Parafermionic Bases of Standard Modules for Twisted Affine Lie Algebras of Type $A^{(2)}_{2l-1}$, $D^{(2)}_{l+1}$, $E^{(2)}_6$ and $D^{(3)}_4$

Masato Okado & Ryo Takenaka

PARAFERMIONIC BASES OF STANDARD MODULES FOR TWISTED AFFINE LIE ALGEBRAS OF TYPE $A_{2l-1}^{(2)}$, $D_{l+1}^{(2)}$, $E_6^{(2)}$ AND $D_4^{(3)}$

MASATO OKADO AND RYO TAKENAKA

ABSTRACT. Using the bases of principal subspaces for twisted affine Lie algebras except $A_{2l}^{(2)}$ by Butorac and Sadowski, we construct bases of the highest weight modules of highest weight $k\Lambda_0$ and parafermionic spases for the same affine Lie algebras. As a result, we obtain their character formulas conjectured in [14].

1. INTRODUCTION

In 1980's, the attempts to obtain combinatorial bases of the integrable highest weight modules or their (coset) subspaces using vertex operators were initiated in the seminal work of Lepowsky and Primc [18]. Let $\mathfrak{g}$ be the simple Lie algebra sl_2 and $\mathfrak{h}$ its Cartan subalgebra. Set $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c, \hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$, where c is the canonical central element. Lepowsky and Primc constructed bases of the coset space $\hat{\mathfrak{g}} \supset \hat{\mathfrak{h}}$ of integrable highest weight modules build upon vertex operator components. This work was generalized by Georgiev to the higher rank case $\mathfrak{g} = sl_{n+1}$. In [12], he first constructed the bases for the pricipal subspaces studied by Feigin and Stoyanovsky [9, 10], and then did for the $\hat{\mathfrak{g}} \supset \hat{\mathfrak{h}}$ coset subspaces or equivalently the parafermionic spaces in [13]. As a result, he obtained a fermionic character formula of the integrable highest weight module $L(k_0\Lambda_0 + k_j\Lambda_j)$.

Recently, based on the works on constructing combinatorial bases of the principal subspaces for other affine algebras (see e.g. [1, 2]), Butorac, Kožić and Primc obtained fermionic formulas for integrable highest weight modules similar to the above highest weight for all untwisted affine Lie algebras in [3], and thereby proved the so called Kuniba-Nakanishi-Suzuki conjecture [16]. Although the untwisted cases are settled, there are also twisted affine Lie algebras and a similar conjecture was stated in [14]. Fortunately, necessary combinatorial bases for the principal subspaces have already been obtained recently in [21, 22, 4]. The purpose of this paper is to make use of their results to settle the conjecture in [14] for the twisted cases.

This paper is organized as follows. In section 2, we review necessary results. In particular, we recall the twisted vertex operator, construction of the level 1 highest weight representation by it, quasi-particles on higher level representations and the result by Butorac and Sadowski on the construction of a basis of the principal subspace by quasi-particles. We then give our main theorem in section 3 on the construction of a basis of the standard module $L(k\Lambda_0)$. In section 4, we consider the so called $\mathcal{Z}$-operator in the twisted case and construct a basis of the parafermionic space. In the final section 5, we obtain the fermionic character formula of the parafermionic space and the standard module $L(k\Lambda_0)$ for the twisted cases. Note that the $A_{2l}^{(2)}$ case is not included, since it is not treated in [4] either. We wish to report also in this case in near future.

2. PRELIMINARIES

2.1. **Notation on simple Lie algebras of type ADE.** Let $\mathfrak{g}$ be a complex simple Lie algebra of type A_ℓ, D_ℓ or E_ℓ ($\ell = 6, 7, 8$ for E_ℓ), and let α_i ($i = 1, 2, \ldots, \ell$) be its simple root. The root lattice of $\mathfrak{g}$ is given by

$$L = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_\ell.$$

Let $\mathfrak{h}$ be the Cartan subalgebra and $\langle \cdot, \cdot \rangle$ a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$. Using this form, we can introduce a symmetric bilinear form $\langle \cdot, \cdot \rangle$ also on $\mathfrak{h}^*$ by identifying an element h of $\mathfrak{h}$ with an element α_h of $\mathfrak{h}^*$ via $\langle \alpha, \alpha_h \rangle = \alpha(h)$. We fix the bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}^*$ so that we have $\langle \alpha, \alpha \rangle = 2$ if α is a root. We take the labeling of the Dynkin diagrams of our Lie algebras as in [4]. Let ν be the Dynkin diagram automorphism of A_{2l-1}, D_{l+1}, E_6 and D_4, where the order r of ν is 2, 2, 2 and 3, respectively. The Dynkin diagrams and the autormorphism ν for each type are given in Table 1.

A_{2l-1}

D_{l+1}

E_6

D_4
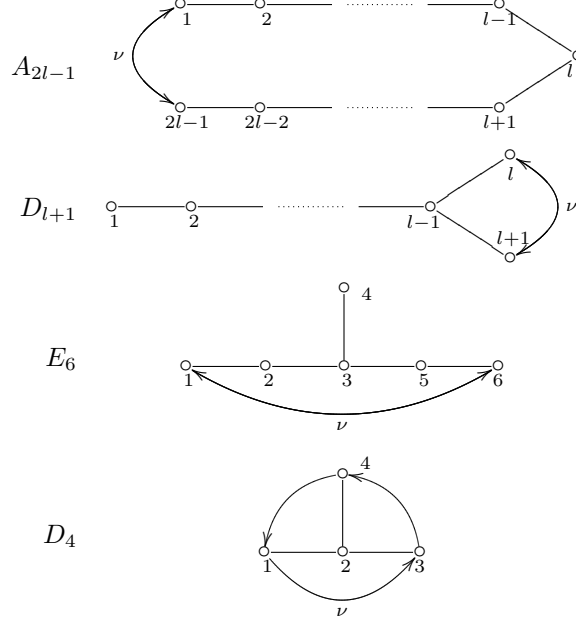
TABLE 1. Dynkin diagrams and the automorphism ν

Remark 1. The results in this paper remain valid when we take $\mathfrak{g}$ to be one of $A_\ell, D_\ell, E_{6,7,8}$ and $\nu = \mathrm{id}$.

Let η be a primitive rth root of unity and set $\eta_0 = (-1)^r \eta$. Following §2 of [5] (see also [4, 21, 22]), we consider two central extensions of L by $\langle \eta_0 \rangle$, denoted by $\hat{L}$ and $\hat{L}_\nu$,

$$1 \longrightarrow \langle \eta_0 \rangle \longrightarrow \hat{L} \ (\text{ or } \hat{L}_\nu) \overset{-}{\longrightarrow} L \longrightarrow 1$$

where $^-$ stands for the projection to L. $\hat{L}$ and $\hat{L}_\nu$ differ from each other when $r = 3$. Define the functions $C_0, C : L \times L \to \mathbb{C}^\times$ by

$$C_0(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle}, \qquad C(\alpha, \beta) = \prod_{j=0}^{r-1} (-\eta^j)^{\langle \nu^j \alpha, \beta \rangle}.$$

The central extension $\hat{L}$ (resp. $\hat{L}_\nu$) is defined by using the commutator map C_0 (resp. C), namely, for $a, b \in \hat{L}$ (resp. $\hat{L}_\nu$) $aba^{-1}b^{-1} = C_0(\bar{a}, \bar{b})$ (resp. $C(\bar{a}, \bar{b})$). Let each commutator map correspond to the 2-cocyle $\epsilon_{C_0}, \epsilon_C$. Namely, for ϵ_{C_0}, it satisfies

$$\epsilon_{C_0}(\alpha, \beta)\epsilon_{C_0}(\alpha + \beta, \gamma) = \epsilon_{C_0}(\beta, \gamma)\epsilon_{C_0}(\alpha, \beta + \gamma), \quad \frac{\epsilon_{C_0}(\alpha, \beta)}{\epsilon_{C_0}(\beta, \alpha)} = C_0(\alpha, \beta).$$

We choose our 2-cocyle ϵ_{C_0} to be

$$\epsilon_{C_0}(\alpha_i, \alpha_j) = \begin{cases} 1 & \text{if } i \leq j \\ (-1)^{\langle \alpha_i, \alpha_j \rangle} & \text{if } i > j \end{cases}.$$

This 2-cocyle satisfies

$$\epsilon_{C_0}(\alpha, \beta)^2 = 1, \quad \epsilon_{C_0}(\alpha, \beta) = \epsilon_{C_0}(\nu\alpha, \nu\beta).$$

The 2-cocycles ϵ_{C_0} and ϵ_C are related by

$$\epsilon_{C_0}(\alpha, \beta) = \left(\prod_{-\frac{r}{2} < j < 0} (-\eta^{-j})^{\langle \nu^j \alpha, \beta \rangle} \right) \epsilon_C(\alpha, \beta). \tag{1}$$

Using the 2-cocycle ϵ_C, we obtain a normalized section $e : L \to \hat{L}_\nu$ by

$$\begin{aligned} e : L &\to \hat{L}_\nu \\ \alpha &\mapsto e_\alpha \end{aligned}$$

with $e_0 = 1$, $\overline{e_\alpha} = \alpha$ and $e_\alpha e_\beta = \epsilon_C(\alpha, \beta) e_{\alpha+\beta}$. According to [5], there exists an automorphism $\hat\nu$ of $\hat{L}_\nu$ such that

$$\overline{\hat\nu a} = \nu \bar{a}, \qquad \hat\nu a = a \text{ if } \nu \bar{a} = \bar{a}. \tag{2}$$

$\hat\nu$ is also an automorphism of $\hat{L}$ satisfying (2). We have $\hat\nu^r = 1$.

For $j \in \mathbb{Z}$, set

$$\mathfrak{h}_{(j)} = \{h \in \mathfrak{h} \mid \nu h = \eta^j h\} \subset \mathfrak{h},$$

so that we have

$$\mathfrak{h} = \bigoplus_{j \in \mathbb{Z}/r\mathbb{Z}} \mathfrak{h}_{(j)}.$$

Here we identify $\mathfrak{h}_{(j \bmod r)}$ with $\mathfrak{h}_{(j)}$. Associated to this decomposition, we define a Lie algebra

$$\hat{\mathfrak{h}}[\nu] = \bigoplus_{m \in \frac{1}{r}\mathbb{Z}} \mathfrak{h}_{(rm)} \otimes t^m \oplus \mathbb{C}c$$

with Lie bracket given by

$$[\alpha \otimes t^m, \beta \otimes t^n] = \langle \alpha, \beta \rangle m \delta_{m+n,0} c, \quad [\hat{\mathfrak{h}}[\nu], c] = 0$$

for $m, n \in \frac{1}{r}\mathbb{Z}$ and $\alpha \in \mathfrak{h}_{(rm)}$ and $\beta \in \mathfrak{h}_{(rn)}$. Consider the subalgebras $\hat{\mathfrak{h}}[\nu]^\pm = \bigoplus_{\pm m > 0} \mathfrak{h}_{(rm)} \otimes t^m$. Then $\hat{\mathfrak{h}}[\nu]_{\frac{1}{r}\mathbb{Z}} = \hat{\mathfrak{h}}[\nu]^+ \oplus \hat{\mathfrak{h}}[\nu]^- \oplus \mathbb{C}c$ is a Heisenberg subalgebra of $\hat{\mathfrak{h}}[\nu]$. We introduce the induced $\hat{\mathfrak{h}}[\nu]$-module

$$S[\nu] = U(\hat{\mathfrak{h}}[\nu]) \otimes_{U(\bigoplus_{m \geq 0} \mathfrak{h}_{(rm)} \otimes t^m \oplus \mathbb{C}c)} \mathbb{C},$$

where $\bigoplus_{m \geq 0} \mathfrak{h}_{(rm)} \otimes t^m$ acts trivially on $\mathbb{C}$ and c as 1. $S[\nu]$ is an irreducible $\hat{\mathfrak{h}}[\nu]_{\frac{1}{r}\mathbb{Z}}$-module, linearly isomorphic to the symmetric algebra $S(\hat{\mathfrak{h}}[\nu]^-)$.

2.2. **Twisted module and twisted vertex operator.** Let P_j be the projection from $\mathfrak{h}$ onto $\mathfrak{h}_{(j)}$ for $j \in \mathbb{Z}/r\mathbb{Z}$. Following [5] and [17], we set

$$N = (1 - P_0)\mathfrak{h} \cap L = \{\alpha \in L \mid \langle \alpha, \mathfrak{h}_{(0)} \rangle = 0\}.$$

Explicitly, we have

$$N = \sum_{i=1}^{\ell} \mathbb{Z}(\alpha_i - \nu\alpha_i)$$

when $r = 2$, and

$$N = \{r_1\alpha_1 + r_3\alpha_3 + r_4\alpha_4 \in L \mid r_1 + r_3 + r_4 = 0\}$$

when $r = 3$. See [4].

Let $\hat{N}$ be the subgroup of $\hat{L}_\nu$ obtained by pulling back the subgroup N of L. Then, by Proposition 6.1 in [17], there is a unique homomorphism

$$\tau : \hat{N} \longrightarrow \mathbb{C}^\times$$

such that

$$\tau(\eta_0) = \eta_0, \quad \tau(a\hat\nu a^{-1}) = \eta^{-\langle \sum_j \nu^j \bar{a}, \bar{a} \rangle / 2}$$

for $a \in \hat{L}_\nu$. Let $\mathbb{C}_\tau$ be the one-dimensional $\hat{N}$-module $\mathbb{C}$ with this character τ and consider the induced $\hat{L}_\nu$-module

$$U_T = \mathbb{C}[\hat{L}_\nu] \otimes_{\mathbb{C}[\hat{N}]} \mathbb{C}_\tau \simeq \mathbb{C}[L/N] \simeq \mathbb{C}[P_0 L].$$

Here the last isomorphism is induced from the projection. The readers are warned that $P_0 L \not\subset L$ but $P_0 L \subset \frac{1}{r}L$. We have

$$U_T = \coprod_{\alpha \in P_0 L} (U_T)_\alpha,$$

where $(U_T)_\alpha = \{u \in U_T \mid hu = \langle h, \alpha \rangle u \text{ for } h \in \mathfrak{h}_{(0)}\}$ and

$$a \cdot (U_T)_\alpha \subset (U_T)_{\alpha + \bar{a}_{(0)}} \quad \text{for } a \in \hat{L}_\nu, \alpha \in P_0 L.$$

$\mathfrak{h}_{(0)}$ acts on U_T as

$$\alpha \cdot b = \langle \alpha, \bar{b} \rangle b \tag{3}$$

for $\alpha \in \mathfrak{h}_{(0)}, b \in \hat{L}_\nu$ and we set $[\alpha, a] = \langle \alpha, \bar{a} \rangle a$. We also define the action of z^α by

$$z^\alpha \cdot b = z^{\langle \alpha, \bar{b} \rangle} b \tag{4}$$

and η^α by $\eta^\alpha \cdot b = \eta^{\langle \alpha, \bar{b} \rangle} b$. Then for $a \in \hat{L}_\nu$, we have $z^\alpha a = a z^{\alpha + \langle \alpha, \bar{a} \rangle}, \eta^\alpha a = a \eta^{\alpha + \langle \alpha, \bar{a} \rangle}$. Moreover, as operators on U_T, we have

$$\hat{\nu} a = a \eta^{-\sum \nu^p \bar{a} - \langle \sum \nu^p \bar{a}, \bar{a} \rangle / 2}. \tag{5}$$

U_T then becomes an $\tilde{\mathfrak{h}}[\nu]$-module by letting the Heisenberg algebra $\tilde{\mathfrak{h}}[\nu]_{\frac{1}{r}\mathbb{Z}}$ act trivially. We set

$$V_L^T = S[\nu] \otimes U_T$$

to be a tensor product of $\tilde{\mathfrak{h}}[\nu]$-module on which $\hat{L}_\nu$ acts by its action on the second component. Set $\tilde{\mathfrak{h}}[\nu] = \hat{\mathfrak{h}}[\nu] \oplus \mathbb{C}d$. d acts on U_T and gives the following grading

$$d \cdot a = -\frac{1}{2r} \langle \sum_p \nu^p \bar{a}, \bar{a} \rangle a \quad \text{for } a \in \hat{L}_\nu.$$

See [17, [7.4]]. Another factor $1/r$ corresponds to the different definition of the vertex operator. V_L^T is called the twisted module.

Next we define the $\hat{\nu}$-twisted vertex operator. We follow §2 of [5]. For each $\alpha \in \mathfrak{h}$ and $m \in \frac{1}{r}\mathbb{Z}$, let $\alpha_{(j)}$ stand for $P_j \alpha \in \mathfrak{h}_{(j)}$. Set

$$\alpha(m) = \alpha_{(rm)} \otimes t^m, \tag{6}$$

$$E^\pm(\alpha, z) = \exp \left(\sum_{\pm m \in \frac{1}{r}\mathbb{Z}_+} \frac{\alpha(m)}{m} z^{-m} \right). \tag{7}$$

Note that from [19, Proposition 3.4], we have the following commutation relation

$$E^+(\alpha, z_1) E^-(\beta, z_2) = E^-(\beta, z_2) E^+(\alpha, z_1) \prod_{i=0}^{r-1} \left(1 - \eta^i \frac{z_2^{\frac{1}{r}}}{z_1^{\frac{1}{r}}} \right)^{\langle \nu^i \alpha, \beta \rangle}. \tag{8}$$

For $a \in \hat{L}_\nu$, as defined in [5, 17], we consider the twisted vertex operator

$$Y^{\hat{\nu}}(a, z) = r^{-\frac{\langle \bar{a}, \bar{a} \rangle}{2}} \sigma(\bar{a}) E^-(-\bar{a}, z) E^+(-\bar{a}, z) a z^{\bar{a}_{(0)} + \frac{\langle \bar{a}_{(0)}, \bar{a}_{(0)} \rangle}{2} - \frac{\langle \bar{a}, \bar{a} \rangle}{2}}, \tag{9}$$

where

$$\sigma(\alpha) = \begin{cases} 2^{\langle \nu \alpha, \alpha \rangle / 2} & (r = 2) \\ (1 - \eta^2)^{\langle \nu \alpha, \alpha \rangle} & (r = 3) \end{cases}$$

acting on V_L^T. (In [5], this vertex operator is defined for an element of the lattice vertex operator algebra V_L. Since we do not use this fact in this paper, we simply defined it for $a \in \hat{L}_\nu$.) We define the component operators $Y_\alpha^{\hat{\nu}}(m)$ for $m \in \frac{1}{r}\mathbb{Z}, \alpha \in L$ by

$$Y^{\hat{\nu}}(e_\alpha, z) = \sum_{m \in \frac{1}{r}\mathbb{Z}} Y_\alpha^{\hat{\nu}}(m) z^{-m - \langle \alpha, \alpha \rangle / 2}. \tag{10}$$

Set

$$\rho_i = \frac{1}{2} \langle (\alpha_i)_{(0)}, (\alpha_i)_{(0)} \rangle. \tag{11}$$

Then for a simple root α_i, we have

$$Y^{\hat{\nu}}(\hat{\nu} e_{\alpha_i}, z) = Y^{\hat{\nu}}(e_{\alpha_i}, z) \big|_{z^{\frac{1}{r}} \to \eta^{-1} z^{\frac{1}{r}}} \tag{12}$$

from (5). For a component operator,

$$Y_{\nu \alpha_i}^{\hat{\nu}}(m) = \eta^{rm} Y_{\alpha_i}^{\hat{\nu}}(m). \tag{13}$$

From this relation, we know if $\nu \alpha_i = \alpha_i$, then $Y_{\alpha_i}^{\hat{\nu}}(m) = 0$ unless $m \in \mathbb{Z}$. By an explicit calculation, we get, if $n \neq 0$,

$$[h(n), Y^{\hat{\nu}}(e_\alpha, z)] = \langle h_{(rn)}, \alpha_{(-rn)} \rangle z^n Y^{\hat{\nu}}(e_\alpha, z). \tag{14}$$
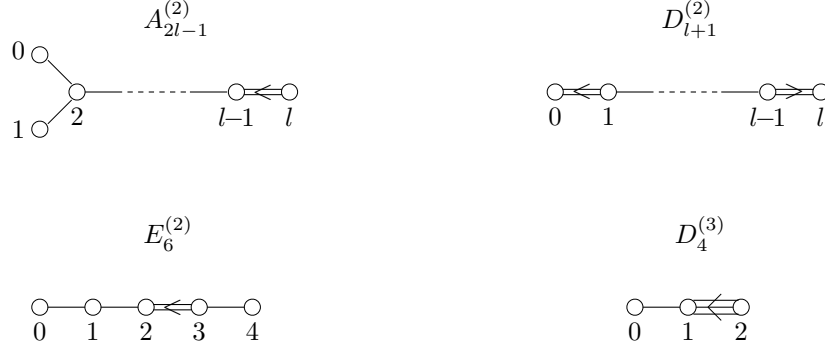
Table 2. Twisted affine Dynkin diagrams

2.3. **Twisted affine Lie algebras.** Recall that $\mathfrak{g}$ is a simple Lie algebra of type A_ℓ, D_ℓ or E_ℓ. Let Δ be its set of roots. Then we have the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathbb{C}x_\alpha.$$

We normalize a root vector x_α so that we have

$$[x_\alpha, x_\beta] = \begin{cases} \epsilon_{C_0}(\alpha, -\alpha)\alpha & \text{if } \alpha + \beta = 0 \\ \epsilon_{C_0}(\alpha, \beta)x_{\alpha+\beta} & \text{if } \alpha + \beta \in \Delta \\ 0 & \text{otherwise.} \end{cases}$$

Then the symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ reads as

$$\langle h, x_\alpha \rangle = 0, \qquad \langle x_\alpha, x_\beta \rangle = \begin{cases} \epsilon_{C_0}(\alpha, -\alpha) & \text{if } \alpha + \beta = 0, \\ 0 & \text{if } \alpha + \beta \neq 0, \end{cases}$$

where $h \in \mathfrak{h}$.

Now, we assume $\mathfrak{g}$ to be one of type A_{2l-1}, D_{l+1}, E_6 or D_4 and ν as in Table 1. Following [4, 5, 17, 21, 22], the automorphism ν of $\mathfrak{h}$ is lifted to an automorphism ν of $\mathfrak{g}$ by

$$\nu x_\alpha = \psi(\alpha) x_{\nu\alpha}$$

where ψ is some function from Δ to $\{\pm 1\}$. For $j \in \mathbb{Z}$ set

$$\mathfrak{g}_{(j)} = \{x \in \mathfrak{g} \mid \nu(x) = \eta^j x\}.$$

The twisted affine Lie algebra $\hat{\mathfrak{g}}[\nu]$ associated to $\mathfrak{g}$ and ν is given by

$$\hat{\mathfrak{g}}[\nu] = \bigoplus_{m \in \frac{1}{r}\mathbb{Z}} \mathfrak{g}_{(rm)} \otimes t^m \oplus \mathbb{C}c$$

with Lie bracket

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + \langle x, y \rangle m \delta_{m+n,0} c, \qquad [c, \hat{\mathfrak{g}}[\nu]] = 0$$

for $m, n \in \frac{1}{r}\mathbb{Z}$, $x \in \mathfrak{g}_{(rm)}$ and $y \in \mathfrak{g}_{(rn)}$. We also define the Lie algebra $\tilde{\mathfrak{g}}[\nu]$ by

$$\tilde{\mathfrak{g}}[\nu] = \hat{\mathfrak{g}}[\nu] \oplus \mathbb{C}d$$

where d is the degree operator such that

$$[d, x \otimes t^n] = n x \otimes t^n$$

for $n \in \frac{1}{r}\mathbb{Z}$, $x \in \mathfrak{g}_{(rn)}$ and $[d, c] = 0$. This Lie algebra $\hat{\mathfrak{g}}$ (or $\tilde{\mathfrak{g}}$) is isomorphic to a twisted affine Lie algebra of type $A_{2l-1}^{(2)}$, $D_{l+1}^{(2)}$, $E_6^{(2)}$ or $D_4^{(3)}$ depending on the choice of L and ν. See Table 2 for their Dynkin diagrams.

Theorem 2. ([5, Theorem 3.1],[11, Theorem 3],[17, Theorem 9.1]) *The representation of $\hat{\mathfrak{h}}[\nu]$ on V_L^T extends uniquely to a Lie algebra representation of $\tilde{\mathfrak{g}}[\nu]$ on V_L^T such that*

$$(x_\alpha)_{(rm)} \otimes t^m \mapsto Y_\alpha^{\hat{\nu}}(m)$$

for all $m \in \frac{1}{r}\mathbb{Z}$ and $\alpha \in L_2 = \{\alpha \in L \mid \langle \alpha, \alpha \rangle = 2\}$. Moreover V_L^T is irreducible as a $\tilde{\mathfrak{g}}[\nu]$-module.

In fact, V_L^T is an integrable highest weight module of highest weight Λ_0, where Λ_0 is the fundamental weight such that $\langle \Lambda_0, c \rangle = 1$ and $\langle \Lambda_0, \mathfrak{h}_{(0)} \rangle = 0 = \langle \Lambda_0, d \rangle$. A highest weight vector $1 \otimes (e_0 \otimes 1) \in V_L^T = S[\nu] \otimes U_T$ is denoted by $\mathbf{1}_T$. We also have the following formulas on V_L^T.

$$e_\alpha d e_\alpha^{-1} = d + \alpha - \frac{1}{2r} \langle \sum_p \nu^p \alpha, \alpha \rangle, \tag{15}$$

$$e_\alpha h e_\alpha^{-1} = h - \alpha(h), \tag{16}$$

$$e_\alpha h(j) e_\alpha^{-1} = h(j) \text{ for } j \neq 0, \tag{17}$$

$$e_\alpha Y_\beta^{\hat{\nu}}(j) e_\alpha^{-1} = C(\alpha, \beta) Y_\beta^{\hat{\nu}}(j - \langle \alpha, \beta_{(0)} \rangle). \tag{18}$$

2.4. **Standard module.** Here we summarize the Cartan subalgebra and simple roots as the twisted affine Lie algebras. Set $l = 4$ for $E_6^{(2)}$ and $l = 2$ for $D_4^{(3)}$. The Cartan subalgebra is identified with

$$\mathfrak{h}_{(0)} \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

Chevalley generators h_i ($0 \leq i \leq l$) are given by $h_i = \alpha_i$ when $\nu\alpha_i = \alpha_i$, $h_i = \sum_{p=0}^{r-1} \nu^p \alpha_i$ otherwise for $i \neq 0$ and $h_0 = -\sum_{p=0}^{r-1} \nu^p \theta^0$ where θ^0 is given by

$$\theta^0 = \begin{cases} \alpha_1 + \cdots + \alpha_{2l-2} & \text{for } A_{2l-1}^{(2)} \\ \alpha_1 + \cdots + \alpha_l & \text{for } D_{l+1}^{(2)} \\ \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 & \text{for } E_6^{(2)} \\ \alpha_1 + \alpha_2 + \alpha_3 & \text{for } D_4^{(3)}. \end{cases}$$

See [15, §8.3]. $\tilde{\mathfrak{g}}[\nu]$ contains a finite-dimensional simple Lie algebra $\tilde{\mathfrak{g}}[\nu]_{\bar{0}}$ (in Kac's notation) whose Dynkin diagram is obtained by removing the node 0 from that of the twisted affine Lie algebra. One can take the set of simple roots of $\tilde{\mathfrak{g}}[\nu]_{\bar{0}}$ as that of the image under the projection P_0 to $\mathfrak{h}_{(0)}$. Because of the labeling of the Dynkin nodes in Table 1, we can take it as $\{(\alpha_1)_{(0)}, \ldots, (\alpha_l)_{(0)}\}$.

We consider the standard $\tilde{\mathfrak{g}}[\nu]$-module $L(k\Lambda_0)$ of higher level k, namely, the integrable highest weight $\tilde{\mathfrak{g}}[\nu]$-module of highest weight $k\Lambda_0$. Since we know $L(\Lambda_0) \simeq V_L^T$, we realize $L(k\Lambda_0)$ as a submodule of the tensor product of k copies of V_L^T as

$$L(k\Lambda_0) \simeq U(\tilde{\mathfrak{g}}[\nu]) \cdot v_0 \subset (V_L^T)^{\otimes k},$$

where $v_0 = \mathbf{1}_T \otimes \cdots \otimes \mathbf{1}_T$ is a highest weight vector of $L(k\Lambda_0)$. On $L(k\Lambda_0)$, elements of $\tilde{\mathfrak{g}}[\nu]$ act through the coproduct

$$\Delta^{(k-1)}(x) = x \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes x \otimes \cdots \otimes 1 + \cdots + 1 \otimes 1 \otimes \cdots \otimes x,$$

where there are k components in each term. It is also true for the twisted vertex operator $Y^{\hat{\nu}}(e_\alpha, z)$. For a simple root α_i and a positive integer n, we set

$$x_{n\alpha_i}^{\hat{\nu}}(z) = x_{\alpha_i}^{\hat{\nu}}(z)^n = [\Delta^{(k-1)}(Y^{\hat{\nu}}(e_{\alpha_i}, z))]^n. \tag{19}$$

Note that $x_{(k+1)\alpha_i}^{\hat{\nu}}(z) = 0$. We also define a component operator $x_{n\alpha_i}^{\hat{\nu}}(m)$ by

$$x_{n\alpha_i}^{\hat{\nu}}(z) = \sum_{m \in \frac{1}{r}\mathbb{Z}} x_{n\alpha_i}^{\hat{\nu}}(m) z^{-m-n}.$$

Later, we will use the following commutation relation which can be shown using (14). If $m \neq 0$,

$$[h(m), x_{n\alpha_i}^{\hat{\nu}}(z)] = n \langle h_{(rm)}, \alpha_{(-rm)} \rangle z^m x_{n\alpha_i}^{\hat{\nu}}(z), \tag{20}$$

or equivalently,

$$[h(m), x_{n\alpha_i}^{\hat{\nu}}(j)] = n \langle h_{(rm)}, \alpha_{(-rm)} \rangle x_{n\alpha_i}^{\hat{\nu}}(j + m).$$

For $\alpha \in L$, $E^\pm(\alpha, z)$ acts on $(V_L^T)^{\otimes k}$ diagonally and hence also on $L(k\Lambda_0)$. One checks

$$[E^\pm(\alpha, z), h(n)] = \begin{cases} k \langle \alpha_{(-rn)}, h_{(rn)} \rangle z^n E^\pm(\alpha, z) & \text{if } \mp n > 0, \\ 0 & \text{othewise.} \end{cases} \tag{21}$$

e_α also acts on $(V_L^T)^{\otimes k}$ diagonally, namely,

$$e_\alpha \mapsto e_\alpha \otimes \cdots \otimes e_\alpha.$$

Set $Q = \bigoplus_{i=1}^l \mathbb{Z}\alpha_i \subset L$. Note that $Q_{(0)} = \bigoplus_{i=1}^l \mathbb{Z}(\alpha_i)_{(0)}$ should be understood as the root lattice of $\tilde{\mathfrak{g}}[\nu]_{\bar{0}}$. Based on the calculations (15)-(18) on V_L^T, we define the adjoint action of e_α on $\tilde{\mathfrak{g}}[\nu]$ by

$$e_\alpha c e_\alpha^{-1} = c, \tag{22}$$

$$e_\alpha d e_\alpha^{-1} = d + \alpha - \frac{1}{2}\langle \alpha_{(0)}, \alpha_{(0)} \rangle c, \tag{23}$$

$$e_\alpha h e_\alpha^{-1} = h - \alpha(h)c \text{ for } h \in \mathfrak{h}_{(0)}, \tag{24}$$

$$e_\alpha h(j) e_\alpha^{-1} = h(j) \text{ for } j \neq 0, \tag{25}$$

$$e_\alpha x_\beta^{\hat{\nu}}(j) e_\alpha^{-1} = C(\alpha, \beta) x_\beta^{\hat{\nu}}(j - \langle \alpha, \beta_{(0)} \rangle). \tag{26}$$

Note that $c = k$ on $L(k\Lambda_0)$ and $\sum_p \langle \nu^p \alpha, \alpha \rangle = \langle \alpha_{(0)}, \alpha_{(0)} \rangle$. e_α corresponds to the translation operator of the affine Weyl group of $\tilde{\mathfrak{g}}[\nu]$. See Section 1.5 of [3] for the untwisted case.

Next we state the vertex operator formula that will be used later. This is a twisted version of (1.27) in [3]. The proof is completely parallel to [18, Theorem 5.6] or [23, Theorem 6.4].

Lemma 3. *For a simple root α_i, renormalize the twisted vertex operator $x_{\alpha_i}^{\hat{\nu}}(z)$ as $\tilde{x}_{\alpha_i}^{\hat{\nu}}(z) = r\sigma(\alpha_i)^{-1}x_{\alpha_i}^{\hat{\nu}}(z)$. Then, for $p, q \geq 0$ such that $p + q = k$, we have*

$$\frac{1}{p!}E^-(\alpha_i, z)(z\tilde{x}_{\alpha_i}^{\hat{\nu}}(z))^p E^+(\alpha_i, z) = \frac{1}{q!}\epsilon_C(\alpha_i, -\alpha_i)^{-q}(z\tilde{x}_{-\alpha_i}^{\hat{\nu}}(z))^q e_{\alpha_i} z^{(\alpha_i)_{(0)}+k\rho_i} \tag{27}$$

as an operator on $(V_L^T)^{\otimes k}$ or $L(k\Lambda_0)$. Furthermore, (27) can be rewritten as

$$E^-(\alpha_i, z)\exp(z\tilde{x}_{\alpha_i}^{\hat{\nu}}(z))E^+(\alpha_i, z) = \exp(\epsilon_C(\alpha_i, -\alpha_i)^{-1}z\tilde{x}_{-\alpha_i}^{\hat{\nu}}(z))e_{\alpha_i} z^{(\alpha_i)_{(0)}+k\rho_i}. \tag{28}$$

Proof. Set $y_\alpha(z) = r^{\frac{\langle \alpha, \alpha \rangle}{2}}\sigma(\alpha)^{-1}Y^{\hat{\nu}}(e_{\alpha_i}, z)$. From (9), we have

$$E^-(\alpha_i, z)y_{\alpha_i}(z)E^+(\alpha_i, z) = e_{\alpha_i} z^{(\alpha_i)_{(0)}+\rho_i-1}$$

on V_L^T. Since $y_{\alpha_i}(z)^2 = 0$ on V_L^T, $\tilde{x}_{\alpha_i}^{\hat{\nu}}(z)^p$ acts on $(V_L^T)^{\otimes k}$ as $p!\sum y_{\alpha_i}(z)^{j_1} \otimes \cdots \otimes y_{\alpha_i}(z)^{j_k}$, where $j_m \in \{0, 1\}$ and $j_1 + \cdots + j_k = p$. Since $E^\pm(\alpha_i, z), e_{\alpha_i}, z^{(\alpha_i)_{(0)}+\rho_i}$ all act grouplike on $(V_L^T)^{\otimes k}$,

$$\text{LHS of (27)} = z^p E^-(\alpha_i, z)(\sum y_{\alpha_i}(z)^{j_1} \otimes \cdots \otimes y_{\alpha_i}(z)^{j_k})E^+(\alpha_i, z)$$

$$= \epsilon_C(\alpha_i, -\alpha_i)^{-q} z^q (\sum y_{-\alpha_i}(z)^{1-j_1} e_{\alpha_i} z^{(\alpha_i)_{(0)}+\rho_i} \otimes \cdots \otimes y_{-\alpha_i}(z)^{1-j_k} e_{\alpha_i} z^{(\alpha_i)_{(0)}+\rho_i})$$

$$= \frac{1}{q!}\epsilon_C(\alpha_i, -\alpha_i)^{-q}(z\tilde{x}_{-\alpha_i}^{\hat{\nu}}(z))^q e_{\alpha_i} z^{(\alpha_i)_{(0)}+k\rho_i}.$$

Here we have used $z^{(\alpha_i)_{(0)}-\rho_i}e_{-\alpha_i}^{-1} = \epsilon_C(\alpha_i, -\alpha_i)^{-1}e_{\alpha_i} z^{(\alpha_i)_{(0)}+\rho_i}$.

Noting $\tilde{x}_{\alpha_i}^{\hat{\nu}}(z)^{k+1} = 0$ on $(V_L^T)^{\otimes k}$, we obtain (28) from (27) immediately. $\square$

2.5. **Principal subspace.** We will introduce the notion of the principal subspace of $L(k\Lambda_0)$ and twisted quasi-particle bases which we will use to construct parafermionic bases. First, denote by Δ_+ the set of positive roots and by

$$\mathfrak{n} = \bigoplus_{\alpha \in \Delta_+} \mathbb{C}x_\alpha$$

the Lie subalgebra of $\mathfrak{g}$ which is the nilradical of a Borel subalgebra. Consider its twisted affinization

$$\hat{\mathfrak{n}}[\nu] = \bigoplus_{m \in \frac{1}{r}\mathbb{Z}} \mathfrak{n}_{(rm)} \otimes t^m \oplus \mathbb{C}c$$

and its subalgebra

$$\bar{\mathfrak{n}}[\nu] = \bigoplus_{m \in \frac{1}{r}\mathbb{Z}} \mathfrak{n}_{(rm)} \otimes t^m.$$

In [4, 9, 10, 21, 22], the principal subspace $W(k\Lambda_0)$ of $L(k\Lambda_0)$ is defined as

$$W(k\Lambda_0) = U(\bar{\mathfrak{n}}[\nu]) \cdot v_0.$$

From [4], we review twisted quasi-particle monomials. We define the twisted quasi-particle of color i, charge n and energy $-m$ for each simple root α_i, $n \in \mathbb{N}$, and $m \in \frac{1}{r}\mathbb{Z}$ as the coefficient $x_{n\alpha_i}^{\hat{\nu}}(m)$ in (19). A twisted quasi-particle monomial is then defined by

$$b = x_{n_{r_l^{(1)},l}\alpha_l}^{\hat{\nu}}(m_{r_l^{(1)},l}) \cdots x_{n_{1,l}\alpha_l}^{\hat{\nu}}(m_{1,l}) \cdots x_{n_{r_1^{(1)},1}\alpha_1}^{\hat{\nu}}(m_{r_1^{(1)},1}) \cdots x_{n_{1,1}\alpha_1}^{\hat{\nu}}(m_{1,1}). \tag{29}$$

The sequence

$$\mathcal{R}' = \left(n_{r_l^{(1)},l}, \ldots, n_{1,l}; \ldots; n_{r_1^{(1)},1} \ldots n_{1,1} \right)$$

is called its *charge-type*. We assume $1 \leq n_{r_i^{(1)},i} \leq \ldots \leq n_{1,i} \leq k$ for each i. The *dual-charge-type*

$$\mathcal{R} = \left(r_l^{(1)}, \ldots, r_l^{(k)}; \ldots; r_1^{(1)}, \ldots, r_1^{(k)} \right)$$

is defined in the way that $(r_i^{(1)}, \ldots, r_i^{(k)})$ is the transposed partition of $(n_{1,i}, \ldots, n_{r_i^{(1)},i})$ for each i. Namely, $r_i^{(s)}$ stands for the number of quasi-particles of color i and charge greater than or equal to s in the monomial b. So we have $r_i^{(1)} \geq r_i^{(2)} \geq \cdots \geq r_i^{(k)} \geq 0$. The *color-type* is defined by

$$\mathcal{C} = (r_l, \ldots, r_1)$$

where

$$r_i = \sum_{p=1}^{r_i^{(1)}} n_{p,i} = \sum_{s=1}^{k} r_i^{(s)}. \tag{30}$$

According to [4], we consider the following conditions $(C1)$-$(C3)$ for the mode m in $x_{n\alpha_i}^{\hat{\nu}}(m)$ in b.

$(C1) \quad m_{p,i} \in \rho_i\mathbb{Z} \quad$ for $1 \leq p \leq r_i^{(1)}, 1 \leq i \leq l$,

$(C2) \quad m_{p,i} \leq -(2p-1)\rho_i n_{p,i} - \langle(\alpha_i)_{(0)}, (\alpha_{i-1})_{(0)}\rangle \sum_{q=1}^{r_{i-1}^{(1)}} \min\{n_{p,i}, n_{q,i-1}\} \quad$ for $1 \leq p \leq r_i^{(1)}$, $1 \leq i \leq l$,

$(C3) \quad m_{p+1,i} \leq m_{p,i} - 2\rho_i n_{p,i}$ if $n_{p+1,i} = n_{p,i} \quad$ for $1 \leq p \leq r_i^{(1)} - 1$, $1 \leq i \leq l$.

Here we understand $r_0^{(1)} = 0$. Set

$$B_W = \bigcup_{\substack{r_1^{(1)} \geq \cdots \geq r_1^{(k)} \geq 0 \\ \vdots \\ r_l^{(1)} \geq \cdots \geq r_l^{(k)} \geq 0}} \{b \text{ as in } (29) \mid b \text{ satisfies } (C1), (C2) \text{ and } (C3)\}$$

We know that the principal subspace $W(k\Lambda_0)$ has a basis consisting from twisted quasi-particle monomials.

Theorem 4. ([4, Theorem 5.1]) *The set $\mathcal{B}_W = \{bv_0 \mid b \in B_W\}$ is a basis of the principal subspace $W(k\Lambda_0)$.*

3. QUASI-PARTICLE BASES OF STANDARD MODULES

3.1. **Spanning sets for standard modules.**

Lemma 5.
(1) $L(k\Lambda_0) = U(\hat{\mathfrak{h}}[\nu]^-)QW(k\Lambda_0)$
(2) $L(k\Lambda_0) = QW(k\Lambda_0)$

Proof. $\hat{\mathfrak{g}}[\nu]$ is generated by $(x_\beta)_{(rm)} \otimes t^m$ acting as $x_\beta^{\hat{\nu}}(m)$ on $L(k\Lambda_0)$ for $m \in \frac{1}{r}\mathbb{Z}$, $\beta \in \Delta$. Since every x_β can be expressed by taking brackets with x_{α_i} for $1 \leq i \leq \ell$ and $x_{\nu\alpha_i}^{\hat{\nu}}(m)$ can also be expressed as $x_{\alpha_i}^{\hat{\nu}}(m)$ by using (12), the standard module $L(k\Lambda_0)$ is spanned by noncommutative monomials in $x_{\pm\alpha_i}^{\hat{\nu}}(m)$, $i = 1, \cdots, l$, $m \in \frac{1}{r}\mathbb{Z}$. By using the vertex operator formula (27), we can express $x_{-\alpha_i}^{\hat{\nu}}(m)$ in terms of $x_{\alpha_i}^{\hat{\nu}}(m')$, e_{α_i} and a polynomial in $U(\hat{\mathfrak{h}}[\nu])$. From (20) and (25), we can move the elements of $\hat{\mathfrak{h}}[\nu]^-$ to the left and the elements of $\hat{\mathfrak{h}}[\nu]^+$ to the right. Since $h(n)v_0 = 0$ for $n > 0$, we see that (1) holds. $U(\hat{\mathfrak{h}}[\nu]^-)$ is spanned by the coefficients of $E^-(-\alpha_i, z)$. Therefore we obtain (2) from (1) by using the vertex operator formula (27) for $q = 0$ and the commutation relation (21). $\square$

The second statement in Lemma 5 implies that

$$\{e_\mu v \mid \mu \in Q, v \in \mathcal{B}_W\}$$

spans $L(k\Lambda_0)$. But this is not a basis.

For the proofs of Proposition 6 and the main theorem, we introduce a linear order on the quasi-particle monomials in B_W following [4]. For two monomials b abd $\bar{b}$ with charge-types $\mathcal{R}'$ and $\bar{\mathcal{R}}' = (\bar{n}_{\bar{r}_l^{(1)},l}, \ldots, \bar{n}_{1,l}; \ldots; \bar{n}_{\bar{r}_1^{(1)},1}, \ldots, \bar{n}_{1,1})$ and with energies $(m_{r_l^{(1)},l}, \ldots, m_{1,1})$ and $(\bar{m}_{\bar{r}_l^{(1)},l}, \ldots, \bar{m}_{1,1})$, we write $b < \bar{b}$ if one of the follwoing conditions holds

(1) $\mathcal{R}' < \bar{\mathcal{R}}'$
(2) $\mathcal{R}' = \bar{\mathcal{R}}'$ and $(m_{r_l^{(1)},l}, \ldots, m_{1,1}) < (\bar{m}_{\bar{r}_l^{(1)},l}, \ldots, \bar{m}_{1,1})$

where we write $\mathcal{R}' < \bar{\mathcal{R}}'$ if there exists i and s such that $r_j^{(1)} = \bar{r}_j^{(1)}, n_{t,j} = \bar{n}_{t,j}$ for $j < i, 1 \leq t \leq r_j^{(1)}$ and $n_{1,i} = \bar{n}_{1,i}, n_{2,i} = \bar{n}_{2,i}, \ldots, n_{s-1,i} = \bar{n}_{s-1,i}, n_{s,i} < \bar{n}_{s,i}$ or $n_{t,i} = \bar{n}_{t,i}$ for $1 \leq t \leq r_i^{(1)}, r_i^{(1)} < \bar{r}_i^{(1)}$. In the case that $\mathcal{R} = \bar{\mathcal{R}}'$, we apply this definition to the sequences of energies to similarly define $(m_{r_l^{(1)},l}, \ldots, m_{1,1}) < (\bar{m}_{\bar{r}_l^{(1)},l}, \ldots, \bar{m}_{1,1})$.

Set

$$B_H = \left\{ h_{\alpha_l} \cdots h_{\alpha_1} \,\middle|\, \begin{array}{l} h_{\alpha_i} = \alpha_i(-m_{t_i,i})^{n_{t_i,i}} \cdots \alpha_i(-m_{1,i})^{n_{1,i}}, i = 1, \ldots l, \\ t_i \in \mathbb{Z}_{\geq 0}, m_{t_i,i} > \cdots > m_{1,i}, m_{p,i} \in \rho_i \mathbb{N}, n_{p,i} \in \mathbb{N} \end{array} \right\}$$

and $B'_W = B_W \cap M'_{QP}$, where M'_{QP} is the set of all quasi-particle monomials with no $x^{\hat{\nu}}_{k\alpha_i}(m)$. The following proposition can be proved in the same way as Lemma 2.3 in [3].

Proposition 6. *The set $\mathcal{B}_L = \{e_\mu hbv_0 \mid \mu \in Q, \ h \in B_H, \ b \in B'_W\}$ spans $L(k\Lambda_0)$.*

Proof. By Lemma 5 (1), the set of vectors

$$\{e_\mu hbv_0 \mid \mu \in Q, h \in B_H, b \in B_W\}$$

spans $L(k\Lambda_0)$. It suffices to check that the arguments used in the proof of [3, Lemma 2.3] also hold for our case. By the vertex operator formula (27), a quasi-particle $x^{\hat{\nu}}_{k\alpha_i}(m)$ is expressed by e_{α_i} and monomials in $U(\hat{\mathfrak{h}}[\nu]^\pm)$. Then we can move e_α and elements of $\hat{\mathfrak{h}}[\nu]^-$ to the left, and elements of $\hat{\mathfrak{h}}[\nu]^+$ to the right by the relations (21) and (25). As a result, we can express $e_\mu hbv_0$ as a linear combination of vectors $e_{\mu'} h'b'v_0$, where $\mu' \in Q, h' \in B_H$, and $b' \in M'_{QP}$. Note that b' contains no quasi-particles $x^{\hat{\nu}}_{k\alpha_i}(m)$, but is not necessarily in B_W. Take any vector $e_{\mu'} h'b'v_0$ that is not in B_W, if it exists. Since $b'v_0 \in W(k\Lambda_0)$, it is expressed as a linear combination of vectors $b''v_0$ such that $b'' \in B_W$ by Theorem 4. b'' may contain $x^{\hat{\nu}}_{k\alpha_i}(m)$, but it is greater than b with respect to the order "$<$" defined above. Since the set B_W is upper bounded with respect to this order, the process of eliminating quasi-particles $x^{\hat{\nu}}_{k\alpha_i}(m)$ end in finitely many steps and b in any term in the final linear combination belongs to $B_W \cap M'_{QP}$. $\qquad\square$

3.2. The main theorem.
Consider the decomposition

$$L(k\Lambda_0) = \bigoplus_{s \in \mathbb{Z}} L(k\Lambda_0)_s, \quad \text{where} \quad L(k\Lambda_0)_s = \bigoplus_{s_2, \cdots, s_l \in \mathbb{Z}} L(k\Lambda_0)_{s_l \alpha_l + \cdots + s_2 \alpha_2 + s\alpha_1}.$$

To prove our main theorem, we use the Georgiev-type projection such that

$$\pi_{\mathcal{R}_{\alpha_1}} : L(k\Lambda_0) \to L(\Lambda_0)_{r_1^{(1)}} \otimes \cdots \otimes L(\Lambda_0)_{r_1^{(k)}},$$

where $\mathcal{R}_{\alpha_1} = (r_1^{(1)}, r_1^{(2)}, \ldots, r_1^{(k)})$ is a fixed dual-charge-type for the color 1 and $r_1 = \sum_{s=1}^k r_1^{(s)}$. This projection is naturally generalized to $L(k\Lambda_0)[[w_{t_l,l}, \ldots, w_{1,1}, z_{r_l^{(1)},l}, \ldots, z_{1,1}]]$. We also denote this generalization by $\pi_{\mathcal{R}_{\alpha_1}}$. Set $\alpha_i(z)_- = \sum_{m<0} \alpha_i(m) z^{-m-1}$. We consider the vector

$$e_\mu \alpha_l(-m'_{t_l,l})^{n'_{t_l,l}} \cdots \alpha_1(-m'_{1,1})^{n'_{1,1}} x^{\hat{\nu}}_{n_{r_l^{(1)},l}\alpha_l}(m_{r_l^{(1)},l}) \cdots x^{\hat{\nu}}_{n_{1,1}\alpha_1}(m_{1,1}) v_0$$

with dual-charge-type $\mathcal{R} = (\mathcal{R}_{\alpha_l}, \cdots, \mathcal{R}_{\alpha_1})$. Recall that the image of this vector with respect to $\pi_{\mathcal{R}_{\alpha_1}}$ coincides with the coefficient of the corresponding projection of the generating function

$$e_\mu \alpha_l(w_{t_l,l})_-^{n'_{t_l,l}} \alpha_1(w_{1,1})_-^{n'_{1,1}} x^{\hat{\nu}}_{n_{r_l^{(1)},l}\alpha_l}(z_{r_l^{(1)},l}) \cdots x^{\hat{\nu}}_{n_{1,1}\alpha_1}(z_{1,1}) v_0.$$

In order to prove the main theorem, we need a generalization of the twisted vertex operator $Y^{\hat{\nu}}(a, z)$ defined in section 2.2 to the case where a belongs to an extension of the weight lattice P of $\mathfrak{g}$. We do not repeat its definition. See [6].

Proposition 7. *Let a, b be elements of an extension $\hat{P}_\nu$ of P such that $\langle \bar{a}, \bar{b} \rangle \in \mathbb{Z}$. Then we have the following commutation relation for the twisted vertex operators.*

$$Y^{\hat{\nu}}(a, z_1) Y^{\hat{\nu}}(b, z_2) = (-1)^{\langle \bar{a}, \bar{b} \rangle} c_\nu(\bar{a}, \bar{b}) Y^{\hat{\nu}}(b, z_2) Y^{\hat{\nu}}(a, z_1)$$

Here $c_\nu(\bar{a}, \bar{b})$ is some constant which belongs to $\mathbb{C}^\times$.

Proof. We use [6, Theorem 5.2] in the case when $\mathbf{h}_* = 0$. The condition $\langle \bar{a}, \bar{b} \rangle \in \mathbb{Z}$ is important, since we use the property $z^m \delta(z) = \delta(z)$ $(m \in \mathbb{Z})$ for the formal delta function. □

Let λ_i $(i = 1, \ldots, \ell)$ be the fundamental weights of $\mathfrak{g}$ and set

$$Y^{\hat{\nu}}(e_{\lambda_i}, z) = \sum_{m \in \frac{1}{r}\mathbb{Z}} A_{\lambda_i}(m) z^{-m + \langle (\lambda_i)_{(0)}, (\lambda_i)_{(0)} \rangle / 2 - \langle \lambda_i, \lambda_i \rangle / 2}.$$

From Proposition 7, we have

$$A_{\lambda_i}(m) Y^{\hat{\nu}}_{\alpha_j}(n) = (-1)^{\delta_{ij}} c_\nu(\lambda_i, \alpha_j) Y^{\hat{\nu}}_{\alpha_j}(n) A_{\lambda_i}(m) \tag{31}$$

on V_P^T, which is the extended space of V_L^T by enlarging the root lattice L to the weight lattice P. We also have

$$[h(n), A_{\lambda_i}(m)] = \langle h_{rn}, (\lambda_i)_{(-rn)} \rangle A_{\lambda_i}(m + n). \tag{32}$$

Moreover,

$$A_{\lambda_i}(m) \mathbf{1}_T \in U(\mathfrak{h}[\nu]^-) e_{\lambda_i} \text{ for } m \geq 0 \text{ and } A_{\lambda_i}(0) \mathbf{1}_T = e_{\lambda_i}. \tag{33}$$

Theorem 8. *The set $\mathcal{B}_L$ is a basis of $L(k\Lambda_0)$.*

Proof. We should prove the linear independence of $\mathcal{B}_L$. We consider a linear combination of vectors in $\mathcal{B}_L$,

$$\sum c_{\mu, h, b} e_\mu h b v_0 = 0 \tag{34}$$

of the fixed degree and $\mathfrak{h}_{(0)}$-weight. From (24), for a $\mathfrak{h}_{(0)}$-weight ρ, the action of e_μ maps the weigt space V_ρ to $V_{\rho + k\mu}$. Hence, we may assume that a summand in (34) with the maximal charge of color 1, $\text{chg}_1 b$, has μ with α_1 coordinate zero. Namely, we assume that summands appear in the form

(A) $e_\mu h b v_0$ with $\text{chg}_1 b = r_1$ and $\mu = c_l \alpha_l + \cdots + c_2 \alpha_2$, or

(B) $e_{\bar{\mu}} \bar{h} \bar{b} v_0$ with $\text{chg}_1 \bar{b} < r_1$ and $\bar{\mu} = \bar{c}_l \alpha_l + \cdots + \bar{c}_1 \alpha_1$, where $\bar{c}_1 > 0$.

Among the vectors $v = e_\mu h b v_0$ with $\text{chg}_1 v = r_1$, we choose a vector with the maximal charge-type $\mathcal{R}'_{\alpha_1}$ and the corresponding dual-charge-type

$$\mathcal{R}_{\alpha_1} = (r_1^{(1)}, \cdots, r_1^{(k-1)})$$

for the color $i = 1$ where $r_1 = r_1^{(1)} + \cdots + r_1^{(k-1)}$. Note that $r_1^{(k)} = 0$ for a vector in $\mathcal{B}_L$. Denote the Georgiev-type projection by $\pi_{\mathcal{R}_{\alpha_1}}$. Since

$$e_{\alpha_1}(\mathbf{1}_T \otimes \cdots \otimes \mathbf{1}_T) = e_{\alpha_1} \mathbf{1}_T \otimes \cdots \otimes e_{\alpha_1} \mathbf{1}_T,$$

we have

$$e_{\bar{\mu}} \bar{h} \bar{v} \in \bigoplus_{\substack{s_1, \cdots, s_{k-1} \in \mathbb{Z} \\ s_k > 0}} L(\Lambda_0)_{s_1} \otimes \cdots \otimes L(\Lambda_0)_{s_k}.$$

Therefore, for the vectors of the form (B) we have $\pi_{\mathcal{R}_{\alpha_1}}(e_{\bar{\mu}} \bar{h} \bar{b}) v_0 = 0$. This means that the $\pi_{\mathcal{R}_{\alpha_1}}$ projection of the sum (34) contains only the summands of the form (A). Applying the same trick for the simple roots $\alpha_2, \ldots, \alpha_l$, we can assume $\mu = 0$ in (34).

Consider a linear combination

$$c_{h,b} h b v_0 + \sum_{b' > b} c_{h', b'} h' b' v_0 = 0. \tag{35}$$

For a monomial $h = h_l(-m_{t_l,l})^{n_{t_l,l}} \cdots h_1(-m_{1,1})^{n_{1,1}}$, set $\bar{h} = h_l(m_{t_l,l})^{n_{t_l,l}} \cdots h_1(m_{1,1})^{n_{1,1}}$ and multiply $\bar{h}$ from left to (35). Using (20), it turns out to be

$$c_{h,b} b v_0 + \sum_{b'' > b} c'_{h'', b''} h'' b'' v_0 = 0$$

up to an overall scalar multiple. Now, using (31),(32) and (33), one can argue in a similar way to the proof of Theorem 5.2 of [12] to prove $c_{h,b} = 0$. For the commutation relation of e_{λ_i} and $Y^{\hat{\nu}}_{\alpha_j}(m)$ is given by substituting $\alpha = \lambda_i$ in (18). In place of (5.25) of [12], we use

$$\pi_{\mathcal{R}_{\alpha_1}} b' x^{\hat{\nu}}_{s\alpha_i}(-s\rho_i)v_0 = \text{const } \pi_{\mathcal{R}_{\alpha_1}} b' \left(1 \otimes \cdots \otimes 1 \otimes \underbrace{e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_1}}_{s \text{ factors}} \right) v_0.$$

Once we show $c_{h,b} = 0$, substitute this relation to (35) and continue the process, then one eventually shows all coefficients are zero. $\qquad\square$

4. Parafermionic bases

4.1. **Vacuum space and twisted $\mathcal{Z}$-algebra.** Denote by $L(k\Lambda_0)^{\hat{\mathfrak{h}}[\nu]^+}$ the vacuum space of the standard module $L(k\Lambda_0)$, i.e.

$$L(k\Lambda_0)^{\hat{\mathfrak{h}}[\nu]^+} = \{v \in L(k\Lambda_0) \mid \hat{\mathfrak{h}}[\nu]^+ \cdot v = 0\}. \tag{36}$$

By the Lepowsky-Wilson theorem [19] (A5.3) we have the canonical isomorphism of d-graded linear spaces

$$U(\hat{\mathfrak{h}}[\nu]^-) \otimes L(k\Lambda_0)^{\hat{\mathfrak{h}}[\nu]^+} \xrightarrow{\cong} L(k\Lambda_0) \tag{37}$$

$$h \otimes u \mapsto h \cdot u$$

where $U(\hat{\mathfrak{h}}[\nu]^-) \simeq S(\hat{\mathfrak{h}}[\nu]^-)$ is the Fock space of level k for the Heisenberg subalgebra $\hat{\mathfrak{h}}[\nu]_{\frac{1}{r}\mathbb{Z}}$ with the action of c being the multiplication by scalar k. We consider the projection

$$\pi^{\hat{\mathfrak{h}}[\nu]^+} : L(k\Lambda_0) \to L(k\Lambda_0)^{\hat{\mathfrak{h}}[\nu]^+}$$

given by the direct decomposition

$$L(k\Lambda_0) = L(k\Lambda_0)^{\hat{\mathfrak{h}}[\nu]^+} \oplus \hat{\mathfrak{h}}[\nu]^- U(\hat{\mathfrak{h}}[\nu]^-) \cdot L(k\Lambda_0)^{\hat{\mathfrak{h}}[\nu]^+}. \tag{38}$$

By (25), we have the projective representation of Q on the vacuum space $L(k\Lambda_0)^{\hat{\mathfrak{h}}[\nu]^+}$.

We set

$$\mathcal{Z}_{n\alpha}(z) = E^-(\alpha, z)^{n/k} x^{\hat{\nu}}_{n\alpha}(z) E^+(\alpha, z)^{n/k}$$

for a quasi-particle of charge n and a root α. It is called the $\mathcal{Z}$-operator. Note that the action of $\mathcal{Z}$-operators commutes with the action of the Heisenberg algebra $\hat{\mathfrak{h}}[\nu]_{\frac{1}{r}\mathbb{Z}}$ on the standard module $L(k\Lambda_0)$. More generally, we need to define the $\mathcal{Z}$-operators for quasi-particles of charge-type $\mathcal{R}' = (n_{r_l^{(1)},l}, \cdots, n_{1,1})$. For $x^{\hat{\nu}}_{\mathcal{R}'}(z_{r_l^{(1)},l}, \cdots, z_{1,1}) = x^{\hat{\nu}}_{n_{r_l^{(1)},l}\alpha_l}(z_{r_l^{(1)},l}) \cdots x^{\hat{\nu}}_{n_{1,1}\alpha_1}(z_{1,1})$ of charge-type $\mathcal{R}'$, we define

$$\mathcal{Z}_{\mathcal{R}'}(z_{r_l^{(1)},l}, \cdots, z_{1,1}) = E^-(\alpha_l, z_{r_l^{(1)},l})^{n_{r_l^{(1)},l}/k} \cdots E^-(\alpha_1, z_{1,1})^{n_{1,1}/k} x^{\hat{\nu}}_{\mathcal{R}'}(z_{r_l^{(1)},l}, \cdots, z_{1,1})$$

$$\times E^+(\alpha_l, z_{r_l^{(1)},l})^{n_{r_l^{(1)},l}/k} \cdots E^+(\alpha_1, z_{1,1})^{n_{1,1}/k}. \tag{39}$$

For convenience, we write this formal Laurent series by

$$\mathcal{Z}_{\mathcal{R}'}(z_{r_l^{(1)},l}, \cdots, z_{1,1}) = \sum_{m_{r_l^{(1)},l}, \cdots, m_{1,1} \in \frac{1}{r}\mathbb{Z}} \mathcal{Z}_{\mathcal{R}'}(m_{r_l^{(1)},l}, \cdots, m_{1,1}) z_{r_l^{(1)},l}^{-m_{r_l^{(1)},l} - n_{r_l^{(1)},l}} \cdots z_{1,1}^{-m_{1,1} - n_{1,1}}.$$

Since $\mathcal{Z}$-operators act on the vacuum space and we can express quasi-particle monomials in terms of $\mathcal{Z}$-operators by reversing (39), we have

$$\pi^{\hat{\mathfrak{h}}[\nu]^+} : x_{\mathcal{R}'}(z_{r_l^{(1)},l}, \cdots, z_{1,1})v_0 \mapsto \mathcal{Z}_{\mathcal{R}'}(z_{r_l^{(1)},l}, \cdots, z_{1,1})v_0.$$

Now, Theorem 8 implies

Theorem 9. *The set of vectors*

$$e_\mu \mathcal{Z}_{\mathcal{R}'}(m_{r_l^{(1)},l}, \cdots, m_{1,1})v_0$$

such that $\mu \in Q$ and the charge-type $\mathcal{R}'$ and the energy-type $(m_{r_l^{(1)},l}, \cdots, m_{1,1})$ satisfy the conditions for B'_W is a basis of the vacuum space $L(k\Lambda_0)^{\hat{\mathfrak{h}}[\nu]^+}$.

The proof is parallel to that of Theorem 3.1 of [3].

4.2. **Parafermionic space and its current.** Recall that the map $\alpha \mapsto e_\alpha$ for $\alpha \in Q$ is extended to a projective representation of Q on $L(k\Lambda_0)$. This gives a diagonal action $\rho(k\alpha) = e_\alpha \otimes \cdots \otimes e_\alpha$ of the sublattice $kQ \subset Q$ such that $L(k\Lambda_0)_\mu^{\hat{\mathfrak{h}}[\nu]^+} \to L(k\Lambda_0)_{\mu+k\alpha}^{\hat{\mathfrak{h}}[\nu]^+}$. We define the parafermionic space of the highest weight $k\Lambda_0$ as the space of kQ-coinvariants in the kQ-module $L(k\Lambda_0)^{\hat{\mathfrak{h}}[\nu]^+}$

$$L(k\Lambda_0)_{kQ}^{\hat{\mathfrak{h}}[\nu]^+} := L(k\Lambda_0)^{\hat{\mathfrak{h}}[\nu]^+} / \mathrm{span}_{\mathbb{C}}\{(\rho(k\alpha)-1)\cdot v \mid \alpha \in Q, v \in L(k\Lambda_0)^{\hat{\mathfrak{h}}[\nu]^+}\}. \tag{40}$$

We have the canonical projection

$$\pi_{kQ}^{\hat{\mathfrak{h}}[\nu]^+} : L(k\Lambda_0)^{\hat{\mathfrak{h}}[\nu]^+} \to L(k\Lambda_0)_{kQ}^{\hat{\mathfrak{h}}[\nu]^+} \tag{41}$$

and denote the composition $\pi_{kQ}^{\hat{\mathfrak{h}}[\nu]^+} \circ \pi^{\hat{\mathfrak{h}}[\nu]^+} : L(k\Lambda_0) \to L(k\Lambda_0)_{kQ}^{\hat{\mathfrak{h}}[\nu]^+}$ by π. Note that in this case, we have

$$L(k\Lambda_0)_{kQ}^{\hat{\mathfrak{h}}[\nu]^+} \simeq \bigoplus_{\mu \in k\Lambda_0 + Q/kQ} L(k\Lambda_0)_\mu^{\hat{\mathfrak{h}}[\nu]^+}.$$

For every root β, we define the parafermionic current of charge n by

$$\Psi_{n\beta}^{\hat\nu}(z) = \mathcal{Z}_{n\beta}(z) z^{-n\beta_{(0)}/k} \epsilon_\beta^{-n/k}, \tag{42}$$

where $\epsilon_\beta : L(k\Lambda_0) \to \mathbb{C}^\times$ is given by

$$\epsilon_\beta u = C(\beta,\mu)u \quad \text{for } u \in L(k\Lambda_0)_\mu.$$

Since $\mathcal{Z}$-operators commute with the action of the Heisenberg subalgebra $\hat{\mathfrak{h}}[\nu]_{\frac{1}{r}\mathbb{Z}}$, the parafermionic current preserves the vacuum space $L(k\Lambda_0)^{\hat{\mathfrak{h}}[\nu]^+}$. The commutation relation (26) can be written as

$$x_\beta^{\hat\nu}(z) e_\alpha = C(\alpha,\beta)^{-1} e_\alpha x_\beta^{\hat\nu}(z) z^{\langle \alpha, \beta_{(0)}\rangle}.$$

From this relation and the one between z^μ and e_α, we have

$$[\rho(k\alpha), \Psi_{n\beta}^{\hat\nu}(z)] = 0.$$

Therefore, $\Psi^{\hat\nu}$ is well-defined on the parafermionic space $L(k\Lambda_0)_{kQ}^{\hat{\mathfrak{h}}[\nu]^+}$. For a quasi-particle of charge-type $\mathcal{R}' = (n_{r_l^{(1)}}, \cdots, n_{1,1})$, we define the parafermionic current of charge-type $\mathcal{R}'$ by

$$\Psi_{\mathcal{R}'}^{\hat\nu}(z_{r_l^{(1)},l}, \cdots, z_{1,1}) = \mathcal{Z}_{\mathcal{R}'}(z_{r_l^{(1)},l}, \cdots, z_{1,1}) z_{r_l^{(1)},l}^{-n_{r_l^{(1)},l}(\alpha_l)_{(0)}/k} \cdots z_{1,1}^{-n_{1,1}(\alpha_1)_{(0)}/k} \epsilon_{\alpha_l}^{-n_{r_l^{(1)},l}/k} \cdots \epsilon_{\alpha_1}^{-n_{1,1}/k}.$$

Note that the parafermionic current of charge-type $\mathcal{R}'$ also commutes with the diagonal action $\rho(k\alpha)$ for $\alpha \in Q$. As in the $\mathcal{Z}$-operator, we set

$$\Psi_{\mathcal{R}'}^{\hat\nu}(z_{r_l^{(1)},l}, \cdots, z_{1,1}) = \sum_{m_{r_l^{(1)},l}, \cdots, m_{1,1}} \psi_{\mathcal{R}'}^{\hat\nu}(m_{r_l^{(1)},l}, \cdots, m_{1,1}) z_{r_l^{(1)},l}^{-m_{r_l^{(1)},l} - n_{r_l^{(1)},l}} \cdots z_{1,1}^{-m_{1,1}-n_{1,1}}$$

where the summation is over all sequences $(m_{r_l^{(1)},l}, \cdots, m_{1,1})$ such that $m_{p,i} \in \rho_i\mathbb{Z} + \frac{n_{p,i}\langle(\alpha_i)_{(0)},\mu\rangle}{k}$ on the μ-weight space $L(k\Lambda_0)_\mu^{\hat{\mathfrak{h}}[\nu]^+}$.

The following lemma associates the coefficients of $\mathcal{Z}$-operators with those of parafermionic currents.

Lemma 10. *For a simple root β, $m \in \frac{1}{r}\mathbb{Z}$ and weight μ we have*

$$\mathcal{Z}_\beta(m)\Big|_{L(k\Lambda_0)_\mu^{\hat{\mathfrak{h}}[\nu]^+}} = C(\beta,\mu)\psi_\beta^{\hat\nu}(m + \langle\beta_{(0)},\mu\rangle/k)\Big|_{L(k\Lambda_0)_\mu^{\hat{\mathfrak{h}}[\nu]^+}}$$

Proof. By applying (42) to the μ-weight space $L(k\Lambda_0)_\mu^{\hat{\mathfrak{h}}[\nu]^+}$, we have

$$\Psi_\beta^{\hat\nu}(z)\Big|_{L(k\Lambda_0)_\mu^{\hat{\mathfrak{h}}[\nu]^+}} = C(\beta,\mu)^{-1/k} \mathcal{Z}_\beta(z) z^{-\langle\beta_{(0)},\mu\rangle/k}\Big|_{L(k\Lambda_0)_\mu^{\hat{\mathfrak{h}}[\nu]^+}}.$$

Hence we obtain the statement by taking the coefficient of $z^{-m-\langle\beta_{(0)},\mu\rangle/k-1}$. $\qquad\qquad\square$

Next we consider the relations between different parafermionic currents. We have the following lemma by direct computation. Noting that $C(\beta,\beta) = 1$ for a simple root β, the proof is parallel to that of Lemma 3.2 of [3].

Lemma 11. *For a simple root β and a positive integer n,*

$$\Psi_{n\beta}^{\hat{\nu}}(z) = \left(\prod_{1 \leq p < s \leq n} \prod_{i=0}^{r-1} (z_s^{\frac{1}{r}} - \eta^i z_p^{\frac{1}{r}})^{\langle \nu^i \beta, \beta \rangle / k} \right) \Psi_{\beta}^{\hat{\nu}}(z_n) \cdots \Psi_{\beta}^{\hat{\nu}}(z_1) \Big|_{z_n = \cdots = z_1 = z}. \tag{43}$$

We set

$$\Psi_{n_t \beta_t, \ldots, n_1 \beta_1}^{\hat{\nu}}(z_t, \ldots, z_1) = \mathscr{Z}_{(n_t, \ldots, n_1)}(z_t, \ldots, z_1) \prod_{i=1}^{t} z_i^{-n_i(\beta_i)_{(0)}/k} \epsilon_{\beta_i}^{1/k}$$

for simplicity for a given simple roots $\beta_t, \ldots, \beta_1$ and charges $n_1, \ldots, n_1$. Analogously to Lemma 11, we obtain the following lemma. The proof is parallel to that of Lemma 3.3 of [3].

Lemma 12. *For given simple roots $\beta_t, \ldots, \beta_1$ and positive integers $n_t, \ldots, n_1$,*

$$\Psi_{n_t \beta_t, \ldots, n_1 \beta_1}^{\hat{\nu}}(z_t, \ldots, z_1)$$

$$= \left(\prod_{1 \leq p < s \leq t} C(\beta_s, \beta_p)^{n_s n_p / k} \prod_{i=0}^{r-1} (z_s^{\frac{1}{r}} - \eta^i z_p^{\frac{1}{r}})^{\langle n_s \nu^i \beta_s, n_p \beta_p \rangle / k} \right) \Psi_{n_t \beta_t}^{\hat{\nu}}(z_t) \cdots \Psi_{n_1 \beta_1}^{\hat{\nu}}(z_1). \tag{44}$$

From Theorem 9, we have

Theorem 13. *For the highest weight $k\Lambda_0$, the set of vectors*

$$\pi_{kQ}^{\hat{\mathfrak{h}}[\nu]^+} \mathscr{Z}_{\mathcal{R}'}(m_{r_l^{(1)}, l}, \cdots, m_{1,1}) v_0 = \psi_{\mathcal{R}'}^{\hat{\nu}}(m_{r_l^{(1)}, l}, \cdots, m_{1,1}) v_0$$

is a basis of the parafermionic space $L(k\Lambda_0)_{kQ}^{\hat{\mathfrak{h}}[\nu]^+}$, where $\mathscr{Z}_{\mathcal{R}'}(m_{r_l^{(1)}, l}, \cdots, m_{1,1}) v_0$ is a vector that appears in a basis of the vacuum space $L(k\Lambda_0)^{\hat{\mathfrak{h}}[\nu]^+}$.

5. CHARACTER FORMULAS

5.1. Grading operator. Since we do not find the coset Virasoro algebra construction [20, §3] for the twisted vertex operator case, we introduce, by hand, a grading operator for our parafermionic space $L(k\Lambda_0)_{kQ}^{\hat{\mathfrak{h}}[\nu]^+}$. Define an operator D acting on the space $L(k\Lambda_0)^{\hat{\mathfrak{h}}[\nu]^+}$ as follows.

$$D = -d - D^{\hat{\mathfrak{h}}[\nu]^+}, \qquad D^{\hat{\mathfrak{h}}[\nu]^+} \Big|_{L(k\Lambda_0)_\mu^{\hat{\mathfrak{h}}[\nu]^+}} = \frac{\langle \mu_{(0)}, \mu_{(0)} \rangle}{2k}.$$

Then, for a simple root $\beta \in L$ and $m \in \mathbb{Z}$, we have

$$[D, x_\beta^{\hat{\nu}}(m)] = \left(-m - \frac{\langle \beta_{(0)}, \beta_{(0)} \rangle}{2k} \right) x_\beta^{\hat{\nu}}(m). \tag{45}$$

On $v \in L(k\Lambda_0)^{\hat{\mathfrak{h}}[\nu]^+}$, we also have

$$[D, \psi_\beta^{\hat{\nu}}(m)] = \left(-m - \frac{\langle \beta_{(0)}, \beta_{(0)} \rangle}{2k} \right) \psi_\beta^{\hat{\nu}}(m).$$

We call the coefficient of the right hand side the conformal energy of $\psi_\beta^{\hat{\nu}}(m)$ and write

$$\text{en } \psi_\beta^{\hat{\nu}}(m) = -m - \frac{\langle \beta_{(0)}, \beta_{(0)} \rangle}{2k}. \tag{46}$$

Now we compute the conformal energies of $\psi_{n\beta}^{\hat{\nu}}(m)$ and $\psi_{n_t \beta_t, \ldots, n_1 \beta_1}^{\hat{\nu}}(m_t, \ldots, m_1)$.

Lemma 14. *For a simple root β and charge n, we have*

$$\text{en } \psi_{n\beta}^{\hat{\nu}}(m) = -m - \frac{n^2 \langle \beta_{(0)}, \beta_{(0)} \rangle}{2k}. \tag{47}$$

Moreover, for simple roots $\beta_t, \cdots, \beta_1$ and charge $n_t, \ldots, n_1$, we have

$$\text{en } \psi_{n_t \beta_t, \ldots, n_1 \beta_1}^{\hat{\nu}}(m_t, \ldots, m_1) = \sum_{i=1}^{t} \left(\text{en } \psi_{n_i \beta_i}^{\hat{\nu}}(m_i) - \sum_{p=1}^{i-1} \frac{\langle n_i(\beta_i)_{(0)}, n_p(\beta_p)_{(0)} \rangle}{k} \right). \tag{48}$$

Proof. Consider the right hand side of (43). This means that $\psi_{n\beta}^{\hat{\nu}}(m)$ can be expressed by $\psi_{\beta}^{\hat{\nu}}(m_i)$ with $i = 1, \cdots, n$. Note that each term $z_i^{\langle\beta_{(0)},\beta_{(0)}\rangle/k}$ reduces the conformal energy by $\langle\beta_{(0)},\beta_{(0)}\rangle/k$. Since (43) contains $n(n-1)/2$ such terms and the energy of $\psi_{\beta}^{\hat{\nu}}(m)$ is give by (46), we have

$$
\mathrm{en}\,\psi_{n\beta}^{\hat{\nu}}(m) = -\sum_{i=1}^{n}\left(m_i + \frac{\langle\beta_{(0)},\beta_{(0)}\rangle}{2k}\right) - \frac{n(n-1)}{2}\cdot\frac{\langle\beta_{(0)},\beta_{(0)}\rangle}{k}
$$

$$
= -m - \frac{n^2\langle\beta_{(0)},\beta_{(0)}\rangle}{2k}
$$

where $m_1 + \cdots + m_n = m$. Now, (48) follows from (44) and the same argument by using the energy of $\psi_{n\beta}^{\hat{\nu}}(m)$ given by (47). $\qquad\square$

Using (23) we can show $[D, \rho(k\alpha)] = 0$ for $(\alpha \in Q)$. Hence, the grading operator D is well defined on the parafermionic space $L(k\Lambda_0)_{kQ}^{\hat{\mathfrak{h}}[\nu]^+}$.

5.2. Parafermionic character formula.
We define the character of the parafermionic space $L(k\Lambda_0)_{kQ}^{\hat{\mathfrak{h}}[\nu]^+}$ by

$$
\mathrm{ch}\,L(k\Lambda_0)_{kQ}^{\hat{\mathfrak{h}}[\nu]^+} = \sum_{m,r_1,\dots r_l \geq 0} \dim(L(k\Lambda_0)_{kQ}^{\hat{\mathfrak{h}}[\nu]^+})_{(m,r_1,\dots,r_l)}q^m y_1^{r_1}\cdots y_l^{r_l},
$$

where $(L(k\Lambda_0)_{kQ}^{\hat{\mathfrak{h}}[\nu]^+})_{(m,r_1\dots,r_l)}$ is the weight space spanned by monomial vectors of conformal energy $-m$ and color-type $(r_1, \dots, r_l)$ (see (30)). Consider an arbitrary quasi-particle monomial

$$
x_{n_{r_l^{(1)},l}\alpha_l}^{\hat{\nu}}(m_{r_l^{(1)},l})\cdots x_{n_{1,l}\alpha_l}^{\hat{\nu}}(m_{1,l})\cdots x_{n_{r_1^{(1)},1}\alpha_1}^{\hat{\nu}}(m_{r_1^{(1)},1})\cdots x_{n_{1,1}\alpha_1}^{\hat{\nu}}(m_{1,1}) \in B_W'. \tag{49}
$$

Denote by

$$
\mathcal{R}' = (n_{r_l^{(1)},l},\cdots, n_{1,1}), \quad \mathcal{R} = (r_l^{(1)},\cdots, r_1^{(k-1)})
$$

its charge-type, dual-charge-type. We define $\mathcal{P}_i = (p_i^{(1)},\dots,p_i^{(k-1)})$ by $p_i^{(s)} = r_i^{(s)} - r_i^{(s+1)}$ $(r_i^{(k)} = 0)$ for $i = 1, \dots, l$, $s = 1, \dots, k-1$, so that $p_i^{(s)}$ stands for the number of quasi-particles of color i and charge s in the monomial (49). Set $\mathcal{P} = (\mathcal{P}_l, \dots, \mathcal{P}_1)$. To emphasize the dependence of k, we also write $\mathcal{P}^{(k-1)}$. We consider the parafermionic space basis given in Theorem 13. From (48), the conformal energy of

$$
\psi_{n_{r_l^{(1)},l}\alpha_l,\dots,n_{1,1}\alpha_1}^{\hat{\nu}}(m_{r_l^{(1)},l},\dots, m_{1,1})
$$

is equal to

$$
-\sum_{i=1}^{l}\left(\sum_{s=1}^{r_i^{(1)}}m_{s,i} + \sum_{s=1}^{r_i^{(1)}}\frac{n_{s,i}^2\rho_i}{k} + \sum_{s=1}^{r_i^{(1)}}\left(\sum_{t=1}^{s-1}\frac{2n_{t,i}n_{s,i}\rho_i}{k} + \sum_{j=1}^{i-1}\sum_{t=1}^{r_j^{(1)}}\frac{\langle n_{s,i}(\alpha_i)_{(0)}, n_{t,j}(\alpha_j)_{(0)}\rangle}{k}\right)\right)
$$

$$
= -\sum_{i=1}^{l}\sum_{s=1}^{r_i^{(1)}}m_{s,i} - \frac{1}{2}\sum_{i,j=1}^{l}\sum_{s=1}^{r_i^{(1)}}\sum_{t=1}^{r_j^{(1)}}\frac{\langle n_{s,i}(\alpha_i)_{(0)}, n_{t,j}(\alpha_j)_{(0)}\rangle}{k}, \tag{50}
$$

where ρ_i is defined in (11). Since

$$
\sum_{s=1}^{r_i^{(1)}}n_{s,i} = \sum_{s=1}^{k-1}sp_i^{(s)},
$$

(50) is further calculated as

$$
-\sum_{i=1}^{l}\sum_{s=1}^{r_i^{(1)}}m_{s,i} - \frac{1}{2}\sum_{i,j=1}^{l}\sum_{s,t=1}^{k-1}\frac{st}{k}\langle(\alpha_i)_{(0)},(\alpha_j)_{(0)}\rangle p_i^{(s)}p_j^{(t)}. \tag{51}
$$

To calculate the character of the parafermionic space, we use the corresponding result of the principal subspace.

Theorem 15. ([4, Theorem 6.1]) *For each of the affine Lie algebras $A_{2l-1}^{(2)}$, $D_{l+1}^{(2)}$, $E_6^{(2)}$, $D_4^{(3)}$, we have*

$$\operatorname{ch} W(k\Lambda_0) = \sum_{\mathcal{P}^{(k)}} \frac{q^{\frac{1}{2}\sum_{i,j=1}^{l}\langle(\alpha_i)_{(0)},(\alpha_j)_{(0)}\rangle\sum_{s,t=1}^{k}\min\{s,t\}p_i^{(s)}p_j^{(t)}}}{\prod_{i=1}^{l}\prod_{s=1}^{k}(q^{\rho_i})_{p_i^{(s)}}}\prod_{i=1}^{l}y_i^{\sum_{s=1}^{k}sp_i^{(s)}}$$

where the sum runs over all sequences $\mathcal{P}^{(k)}$ of lk nonnegative integers.

Note that in this formula, we use the index set $\mathcal{P}^{(k)}$ rather than $\mathcal{P}^{(k-1)}$, since the basis vectors contain quasi-particles of charge k by Theorem 4. Now we can obtain the character of the parafermionic space $L(k\Lambda_0)_{kQ}^{\hat{\mathfrak{h}}[\nu]^+}$.

Theorem 16. *For each of the affine Lie algebras $A_{2l-1}^{(2)}$, $D_{l+1}^{(2)}$, $E_6^{(2)}$, $D_4^{(3)}$, we have*

$$\operatorname{ch} L(k\Lambda_0)_{kQ}^{\hat{\mathfrak{h}}[\nu]^+} = \sum_{\mathcal{P}^{(k-1)}} \frac{q^{\frac{1}{2}\sum_{i,j=1}^{l}\langle(\alpha_i)_{(0)},(\alpha_j)_{(0)}\rangle\sum_{s,t=1}^{k-1}D_{s,t}^{(k)}p_i^{(s)}p_j^{(t)}}}{\prod_{i=1}^{l}\prod_{s=1}^{k-1}(q^{\rho_i})_{p_i^{(s)}}}\prod_{i=1}^{l}y_i^{\sum_{s=1}^{k-1}sp_i^{(s)}}$$

where the sum runs over all sequences $\mathcal{P}^{(k-1)}$ of $l(k-1)$ nonnegative integers and

$$D_{s,t}^{(k)} = \min\{s,t\} - \frac{st}{k}.$$

Proof. Comparing Theorems 4 and 15, one can readily calculate the generating function with weight given by the first term of (51) as

$$\sum_{\mathcal{P}^{(k-1)}} \frac{q^{\frac{1}{2}\sum_{i,j=1}^{l}\langle(\alpha_i)_{(0)},(\alpha_j)_{(0)}\rangle\sum_{s,t=1}^{k-1}\min\{s,t\}p_i^{(s)}p_j^{(t)}}}{\prod_{i=1}^{l}\prod_{s=1}^{k-1}(q^{\rho_i})_{p_i^{(s)}}}\prod_{i=1}^{l}y_i^{\sum_{s=1}^{k-1}sp_i^{(s)}}.$$

Taking the conformal shift, namely the second term of (51), into account, we obtain the desired formula. $\square$

5.3. **Character formula of the standard module.** Finally, we prove the following character formula of $L(k\Lambda_0)$ that was conjectured in [14, Conjecture 5.3].

Theorem 17. *For each of the affine Lie algebras $A_{2l-1}^{(2)}$, $D_{l+1}^{(2)}$, $E_6^{(2)}$, $D_4^{(3)}$, we have*

$$\operatorname{ch} L(k\Lambda_0) = \frac{1}{\prod_{i=1}^{l}(q^{\rho_i})_\infty}\sum_{\eta\in Q_{(0)}}q^{\langle\eta,\eta\rangle/2k}\prod_{i=1}^{l}y_i^{\eta_i}\sum_{\mathcal{P}^{(k-1)}}\frac{q^{\frac{1}{2}\sum_{i,j=1}^{l}\langle(\alpha_i)_{(0)},(\alpha_j)_{(0)}\rangle\sum_{s,t=1}^{k-1}D_{s,t}^{(k)}p_i^{(s)}p_j^{(t)}}}{\prod_{i=1}^{l}\prod_{s=1}^{k-1}(q^{\rho_i})_{p_i^{(s)}}}$$

where the sum $\sum_{\mathcal{P}^{(k-1)}}$ runs over all sequences $\mathcal{P}^{(k-1)}$ of $l(k-1)$ nonnegative integers satisfying

$$\sum_{i=1}^{l}\sum_{s=1}^{k-1}sp_i^{(s)}(\alpha_i)_{(0)} \in \eta + kQ_{(0)}.$$

Proof. By Theorem 8 and (37), we have

$$\operatorname{ch} L(k\Lambda_0) = \frac{1}{\prod_{i=1}^{l}(q^{\rho_i})_\infty}\operatorname{ch} L(k\Lambda_0)^{\hat{\mathfrak{h}}[\nu]^+}, \quad L(k\Lambda_0)^{\hat{\mathfrak{h}}[\nu]^+} = \operatorname{span}\{e_\mu bv_0 \mid \mu\in Q, b\in B_W'\}.$$

By $\operatorname{wt} v$, we denote the $\mathfrak{h}_{(0)}$-weight of a weight vector v. From (24) we have

$$\operatorname{wt} e_\mu bv_0 = \operatorname{wt} bv_0 + k\mu_{(0)}. \tag{52}$$

From (23) we also obtain

$$-d\cdot e_\mu bv_0 = -d\cdot bv_0 + \langle\operatorname{wt} bv_0,\mu_{(0)}\rangle + \frac{k}{2}\langle\mu_{(0)},\mu_{(0)}\rangle. \tag{53}$$

Here we note $\langle\lambda,\mu\rangle = \langle\lambda,\mu_{(0)}\rangle$ for a $\mathfrak{h}_{(0)}$-weight λ and $\mu\in Q$. From (52),(53) and Theorem 15 we obtain

$$\operatorname{ch} L(k\Lambda_0)^{\hat{\mathfrak{h}}[\nu]^+} = \sum_{\lambda\in Q_{(0)},\mu\in Q}\prod_i y_i^{\lambda_i+k(\mu_{(0)})_i}q^{\langle\lambda,\mu_{(0)}\rangle+\frac{k}{2}\langle\mu_{(0)},\mu_{(0)}\rangle}\operatorname{ch} W((k-1)\Lambda_0)_\lambda \tag{54}$$

where $\operatorname{ch} W((k-1)\Lambda_0)_\lambda$ is the coefficient of $\prod_i y_i^{\lambda_i}$ ($\lambda = \sum_i\lambda_i(\alpha_i)_{(0)}$) in $\operatorname{ch} W((k-1)\Lambda_0)$. Setting $\lambda + k\mu_{(0)} = \eta$, the coefficient of $\prod_i y_i^{\eta_i}$ in (54) is equal to

$$q^{\langle\eta,\eta\rangle/2k}\sum_{\lambda\in\eta+kQ_{(0)}}q^{-\langle\lambda,\lambda\rangle/2k}\operatorname{ch} W((k-1)\Lambda_0)_\lambda.$$

In view of Theorem 15, the factor $q^{-\langle\lambda,\lambda\rangle/2k}$ can be removed by replacing $\min\{s,t\}$ with $D_{s,t}^{(k)}$ in the power of q and we obtain the desired formula. $\qquad\square$

Remark 18. We compare this result with [14, Conjecture 5.3]. First, note that due to the difference of the normalization of the bilinear form $\langle\,,\rangle$ on the root lattice $Q_{(0)}$, q in this paper should be replaced with q^r to compare with [14]. Recalling $r\rho_i = t_i^\vee$, we see that the above formula is consistent with Conjecture 5.3 in [14].

Acknowledgments

Data Availability

Data sharing not applicable to this article.

Ethical Statement/Conflict on Interest

Not applicable to this article.

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OSAKA CITY UNIVERSITY ADVANCED MATHEMATICAL INSTITUTE & DEPARTMENT OF MATHEMATICS, OSAKA CITY UNIVERSITY, 3-3-138 SUGIMOTO, SUMIYOSHI-KU, OSAKA, 558-8585, JAPAN
  Email address: `okado@osaka-cu.ac.jp`

DEPARTMENT OF MATHEMATICS, OSAKA CITY UNIVERSITY, 3-3-138 SUGIMOTO, SUMIYOSHI-KU, OSAKA, 558-8585, JAPAN
  Email address: `m20sa023@zp.osaka-cu.ac.jp`