminus B},$$

and also

$$\int_{B\cap\mathbb{R}^2_+} \psi_n\omega_n \mathrm{d}x = \int_{B\cap\mathbb{R}^2_+} \omega(x)\mathrm{d}x \int_{\mathbb{R}^2_+} G(x,y)\omega_n(y)\mathrm{d}y = \int_{B\cap\mathbb{R}^2_+}\int_{B\cap\mathbb{R}^2_+} + \int_{B\cap\mathbb{R}^2_+}\int_{\mathbb{R}^2_+\setminus B}.$$

Observe that by $G(x,y)=G(y,x)$,

$$
\begin{aligned}
\int_{B\cap\mathbb{R}^2_+}\omega_n(x)\mathrm{d}x\int_{\mathbb{R}^2_+\backslash B}G(x,y)\omega_n(y)\mathrm{d}y &= \int_{B\cap\mathbb{R}^2_+}\omega_n(y)\mathrm{d}y\int_{\mathbb{R}^2_+\backslash B}G(x,y)\omega_n(x)\mathrm{d}x \\
&\leq \int_{\mathbb{R}^2_+\backslash B}\omega_n(x)\mathrm{d}x\int_{\mathbb{R}^2_+}G(x,y)\omega_n(y)\mathrm{d}y \\
&= \int_{\mathbb{R}^2_+\backslash B}\psi_n(x)\omega_n(x)\mathrm{d}x.
\end{aligned}
$$

Hence

$$
\left|2E[\omega_n]-\int_{B\cap\mathbb{R}^2_+}\int_{B\cap\mathbb{R}^2_+}G(x,y)\omega_n(x)\omega_n(y)\mathrm{d}x\mathrm{d}y\right|\leq 2\int_{\mathbb{R}^2_+\backslash B}\psi_n(x)\omega_n(x)\mathrm{d}x.
$$

By

$$
\int_{\mathbb{R}^2_+\backslash B}\psi_n(x)\omega_n(x)\mathrm{d}x\leq\left\|\frac{\psi_n}{x_2^{1/2}}\right\|_\infty\left(\int_{\mathbb{R}^2_+\backslash B}x_2\omega_n\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{R}^2_+\backslash B}\omega_n\mathrm{d}x\right)^{1/2}\leq C\varepsilon^{1/2},
$$

and estimating $E[\omega]$ in the same way,

$$
2\left|E[\omega_n]-E[\omega]\right|\leq\left|\int_{B\cap\mathbb{R}^2_+}\int_{B\cap\mathbb{R}^2_+}G(x,y)\left(\omega_n(x)\omega_n(y)-\omega(x)\omega(y)\right)\mathrm{d}x\mathrm{d}y\right|+C\varepsilon^{1/2}.
$$

Since $G(x,y)\in L^2(B\times B)$ and $\omega_n(x)\omega_n(y)\rightharpoonup\omega(x)\omega(y)$ in $L^2(B\times B)$, sending $n\to\infty$ and $\varepsilon\to 0$ yields $\lim_{n\to\infty}E[\omega_n]=E[\omega]$. The proof is now complete. □

5. Orbital stability

We prove Theorem 1.4. We first show existence of global weak solutions of (1.1) satisfying the conservations (1.8). To see this, we recall renormalized solutions of DiPerna-Lions [18].

5.1. **Existence of global weak solutions.** We consider the linear transport equation

$$
\begin{aligned}
\partial_t\xi+b\cdot\nabla\xi&=0\quad\text{in }\mathbb{R}^2\times(0,T),\\
\xi(x,0)&=\xi_0\quad\text{on }\mathbb{R}^2\times\{t=0\},
\end{aligned}\tag{5.1}
$$

with the divergence-free drift b, i.e., div $b=0$, satisfying

$$
b \in L^1(0,T;W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)),
\quad
\frac{b}{1+|x|} \in L^1(0,T;L^1+L^\infty(\mathbb{R}^2)). \tag{5.2}
$$

We denote by L^0 the set of all measurable functions f such that $|\{|f|>\alpha\}|<\infty$ for each $\alpha\in(0,\infty)$. We say that $\xi\in L^\infty(0,T;L^0)$ is a renormalized solution of $(5.1)_1$ if ξ satisfies

$$
\partial_t\beta(\xi)+b\cdot\nabla\beta(\xi)=0 \quad \text{in } \mathbb{R}^2\times(0,T), \tag{5.3}
$$

for all $\beta\in C^1\cap L^\infty(\mathbb{R})$ vanishing near zero, in the sense of distribution. It is proved in [18, Theorem II. 3] under the condition (5.2) that for $\xi_0\in L^0$ there exists a unique renormalized solution $\xi\in C([0,T];L^0)$ of (5.1) and if $\xi_0\in L^q(\mathbb{R}^2)$, $q\in[1,\infty]$, the renormalized solution satisfies $\xi\in C([0,T];L^q(\mathbb{R}^2))$ and

$$
\|\xi\|_q(t)=\|\xi_0\|_q \quad \text{for all } t\geq 0. \tag{5.4}
$$

It is proved in [33, p.357] that every global weak solution of (1.1) for $\zeta_0\in L^q\cap L^1(\mathbb{R}^2)$, $q\in(1,\infty)$, constructed by approximation of ζ_0 is a renormalized solution of (5.1) for $b=k*\zeta$, see also [17]. Thus the conservation $(1.8)_1$ holds for the weak solutions by (5.4).

Proposition 5.1. *For symmetric initial data $\zeta_0\in L^2\cap L^1(\mathbb{R}^2)$ such that $x_2\zeta_0\in L^1(\mathbb{R}^2)$ and $\zeta_0\geq 0$ for $x_2\geq 0$, i.e., $\zeta_0(x_1,x_2)=-\zeta_0(x_1,-x_2)$, there exists a symmetric global weak solution $\zeta\in BC([0,\infty);L^2\cap L^1(\mathbb{R}^2))$ of (1.1) such that $x_2\zeta\in BC([0,\infty);L^1(\mathbb{R}^2))$, $\zeta\geq 0$ for $x_2\geq 0$,*

$$
\int_0^\infty\int_{\mathbb{R}^2}\zeta(\partial_t\varphi+v\cdot\nabla\varphi)dxdt=-\int_{\mathbb{R}^2}\zeta_0(x)\varphi(x,0)dx \tag{5.5}
$$

for $v=k*\zeta$ and all $\varphi\in C_c^\infty(\mathbb{R}^2\times[0,\infty))$. This weak solution ζ satisfies the conservations (1.8).

Proof. For smooth and symmetric initial data $\zeta_0\in C_c^\infty$, there exists a symmetric classical solution $\zeta\in BC([0,\infty);L^2\cap L^1)$ of (1.1) [34]. Since ζ is conserved on the particle trajectory map that globally exists and is homeomorphism of $\mathbb{R}^2$ onto $\mathbb{R}^2$ [35, Proposition 4.1, Corollary 4.1], $\zeta(\cdot,t)$ is compactly supported in $\mathbb{R}^2$. By the conservations (1.8) and the Biot-Savart law $v=k*\zeta$, the solution satisfies

$$
\begin{aligned}
&\zeta\in L^\infty(0,\infty;L^2\cap L^1),\\
&x_2\zeta\in L^\infty(0,\infty;L^1),\\
&v\in L^\infty(0,\infty;L^p),\ 2\leq p<\infty,\\
&\nabla v\in L^\infty(0,\infty;L^q),\ 1<q\leq 2.
\end{aligned} \tag{5.6}
$$

By $(5.6)_3$ and $(5.6)_4$, $v\zeta \in L^\infty(0,\infty;L^r)$, $1 < r < 2$. For arbitrary $\varphi \in C_c^\infty(\mathbb{R}^2)$, we have $\|\nabla k * \varphi\|_{L^{r'}} \leq C\|\varphi\|_{L^{r'}}$ for $1/r + 1/r' = 1$ by the estimate of the singular integral operator. By Fubini's theorem,

$$\int_{\mathbb{R}^2} v(x,t)\varphi(x)\mathrm{d}x = \int_{\mathbb{R}^2} (k * \zeta)(x,t)\varphi(x)\mathrm{d}x = -\int_{\mathbb{R}^2} \zeta(x,t)(k * \varphi)(x)\mathrm{d}x.$$

Since $\zeta(\cdot,t)$ is compactly supported in $\mathbb{R}^2$ and a smooth solution to $\partial_t\zeta + v \cdot \nabla\zeta = 0$, by differentiating the above equation in t,

$$\int_{\mathbb{R}^2} \partial_t v(x,t)\varphi(x)\mathrm{d}x = -\int_{\mathbb{R}^2} \partial_t\zeta(x,t)(k * \varphi)(x)\mathrm{d}x = -\int_{\mathbb{R}^2} v(x,t)\zeta(x,t) \cdot \nabla(k * \varphi)(x)\mathrm{d}x.$$

Thus

$$\left|\int_{\mathbb{R}^2} \partial_t v(x,t)\varphi(x)\mathrm{d}x\right| \leq C\|v\zeta\|_{L^r}\|\varphi\|_{L^{r'}}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^2)$. By density of $C_c^\infty(\mathbb{R}^2)$ in $L^{r'}(\mathbb{R}^2)$, the functional $\varphi \longmapsto \int_{\mathbb{R}^2} \partial_t v\varphi \mathrm{d}x$ is uniquely extendable to that on $L^{r'}(\mathbb{R}^2)$. By Riesz representation theorem, $\partial_t v(\cdot,t) \in L^r(\mathbb{R}^2)$ and the above inequality holds for all $\varphi \in L^{r'}(\mathbb{R}^2)$. Thus by duality, $\|\partial_t v\|_{L^r} \leq C\|v\zeta\|_{L^r}$ and

$$\partial_t v \in L^\infty(0,\infty;L^r),\ 1 < r < 2. \tag{5.7}$$

We set $\phi(x,t) = \int_{\mathbb{R}^2_+} G(x,y)\zeta(y,t)\mathrm{d}y$ by (1.5). We have $v = \nabla^\perp\phi$ for $\nabla^\perp = {}^t(\partial_2, -\partial_1)$. By (5.7) and applying the Sobolev inequality $\|\varphi\|_{L^s} \leq C\|\nabla\varphi\|_{L^r}$, $1/s = 1/r - 1/2$, $1 < r < 2$ for $\varphi = \partial_t\phi$,

$$\partial_t\phi \in L^\infty(0,\infty;L^s),\ 2 < s < \infty. \tag{5.8}$$

We will use (5.7) and (5.8) to obtain the equality $(1.8)_3$.

The function v satisfies the condition (5.2). Indeed, by $v = k * \zeta$, $k = k1_B + k1_{B^c} = k_1 + k_2$, $B = B(0,1)$, and Young's inequality,

$$\|v\|_{L^1+L^\infty} \leq \|k_1 * \zeta\|_{L^1} + \|k_2 * \zeta\|_{L^\infty} \leq (\|k_1\|_{L^1} + \|k_2\|_{L^\infty})\|\zeta\|_{L^1}.$$

Hence

$$v \in L^\infty(0,\infty;L^1 + L^\infty). \tag{5.9}$$

The existence of a global weak solution of (1.1) satisfying (5.5)-(5.8) for symmetric $\zeta_0 \in L^2 \cap L^1$, $x_2\zeta_0 \in L^1$, $\zeta_0 \geq 0$ for $x_2 \geq 0$, follows by an approximation of ζ_0 by elements of C_c^∞, e.g., [35]. By the conditions $(5.6)_4$, (5.9) and the consistency [18, Theorem II.3 (1)], the constructed global weak solution ζ is a renormalized solution of (5.1). Hence $\zeta \in BC([0,\infty); L^2 \cap L^1)$ and $(1.8)_1$ holds.

The conservations $(1.8)_2$ and $(1.8)_3$ follow from the weak form (5.5). To see this, we take a cut-off function $\theta \in C^\infty(\mathbb{R})$, satisfying $\theta \equiv 1$ in $(-\infty, 1]$ and $\theta \equiv 0$ in $[2,\infty)$ and set $\theta_R(x) = \theta(|x|/R)$, $R \geq 1$, and $\eta_m(t) = \theta(m(t-T)+1)$, $m \geq 1$, $T > 0$. For arbitrary $f \in BC[0,\infty)$,

$$\int_0^\infty f(t)\partial_t \eta_m(t)\mathrm{d}t = \int_1^2 f\left(\frac{s-1}{m}+T\right)\dot{\theta}(s)\mathrm{d}s \to f(T)\int_1^2 \dot{\theta}(s)\mathrm{d}s = -f(T) \quad \text{as } m \to \infty.$$

Thus by substituting $\varphi = x_2\theta_R\eta_m$ into (5.5) and sending $m \to \infty$,

$$\int_0^T \int_{\mathbb{R}^2} \zeta v \cdot \nabla(x_2\theta_R)\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^2} x_2\zeta(x,T)\theta_R(x)\mathrm{d}x - \int_{\mathbb{R}^2} x_2\zeta_0(x)\theta_R(x)\mathrm{d}x.$$

Since

$$\zeta v \cdot \nabla(x_2\theta_R) = \left(\partial_1\left(\frac{1}{2}\left(|v^2|^2 - |v^1|^2\right)\right) - \partial_2(v^1v^2)\right)\theta_R + \zeta v x_2 \cdot \nabla\theta_R,$$

sending $R \to \infty$ implies $(1.8)_2$.

To prove $(1.8)_3$, it suffices to show the conservation of the kinetic energy

$$\int_{\mathbb{R}^2} |v(x,T)|^2\mathrm{d}x = \int_{\mathbb{R}^2} |v_0(x)|^2\mathrm{d}x. \tag{5.10}$$

Since $2E[\omega] = \|v\|_2^2$ by (2.6), $(1.8)_1$ and (5.10) imply $(1.8)_3$. By (5.6) and (5.7), observe that

$$2\int_0^T \int_{\mathbb{R}^2} v \cdot \partial_t v\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^2} |v(x,T)|^2\mathrm{d}x - \int_{\mathbb{R}^2} |v_0(x)|^2\mathrm{d}x. \tag{5.11}$$

By (5.6) and approximation of the test functions in (5.5), we have

$$\int_0^T \int_{\mathbb{R}^2} \zeta(\partial_t\varphi + v \cdot \nabla\varphi)\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^2} \zeta(x,T)\varphi(x,T)\mathrm{d}x - \int_{\mathbb{R}^2} \zeta_0(x)\varphi(x,0)\mathrm{d}x$$

for all $\varphi \in L^\infty(\mathbb{R}^2 \times (0,T))$ satisfying $\nabla\varphi, \partial_t\varphi \in L^\infty(0,T;L^s)$, $2 < s < \infty$. By $(5.6)_1$, $(5.6)_2$ and Proposition 2.2, $\phi \in L^\infty(\mathbb{R}^2 \times (0,T))$. By $(5.6)_3$ and (5.8), $\nabla\phi, \partial_t\phi \in L^\infty(0,T;L^s)$. Thus by substituting ϕ into the above and applying Proposition 2.2,

$$\int_0^T \int_{\mathbb{R}^2} v \cdot \partial_t v\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^2} |v(x,T)|^2\mathrm{d}x - \int_{\mathbb{R}^2} |v_0(x)|^2\mathrm{d}x.$$

By (5.11), we have obtained (5.10). The proof is complete. □

5.2. **An application to stability.** We now apply Theorem 1.3 for:

Proof of Theorem 1.4. We give a proof for the case $0 < \mu < \infty$, $\nu = \lambda = 1$. The proof is also applied to the general case $0 < \mu, \nu, \lambda < \infty$ by replacing K_μ, I_μ, S_μ by $K_{\mu,\nu}, I_{\mu,\nu,\lambda}, S_{\mu,\nu,\lambda}$, respectively. Suppose that (1.10) were false. Then there exists $\varepsilon_0 > 0$ such that for $n \geq 1$, there exist $\zeta_{0,n} \in L^2 \cap L^1$ satisfying $\zeta_{0,n} \geq 0$, $\|\zeta_{0,n}\|_1 \leq 1$ and $t_n \geq 0$ such that a global weak solution in Proposition 5.1 satisfies

$$\begin{aligned}
&\inf_{\omega\in S_\mu} \{\|\zeta_{0,n} - \omega\|_2 + \|x_2(\zeta_{0,n} - \omega)\|_1\} \leq \frac{1}{n},\\
&\inf_{\omega\in S_\mu} \{\|\zeta_n(t_n) - \omega\|_2 + \|x_2(\zeta_n(t_n) - \omega)\|_1\} \geq \varepsilon_0.
\end{aligned}$$

We write $\zeta_n = \zeta_n(t_n)$ by suppressing t_n. We take $\omega_n \in S_\mu$ such that $\|\zeta_{0,n} - \omega_n\|_2 + \|x_2(\zeta_{0,n} - \omega)\|_1 \to 0$. By (2.4),

$$\left|E_2[\zeta_{0,n}] + I_\mu\right| = \left|E_2[\zeta_{0,n}] - E_2[\omega_n]\right| \to 0 \quad \text{as } n \to \infty.$$

Thus $\{\zeta_{0,n}\}$ is a minimizing sequence such that $\zeta_{0,n} \in K_{\mu_n}$, $\mu_n = \int x_2\zeta_{0,n}\mathrm{d}x \to \mu$ and $-E_2[\zeta_{0,n}] \to I_\mu$ as $n \to \infty$.

By the conservations (1.8), $\zeta_n \in K_{\mu_n}$ and

$$\left|E_2[\zeta_n] + I_\mu\right| = \left|E_2[\zeta_{0,n}] + I_\mu\right| \to 0 \quad \text{as } n \to \infty.$$

Hence $\{\zeta_n\}$ is also a minimizing sequence such that $\zeta_n \in K_{\mu_n}$, $\mu_n \to \mu$ and $-E_2[\zeta_n] \to I_\mu$. By Theorem 1.3, there exists a sequence $\{y_n\} \subset \partial\mathbb{R}^2_+$ such that, by choosing a subsequence (still denoted by $\{\zeta_n\}$), there exists $\zeta \in L^2 \cap L^1$ such that

$$\begin{aligned}
\zeta_n(\cdot + y_n) &\to \zeta \quad \text{in } L^2(\mathbb{R}^2_+),\\
x_2\zeta_n(\cdot + y_n) &\to x_2\zeta \quad \text{in } L^1(\mathbb{R}^2_+),
\end{aligned}$$

and the limit $\zeta \in K_\mu$ is a minimizer of I_μ, i.e., $\zeta \in S_\mu$. Sending $n \to \infty$ implies

$$\begin{aligned}
0 = \inf_{\omega\in S_\mu} \{\|\zeta - \omega\|_2 + \|x_2(\zeta - \omega)\|_1\} &= \inf_{\omega\in S_\mu} \left(\lim_{n\to\infty} \{\|\zeta_n - \omega\|_2 + \|x_2(\zeta_n - \omega)\|_1\}\right)\\
&\geq \liminf_{n\to\infty} \left(\inf_{\omega\in S_\mu} \{\|\zeta_n - \omega\|_2 + \|x_2(\zeta_n - \omega)\|_1\}\right) \geq \varepsilon_0.
\end{aligned}$$

We obtained a contradiction. □

Remarks 5.2. (i) It is observed from the above proof that the assertion of Theorem 1.4 holds even if impulse of initial data is merely close to μ, i.e., for $\varepsilon > 0$ there exists $\delta > 0$ such that for $\zeta_0 \in L^2 \cap L^1(\mathbb{R}^2_+)$ satisfying $\zeta_0 \geq 0$, $\|\zeta_0\|_1 \leq \nu$ and

$$\inf_{\omega \in S_{\mu,\nu,\lambda}} \|\zeta_0 - \omega\|_2 + \left| \int_{\mathbb{R}^2_+} x_2 \zeta_0 \mathrm{d}x - \mu \right| \leq \delta, \tag{5.12}$$

there exists a global weak solution of (1.1) satisfying (1.10).
(ii) In [12], orbital stability by the L^2-norm is proved if initial data ζ_0 is close to a set of minimizers in the same topology as (5.12).

6. Uniqueness of the Lamb dipole

We prove Theorem 1.5. For minimizers $\omega \in S_\mu$, the associated stream functions are positive solutions of (2.19) for $W > 0$ and $\gamma = 0$, provided that $0 < \mu \leq M_1$ as in Lemma 2.9. Our goal is to prove that such solutions are only translations of the Lamb dipole (1.3) for $\lambda = 1$.

6.1. A decay estimate.

We consider positive solutions $\psi > 0$ of the problem:

$$\begin{aligned} -\Delta\psi(x) &= f(\psi - Wx_2) \quad \text{in } \mathbb{R}^2_+, \\ \psi &= 0 \quad \text{on } \partial\mathbb{R}^2_+, \\ \psi &\to 0 \quad \text{as } |x| \to \infty, \end{aligned} \tag{6.1}$$

for some constant $W > 0$.

Theorem 6.1. *Let $\psi \in BUC^{2+\alpha}(\overline{\mathbb{R}^2_+})$, $0 < \alpha < 1$, be a positive solution of (6.1) for some $W > 0$ such that $\psi/x_2 \in BUC^{1+\alpha}(\overline{\mathbb{R}^2_+})$, $\psi/x_2 \to 0$ as $|x| \to \infty$ and for $\Omega = \{x \in \mathbb{R}^2_+ \mid \psi(x) - Wx_2 > 0\}$, $\overline{\Omega}$ is compact in $\overline{\mathbb{R}^2_+}$. Then, $\psi(x_1, x_2) = \psi_L(x_1 + q, x_2)$ for some $q \in \mathbb{R}$, where $\psi_L = \Psi_L + Wx_2$ and Ψ_L is the Lamb dipole (1.3) for $\lambda = 1$ and the given $W > 0$.*

The uniqueness (up to translations) of weak solutions $\psi \in \dot{H}^1_0(\mathbb{R}^2_+)$ to (6.1) for $W > 0$ is proved by Burton [9, Theorem 2.1] by applying a symmetry result of Fraenkel [20, Theorem 4.2] for positive solutions to semi-linear elliptic problems, see also [25, Theorem 4, 2.3. Remark 1]. His proof is based on the fact [50, Lemma 1] that $\dot{H}^1_0(\mathbb{R}^2_+)$ is isometrically isomorphic to a subspace of axisymmetric functions in $\dot{H}^1(\mathbb{R}^4)$ by the transform

$$\psi(x_1, x_2) \longmapsto \varphi(y) = \frac{\psi(x_1, x_2)}{x_2}, \quad y = {}^t(y', y_4) \in \mathbb{R}^4,\ x_1 = y_4,\ x_2 = |y'|. \tag{6.2}$$

This reduces weak solutions of (6.1) to those of

$$
(6.3) \qquad \begin{aligned} -\Delta_y \varphi &= f(\varphi - W) \quad \text{in } \mathbb{R}^4, \\ \varphi &\to 0 \quad \text{as } |y| \to \infty. \end{aligned}
$$

By the Sobolev embedding $\dot{H}^1(\mathbb{R}^4) \subset L^4(\mathbb{R}^4)$ and differentiability of weak solutions to the Poisson equation, they are in $W^{2,4}(\mathbb{R}^4) \subset BUC^{\alpha}(\mathbb{R}^4)$, $0 < \alpha < 1$ and satisfy the decay $(6.3)_2$. The decay implies that $f(\varphi - W)$ is compactly supported and thus for

$$
(6.4) \qquad \Xi = \left\{ y \in \mathbb{R}^4 \;\middle|\; \varphi(y) - W > 0 \right\},
$$

$\overline{\Xi}$ is compact in $\mathbb{R}^4$. By uniqueness of the Poisson equation, φ is expressed in terms of the Newton potential. The potential representation implies that $\varphi \in BUC^{2+\alpha}(\mathbb{R}^4)$ is a positive solution to (6.3) and satisfies the admissible asymptotic behavior for the application of the moving plane method [20], see below (6.6).

The uniqueness in Theorem 6.1 will be proved without the isometry since the solution $\psi \in BUC^{2+\alpha}(\overline{\mathbb{R}^2_+})$, $0 < \alpha < 1$, is a classical solution to (6.1) with compactly supported $\overline{\Omega} \subset \overline{\mathbb{R}^2_+}$. By the transform (6.2), $\varphi \in BUC^{1+\alpha}(\mathbb{R}^4)$ is a solution of (6.3) in $\mathbb{R}^4 \backslash \{y' = 0\}$ with compactly supported $\overline{\Xi} \subset \mathbb{R}^4$.

Following [1, Lemma 2.2], we take a function $\theta \in C^{\infty}(\mathbb{R})$ such that $\theta(t) = 0$ for $t \leq 1$ and $\theta(t) = 1$ for $t \geq 2$. For arbitrary $\xi \in C^1_c(\mathbb{R}^4)$ and $\delta > 0$, we set $\xi_\delta(y) = \xi(y)\theta(|y'|/\delta)$ so that $\operatorname{spt} \xi_\delta \cap \{y' = 0\} = \emptyset$. The function $\xi_\delta \in C^1_c(\mathbb{R}^4)$ satisfies $\xi_\delta \to \xi$ in $W^{1,1}(\mathbb{R}^4)$ as $\delta \to 0$. Since φ is $C^{2+\alpha}$ in $\mathbb{R}^4 \backslash \{y' = 0\}$ and satisfies $(6.3)_1$ in a classical sense, multiplying ξ_δ by $(6.3)_1$ in $\mathbb{R}^4 \backslash \{y' = 0\}$ and integration by parts,

$$
\int_{\mathbb{R}^4} \nabla \varphi \cdot \nabla \xi_\delta \mathrm{d}y = \int_{\mathbb{R}^4} f(\varphi - W) \xi_\delta \mathrm{d}y.
$$

Sending $\delta \to 0$ implies that $\varphi \in BUC^{1+\alpha}(\mathbb{R}^4)$ is a weak solution of the Poisson equation in $\mathbb{R}^4$ in the sense that for all $\xi \in C^1_c(\mathbb{R}^4)$,

$$
\int_{\mathbb{R}^4} \nabla \varphi \cdot \nabla \xi \mathrm{d}y = \int_{\mathbb{R}^4} f(\varphi - W) \xi \mathrm{d}y.
$$

Thus by differentiability of weak solutions [26, Theorem 8.8], we have $\varphi \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^4)$. By the compactness of $\overline{\Xi} \subset \mathbb{R}^4$ and uniqueness of the Poisson equation under the decay condition $\varphi(y) \to 0$ as $|y| \to \infty$, φ is expressed with $\Gamma(y) = (4\pi^2)^{-1}|y|^{-2}$ as

$$
(6.5) \qquad \varphi(y) = \int_{\Xi} \Gamma(y - z) f(\varphi - W) \mathrm{d}z.
$$

This implies that $\varphi \in BUC^{2+\alpha}(\mathbb{R}^4)$ is a positive solution to (6.3).

Lemma 6.2. *Let φ be as in (6.5). There exists $p > 0$ and $q \in \mathbb{R}$ such that*

$$
\begin{aligned}
&\varphi(y', y_4 + q) = \frac{p}{|y|^2} + g(y), \\
&|g(y)| \leq \frac{C}{|y|^4}, \quad |\nabla g(y)| \leq \frac{C}{|y|^5}, \quad \textit{for } |y| \geq 2R + |q|,
\end{aligned} \tag{6.6}
$$

for some $R > 0$ such that $\Xi \subset B(0, R)$ with some constant C, where $B(0, R)$ is an open ball in $\mathbb{R}^4$.

Proof. By (6.5),

$$
\begin{aligned}
&\varphi(y) = \Gamma(y) \int_{\Xi} f(\varphi - W) \mathrm{d}z - \nabla_y \Gamma(y) \cdot \left(\int_{\Xi} z f(\varphi - W) \mathrm{d}z \right) + g_0(y), \\
&|g_0(y)| \leq \frac{C}{|y|^4}, \quad |\nabla g_0(y)| \leq \frac{C}{|y|^5}, \quad \text{for } |y| \geq 2R.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\varphi(y) = \frac{p}{|y|^2} + \sum_{j=1}^{4} \frac{p_j y_j}{|y|^4} + g_0(y), \\
&p = \frac{1}{4\pi^2} \int_{\Xi} f(\varphi - W) \mathrm{d}z, \quad p_j = \frac{1}{2\pi^2} \int_{\Xi} z_j f(\varphi - W) \mathrm{d}z, \quad j = 1, 2, 3, 4.
\end{aligned}
$$

Since Ξ and φ are symmetric for $y' = 0$, $p_j = 0$ for $j = 1, 2, 3$. By taking $q = p_4/(2p)$, (6.6) follows. □

6.2. **Moving plane method.** The decay (6.6) is the admissible asymptotic behavior [20, Definition 4.1 (C)] for the application of the moving plane method.

Proof of Theorem 6.1. We apply a symmetry result for positive solutions to (6.3) satisfying (6.6) [20, Theorem 4.2] and deduce that $\phi(y) = \varphi(y', y_4 + q)$ is radially symmetric in $\mathbb{R}^4$ and decreasing in the radial direction. Since $|y| = |x|$ and $\phi(y) = \phi(|y|)$, we deduce that

$$
\frac{\psi(x_1 + q, x_2)}{x_2} = \varphi(y', y_4 + q) = \phi(y', y_4) = \phi\left(|x|\right).
$$

By translation of ψ for the x_1-variable, we may assume that $q = 0$, i.e., $\psi(x_1, x_2)/x_2 = \phi(|x|)$. In polar coordinates defined by $x_1 = r \cos\theta$, $x_2 = r \sin\theta$, we set

$$
\Psi(x) = \psi(x) - W x_2 = (\phi(r) - W) r \sin\theta =: \eta(r) \sin\theta.
$$

We prove $\Psi = \Psi_L$. By (6.1), Ψ satisfies

$$
(6.7)\qquad
\begin{aligned}
-\Delta\Psi &= \Psi \quad \text{in } \Omega,\\
-\Delta\Psi &= 0 \quad \text{in } \mathbb{R}^2_+\backslash\Omega,\\
\Psi &= 0 \quad \text{on } \partial\mathbb{R}^2_+ \cup \partial\Omega,\\
\partial_{x_1}\Psi \to 0,\ \partial_{x_2}\Psi &\to -W \quad \text{as } |x| \to \infty.
\end{aligned}
$$

Since $\phi(r)$ is decreasing for $r > 0$ and $\Psi = 0$ on $\partial\Omega$, there exists some $a > 0$ such that $\phi(a) = W$ and $\Omega = B(0, a) \cap \mathbb{R}^2_+$. Substituting $\Psi = \eta(r)\sin\theta$ into $(6.7)_1$ implies that $\eta(r)$ is a solution of the Bessel's differential equation:

$$
(6.8)\qquad
\begin{aligned}
\ddot{\eta} + \frac{1}{r}\dot{\eta} - \frac{1}{r^2}\eta + \eta = 0,\ \eta > 0, \quad 0 < r < a,\\
\eta(a) = 0.
\end{aligned}
$$

Solutions of (6.8) are given by a linear combination of the Bessel functions of the first and second kind of order one. Since $\eta(r) > 0$ is bounded at $r = 0$ and $\eta(a) = 0$,

$$
\begin{aligned}
&\eta(r) = C_1 J_1(r),\\
&a = c_0,
\end{aligned}
$$

for some constant C_1, where c_0 is the first zero point of J_1. Hence, $\Psi(x) = C_1 J_1(r)\sin\theta$ for $r \leq a$.

In a similar way, we consider the region $r \geq a$. Since Ψ is harmonic for $r > a$, $\eta = C_2/r + C_3 r$ with some constants C_2, C_3. Since $\nabla\Psi = (C_2/r^2)^t(-\sin 2\theta, \cos 2\theta) + {}^t(0, C_3)$, sending $r \to \infty$ implies that $C_3 = -W$. By $\Psi = 0$ for $r = a$, $C_2 = Wa^2$. Hence $\Psi(x) = -W(r - a^2/r)\sin\theta$ for $r > a$.

The constant C_1 is determined by continuity of $\partial_r\Psi$ at $r = a$, i.e., $\lim_{r\to a+0}\partial_r\Psi = \lim_{r\to a-0}\partial_r\Psi$. By using $\dot{J}_1(c_0) = J_0(c_0)$, $C_1 = -2W/J_0(c_0) = C_L$ follows. We have proved $\Psi = \Psi_L$. □

Proof of Theorem 1.5. By the scaling (1.12), we reduce to the case $\nu = \lambda = 1$. By Theorem 1.3, S_μ is not empty, i.e., $S_\mu \neq \emptyset$. Let $0 < \mu \leq M_1$ for the constant $M_1 > 0$ as in Remarks 2.6 (iii). For an arbitrary $\omega \in S_\mu$, the associated stream function ψ is a positive solution of (6.1) for some $W > 0$ satisfying ψ, $\psi/x_2 \to 0$ as $|x| \to \infty$ and for $\Omega = \{\psi - Wx_2 > 0\}$, $\overline{\Omega}$ is compact in $\overline{\mathbb{R}^2_+}$ by Lemma 2.9. Applying Theorem 6.1 and $\omega \in K_\mu$ imply that ω is translation of the Lamb dipole ω_L for $W = \mu/(c_0^2\pi)$. Hence $S_\mu \subset \{\omega_L(\cdot + y) \mid y \in \partial\mathbb{R}^2_+\}$.

Since $S_\mu \neq \emptyset$, there exists $\omega \in S_\mu$ and $y_0 \in \partial\mathbb{R}^2_+$ such that $\omega = \omega_L(\cdot + y_0)$ for the Lamb dipole ω_L for $W = \mu/(c_0^2\pi)$. By translation invariance of E_2 for the x_1-variable, $\{\omega_L(\cdot + y) \mid y \in \partial\mathbb{R}^2_+\} \subset S_\mu$ follows. We have proved (1.11). The proof is now complete. □

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