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Improving Upper and Lower Bounds for the Total Number of Edge Crossings of Euclidean Minimum Weight Laman Graphs*

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Abstract. We investigate the total number of edge crossings (i.e., the crossing number) of the Euclidean minimum weight Laman graph $\mathsf{MLG}(P)$ on a planar point set P. Bereg et al. (2016) showed that the upper and lower bounds for the crossing number of $\mathsf{MLG}(P)$ are $6|P|-9$ and $|P|-3$, respectively. In this paper, we improve these upper and lower bounds given by Bereg et al. (2016) to $2.5|P|-5$ and $(1.25-\varepsilon)|P|$ for any $\varepsilon > 0$, respectively. Especially, for improving the upper bound, we introduce a novel counting scheme based on some geometric observations.

Keywords: Laman graphs, Sparse and tight graphs, Plane graphs, Geometric graphs, Edge crossings

1 Introduction

A graph $G = (V, E)$ is called *Laman* if $|E| = 2|V| - 3$ and $|E(H)| \leq 2|V(H)| - 3$ for any subgraph H of G with $E(H) \neq \emptyset$. A Laman graph has a property of being *minimally rigid* in the plane if it is realized as a *generic bar-joint framework* [5, 8]. A bar-joint framework is a straight-line realization of a graph in the plane, and by regarding each edge as a bar and each point as a joint the rigidity of such a graph can be defined in a natural way (see, e.g., [5]). One of the most fundamental results in combinatorial rigidity theory asserts that a graph G realized on a generic point set (i.e., the set of the coordinates is algebraically independent over the rational field) is rigid if and only if G contains a spanning Laman subgraph [8]. Laman graphs appear in a wide range of applications, not only statics but also mechanical design such as linkages, design of CAD systems, analysis of protein flexibility, and sensor network localization [9, 10].

Given a set P of n points in the Euclidean plane, let $G(P)$ denote a *geometric graph* on P, i.e., $G(P) = (P, E)$ where E is a set of edges each of which is drawn as a segment between two points in P. Throughout the paper, we assume that no three points in P are collinear and all interpoint distances are distinct. The point

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set satisfying these assumptions is called *semi-generic*. A two-dimensional bar-joint framework is considered as a geometric graph, thus in this paper, we deal with geometric graphs where the underlying graphs are Laman, called *Euclidean Laman graphs*. It is then natural to consider the Euclidean Laman graph on a planar point set P with the minimum total edge-length over all Euclidean Laman graphs on P, i.e., the *Euclidean minimum weight Laman graph* on P denoted by $\mathsf{MLG}(P)$.

In order to realize a geometric graph as a bar-joint framework in the real world, it is important to consider the *crossing property* of the geometric graph. A geometric graph is called *plane* (or *non-crossing*) if any two edges do not have a crossing except possibly at their endpoints. In fact, the Euclidean minimum spanning tree on a semi-generic planar point set P ($\mathsf{MST}(P)$ for short) is plane. Observe that both Laman graphs and spanning trees are characterized by similar *sparsity conditions*: A graph G is called (k,l)-*sparse* if $|E(H)| \leq k|V(H)| - l$ for any subgraph H of G with $E(H) \neq \emptyset$, and a (k,l)-sparse graph is called (k,l)-*tight* if it has exactly $k|V(H)| - l$ edges (see, e.g., [8]). A spanning tree is a $(1,1)$-tight graph while a Laman graph is a $(2,3)$-tight graph. Since (k,l)-sparse graphs have several common combinatorial properties such as being independent sets of a matroid, a natural question is whether the Euclidean minimum weight (k,l)-tight graph on a point set has a nice crossing property as does the Euclidean minimum weight $(1,1)$-tight graphs.

Bereg et al. [3] studied crossing properties of $\mathsf{MLG}(P)$. They proved as the main results that $\mathsf{MLG}(P)$ is 6-planar, i.e., each edge in $\mathsf{MLG}(P)$ has at most six crossings, and $\mathsf{MLG}(P)$ is also *quasi-planar*, i.e., no three edges in $\mathsf{MLG}(P)$ pairwise cross. In addition, they showed an instance P for which there exists an edge that has six crossings in $\mathsf{MLG}(P)$.

In the following, we use the terminology *crossing number* to denote the total number of crossings. According to the results by Bereg et al. [3], it is easy to see that the crossing number of $\mathsf{MLG}(P)$ is at most $6 \times (2|P| - 3)/2 = 6|P| - 9$. Bereg et al. [3] also provided an instance P for which the crossing number of $\mathsf{MLG}(P)$ is $|P| - 3$ (as shown in Fig. 3), therefore, there has been a gap between upper and lower bounds for the crossing number of $\mathsf{MLG}(P)$. In this paper, we improve these upper and lower bounds given by Bereg et al. [3] to $2.5|P| - 5$ and $(1.25 - \varepsilon)|P|$ for any $\varepsilon > 0$, respectively. Especially, for improving the upper bound, we introduce a novel counting scheme based on some geometric observations, which is the most important contribution presented in the paper.

As for the crossing number of geometric graphs, several classes of *proximity graphs* are studied by Ábrego et al. [1], e.g., *nearest neighbor graphs*, *relative neighborhood graphs*, *Gabriel graphs* and *Delaunay graphs*. In a *k-nearest neighbor graph* on a point set P (k-$\mathsf{NNG}(P)$ for short), for $p, q \in P$, pq [3] is included if and only if p is the i-th closest point among p from q for some $i \geq k$ or vice versa. In a *k-relative neighborhood graph* on a point set P (k-$\mathsf{RNG}(P)$ for short), for $p, q \in P$, pq is included if and only if $D_p(pq) \cap D_q(pq)$ (where $D_p(r)$

[3] Throughout the paper, for two points p, q, we abuse the notation pq to denote the line segment between p and q or the length of itself, depending on the context.

denotes the closed disk with center p and radius r) contains at most k points among $P \setminus \{p,q\}$. In a *k-Gabriel graph* on a point set P (k-$\mathsf{GG}(P)$ for short), for $p,q \in P$, pq is included if and only the circle through p and q with diameter pq contains at most k points among $P \setminus \{p,q\}$. In a *k-Delaunay graphs* on a point set P (k-$\mathsf{DG}(P)$ for short), for $p,q \in P$, pq is included if and only if there is a circle through p and q that contains at most k other points. Ábrego et al. [1] proved that for any set P of n points, k-$\mathsf{NNG}(P)$ has at most k^3n crossings, k-$\mathsf{RNG}(P)$ has at most $9k^3n$ crossings, k-$\mathsf{GG}(P)$ has at most $3k^2n^2$ crossings, and k-$\mathsf{DG}(P)$ has at most $3k^2n^2$ crossings. Note that Bereg et al. [3] showed the relation among k-$\mathsf{NNG}(P)$, k-$\mathsf{RNG}(P)$, k-$\mathsf{GG}(P)$ and $\mathsf{MST}(P)$. See Lemma 1 for the details.

The rest of the paper is organized as follows. In Section 2, we introduce some notations and properties of $\mathsf{MLG}(P)$ given by Bereg et al. [3], that are used throughout the paper. In Section 3, we provide new geometric observations and give an efficient counting scheme for improving the upper bound based on the shown observations. In Section 4, we show how to construct an instance which achieves the improved lower bound. In Section 5, we discuss future works, which concludes the paper.

2 Preliminaries

First of all, we introduce some notations used throughout the paper. The closed disk (resp. circle) with center p and radius r is denoted $D_p(r)$ (resp. $C_p(r)$). Consider two points p,q in the plane. Let $\texttt{Lens}(pq) = D_p(pq) \cap D_q(pq)$. Let $\texttt{bisect}(pq)$ denote the perpendicular bisector of segment pq. Let $\texttt{Up\_Lens}(pq)$ (resp. $\texttt{Low\_Lens}(pq)$) denote the intersection of $\texttt{Lens}(pq)$ and the halfplane determined by $\texttt{bisect}(pq)$ that contains p (resp. q). Let $\texttt{L\_Lens}(pq)$ (resp. $\texttt{R\_Lens}(pq)$) denote the intersection of $\texttt{Lens}(pq)$ and the halfplane determined by the supporting line of segment pq that contains a point p' such that p, q, and p' are arranged on triangle pqp' in clockwise (resp. counterclockwise) order. For a point p and two half lines ℓ and ℓ' starting at p in the plane, let $\texttt{angle}_p(\ell,\ell')$ denote the smaller angle between ℓ and ℓ', and $\texttt{Cone}_p(\ell,\ell')$ denote the cone with apex at p delimited by ℓ and ℓ', which corresponds to $\texttt{angle}_p(\ell,\ell')$.

In the rest of this section, we introduce several lemmas and theorems shown by Bereg et al. [3] since those are useful to prove our main lemmas provided in the next section. In the following, let P be a set of semi-generic n points in the Euclidean plane, and for a geometric graph $G(P)$, we abuse notation $G(P)$ to denote a set of edges in $G(P)$.

Let us start with a property based on which our counting scheme is.

Lemma 1. (Theorem 1.1 in [3]) *It holds*

$$\mathsf{MST}(P) \cup 2\text{-}\mathsf{NNG}(P) \subseteq \mathsf{MLG}(P) \subseteq 1\text{-}\mathsf{GG}(P) \cap 2\text{-}\mathsf{RNG}(P).$$

Focusing on $\mathsf{MST}(P) \subseteq \mathsf{MLG}(P)$, we classify the edges in $\mathsf{MLG}(P)$ into ones in $\mathsf{MST}(P)$ and ones not in $\mathsf{MST}(P)$. The details will be given in the next section.

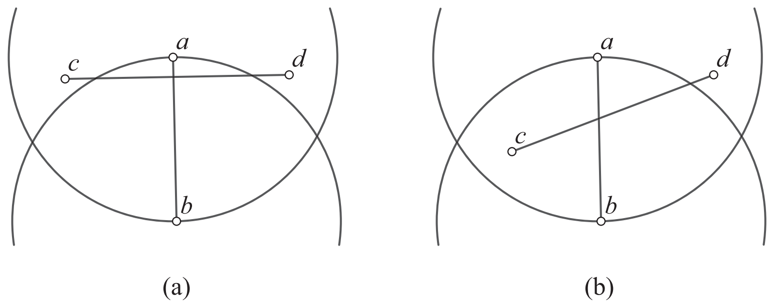


Fig. 1. (a) A lens-crossing edge cd for ab. (b) A fan-crossing edge cd for ab.

The following lemma shows properties which points of P in $\texttt{Lens}(ab)$ for $ab \in \mathsf{MLG}(P)$ satisfy.

Lemma 2. (Lemma 2.3 in [3]) *Consider an edge* $ab \in \mathsf{MLG}(P)$.
(i) There exists at most one point of P *in each of* $\texttt{L\_Lens}(ab)$ *and* $\texttt{R\_Lens}(ab)$.
(ii) If there exists one point of P *in each of* $\texttt{L\_Lens}(ab)$ *and* $\texttt{R\_Lens}(ab)$, *i.e.,* $c \in \texttt{L\_Lens}(ab)$ *and* $d \in \texttt{R\_Lens}(ab)$, *it then holds that* $ab < cd$ *and* $cd \notin \mathsf{MLG}(P)$.

We introduce key concepts both in our paper and [3]. See also Fig. 1.

Definition 1. (lens-crossing edge) *For four points* $a, b, c, d \in P$, *suppose that segments* ab *and* cd *cross each other, and* $c, d \notin \texttt{Lens}(ab)$. *Then,* cd *is called a lens-crossing edge for* ab.

Definition 2. (fan-crossing edge) *For four points* $a, b, c, d \in P$, *suppose that segments* ab *and* cd *cross each other, and* $c \in \texttt{Lens}(ab)$ *and* $d \notin \texttt{Lens}(ab)$. *Then,* cd *is called a fan-crossing edge for* ab.

The following two lemmas show properties on lens-crossing edges for $ab \in \mathsf{MLG}(P)$.

Lemma 3. (Lemma 3.3 in [3]) *For four points* $a, b, c, d \in P$, *suppose that segment* cd *is a lens-crossing edge for segment* ab, *and* cd *cuts only* $\texttt{Up\_Lens}(ab)$ *(i.e., it does not cut* $\texttt{Low\_Lens}(ab)$*). Then, it holds* $a \in \texttt{Lens}(cd)$.

Lemma 4. (Lemma 4.1 in [3] [4]) *Consider an edge* $ab \in \mathsf{MLG}(P)$. *Then,* $\mathsf{MLG}(P)$ *includes*
(i) at most one lens-crossing edge for ab *that cuts only* $\texttt{Up\_Lens}(ab)$,
(ii) at most one lens-crossing edge for ab *that cuts only* $\texttt{Low\_Lens}(ab)$, *and*
(iii) no lens-crossing edge for ab *that cuts both* $\texttt{Up\_Lens}(ab)$ *and* $\texttt{Low\_Lens}(ab)$.

Lemma 4 means that $\mathsf{MLG}(P)$ includes at most two lens-crossing edges for $ab \in \mathsf{MLG}(P)$. Especially, it is easy to see the proof of Lemma 4(iii) as follows: Suppose

[4] Lemma 4.1 in [3] corresponds to Lemma 4(i)(ii).

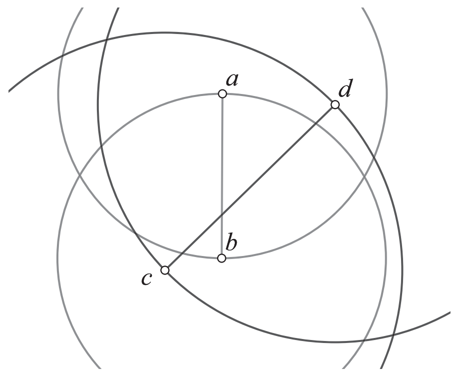


Fig. 2. Illustration of Lemma 4(iii).

that a lens-crossing edge for ab, say cd, is also included in $\mathsf{MLG}(P)$, and cd cuts both $\texttt{Up\_Lens}(ab)$ and $\texttt{Low\_Lens}(ab)$ as shown in Fig. 2. Then, it holds $ab < cd$ and $a, b \in \texttt{Lens}(cd)$, which contradicts Lemma 2(ii).

As for fan-crossing edges for $ab \in \mathsf{MLG}(P)$, we have the following lemma.

Lemma 5. (Lemma 4.3 in [3]) *Consider an edge $ab \in \mathsf{MLG}(P)$. Then, $\mathsf{MLG}(P)$ includes at most four fan-crossing edges for ab.*

Based on the proof of Lemma 5 written in [3], we analyze more details and obtain Lemmas 10 and 11 provided in the next section. Note that, indeed, the proof of Lemma 5 for the case where only one point exists in $\texttt{Lens}(ab)$ immediately follows from the proof of Lemma 10.

Let $\sigma(P)$ denote the crossing number of $\mathsf{MLG}(P)$. By Lemmas 4 and 5, we see that every edge in $\mathsf{MLG}(P)$ has at most six crossings. Therefore, $\sigma(P)$ is at most $6 \times (2n - 3)/2$.

Theorem 1. [3] *For any set of semi-generic points P, it holds $\sigma(P) \leq 6|P| - 9$.*

Bereg et al. [3] also provide an instance P whose crossing number is $|P| - 3$ as shown in Fig. 3.

Theorem 2. [3] *There exists a set of semi-generic points P such that $\sigma(P) \geq |P| - 3$.*

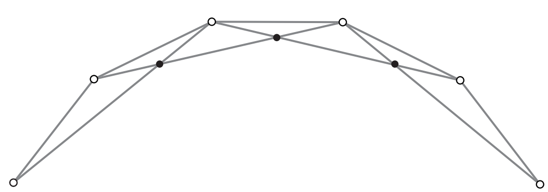


Fig. 3. $\mathsf{MLG}(P)$ that has $|P| - 3$ crossings (indicated by black-colored circles).

To conclude this section, we introduce the following useful lemmas implicitly shown in [11] and [3].

Lemma 6. [11] *For three points $a, b, c \in P$, suppose that $\mathtt{angle}_a(\ell, \ell') < 60°$ and $b, c \in \mathtt{Cone}_a(\ell, \ell')$. Then, the longer of ab and ac is not included in $\mathsf{MST}(P)$.*

Lemma 7. [3] *For four points $a, b, c, d \in P$, suppose that $\mathtt{angle}_a(\ell, \ell') < 60°$ and $b, c, d \in \mathtt{Cone}_a(\ell, \ell')$. Then, the longest of ab, ac, and ad is not included in $\mathsf{MST}(P)$.*

3 Improved upper bound for $\sigma(P)$

In this section, we show a novel counting scheme based on some geometric observations, which improves the upper bound for $\sigma(P)$ shown in Theorem 1. Recall that $\mathsf{MST}(P) \subseteq \mathsf{MLG}(P)$ as shown in Lemma 1. In the following, let $\overline{\mathsf{MST}}(P) = \mathsf{MLG}(P) \setminus \mathsf{MST}(P)$. For counting $\sigma(P)$, we basically classify the edges in $\mathsf{MLG}(P)$ into ones in $\mathsf{MST}(P)$ and ones in $\overline{\mathsf{MST}}(P)$.

Let us first see the following lemma.

Lemma 8. *For an edge $ab \in \mathsf{MST}(P)$, there is no fan-crossing edge.*

Proof. We prove by contradiction that there exists no point of P in $\mathtt{Lens}(ab)$: Suppose that $c \in P$ lies in $\mathtt{Lens}(ab)$. We then have $\max\{ab, bc, ca\} = ab$. Since for any triangle whose vertices are points in P the longest edge is not in $\mathsf{MST}(P)$, it holds $ab \notin \mathsf{MST}(P)$, a contradiction. This completes the proof. $\square$

At this point, it is easy to see $\sigma(P) \leq 4n-7$ as follows: By Lemmas 4 and 8, an edge in $\mathsf{MST}(P)$ has at most two crossings in $\mathsf{MLG}(P)$. Therefore, we obtain

$$\sigma(P) \leq \frac{2|\mathsf{MST}(P)| + 6|\overline{\mathsf{MST}}(P)|}{2} = \frac{2(n-1) + 6(n-2)}{2} = 4n - 7.$$

In the rest of this section, we further improve this upper bound.

Next see the following lemma.

Lemma 9. *For four points $a, b, c, d \in P$, suppose that cd is a lens-crossing edge for ab. Then, ab is a fan-crossing edge for cd.*

Proof. By Lemma 4, without loss of generality, cd cuts $\mathtt{Up\_Lens}(ab)$ and does not cut $\mathtt{Low\_Lens}(ab)$ (see Fig. 4). Then, by Lemma 3, we have $a \in \mathtt{Lens}(cd)$. On the other hand, it holds $b \notin \mathtt{Lens}(cd)$ by Lemma 2(ii). This completes the proof. $\square$

We now consider classifying crossings in $\mathsf{MLG}(P)$ into two cases.

Definition 3. (f-f crossing/f-l crossing) *For four points $a, b, c, d \in P$, suppose that two segments ab and cd intersect each other.*
(i) If cd is a fan-crossing edge for ab, and ab is a fan-crossing edge for cd, we call the crossing between ab and cd an f-f crossing.
(ii) If cd is a fan-crossing edge for ab, and ab is a lens-crossing edge for cd, we call the crossing between ab and cd an f-l crossing.

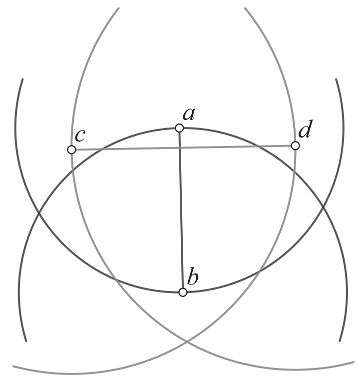


Fig. 4. Illustration of the proof of Lemma 9.

Note that if cd is a lens-crossing edge for ab, by Lemma 9, ab must be a fan-crossing edge for cd. Therefore, every crossing in $\mathsf{MLG}(P)$ is an f-f crossing or an f-l crossing. Furthermore, since there is no fan-crossing edge for any edge in $\mathsf{MST}(P)$ by Lemma 8, every crossing in $\mathsf{MLG}(P)$ is a crossing between an edge $e \in \overline{\mathsf{MST}}(P)$ and a fan-crossing edge for e, which implies that $\sigma(P)$ can be counted only by checking fan-crossing edges for edges in $\overline{\mathsf{MST}}(P)$.

Prior to details of our counting scheme, we show the following two lemmas.

Lemma 10. *Consider an edge $ab \in \overline{\mathsf{MST}}(P)$. Among the crossings between ab and fan-crossing edges for ab in $\mathsf{MLG}(P)$, there exist at most two f-l crossings.*

Proof. First of all, if there exists no point of P in $\mathtt{Lens}(ab)$, the statement clearly holds. According to Lemma 2, we consider other two cases: [Case 1] There exists one point of P in $\mathtt{Lens}(ab)$. [Case 2] There exist two points of P in $\mathtt{Lens}(ab)$.
Case 1: Without loss of generality, $c \in P$ lies in $\mathtt{L\_Lens}(ab)$. Let ℓ_a be a half line emanating from c to a and ℓ'_a be a half line emanating from c such that $\mathtt{angle}_c(\ell_a, \ell'_a) = 60°$ and ℓ'_a cuts $\mathtt{R\_Lens}(ab)$. Similarly, let ℓ_b be a half line emanating from c to b and ℓ'_b be a half line emanating from c such that $\mathtt{angle}_c(\ell_b, \ell'_b) = 60°$ and ℓ'_b cuts $\mathtt{R\_Lens}(ab)$.

Let d be a point of P such that cd is a fan-crossing edge for ab and cd lies in $\mathtt{Cone}_c(\ell_a, \ell'_a)$ (see Fig. 5(a)). Let d' be the crossing between segment cd and the boundary of $\mathtt{Lens}(ab)$. Since $\mathtt{bisect}(ad')$ passes through b since ad' is a chord of $C_b(ab)$, it is easy to see that a and c lie in the same side of $\mathtt{bisect}(ad')$, which means $ca < cd'$. By $cd' \leq cd$, we obtain $ca < cd$. On the other hand, since $\angle dca \leq 60° < \angle dac$, we have $ad < cd$. Hence, it holds $a \in \mathtt{Lens}(cd)$, i.e., a crossing between ab and cd is an f-f crossing. In a symmetric manner, we obtain the same conclusion even if cd lies in $\mathtt{Cone}_c(\ell_b, \ell'_b)$. Therefore, only if cd lies in $\mathtt{Cone}_c(\ell'_a, \ell'_b)$, a crossing between ab and cd can be an f-l crossing. By the fact of $\mathtt{angle}_c(\ell'_a, \ell'_b) < 60°$ and Lemma 7, $\mathsf{MLG}(P)$ includes at most two fan-crossing edges lying in $\mathtt{Cone}_c(\ell'_a, \ell'_b)$, which completes the proof for Case 1.
Case 2: According to Lemma 2, without loss of generality, $c, d \in P$ lie in $\mathtt{L\_Lens}(ab)$ and $\mathtt{R\_Lens}(ab)$, respectively. Let p (resp. q) be the intersection point

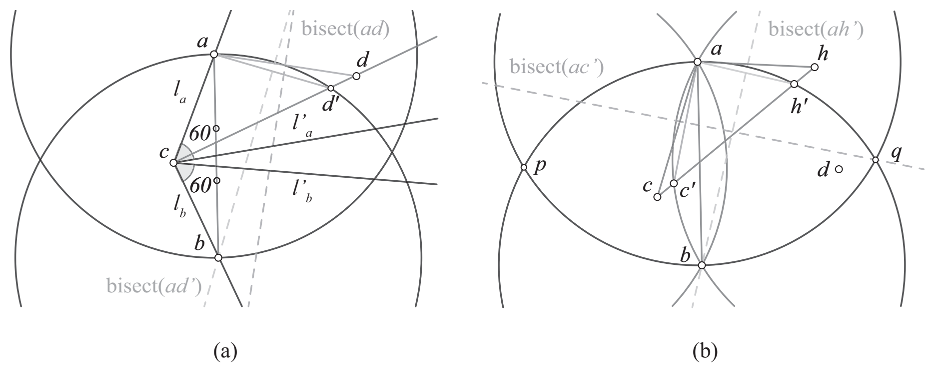


Fig. 5. (a) Illustration of Case 1 in the proof of Lemma 10. (b) Illustration of Case 2 in the proof of Lemma 10.

of two circles $C_a(ab)$ and $C_b(ab)$ in $\texttt{L\_Lens}(ab)$ (resp. $\texttt{R\_Lens}(ab)$) (see Fig. 5(b)). We can see $c \notin D_q(ab)$ and $d \notin D_p(ab)$ since otherwise $c \in D_q(ab)$ or $d \in D_p(ab)$ holds, and then $cd < ab$ holds, which contradicts Lemma 2.

We consider only fan-crossing edges for ab emanating from c since ones emanating from d are symmetric. Let ℓ_a, ℓ_b, ℓ_q be a half line emanating from c to a, b, q respectively. Let h be a point of P such that ch is a fan-crossing edge for ab and ch lies in $\texttt{Cone}_c(\ell_a, \ell_q)$. Let h' be the crossing between segment ch and the boundary of $\texttt{Lens}(ab)$, and c' be the crossing (inside $\texttt{Lens}(ab)$) between segment ch and $C_q(ab)$. Since $\texttt{bisect}(ah')$ passes through b, it is easy to see that a and c lie in the same side of $\texttt{bisect}(ah')$, which means $ca < ch'$. By $ch' \leq ch$, we obtain $ca < ch$. Similarly, by considering $\texttt{bisect}(ac')$, we obtain $ah < ch$. Hence, it holds $a \in \texttt{Lens}(ch)$, i.e., a crossing between ab and ch is an f-f crossing. In a symmetric manner, we obtain the same conclusion even if ch lies in $\texttt{Cone}_c(\ell_b, \ell_q)$, which implies that there exists no f-l crossing between ab and fan-crossing edges for ab. This completes the proof for Case 2. □

Lemma 11. *For an edge $ab \in \overline{\mathsf{MST}}(P)$, at most one fan-crossing edge is included in $\mathsf{MST}(P)$.*

Proof. Consider the same cases as in the proof of Lemma 10.
Case 1: As shown in the proof of Lemma 10, if cd is a fan-crossing edge for ab and cd lies in $\texttt{Cone}_c(\ell_a, \ell'_a)$ or $\texttt{Cone}_c(\ell_b, \ell'_b)$, $\texttt{Lens}(cd)$ includes a or b, respectively, i.e., $cd \notin \mathsf{MST}(P)$. Therefore, only if cd lies in $\texttt{Cone}_c(\ell'_a, \ell'_b)$, cd can be included in $\mathsf{MST}(P)$. By the fact of $\texttt{angle}_c(\ell'_a, \ell'_b) < 60°$ and Lemma 6, $\mathsf{MST}(P)$ includes at most one fan-crossing edges lying in $\texttt{Cone}_c(\ell'_a, \ell'_b)$, which completes the proof for Case 1.
Case 2: As shown in the proof of Lemma 10, if ch is a fan-crossing edge for ab and ch lies in $\texttt{Cone}_c(\ell_a, \ell_q)$ or $\texttt{Cone}_c(\ell_b, \ell_q)$, $\texttt{Lens}(cd)$ includes a or b, respectively, i.e., $ch \notin \mathsf{MST}(P)$. Therefore, no fan-crossing edge is included in $\mathsf{MST}(P)$. This completes the proof for Case 2. □

3.1 Counting scheme

Recall that every crossing in $\mathsf{MLG}(P)$ is a crossing between an edge $e \in \overline{\mathsf{MST}}(P)$ and a fan-crossing edge for e. In the following, we classify each edge $e \in \overline{\mathsf{MST}}(P)$ into two types: [Type 1] There exists at least one f-l crossing among the crossings between e and fan-crossing edges for e in $\mathsf{MLG}(P)$. [Type 2] There exists no f-l crossing among the crossings between e and fan-crossing edges for e in $\mathsf{MLG}(P)$. Let m_i be the number of edges of Type i in $\overline{\mathsf{MST}}(P)$. Clearly, it holds

$$m_1 + m_2 = |\overline{\mathsf{MST}}(P)| = n - 2. \tag{1}$$

We then consider numbering edges in Type i from 1 to m_i in any order, and use e_{ij} to denote the j-th edge in Type i. Recall that every crossing in $\mathsf{MLG}(P)$ is an f-f crossing or an f-l crossing. For every edge $e_{ij} \in \overline{\mathsf{MST}}(P)$, let $\sigma_{ij}^{\text{f-f}}$ (resp. $\sigma_{ij}^{\text{f-l}}$) be the number of f-f (resp. f-l) crossings between e_{ij} and fan-crossing edges for e_{ij} in $\mathsf{MLG}(P)$. By Lemma 10, it holds

$$\sigma_{1j}^{\text{f-l}} \leq 2 \quad \text{for } j = 1, \ldots, m_1. \tag{2}$$

Also, we have by the definition

$$\sigma_{2j}^{\text{f-l}} = 0 \quad \text{for } j = 1, \ldots, m_2. \tag{3}$$

Let $\sigma^{\text{f-f}}$ (resp. $\sigma^{\text{f-l}}$) denote the number of f-f (resp. f-l) crossings in $\mathsf{MLG}(P)$. Clearly, it holds

$$\sigma(P) = \sigma^{\text{f-f}} + \sigma^{\text{f-l}}. \tag{4}$$

First, we consider counting $\sigma^{\text{f-f}}$ and $\sigma^{\text{f-l}}$ by checking fan-crossing edges for every $e_{ij} \in \overline{\mathsf{MST}}(P)$. While counting, each f-f crossing is counted exactly twice. We thus have

$$\sigma^{\text{f-f}} = \frac{1}{2}\left(\sum_{i=1}^{2}\sum_{j=1}^{m_i}\sigma_{ij}^{\text{f-f}}\right). \tag{5}$$

On the other hand, each f-l crossing is counted exactly once. We thus have

$$\sigma^{\text{f-l}} = \sum_{i=1}^{2}\sum_{j=1}^{m_i}\sigma_{ij}^{\text{f-l}} = \sum_{j=1}^{m_1}\sigma_{1j}^{\text{f-l}}. \tag{6}$$

Note that the second equality in Eq. (6) holds by applying Eq. (3). Summarizing Eq. (4), Eq. (5) and Eq. (6), we obtain

$$\begin{aligned}
\sigma(P) &= \frac{1}{2}\left(\sum_{i=1}^{2}\sum_{j=1}^{m_i}\sigma_{ij}^{\text{f-f}}\right) + \sum_{j=1}^{m_1}\sigma_{1j}^{\text{f-l}} \\
&= \frac{1}{2}\left(\sum_{i=1}^{2}\sum_{j=1}^{m_i}\sigma_{ij}^{\text{f-f}} + \sum_{j=1}^{m_1}\sigma_{1j}^{\text{f-l}}\right) + \frac{1}{2}\sum_{j=1}^{m_1}\sigma_{1j}^{\text{f-l}}.
\end{aligned} \tag{7}$$

Next, consider counting the number of fan-crossing edges in $\overline{\mathsf{MST}}(P)$, say α, for every $e_{ij} \in \overline{\mathsf{MST}}(P)$. Recall that the crossing between e_{ij} and an edge in $\mathsf{MST}(P)$ is always an f-l crossing. Hence by Lemma 11, for an edge e_{1j}, the number of fan-crossing edges in $\overline{\mathsf{MST}}(P)$ is at least $\sigma_{1j}^{\text{f-f}} + \sigma_{1j}^{\text{f-l}} - 1$, and for an edge e_{2j}, it is exactly $\sigma_{2j}^{\text{f-f}} + \sigma_{2j}^{\text{f-l}} = \sigma_{2j}^{\text{f-f}}$ (by Eq. (3)), i,e,. it holds

$$\alpha \geq \sum_{j=1}^{m_1} (\sigma_{1j}^{\text{f-f}} + \sigma_{1j}^{\text{f-l}} - 1) + \sum_{j=1}^{m_2} \sigma_{2j}^{\text{f-f}} = \sum_{i=1}^{2} \sum_{j=1}^{m_i} \sigma_{ij}^{\text{f-f}} + \sum_{j=1}^{m_1} \sigma_{1j}^{\text{f-l}} - m_1. \tag{8}$$

On the other hand, since each edge in $\overline{\mathsf{MST}}(P)$ is counted at most twice, and $|\overline{\mathsf{MST}}(P)| = n - 2$, we have

$$\alpha \leq 2(n-2). \tag{9}$$

Summarizing Eq. (8) and Eq. (9), we obtain

$$\sum_{i=1}^{2} \sum_{j=1}^{m_i} \sigma_{ij}^{\text{f-f}} + \sum_{j=1}^{m_1} \sigma_{1j}^{\text{f-l}} \leq 2(n-2) + m_1. \tag{10}$$

Hence, we have an improved upper bound for $\sigma(P)$ as follows:

$$\begin{aligned}
\sigma(P) &\leq \frac{2(n-2) + m_1}{2} + \frac{1}{2} \sum_{j=1}^{m_1} \sigma_{1j}^{\text{f-l}} && \text{(by substituting Eq. (10) into Eq. (7))} \\
&\leq \frac{2(n-2) + m_1}{2} + \frac{1}{2} \cdot 2m_1 && \text{(by Eq. (2))} \\
&\leq \frac{5}{2}(n-2) && \text{(since } m_1 \leq n-2 \text{ by Eq. (1)).}
\end{aligned}$$

Theorem 3. *For a set of any semi-generic points P, it holds $\sigma(P) \leq 2.5|P| - 5$.*

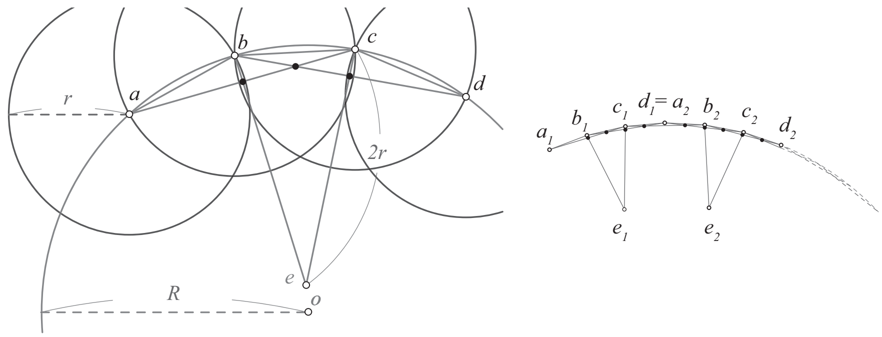
4 Improved lower bound for $\boldsymbol{\sigma(P)}$

In this section, we show how to construct an instance P for which there exist more crossings in $\mathsf{MLG}(P)$ than one shown by Bereg et al. [3].

Consider a set of five points $\{a, b, c, d, e\}$ as shown in Fig. 6(a). For a point o, and two real numbers R and r with $R > r > 0$, points a, b, c, d are arranged on $C_o(R)$ in this order so that $ab = bc = cd = r$, and point e is located in $D_o(R)$ such that $be = ce = 2r$. We call such a set of five points *unit* w.r.t. (o, R, r) in the following.

For an integer $t > 0$, let us consider t numbered units w.r.t. (o, R, r). We identify points a, b, c, d, e of i-th unit as a_i, b_i, c_i, d_i, e_i, respectively. We now put t units so that $d_i = a_{i+1}$ (regarded as one point) for $i = 1, \ldots, t-1$ as shown in Fig. 6(b). Let $P(o, R, r, t)$ denote a set of points constructed in the above manner. It is then easy to see

$$|P(o, R, r, t)| = 4t + 1. \tag{11}$$



(a) (b)

Fig. 6. (a) A unit w.r.t. (o, R, r). (b) Illustration of how to connect two units.

Let us consider $\mathsf{MLG}(P(o, R, r, t))$. In the following, we take values R, r, t so that $R/t > 1 \gg r$. It is then easy to see that

$$\begin{aligned} a_i b_i = b_i c_i = c_i d_i (= r) < a_i c_i = b_i d_i = c_i b_{i+1} (\simeq 2r) \quad &< b_i e_i = c_i e_i (= 2r) \\ < a_i e_i = d_i e_i (\simeq \sqrt{6} r) \quad &< e_i e_{i+1} (\simeq 3r), \end{aligned}$$

thus $\mathsf{MLG}(P(o, R, r, t))$ consists of edges $a_i b_i$, $b_i c_i$, $c_i d_i$, $a_i c_i$, $b_i d_i$, $b_i e_i$, $c_i e_i$ for $i = 1, \ldots, t$, and edges $c_i b_{i+1}$ for $i = 1, \ldots, t-1$. Since there are three crossings in the i-th unit, and each edge $c_i b_{i+1}$ has two crossings, it holds

$$\sigma(P(o, R, r, t)) = 3t + 2(t-1) = 5t - 2. \tag{12}$$

By Eq. (11) and Eq.(12), we have

$$\frac{\sigma(P(o, R, r, t))}{|P(o, R, r, t)|} = \frac{5t-2}{4t+1} = \frac{5}{4} - \frac{13}{16t+4},$$

which can be larger than $5/4 - \varepsilon$ for any $\varepsilon > 0$ by taking t as a sufficiently large integer. Notice that $P(o, R, r, t)$ is not semi-generic, however by moving each point in $P(o, R, r, t)$ infinitesimally, we can obtain a set of semi-generic points P such that the topology of $\mathsf{MLG}(P)$ is the same as one of $\mathsf{MLG}(P(o, R, r, t))$ and $\sigma(P) = \sigma(P(o, R, r, t))$.

Theorem 4. *For any $\varepsilon > 0$, there exists a set of semi-generic points P such that $\sigma(P) \geq (1.25 - \varepsilon)|P|$.*

5 Future works

Several problems related to the crossing number of $\mathsf{MLG}(P)$ remain open.

One problem is to further improve upper or lower bounds for $\sigma(P)$. Although in this paper, we have improved upper and lower bounds for $\sigma(P)$ as shown in Theorems 3 and 4, respectively, there is still a gap.

Another interesting problem is to analyze the *thickness* of $\mathsf{MLG}(P)$. The thickness of a geometric graph $G(P)$[5] is the smallest number of layers necessary to partition the edges of $G(P)$ into layers in such a way that no two edges of the same layer cross. It is easy to see that the thickness of $\mathsf{MLG}(P)$ is at most 4 since it holds $\mathsf{MLG}(P) \subseteq 1\text{-}\mathsf{GG}(P)$ by Lemma 1, and it is shown by Bose et al. [4] that the thickness of $1\text{-}\mathsf{GG}(P)$ is at most 4. Therefore, a problem of whether the thickness of $\mathsf{MLG}(P)$ is at most 3 naturally arises. In order to prove this, one direction worth considering is as follows: Define a graph $H = (W, F)$ for $\mathsf{MLG}(P)$ such that each vertex $e \in W$ corresponds to each edge $e \in \mathsf{MLG}(P)$, and for two vertices $e, e' \in W$, edge (e, e') is included in F if and only if edges e and e' cross each other in $\mathsf{MLG}(P)$. We then notice that the thickness of $\mathsf{MLG}(P)$ is equal to the chromatic number of H. It is proved by Grötzsch [6] that a planar triangle-free graph is 3-colorable. On the other hand, Bereg et al. [3] show the quasi-planarity of $\mathsf{MLG}(P)$, i.e., no three edges in $\mathsf{MLG}(P)$ pairwise cross, which means that H is triangle-free. Hence, once we prove the planarity of H, the claim immediately holds.

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[5] Another terminology *constrained geometric thickness of a graph* is used in [4].