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<b>Citation</b>	Communications in Algebra.50(8); 3608-3617
<b>Published</b>	2022-02-18
<b>Type</b>	Journal Article
<b>Textversion</b>	Author
<b>Rights</b>	This is an Accepted Manuscript of an article published by Taylor & Francis in Communications in Algebra on 18 Feb 2022, available at: <a href="https://doi.org/10.1080/00927872.2022.2036749">https://doi.org/10.1080/00927872.2022.2036749</a>
<b>DOI</b>	10.1080/00927872.2022.2036749

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# SOLUTION TO THE REFLECTION EQUATION RELATED TO THE $\imath$ QUANTUM GROUP OF TYPE AII

HIROTO KUSANO AND MASATO OKADO

ABSTRACT. A solution to the reflection equation associated to a coideal subalgebra of  $U_q(A_{2n-1}^{(1)})$  of type AII in the symmetric tensor representations is presented. If parameters of the coideal subalgebra are suitably chosen, the  $K$  matrix does not depend on the quantum parameter  $q$  and still agrees with a solution in [8] at  $q = 0$ .

## 1. INTRODUCTION

Reflection equation assures the integrability in one-dimensional quantum systems or two-dimensional statistical models with boundaries. In the context of quantum integrability, it is an equation involving two kinds of linear operators, called quantum  $R$  and  $K$  matrices, on the twofold tensor product of vector spaces. The mathematical framework to construct its solution lies in considering a pair of a quantum group and its coideal subalgebra. They are called a quantum symmetric pair [10] or an  $\imath$ quantum group [3] and known to be classified by Satake diagrams [10, 7]. In such a situation,  $R$  and  $K$  matrices contain the quantum parameter  $q$ . Moreover, if the representations have crystal bases in the sense of Kashiwara [6], one can take the limit where  $q$  goes to 0, and we obtain bijections between sets that still satisfy a combinatorial version of the reflection equation.

In [8], from the motivation of constructing a so-called box-ball system with boundary, we found three solutions of the combinatorial  $K$  matrix where the combinatorial  $R$  matrix in the reflection equation comes from the crystal basis of the symmetric tensor representation of the quantum affine algebra of type  $A$ . See (2.10)-(2.12) of [8]. They were called ‘‘Rotateleft’’, ‘‘Switch<sub>12</sub>’’ and ‘‘Switch<sub>1n</sub>’’. However, their quantum versions, namely, solutions of quantum  $K$  matrices, were not found for a long time. Only recently, in [9] the solution corresponding to ‘‘Rotateleft’’ were found. The purpose of this note is to find the origin of the other two solutions ‘‘Switch<sub>12</sub>’’ and ‘‘Switch<sub>1n</sub>’’ from the list of  $\imath$ quantum groups. The correct one was found to be the affine version of type AII. See e.g. [10, 7, 12]. Rather surprisingly, if we choose parameters in our  $\imath$ quantum group suitably, the  $K$  matrices does not depend on  $q$ , although the  $R$  matrices do.

There are many  $\imath$ quantum groups other than affine type AII which we dealt with in this note, and there also exists a notion of the universal (or quasi)  $K$  matrix [2, 3, 4, 1] as with the universal (quasi)  $R$  matrix of a quantum group. We hope to report more solutions of the reflection equation that become combinatorial upon taking the limit  $q \rightarrow 0$  in near future.

## 2. $U_q(A_{2n-1}^{(1)})$ AND RELEVANT $R$ MATRICES

**2.1.  $U_q(A_{2n-1}^{(1)})$  and relevant representations.** Let  $\mathbf{U} = U_q(A_{2n-1}^{(1)})$  be the Drinfeld-Jimbo quantum affine algebra (without the derivation operator). In this note, we assume  $n \geq 2$ .  $\mathbf{U}$  is generated by  $e_i, f_i, k_i^{\pm 1}$  ( $i \in \mathbb{Z}_{2n}$ ) obeying the relations

$$\begin{aligned} k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad [k_i, k_j] = 0, \quad k_i e_j k_i^{-1} = q^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = q^{-a_{ij}} f_j, \quad [e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}}, \\ \sum_{\nu=0}^{1-a_{ij}} (-1)^\nu e_i^{(1-a_{ij}-\nu)} e_j e_i^{(\nu)} = 0, \quad \sum_{\nu=0}^{1-a_{ij}} (-1)^\nu f_i^{(1-a_{ij}-\nu)} f_j f_i^{(\nu)} = 0 \quad (i \neq j), \end{aligned} \tag{1}$$

where  $e_i^{(\nu)} = e_i^\nu / [\nu]!$ ,  $f_i^{(\nu)} = f_i^\nu / [\nu]!$  and  $[m]! = \prod_{j=1}^m [j]$ . The Cartan matrix  $(a_{ij})_{i,j \in \mathbb{Z}_{2n}}$  is given by  $a_{ij} = 2\delta_{i,j} - \delta_{i,j+1} - \delta_{i,j-1}$ . It is well known that  $\mathbf{U}$  is a Hopf algebra. We employ the coproduct  $\Delta$  of the form

$$\Delta(k_i^{\pm 1}) = k_i^{\pm 1} \otimes k_i^{\pm 1}, \quad \Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i. \tag{2}$$

We will be concerned with the two irreducible representations of  $\mathbf{U}$  labeled with a positive integer  $l$ :

$$\pi_{l,x} : \mathbf{U} \rightarrow \text{End}(V_{l,x}), \quad V_{l,x} = \bigoplus_{\alpha \in B_l} \mathbb{Q}(q)v_\alpha, \quad (3)$$

$$\pi_{l,x}^* : \mathbf{U} \rightarrow \text{End}(V_{l,x}^*), \quad V_{l,x}^* = \bigoplus_{\alpha \in B_l} \mathbb{Q}(q)v_\alpha^*, \quad (4)$$

where  $x$  is a spectral parameter in  $\mathbb{Q}(q)$  and

$$B_l = \{\alpha = (\alpha_1, \dots, \alpha_{2n}) \in \mathbb{Z}_{\geq 0}^{2n} \mid |\alpha| = l\}. \quad (5)$$

Here  $|\alpha| = \sum_{i=1}^{2n} \alpha_i$ . The actions of the generators of  $\mathbf{U}$  on these representations are given by

$$e_i v_\alpha = x^{\delta_{i,0}} [\alpha_{i+1}] v_{\alpha + e_i - e_{i+1}}, \quad e_i v_\alpha^* = x^{\delta_{i,0}} [\alpha_i] v_{\alpha - e_i + e_{i+1}}^*, \quad (6)$$

$$f_i v_\alpha = x^{-\delta_{i,0}} [\alpha_i] v_{\alpha - e_i + e_{i+1}}, \quad f_i v_\alpha^* = x^{-\delta_{i,0}} [\alpha_{i+1}] v_{\alpha + e_i - e_{i+1}}^*, \quad (7)$$

$$k_i v_\alpha = q^{\alpha_i - \alpha_{i+1}} v_\alpha, \quad k_i v_\alpha^* = q^{-\alpha_i + \alpha_{i+1}} v_\alpha^*. \quad (8)$$

Here  $e_i$  is the  $i$ -th standard basis vector and the index  $j$  of the Chevalley generators or  $\alpha$  should be understood as elements of  $\mathbb{Z}_{2n}$ .  $V_{l,x}$  is the  $l$ -th symmetric tensor representation of  $\mathbf{U}$ .  $V_{l,x}^*$  is constructed on the dual space of  $V_{l,x}$  by using the anti-automorphism  $*$  of  $\mathbf{U}$  defined on the generators as

$$e_i^* = e_i, \quad f_i^* = f_i, \quad k_i^* = k_i^{-1},$$

and by defining actions on  $V_{l,x}^*$  as  $\langle uv^*, v \rangle = \langle v^*, u^*v \rangle$  for  $u \in \mathbf{U}, v \in V_{l,x}, v^* \in V_{l,x}^*$ . Our basis  $\{v_\alpha^*\}$  of  $V_{l,x}^*$  is changed from the dual basis of  $\{v_\alpha\}$  by multiplying  $\prod_j [\alpha_j]!^{-1}$  on each dual basis vector, so it turns out that when  $x = 1$  both  $\{v_\alpha\}$  and  $\{v_\alpha^*\}$  are upper crystal bases [6]. At  $q = 0$ , the former gives the crystal  $B_l$  and the latter its dual  $B_l^\vee$  in [8].

**2.2.  $R$  matrices.** We consider the following three  $R$  matrices  $R, R^*, R^{**}$  that are defined as intertwiners between the tensor product representations below.

$$R(x/y) : V_{l,x} \otimes V_{m,y} \rightarrow V_{m,y} \otimes V_{l,x}, \quad (\pi_{m,y} \otimes \pi_{l,x}) \Delta(u) R(x/y) = R(x/y) (\pi_{l,x} \otimes \pi_{m,y}) \Delta(u), \quad (9)$$

$$R^*(x/y) : V_{l,x}^* \otimes V_{m,y} \rightarrow V_{m,y} \otimes V_{l,x}^*, \quad (\pi_{m,y} \otimes \pi_{l,x}^*) \Delta(u) R^*(x/y) = R^*(x/y) (\pi_{l,x}^* \otimes \pi_{m,y}) \Delta(u), \quad (10)$$

$$R^{**}(x/y) : V_{l,x}^* \otimes V_{m,y} \rightarrow V_{m,y}^* \otimes V_{l,x}, \quad (\pi_{m,y}^* \otimes \pi_{l,x}^*) \Delta(u) R^{**}(x/y) = R^{**}(x/y) (\pi_{l,x}^* \otimes \pi_{m,y}^*) \Delta(u), \quad (11)$$

where  $u \in \mathbf{U}$ . They satisfy the Yang-Baxter equations:

$$(1 \otimes R(x))(R(xy) \otimes 1)(1 \otimes R(y)) = (R(y) \otimes 1)(1 \otimes R(xy))(R(x) \otimes 1), \quad (12)$$

$$(1 \otimes R^*(x))(R^*(xy) \otimes 1)(1 \otimes R(y)) = (R(y) \otimes 1)(1 \otimes R^*(xy))(R^*(x) \otimes 1), \quad (13)$$

$$(1 \otimes R^{**}(x))(R^{**}(xy) \otimes 1)(1 \otimes R^*(y)) = (R^*(y) \otimes 1)(1 \otimes R^{**}(xy))(R^{**}(x) \otimes 1), \quad (14)$$

$$(1 \otimes R^{**}(x))(R^{**}(xy) \otimes 1)(1 \otimes R^{**}(y)) = (R^{**}(y) \otimes 1)(1 \otimes R^{**}(xy))(R^{**}(x) \otimes 1). \quad (15)$$

### 3. REFLECTION EQUATION AND ITS SOLUTION

**3.1. Coideal subalgebra.** We consider two coideal subalgebras  $\mathbf{U}_\varepsilon^l$  ( $\varepsilon = 0, 1$ ) of  $\mathbf{U}$ . Set  $I = \{0, 1, \dots, 2n-1\}$ . An element of  $I$  is considered to correspond to a vertex of the Dynkin diagram of  $A_{2n-1}^{(1)}$ . In view of this, we identify  $I$  with  $\mathbb{Z}_{2n}$ . For each  $\varepsilon = 0, 1$ , set

$$I_\circ = \{\varepsilon, 2 + \varepsilon, \dots, 2n - 2 + \varepsilon\}, \quad I_\bullet = I \setminus I_\circ.$$

We define two subalgebras  $\mathbf{U}_\varepsilon^l$  of  $\mathbf{U}$  for  $\varepsilon = 0, 1$ . Each one is generated by  $e_i, f_i, k_i$  ( $i \in I_\bullet$ ),  $b_i$  ( $i \in I_\circ$ ) where

$$b_i = f_i + \gamma_i T_{w_\bullet}(e_i) k_i^{-1},$$

$$T_{w_\bullet}(e_i) = e_{i+1} e_{i-1} e_i - q^{-1} (e_{i+1} e_i e_{i-1} + e_{i-1} e_i e_{i+1}) + q^{-2} e_i e_{i-1} e_{i+1}.$$

Here  $\gamma_i$  is a constant. Then, we have the following facts which are well known. See [10, 7, 12] for instance.

**Proposition 1.** For  $i \in I_\circ$ ,  $e_{i\pm 1} b_i = b_i e_{i\pm 1}$ .

**Proposition 2.**  $\mathbf{U}_\varepsilon^l$  is a right coideal subalgebra of  $\mathbf{U}$ . Namely, we have  $\Delta(\mathbf{U}_\varepsilon^l) \subset \mathbf{U}_\varepsilon^l \otimes \mathbf{U}$ .

We also use the following result later.

TABLE 1. Satake diagrams of  $\mathbf{U}_0^e$  and  $\mathbf{U}_1^e$ 

**Lemma 3.** For  $i \in I_o$ , the action of  $b_i$  on  $V_{l,x}$  or  $V_{l,x}^*$  is given by

$$\begin{aligned} b_i v_\alpha &= x^{-\delta_{i,0}} [\alpha_i] v_{\alpha - e_i + e_{i+1}} - x^{\delta_{i,0} + \delta_{i,1} + \delta_{i,-1}} q^{-1} \gamma_i [\alpha_{i+2}] v_{\alpha + e_{i-1} - e_{i+2}}, \\ b_i v_\alpha^* &= x^{-\delta_{i,0}} [\alpha_{i+1}] v_{\alpha^* + e_i - e_{i+1}} - x^{\delta_{i,0} - \delta_{i,1} - \delta_{i,-1}} q^{-1} \gamma_i [\alpha_{i-1}] v_{\alpha^* - e_{i-1} + e_{i+2}}. \end{aligned}$$

**3.2.  $K$  matrix and the reflection equation.** For each  $\varepsilon = 0, 1$ , consider a linear map  $K(x) : V_{l,x} \rightarrow V_{l,x-1}^*$  satisfying

$$K(x) \pi_{l,x}(a) = \pi_{l,x-1}^*(a) K(x) \quad \text{for any } a \in \mathbf{U}_\varepsilon^e. \quad (16)$$

To describe the solution, we introduce a particular permutation  $\sigma^{(\varepsilon)}$  of entries of  $\alpha$  for  $\varepsilon = 0, 1$ .  $\sigma^{(\varepsilon)}$  switches  $\alpha_{i-1}$  and  $\alpha_i$  whenever  $i \equiv \varepsilon \pmod{2}$ . For instance, when  $n = 3$  we have

$$\sigma^{(0)}(\alpha) = (\alpha_2, \alpha_1, \alpha_4, \alpha_3, \alpha_6, \alpha_5), \quad \sigma^{(1)}(\alpha) = (\alpha_6, \alpha_3, \alpha_2, \alpha_5, \alpha_4, \alpha_1).$$

**Proposition 4.** For each  $\varepsilon = 0, 1$ , the intertwining relation (16) has a solution if and only if

$$\prod_{j \in I_o} \gamma_j = (-q)^n,$$

in which case the solution is unique up to scalar multiple and given by

$$K(x) v_\alpha = x^{\varepsilon(\alpha_1 - \alpha_{2n})} \prod_{j=\varepsilon, 2+\varepsilon, \dots, 2n-2+\varepsilon} (-q^{-1} \gamma_j)^{-\sum_{i=1+\varepsilon}^j \alpha_i} v_{\sigma^{(\varepsilon)}(\alpha)}^*.$$

*Proof.* In the proof we assume  $i \in I_\bullet, j \in I_o$ . Define  $K_\alpha^\beta$  by  $K(x) v_\alpha = \sum_\beta K_\alpha^\beta v_\beta^*$ . Note that  $K_\alpha^\beta$  also depends on  $x$ . Comparing the coefficients of  $v_\beta^*$  in  $K(x) \pi_{l,x}(a) v_\alpha = \pi_{l,x-1}^*(a) K(x) v_\alpha$  with  $k_i, e_i, f_i, b_j$  we obtain

$$K_\alpha^\beta \neq 0 \Rightarrow \alpha_i - \alpha_{i+1} = -\beta_i + \beta_{i+1}, \quad (17)$$

$$[\beta_i + 1] K_\alpha^{\beta + e_i - e_{i+1}} = x^{2\delta_{i,0}} [\alpha_{i+1}] K_{\alpha + e_i - e_{i+1}}^\beta, \quad (18)$$

$$[\alpha_i + 1] K_\alpha^{\beta + e_i - e_{i+1}} = x^{2\delta_{i,0}} [\beta_{i+1}] K_{\alpha + e_i - e_{i+1}}^\beta, \quad (19)$$

$$\begin{aligned} x^{\delta_{j,0}} [\beta_{j+1} + 1] K_\alpha^{\beta - e_j + e_{j+1}} - x^{-\delta_{j,0} - \delta_{j,1} - \delta_{j,-1}} q^{-1} \gamma_j [\beta_{j-1} + 1] K_\alpha^{\beta + e_{j-1} - e_{j+2}} \\ = x^{-\delta_{j,0}} [\alpha_j] K_\alpha^{\beta - e_j + e_{j+1}} - x^{\delta_{j,0} + \delta_{j,1} + \delta_{j,-1}} q^{-1} \gamma_j [\alpha_{j+2}] K_\alpha^{\beta + e_{j-1} - e_{j+2}}. \end{aligned} \quad (20)$$

Since we look for a nontrivial solution, we assume the right hand side of (17). This condition together with (18),(19) implies

$$\alpha_i = \beta_{i+1}, \quad \beta_i = \alpha_{i+1} \quad (21)$$

or equivalently  $\beta = \sigma^{(\varepsilon)}(\alpha)$ . Then by taking  $\beta$  to  $\sigma^{(\varepsilon)}(\alpha) - e_i + e_{i+1}$  in (18) or (19), we have

$$K_\alpha^{\sigma^{(\varepsilon)}(\alpha)} = x^{2\delta_{i,0}} K_{\alpha + e_i - e_{i+1}}^{\sigma^{(\varepsilon)}(\alpha + e_i - e_{i+1})}. \quad (22)$$

Similarly, assuming (21), (20) reduces to

$$\begin{aligned} x^{\delta_{j,0}} [\alpha_{j+2}] (K_\alpha^{\beta - e_j + e_{j+1}} + x^{\delta_{j,1} + \delta_{j,-1}} q^{-1} \gamma_j K_\alpha^{\beta + e_{j-1} - e_{j+2}}) \\ = x^{-\delta_{j,0}} [\alpha_j] (K_\alpha^{\beta - e_j + e_{j+1}} + x^{-\delta_{j,1} - \delta_{j,-1}} q^{-1} \gamma_j K_\alpha^{\beta + e_{j-1} - e_{j+2}}). \end{aligned}$$

If  $\beta = \sigma^{(\varepsilon)}(\alpha) + e_j - e_{j+1}$ , the right hand side vanishes, whereas if  $\beta = \sigma^{(\varepsilon)}(\alpha - e_j + e_{j+1})$ , the left one does. Under (22), both conditions reduce to

$$K_{\alpha - e_{j-1} + e_{j+1}}^{\sigma^{(\varepsilon)}(\alpha - e_{j-1} + e_{j+1})} / K_\alpha^{\sigma^{(\varepsilon)}(\alpha)} = -x^{\delta_{j,1} - \delta_{j,-1}} q^{-1} \gamma_j. \quad (23)$$

Replacing  $\alpha$  with  $\alpha + e_{j-1}$  in this equation and multiplying it for  $j = \varepsilon, 2 + \varepsilon, \dots, 2n - 2 + \varepsilon$ , we obtain the condition for  $K$  to exist. Set  $K_\alpha^{\sigma^{(\varepsilon)}(\alpha)} = x^{\varepsilon(\alpha_1 - \alpha_{2n})} \mathcal{K}_\alpha$ . Then (22) and (23) reduce to

$$\mathcal{K}_\alpha = \mathcal{K}_{\alpha + e_i - e_{i+1}}, \quad \mathcal{K}_{\alpha - e_{j-1} + e_{j+1}} / \mathcal{K}_\alpha = -q^{-1} \gamma_j.$$

Recall  $i \in I_\bullet, j \in I_\circ$ . From the first relation, one notices that  $\mathcal{K}_\alpha$  depends only on  $\alpha_i + \alpha_{i+1}$  ( $i \in I_\bullet$ ). The second one then determines  $\mathcal{K}_\alpha$  uniquely up to scalar multiple.  $\square$

**Remark 5.** The  $K$  matrix corresponding to the same Satake diagram to ours but in the vector representation  $V_1$  was considered in [11, Result 9.9].

In view of this proposition, we set  $\gamma_j = -q$  for any  $j \in I_\circ$  later in this note.

**Theorem 6.** *The reflection equation*

$$K_1(x)R^*((xy)^{-1})K_1(y)R(xy^{-1}) = R^*(xy^{-1})K_1(y)R^*((xy)^{-1})K_1(x) \quad (24)$$

holds as a linear map  $V_{l,x} \otimes V_{m,y} \rightarrow V_{l,x^{-1}}^* \otimes V_{m,y^{-1}}^*$ . Here  $K_1(x) = K(x) \otimes 1$ .

The proof is completely the same as that in [9, Theorem 1], although a similar reasoning goes back to [5] at the latest, under the assumption that  $V_{l,x} \otimes V_{m,y}$  is irreducible as a  $\mathbf{U}_\varepsilon^\varepsilon$ -module, which is shown in next section.

#### 4. PROOF OF THE IRREDUCIBILITY OF $V_{l,x} \otimes V_{m,y}$

To show that the reflection equation holds (Theorem 6), we need to prove

**Theorem 7.** *As a  $\mathbf{U}_\varepsilon^\varepsilon$ -module,  $V_{l,x} \otimes V_{m,y}$  is irreducible.*

Actually, even when the spectral parameters  $x, y$  are specialized to 1, it is irreducible as we will see below. Hence, in this section we set  $x = y = 1$ , since it is enough to show the theorem.  $V_{l,1}$  will be denoted by  $V_l$ . We can also restrict our proof to the  $\varepsilon = 0$  case, since the consideration for the  $\varepsilon = 1$  case is just the repetition by shifting the index  $i$  of the generators or the entries of  $\alpha$ . Finally, in view of Proposition 4, we specialize  $\gamma_i$  for  $i \in I_\circ$  to be  $-q$ .

**4.1. Representation theory of  $U_q(sl_2)$ .**  $U_q(sl_2)$  is the subalgebra of  $\mathbf{U}$  generated only by  $e_1, f_1, k_1$ . Its irreducible representations are parametrized by their dimensions which run positive integers. Let  $U_l$  be the  $(l+1)$ -dimensional module of  $U_q(sl_2)$ . As a basis of  $U_l$ , one can take  $\{v_\alpha \mid |\alpha| = l\}$  in (3) with  $n = 1$ . The actions of the generators  $e_1, f_1, k_1$  are given by (6)-(8). It is well known that  $U_l \otimes U_m$  decomposes into  $\min(l, m) + 1$  components as

$$U_l \otimes U_m \simeq \bigoplus_{j=0}^{\min(l,m)} U_{l+m-2j}$$

where a highest weight vector of  $U_{l+m-2j}$  is given by

$$w_j^{(l,m)} = \sum_{p=0}^j (-1)^p q^{p(l-p+1)} \begin{bmatrix} j \\ p \end{bmatrix} v_{(l-p,p)} \otimes v_{(m-j+p,j-p)}. \quad (25)$$

Here  $\begin{bmatrix} j \\ p \end{bmatrix}$  is the  $q$ -binomial coefficient defined by  $\frac{[j]!}{[p]![j-p]!}$ .

Now consider the subalgebra  $\mathbf{U}(I_\bullet)$  of  $\mathbf{U}^\varepsilon$  generated by  $e_i, f_i, k_i$  ( $i \in I_\bullet$ ). Recall  $I_\bullet = \{1, 3, \dots, 2n-1\}$ .  $\mathbf{U}(I_\bullet)$  is isomorphic to  $U_q(sl_2)^{\otimes n}$ . We want to construct a basis of  $V_l \otimes V_m$  using its  $\mathbf{U}(I_\bullet)$ -module structure. To parametrize the highest weight vectors of  $V_l \otimes V_m$ , we introduce  $n$ -tuples of nonnegative integers  $\mathbf{l} = (l_1, \dots, l_n), \mathbf{m} = (m_1, \dots, m_n)$  such that  $|\mathbf{l}| = l, |\mathbf{m}| = m$ . Here we use the notation  $|\mathbf{l}|$  to signify the sum of entries of the vector  $\mathbf{l}$  irrespective of the number of entries. Let

$$\iota : \bigoplus_{\mathbf{l}, \mathbf{m}} (U_{l_1} \otimes U_{m_1}) \otimes \cdots \otimes (U_{l_n} \otimes U_{m_n}) \longrightarrow V_l \otimes V_m$$

be the linear map sending  $(v_{(\alpha_1, \alpha_2)} \otimes v_{(\beta_1, \beta_2)}) \otimes \cdots \otimes (v_{(\alpha_{2n-1}, \alpha_{2n})} \otimes v_{(\beta_{2n-1}, \beta_{2n})})$  to  $v_\alpha \otimes v_\beta$ . Note that  $U_{l_i} \otimes U_{m_i}$  is the tensor product of the irreducible highest weight modules  $U_{l_i}, U_{m_i}$  of the  $i$ -th  $U_q(sl_2)$  of  $U_q(sl_2)^{\otimes n}$  generated by  $e_{2i-1}, f_{2i-1}, k_{2i-1}$ . Since  $U_q(sl_2)$  in different positions commute with each other, one obtains the following proposition.

**Proposition 8.** *For any  $\mathbf{l}, \mathbf{m}$  and  $\mathbf{j} = (j_1, \dots, j_n)$  such that  $0 \leq j_i \leq \min(l_i, m_i)$  for  $1 \leq i \leq n$ ,*

$$\mathbf{w}_{\mathbf{j}}^{(\mathbf{l}, \mathbf{m})} = \iota(w_{j_1}^{(l_1, m_1)} \otimes \cdots \otimes w_{j_n}^{(l_n, m_n)})$$

is a  $\mathbf{U}(I_\bullet)$ -highest weight vector, and we have  $\bigoplus_{\mathbf{l}, \mathbf{m}, \mathbf{j}} \mathbf{U}(I_\bullet) \mathbf{w}_{\mathbf{j}}^{(\mathbf{l}, \mathbf{m})} = V_l \otimes V_m$ .

4.2. **Necessary formulas.** In what follows, we assume  $i \in I_o = \{0, 2, \dots, 2n-2\}$  and set  $i = 2s$ . By abuse of notation, we denote by  $e_s$  ( $s = 1, \dots, n$ ) the  $s$ -th standard basis vector of the  $n$ -dimensional space, although we have used it in section 2 for the  $2n$ -dimensional space.  $e_0$  should be understood as  $e_n$ . For the action of  $\mathbf{U}$  on the tensor product, we abbreviate  $\Delta$ .

**Proposition 9.** *On  $V_l \otimes V_m$ , we have*

$$b_i \mathbf{w}_j^{(l, m)} = D'_1 \mathbf{w}_{j-e_s}^{(l-e_s+e_{s+1}, m)} + D'_2 \mathbf{w}_{j-e_s}^{(l, m-e_s+e_{s+1})} + D'_3 \mathbf{w}_{j-e_{s+1}}^{(l+e_s-e_{s+1}, m)} + D'_4 \mathbf{w}_{j-e_{s+1}}^{(l, m+e_s-e_{s+1})},$$

where

$$\begin{aligned} D'_1 &= -q^{-j_s-j_{s+1}+l_s+m_{s+1}+1} [j_s], & D'_2 &= [j_s], \\ D'_3 &= -q^{-j_s-j_{s+1}+l_{s+1}+m_{s+1}+1} [j_{s+1}], & D'_4 &= q^{-2j_s-2j_{s+1}+l_s+l_{s+1}+2m_{s+1}+2} [j_{s+1}]. \end{aligned}$$

*Proof.* Using Proposition 1, one finds that  $b_i \mathbf{w}_j^{(l, m)}$  is a  $\mathbf{U}(I_\bullet)$ -highest weight vector. By the weight consideration, it should be a linear combination of the following vectors.

$$\mathbf{w}_{j-e_s}^{(l-e_s+e_{s+1}, m)}, \mathbf{w}_{j-e_s}^{(l, m-e_s+e_{s+1})}, \mathbf{w}_{j-e_{s+1}}^{(l+e_s-e_{s+1}, m)}, \mathbf{w}_{j-e_{s+1}}^{(l, m+e_s-e_{s+1})}.$$

The four coefficients can be calculated directly. □

**Proposition 10.** *On  $V_l \otimes V_m$ , we have*

$$\begin{aligned} b_i f_{i-1} \mathbf{w}_j^{(l, m)} &= \frac{[l_s + m_s - j_s + 1]}{[l_s + m_s - 2j_s + 1]} (B'_1 \mathbf{w}_j^{(l-e_s+e_{s+1}, m)} + B'_2 \mathbf{w}_j^{(l, m-e_s+e_{s+1})}) \\ &\quad + \frac{[j_{s+1}]}{[l_s + m_s - 2j_s + 1]} (B'_3 \mathbf{w}_{j+e_s-e_{s+1}}^{(l+e_s-e_{s+1}, m)} + B'_4 \mathbf{w}_{j+e_s-e_{s+1}}^{(l, m+e_s-e_{s+1})}) \\ &\quad + \frac{[l_s + m_s - 2j_s]}{[l_s + m_s - 2j_s + 1]} (D'_1 f_{i-1} \mathbf{w}_{j-e_s}^{(l-e_s+e_{s+1}, m)} + D'_2 f_{i-1} \mathbf{w}_{j-e_s}^{(l, m-e_s+e_{s+1})}) \\ &\quad + D'_3 f_{i-1} \mathbf{w}_{j-e_{s+1}}^{(l+e_s-e_{s+1}, m)} + D'_4 f_{i-1} \mathbf{w}_{j-e_{s+1}}^{(l, m+e_s-e_{s+1})}, \end{aligned}$$

$$\begin{aligned} b_i f_{i+1} \mathbf{w}_j^{(l, m)} &= \frac{[j_s]}{[l_{s+1} + m_{s+1} - 2j_{s+1} + 1]} (C'_1 \mathbf{w}_{j-e_s+e_{s+1}}^{(l-e_s+e_{s+1}, m)} + C'_2 \mathbf{w}_{j-e_s+e_{s+1}}^{(l, m-e_s+e_{s+1})}) \\ &\quad + \frac{[l_{s+1} + m_{s+1} - j_{s+1} + 1]}{[l_{s+1} + m_{s+1} - 2j_{s+1} + 1]} (C'_3 \mathbf{w}_j^{(l+e_s-e_{s+1}, m)} + C'_4 \mathbf{w}_j^{(l, m+e_s-e_{s+1})}) \\ &\quad + \frac{[l_{s+1} + m_{s+1} - 2j_{s+1}]}{[l_{s+1} + m_{s+1} - 2j_{s+1} + 1]} (D'_1 f_{i+1} \mathbf{w}_{j-e_s}^{(l-e_s+e_{s+1}, m)} + D'_2 f_{i+1} \mathbf{w}_{j-e_s}^{(l, m-e_s+e_{s+1})}) \\ &\quad + D'_3 f_{i+1} \mathbf{w}_{j-e_{s+1}}^{(l+e_s-e_{s+1}, m)} + D'_4 f_{i+1} \mathbf{w}_{j-e_{s+1}}^{(l, m+e_s-e_{s+1})}, \end{aligned}$$

where

$$\begin{aligned} B'_1 &= q^{j_s-j_{s+1}-m_s+m_{s+1}} [l_s - j_s], & B'_2 &= [m_s - j_s], \\ B'_3 &= -q^{j_s-j_{s+1}-l_s-m_s+l_{s+1}+m_{s+1}} [m_s - j_s], & B'_4 &= -q^{2j_s-2j_{s+1}-l_s-2m_s+l_{s+1}+2m_{s+1}} [l_s - j_s], \\ C'_1 &= -q^{-j_s+j_{s+1}+l_s-l_{s+1}} [m_{s+1} - j_{s+1}], & C'_2 &= -[l_{s+1} - j_{s+1}], \\ C'_3 &= q^{-j_s+j_{s+1}} [l_{s+1} - j_{s+1}], & C'_4 &= q^{-2j_s+2j_{s+1}+l_s-l_{s+1}} [m_{s+1} - j_{s+1}], \end{aligned}$$

*Proof.* Using Proposition 1, we have

$$\begin{aligned} e_{i-1} b_i f_{i-1} \mathbf{w}_j^{(l, m)} &= b_i e_{i-1} f_{i-1} \mathbf{w}_j^{(l, m)} = b_i \{k_{i-1}\} \mathbf{w}_j^{(l, m)} = [l_s + m_s - 2j_s] b_i \mathbf{w}_j^{(l, m)}, \\ e_{i+1} b_i f_{i+1} \mathbf{w}_j^{(l, m)} &= b_i e_{i+1} f_{i+1} \mathbf{w}_j^{(l, m)} = b_i \{k_{i+1}\} \mathbf{w}_j^{(l, m)} = [l_{s+1} + m_{s+1} - 2j_{s+1}] b_i \mathbf{w}_j^{(l, m)}, \end{aligned}$$

where  $\{k_i\} = \frac{k_i - k_i^{-1}}{q - q^{-1}}$ . Thus same in Lemma 9,  $e_{i+1} b_i f_{i+1}$  and  $e_{i-1} b_i f_{i-1}$  are a linear combination of the following vectors.

$$\mathbf{w}_{j-e_s}^{(l-e_s+e_{s+1}, m)}, \mathbf{w}_{j-e_s}^{(l, m-e_s+e_{s+1})}, \mathbf{w}_{j-e_{s+1}}^{(l+e_s-e_{s+1}, m)}, \mathbf{w}_{j-e_{s+1}}^{(l, m+e_s-e_{s+1})}.$$

By considering weight, one find that  $b_i f_{i-1}$  and  $b_i f_{i+1}$  are a linear combination like a assertion, and coefficients can be calculated directly. □

**Corollary 11.** *On  $V_l \otimes V_m$ , we have*

$$\begin{aligned} b_i f_{i+1} \mathbf{w}_o^{(l, \mathbf{m})} &= [l_{s+1}] \mathbf{w}_o^{(l+e_s-e_{s+1}, \mathbf{m})} + q^{l'-l_{s+1}} [m_{s+1}] \mathbf{w}_o^{(l, \mathbf{m}+e_s-e_{s+1})}, \\ b_i f_{i-1} \mathbf{w}_o^{(l, \mathbf{m})} &= q^{m_{s+1}-m_s} [l_s] \mathbf{w}_o^{(l-e_s+e_{s+1}, \mathbf{m})} + [m_s] \mathbf{w}_o^{(l, \mathbf{m}-e_s+e_{s+1})}. \end{aligned}$$

**Proposition 12.** *On  $V_l \otimes V_m$ , we have*

$$\begin{aligned} & b_i f_{i-1} f_{i+1} \mathbf{w}_j^{(l, \mathbf{m})} \\ &= A_1 \mathbf{w}_{j+e_{s+1}}^{(l-e_s+e_{s+1}, \mathbf{m})} + B_1 f_{i+1} \mathbf{w}_j^{(l-e_s+e_{s+1}, \mathbf{m})} + C_1 f_{i-1} \mathbf{w}_j^{(l-e_s+e_{s+1}, \mathbf{m})} + D_1 f_{i-1} f_{i+1} \mathbf{w}_{j-e_s}^{(l-e_s+e_{s+1}, \mathbf{m})} \\ &+ A_2 \mathbf{w}_{j+e_{s+1}}^{(l, \mathbf{m}-e_s+e_{s+1})} + B_2 f_{i+1} \mathbf{w}_j^{(l, \mathbf{m}-e_s+e_{s+1})} + C_2 f_{i-1} \mathbf{w}_j^{(l, \mathbf{m}-e_s+e_{s+1})} + D_2 f_{i-1} f_{i+1} \mathbf{w}_{j-e_s}^{(l, \mathbf{m}-e_s+e_{s+1})} \\ &+ A_3 \mathbf{w}_{j+e_s}^{(l+e_s-e_{s+1}, \mathbf{m})} + B_3 f_{i+1} \mathbf{w}_{j+e_s-e_{s+1}}^{(l+e_s-e_{s+1}, \mathbf{m})} + C_3 f_{i-1} \mathbf{w}_j^{(l+e_s-e_{s+1}, \mathbf{m})} + D_3 f_{i-1} f_{i+1} \mathbf{w}_{j-e_{s+1}}^{(l+e_s-e_{s+1}, \mathbf{m})} \\ &+ A_4 \mathbf{w}_{j+e_s}^{(l, \mathbf{m}+e_s-e_{s+1})} + B_4 f_{i+1} \mathbf{w}_{j+e_s-e_{s+1}}^{(l, \mathbf{m}+e_s-e_{s+1})} + C_4 f_{i-1} \mathbf{w}_j^{(l, \mathbf{m}+e_s-e_{s+1})} + D_4 f_{i-1} f_{i+1} \mathbf{w}_{j-e_{s+1}}^{(l, \mathbf{m}+e_s-e_{s+1})}, \end{aligned} \tag{26}$$

where

$$\begin{aligned} A_1 &= q^{j_s+j_{s+1}-l_{s+1}-m_s-1} \frac{[l_s-j_s][m_{s+1}-j_{s+1}][l_s+m_s-j_s+1]}{[l_s+m_s-2j_s+1][l_{s+1}+m_{s+1}-2j_{s+1}+1]} \\ A_2 &= -\frac{[l_{s+1}-j_{s+1}][m_s-j_s][l_s+m_s-j_s+1]}{[l_s+m_s-2j_s+1][l_{s+1}+m_{s+1}-2j_{s+1}+1]} \\ A_3 &= q^{j_s+j_{s+1}-l_s-m_s-1} \frac{[l_{s+1}-j_{s+1}][m_s-j_s][l_{s+1}+m_{s+1}-j_{s+1}+1]}{[l_s+m_s-2j_s+1][l_{s+1}+m_{s+1}-2j_{s+1}+1]} \\ A_4 &= -q^{2j_s+2j_{s+1}-l_s-l_{s+1}-2m_s-2} \frac{[l_s-j_s][m_{s+1}-j_{s+1}][l_{s+1}+m_{s+1}-j_{s+1}+1]}{[l_s+m_s-2j_s+1][l_{s+1}+m_{s+1}-2j_{s+1}+1]}, \\ B_j &= B'_j \frac{[l_s+m_s-j_s+1][l_{s+1}+m_{s+1}-2j_{s+1}]}{[l_s+m_s-2j_s+1][l_{s+1}+m_{s+1}-2j_{s+1}+1]} \quad (j=1, 2), \\ &= B'_j \frac{[j_{s+1}][l_{s+1}+m_{s+1}-2j_{s+1}]}{[l_s+m_s-2j_s+1][l_{s+1}+m_{s+1}-2j_{s+1}+1]} \quad (j=3, 4), \\ C_j &= C'_j \frac{[j_s][l_s+m_s-2j_s]}{[l_s+m_s-2j_s+1][l_{s+1}+m_{s+1}-2j_{s+1}+1]} \quad (j=1, 2), \\ &= C'_j \frac{[l_s+m_s-2j_s][l_{s+1}+m_{s+1}-j_{s+1}+1]}{[l_s+m_s-2j_s+1][l_{s+1}+m_{s+1}-2j_{s+1}+1]} \quad (j=3, 4), \\ D_j &= D'_j \frac{[l_s+m_s-2j_s][l_{s+1}+m_{s+1}-2j_{s+1}]}{[l_s+m_s-2j_s+1][l_{s+1}+m_{s+1}-2j_{s+1}+1]} \quad (j=1, 2, 3, 4). \end{aligned}$$

*Proof.* Similar to Proposition 9 and 10,  $b_i f_{i-1} f_{i+1} \mathbf{w}_j^{(l, \mathbf{m})}$  can be expressed with suitable scalars  $A_j, B_j, C_j, D_j$  ( $1 \leq j \leq 4$ ) as (26). By applying  $e_{i-1} e_{i+1}$  on both sides, the first to third terms in each line of the right hand side vanish. So by Proposition 9,  $D_j$  ( $1 \leq j \leq 4$ ) is determined. Then, by applying  $e_{i+1}$  on both sides of (26),  $B_j$  ( $1 \leq j \leq 4$ ) is determined, and by applying  $e_{i-1}$ ,  $C_j$  ( $1 \leq j \leq 4$ ) is done by Proposition 10. Finally,  $A_j$  ( $1 \leq j \leq 4$ ) is determined by a direct calculation.  $\square$

**Corollary 13.** *On  $V_l \otimes V_m$ , we have*

$$\begin{aligned} b_i f_{i-1} f_{i+1} \mathbf{w}_j^{(l, \mathbf{m})} &= A_1 \mathbf{w}_{j+e_{s+1}}^{(l-e_s+e_{s+1}, \mathbf{m})} + A_2 \mathbf{w}_{j+e_{s+1}}^{(l, \mathbf{m}-e_s+e_{s+1})} + A_3 \mathbf{w}_{j+e_s}^{(l+e_s-e_{s+1}, \mathbf{m})} + A_4 \mathbf{w}_{j+e_s}^{(l, \mathbf{m}+e_s-e_{s+1})} \\ &+ (\text{other terms}), \end{aligned}$$

where  $A_j$  ( $j=1, 2, 3, 4$ ) is given in Proposition 12 and (other terms) stands for the linear combination of vectors of the form  $\mathbf{w}_j^{(l', \mathbf{m}'')}$  possibly applied by  $f_{i-1}, f_{i+1}$  with  $(l', \mathbf{m}'')$  appearing in the right hand side and  $j'_k \leq j_k$  for  $1 \leq k \leq n$ .

**4.3. Proof of Theorem 7.** We prove Theorem 7 when  $\varepsilon = 0$ . Suppose  $W$  is a nonzero  $\mathbf{U}^2$ -invariant subspace of  $V_l \otimes V_m$ . Note that  $\mathbf{U}^2$  contains  $\mathbf{U}(I_\bullet)$ . In view of Proposition 8, one can assume that  $W$  contains a vector of the form

$$\sum_{l, \mathbf{m}, j} c(l, \mathbf{m}, j) \mathbf{w}_j^{(l, \mathbf{m})} \tag{27}$$

where  $c(\mathbf{l}, \mathbf{m}, \mathbf{j}) \in \mathbb{Q}(q)$  and  $\mathbf{l}, \mathbf{m}, \mathbf{j}$  run over all possible integer vectors such that  $l_s + m_s - 2j_s$  is constant for any  $s = 1, \dots, n$ . By applying  $b_i$  ( $i \in I_\circ$ ) in a suitable order, from Proposition 9 one can assume  $\mathbf{j} = \mathbf{o}$  in (27). Then by Corollary 11, one can eventually assume  $\mathbf{l} = l\mathbf{e}_1, \mathbf{m} = m\mathbf{e}_1$  where  $l = |\mathbf{l}|, m = |\mathbf{m}|$ . Hence, we have  $\mathbf{w}_\circ^{(l\mathbf{e}_1, m\mathbf{e}_1)} \in W$ .

Next show  $\mathbf{w}_\circ^{(l_1\mathbf{e}_1+l_2\mathbf{e}_2, m_1\mathbf{e}_1+m_2\mathbf{e}_2)} \in W$  for any  $l_1, l_2, m_1, m_2$  such that  $l_1 + l_2 = l, m_1 + m_2 = m$ . We do it by induction on  $k = l_2 + m_2$ . The  $k = 0$  case is done. Assume  $\mathbf{w}_\circ^{(l_1\mathbf{e}_1+l_2\mathbf{e}_2, m_1\mathbf{e}_1+m_2\mathbf{e}_2)} \in W$  for  $l_2 + m_2 = k$ . By Corollary 11, we have

$$\begin{aligned} b_2 f_1 \mathbf{w}_\circ^{(l_1\mathbf{e}_1+l_2\mathbf{e}_2, m_1\mathbf{e}_1+m_2\mathbf{e}_2)} &= q^{m_2-m_1} [l_1] \mathbf{w}_\circ^{((l_1-1)\mathbf{e}_1+(l_2+1)\mathbf{e}_2, m_1\mathbf{e}_1+m_2\mathbf{e}_2)} + [m_1] \mathbf{w}_\circ^{(l_1\mathbf{e}_1+l_2\mathbf{e}_2, (m_1-1)\mathbf{e}_1+(m_2+1)\mathbf{e}_2)}, \\ (b_4 f_5) \cdots (b_{2n-2} f_{2n-1}) (b_0 f_1) \mathbf{w}_\circ^{(l_1\mathbf{e}_1+l_2\mathbf{e}_2, m_1\mathbf{e}_1+m_2\mathbf{e}_2)} & \\ &= [l_1] \mathbf{w}_\circ^{((l_1-1)\mathbf{e}_1+(l_2+1)\mathbf{e}_2, m_1\mathbf{e}_1+m_2\mathbf{e}_2)} + q^{l_2-l_1} [m_1] \mathbf{w}_\circ^{(l_1\mathbf{e}_1+l_2\mathbf{e}_2, (m_1-1)\mathbf{e}_1+(m_2+1)\mathbf{e}_2)}. \end{aligned}$$

If  $l_1 + m_1 \neq l_2 + m_2$ , these two vectors are linearly independent. Hence the induction proceeds up to  $k \leq l_1 + m_1$ . When  $l_2 + m_2 \geq l_1 + m_1$ , we first recognize that  $\mathbf{w}_\circ^{(l_2, m_2)} \in W$  by applying  $(b_2 f_1)^{l+m}$  to  $\mathbf{w}_\circ^{(l\mathbf{e}_1, m\mathbf{e}_1)}$ . We then do the same exercise as before.

Let us now show  $W$  contains  $\mathbf{w}_\circ^{(\mathbf{l}, \mathbf{m})}$  for any possible  $\mathbf{l}$  and  $\mathbf{m}$ . From the previous paragraph, we know  $\mathbf{w}_\circ^{(l_1\mathbf{e}_1+l_2\mathbf{e}_2, m_1\mathbf{e}_1+m_2\mathbf{e}_2)} \in W$ . Applying  $b_i f_{i-1}$  ( $i = 4, \dots, 2n-2$ ) suitable times, we know  $\mathbf{w}_\circ^{(l, m\mathbf{e}_1)} \in W$  for any  $l$ . Then by doing similarly including  $i = 2$ , we know  $\mathbf{w}_\circ^{(\mathbf{l}, \mathbf{m})} \in W$  for any  $\mathbf{l}, \mathbf{m}$ .

By Proposition 8, it is enough to show  $W$  contains  $\mathbf{w}_\mathbf{j}^{(\mathbf{l}, \mathbf{m})}$  for any possible  $\mathbf{l}, \mathbf{m}, \mathbf{j}$ . From the considerations so far, it is true when  $|\mathbf{j}| = 0$ . The following proposition makes the induction on  $|\mathbf{j}|$  work and finishes the proof of Theorem 7.

**Proposition 14.** *Consider the following matrix  $C$  depending on  $l, m, j$ . Its row index runs over all  $(i, \mathbf{l}, \mathbf{m}, \mathbf{j})$  with  $i = 0, 2, \dots, 2n-2$  and  $|\mathbf{l}| = l, |\mathbf{m}| = m, |\mathbf{j}| = j$ , and its column index runs over all  $(\mathbf{l}', \mathbf{m}', \mathbf{j}')$  with  $|\mathbf{l}'| = l, |\mathbf{m}'| = m, |\mathbf{j}'| = j+1$ . The entry for the pair  $((i, \mathbf{l}, \mathbf{m}, \mathbf{j}), (\mathbf{l}', \mathbf{m}', \mathbf{j}'))$  is given by the coefficient of  $\mathbf{w}_{\mathbf{j}'}^{(\mathbf{l}', \mathbf{m}')}$  in  $b_i f_{i-1} f_{i+1} \mathbf{w}_\mathbf{j}^{(\mathbf{l}, \mathbf{m})}$  in the previous proposition. Then  $C$  is of full rank. Note that the rank does not depend on the orders of the index sets.*

*Proof.* Let  $A$  be the subring of  $\mathbb{Q}(q)$  defined by  $A = \{f(q) \in \mathbb{Q}(q) \mid f(q) \text{ is regular at } q=0\}$ . Let  $\alpha_t$  ( $t = 1, 2, 3, 4$ ) be the largest integer such that  $A_t$  in Corollary 13 belongs to  $q^{\alpha_t} A$ . We have

$$\begin{aligned} \alpha_1 - \alpha_2 &= \alpha_4 - \alpha_3 = j_s + j_{s+1} - l_s - m_{s+1} - 1 < 0, \\ \alpha_4 - \alpha_1 &= 2j_{s+1} - l_{s+1} - m_{s+1} - 1 < 0, \end{aligned}$$

since  $j_t \leq \min(l_t, m_t)$  ( $t = s, s+1$ ). Therefore,  $\alpha_4$  is minimal and the others are strictly larger.

For  $\mathbf{w}_{\mathbf{j}'}^{(\mathbf{l}', \mathbf{m}')}$  such that  $|\mathbf{l}'| = l, |\mathbf{m}'| = m, |\mathbf{j}'| = j+1$ , choose the minimal  $s$  such  $j'_s > 0$  and consider  $b_i f_{i-1} f_{i+1} \mathbf{w}_{\mathbf{j}'-e_s}^{(\mathbf{l}', \mathbf{m}'-e_s+e_{s+1})}$  with  $i = 2s$ . By Proposition 12 the fourth term of the above is nonzero. Consider the row of  $C$  corresponding to the index  $(i, \mathbf{l}', \mathbf{m}' - e_s + e_{s+1}, \mathbf{j}' - e_s)$ . By multiplying a suitable scalar to this row, one can make the  $((i, \mathbf{l}', \mathbf{m}' - e_s + e_{s+1}, \mathbf{j}' - e_s), (\mathbf{l}', \mathbf{m}', \mathbf{j}'))$ -entry of  $C$  be 1, and the other three nonzero entries in the same row belong to  $qA$ . Consider the square matrix  $C'$  obtained by varying all possible  $(\mathbf{l}', \mathbf{m}', \mathbf{j}')$  and picking the corresponding renormalized rows. Then from the construction,  $\det C'$  belongs to  $\{\pm 1\} + qA$ . Hence the assertion is confirmed.  $\square$

#### ACKNOWLEDGMENTS

The authors thank Atsuo Kuniba, Hideya Watanabe, Yasuhiko Yamada and Akihito Yoneyama for comments and giving us references. They also thank anonymous referee for careful reading of the manuscript. M.O. is supported by Grants-in-Aid for Scientific Research No. 19K03426 and No. 16H03922 from JSPS. This work was partly supported by Osaka City University Advanced Mathematical Institute (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics JPMXP0619217849).

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