A Payne-Rayner type inequality for the Robin problem on arbitrary minimal surfaces in $\mathbb{R}^N$

Futoshi Takahashi and Akinobu Uegaki

<table>
<thead>
<tr>
<th>Citation</th>
<th>OCAMI Preprint Series</th>
</tr>
</thead>
<tbody>
<tr>
<td>Issue Date</td>
<td>2010</td>
</tr>
<tr>
<td>Type</td>
<td>Preprint</td>
</tr>
<tr>
<td>Textversion</td>
<td>Author</td>
</tr>
<tr>
<td>Rights</td>
<td>For personal use only. No other uses without permission.</td>
</tr>
<tr>
<td>Relation</td>
<td>This is a pre-print of an article published in Results in Mathematics. The final authenticated version is available online at: <a href="https://doi.org/10.1007/s00025-010-0064-y">https://doi.org/10.1007/s00025-010-0064-y</a>.</td>
</tr>
</tbody>
</table>

From: Osaka City University Advanced Mathematical Institute

A Payne-Rayner type inequality for the Robin problem on arbitrary minimal surfaces in $\mathbb{R}^N$

Futoshi Takahashi and Akinobu Uegaki

Abstract. We prove a Payne-Rayner type inequality for the first eigenfunction of the Laplacian with Robin boundary condition on any compact minimal surface with boundary in $\mathbb{R}^N$. We emphasize that no topological condition is necessary on the boundary.


Keywords. Robin problem, Payne-Rayner type inequality.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial \Omega$, and let $\lambda_1(\Omega)$ and $\psi$ denote the first eigenvalue and the corresponding first eigenfunction, respectively, to the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega. \end{cases}$$

In [7], Payne and Rayner proved the following inequality

$$\left( \int_{\Omega} \psi^2 \, dx \right) \leq \frac{\lambda_1(\Omega)}{4\pi} \left( \int_{\Omega} \psi \, dx \right)^2.$$

A remarkable point of this inequality is that it gives an exact lower-bound of the first eigenvalue by means of some integral-norms of the first eigenfunction, on one hand, and on the other hand, it also says that the first eigenfunction satisfies a reverse Hölder type inequality. Actually, the $L^2$ norm of $\psi$ is bounded by the $L^1$ norm of $\psi$.

In this paper, we extend the above result, known to hold on a flat domain with the Dirichlet boundary condition, to a more general setting. Namely, let $\Sigma$ be

The first author acknowledges the support by JSPS Grant-in-Aid for Scientific Research (C), No. 20540216.
a compact minimal surface in $\mathbb{R}^N (N \geq 3)$ with smooth boundary $\partial \Sigma$. We consider the following eigenvalue problem with the Robin boundary condition:

$$
\begin{align*}
&-\Delta \Sigma u = \lambda u \quad \text{in } \Sigma, \\
&\frac{\partial u}{\partial \nu} + \beta u = 0 \quad \text{on } \partial \Sigma,
\end{align*}
$$

(1.1)

where $\Delta \Sigma$ is the Laplace-Beltrami operator on $\Sigma$, $\beta$ is a positive constant and $\nu$ is the outer unit normal to $\partial \Sigma$. Let $\lambda_1^\beta(\Sigma)$ denote the first eigenvalue of (1.1), given by the variational formula

$$
\lambda_1^\beta(\Sigma) = \min_{u \in H^1(\Sigma)} \frac{\int_\Sigma |\nabla \Sigma u|^2 d\mathcal{H}^2 + \beta \int_{\partial \Sigma} u^2 d\mathcal{H}^1}{\int_\Sigma u^2 d\mathcal{H}^2},
$$

where $\nabla \Sigma$ is the gradient operator on $\Sigma$ and $\mathcal{H}^k$ denotes the $k$-dimensional Hausdorff measure in $\mathbb{R}^N$. It is well known that $\lambda_1^\beta(\Sigma)$ is simple and isolated, and the corresponding eigenfunction $\psi_\beta$ is smooth, positive, and unique up to multiplication by constants. (see, for example, [3]).

Now, let us consider the auxiliary problem

$$
\begin{align*}
&-\Delta \Sigma f = 2 \quad \text{in } \Sigma, \\
&f = 0 \quad \text{on } \partial \Sigma.
\end{align*}
$$

(1.2)

Our main result is the following Payne-Rayner type inequality.

**Theorem 1.1.** Let $\lambda_1^\beta(\Sigma)$ be the first eigenvalue of (1.1) and $\psi_\beta$ be the eigenfunction corresponding to $\lambda_1^\beta(\Sigma)$. Then

$$
\int_\Sigma \psi_\beta^2 d\mathcal{H}^2 \leq \frac{\lambda_1^\beta(\Sigma)}{\sqrt{2\pi}} \left( \int_\Sigma \psi_\beta^2 d\mathcal{H}^2 \right)^2 + \frac{1}{2} \int_{\partial \Sigma} \psi_\beta^2 \left( \frac{\partial f_\Sigma}{\partial \nu} \right) d\mathcal{H}^1 + \frac{1}{\sqrt{2\pi}} \mathcal{H}^1(\partial \Sigma)(M^2 - m_*^2)
$$

holds, where $M = \max_{\partial \Sigma} \psi_\beta$, $m_* = \min_{\Sigma \cup \partial \Sigma} \psi_\beta$, and $f_\Sigma$ is the unique solution to the problem (1.2).

As for the Dirichlet eigenvalue problem

$$
\begin{align*}
&-\Delta \Sigma u = \lambda u \quad \text{in } \Sigma, \\
u = 0 \quad \text{on } \partial \Sigma,
\end{align*}
$$

(1.3)

the same proof of Theorem 1.1 works well and we obtain

**Theorem 1.2.** Let $\lambda_1^D(\Sigma)$ be the first eigenvalue of (1.3) and $\psi_D$ be the eigenfunction corresponding to $\lambda_1^D(\Sigma)$. Then we have

$$
\int_\Sigma \psi_D^2 d\mathcal{H}^2 \leq \frac{\lambda_1^D(\Sigma)}{2\sqrt{2\pi}} \left( \int_\Sigma \psi_D^2 d\mathcal{H}^2 \right)^2.
$$

Under the assumption that the boundary $\partial \Sigma$ is weakly connected (see Li-Schoen-Yau [6]), Wang and Xia [8] recently proved the sharp inequality

$$
\int_\Sigma \psi_D^2 d\mathcal{H}^2 \leq \frac{\lambda_1^D(\Sigma)}{4\pi} \left( \int_\Sigma \psi_D^2 d\mathcal{H}^2 \right)^2.
$$
for the first eigenfunction to (1.3), with the equality holds if and only if $\Sigma$ is a flat
disc on an affine 2-plane in $\mathbb{R}^N$.

Our method of proof is strongly related to that of [8], which in turn goes back
to the work [7]. However, in our case, we cannot apply the sharp isoperimetric
inequality by Li-Schoen-Yau [6] directly to level sets of the first eigenfunction,
since we put no topological assumptions on the boundary. Instead, we use a weaker
version of the isoperimetric inequality due to A. Stone ([1]: Lemma 4.3):

Let $\Sigma$ be a compact minimal surface in $\mathbb{R}^N$ with boundary $\partial \Sigma$. Let $A$ denote
the area of $\Sigma$ and $L$ the length of $\partial \Sigma$. Then the inequality
\begin{equation}
2\sqrt{2}\pi A \leq L^2
\end{equation}
holds.

Though the constant $2\sqrt{2}\pi$ in front of $A$ is not the best possible value $4\pi$,
this weaker inequality is valid for any compact minimal surface in $\mathbb{R}^N$ with boundary.
Thanks to this, we do not need any topological assumption such as weak connect-
edness on the boundary in Theorem 1.1 and Theorem 1.2.

In case $\Sigma = \Omega \subset \mathbb{R}^2$ is a bounded smooth domain in (1.1), we can appeal to
the classical sharp isoperimetric inequality $4\pi A \leq L^2$ on the plane, then we obtain

**Theorem 1.3.** Let $\Sigma = \Omega$ is a smooth bounded domain in $\mathbb{R}^2$. Then we have
\begin{equation}
\int_\Omega \psi^2 dx \leq \frac{\lambda_1^2(\Omega)}{2\pi} \left( \int_\Omega \psi^2 dx \right)^2 + \frac{1}{2} \int_{\partial \Omega} \frac{1}{\partial \nu} \left( \frac{\partial f}{\partial \nu} \right) d\mathcal{H}^1 + \frac{1}{2\pi} \mathcal{H}^1(\partial \Sigma)^2(M^2 - m_*^2)
\end{equation}

We do not repeat the proof of Theorem 1.2 and Theorem 1.3 here, since it
needs only a trivial change in the proof of Theorem 1.1.

2. Proof of Theorem 1.1

First, we set
\begin{align*}
U(t) &= \{x \in \Sigma : \psi_\beta(x) > t\}, \\
S(t) &= \Sigma \cap \partial U(t), \\
\Gamma(t) &= \partial \Sigma \cap \partial U(t)
\end{align*}
for $t > 0$. Then $\partial U(t) = S(t) \cup \Gamma(t)$ is a disjoint union. Since $\psi_\beta$ is smooth up
to the boundary ([5]), Sard’s lemma implies that $|\nabla_\Sigma \psi_\beta| \neq 0$ on $S(t)$, $S(t)$ is a smooth hypersuface and can be written as $S(t) = \{x \in \Sigma : \psi_\beta(x) = t\}$ for a.e.
t $> 0$. Recall $M = \max_{\Sigma} \psi_\beta$ and $m_* = \min_{\Sigma \cup \partial \Sigma} \psi_\beta$. We claim that $\min_{\partial \Sigma} \psi_\beta > 0$.
Indeed, if $\psi_\beta(x_0) = 0$ for some $x_0 \in \partial \Sigma$, then the boundary condition implies that
$\frac{\partial \psi_\beta}{\partial \nu}(x_0) = 0$ also holds. On the other hand, by the positivity of $\psi_\beta$ and Hopf’s
lemma, we have $\frac{\partial \psi_\beta}{\partial \nu}(x_0) < 0$, which is a contradiction. Since $\psi_\beta$ is positive on $\Sigma$, the
above claim yields $m_* > 0$, and then $U(t) = \Sigma$ for any $0 < t < m_*$. Also we
note that $\Gamma(t) = \emptyset$ if $t > M$. 
As in the proof of [2], [3], [8], our main tool is the following co-area formula, asserting that for every $w \in L^1(\Sigma)$, it holds
\[ \int_{U(t)} w dH^2 = \int_t^\infty \int_{S(\tau)} \frac{w}{|\nabla \Sigma \psi\beta|} dH^1 d\tau, \]
\[ \frac{d}{dt} \int_{U(t)} w dH^2 = - \int_{S(t)} \frac{w}{|\nabla \Sigma \psi\beta|} dH^1. \]

See, for instance, [4]. Note that in the right hand side, the integral over $\Gamma(t)$ does not appear.

We define the following two functions $g$ and $h$ as
\[ g(t) = \int_{U(t)} \psi\beta dH^2 = \int_t^\infty \int_{S(\tau)} \frac{\psi\beta}{|\nabla \Sigma \psi\beta|} dH^1 d\tau, \]
\[ h(t) = - \int_{U(t)} \langle \nabla \Sigma \left( \frac{1}{2} \psi^2 \right), \nabla \Sigma f \rangle dH^2 \]
\[ = - \int_t^\infty \int_{S(\tau)} \frac{\psi\beta \langle \nabla \Sigma \psi\beta, \nabla \Sigma f \rangle}{|\nabla \Sigma \psi\beta|} dH^1 ds, \]
where $f$ is the unique solution of the problem (1.2).

Differentiating $g$ and $h$, we have
\[ g'(t) = - \int_{S(t)} \frac{1}{|\nabla \Sigma \psi\beta|} dH^1, \quad (2.1) \]
\[ h'(t) = t \int_{S(t)} \frac{\langle \nabla \Sigma \psi\beta, \nabla \Sigma f \rangle}{|\nabla \Sigma \psi\beta|} dH^1 = - t \int_{S(t)} \langle \nabla \Sigma f, \nu \rangle dH^1 \]
\[ = - t \int_{S(t)} \frac{\partial f}{\partial \nu} dH^1 \quad (2.2) \]
for a.e. $t > 0$, since $-\nabla \Sigma \psi\beta / |\nabla \Sigma \psi\beta| \bigg|_{S(t)}$ is outward unit normal vector field $\nu$ of $S(t)$.

On the other hand, integrating both sides of $-\Delta \Sigma \psi\beta = \lambda^2_1(\Sigma) \psi\beta$ over $U(t)$, we have
\[ \lambda^2_1(\Sigma) g(t) = \lambda^2_1(\Sigma) \int_{U(t)} \psi\beta dH^2 = - \int_{U(t)} \Delta \Sigma \psi\beta dH^2 \]
\[ = \int_{S(t)} |\nabla \Sigma \psi\beta| dH^1 - \int_{\Gamma(t)} \frac{\partial \psi\beta}{\partial \nu} dH^1 \]
\[ = \int_{S(t)} |\nabla \Sigma \psi\beta| dH^1 + \beta \int_{\Gamma(t)} \psi\beta dH^1 \]
\[ \geq \int_{S(t)} |\nabla \Sigma \psi\beta| dH^1, \quad (2.3) \]
since $-\frac{\partial \psi\beta}{\partial \nu} = \beta \psi\beta > 0$ on $\Gamma(t) \subset \partial \Sigma$. 


Also, we see
\[2\mathcal{H}^2(U(t)) = \int_{U(t)} 2d\mathcal{H}^2 = \int_{U(t)} \Delta f d\mathcal{H} = \int_{\partial U(t)} \frac{\partial f}{\partial \nu} d\mathcal{H}^1\]
\[= \int_{S(t)} \frac{\partial f}{\partial \nu} d\mathcal{H}^1 + \int_{\Gamma(t)} \frac{\partial f}{\partial \nu} d\mathcal{H}^1\]
\[\geq \int_{S(t)} \frac{\partial f}{\partial \nu} d\mathcal{H}^1 = \frac{-1}{t} h'(t)\]  \hfill (2.4)
by (2.2). The last inequality follows by the fact \(\frac{\partial f}{\partial \nu} > 0\) on \(\Gamma(t) \subset \partial \Sigma\), which in turn is assured by the Hopf lemma.

From the weak isoperimetric inequality (1.4) applied to \(U(t)\), we have
\[2\sqrt{2\pi} \mathcal{H}^2(U(t)) \leq \mathcal{H}^1(\partial U(t))^2\]
\[\leq (\mathcal{H}^1(S(t)) + \mathcal{H}^1(\Gamma(t)))^2\]
\[\leq 2\mathcal{H}^1(S(t))^2 + 2\mathcal{H}^1(\Gamma(t))^2.\]  \hfill (2.5)
Now, Schwarz’s inequality, (2.1) and (2.3) imply
\[\mathcal{H}^1(S(t))^2 = \left(\int_{S(t)} 1 d\mathcal{H}^1\right)^2 \leq \left(\int_{S(t)} |\nabla \Sigma| d\mathcal{H}^1\right) \left(\int_{S(t)} \frac{1}{|\nabla \Sigma|} d\mathcal{H}^1\right)\]
\[\leq \lambda^1_1(\Sigma) g(t) \cdot \left(-\frac{g'(t)}{t}\right).\]
Therefore, by (2.4) and (2.5), we obtain
\[-\sqrt{2\pi} h'(t) \leq 2\sqrt{2\pi} \mathcal{H}^2(U(t)) \leq 2\lambda^1_1(\Sigma) g(t) \cdot \left(-\frac{g'(t)}{t}\right) + 2\mathcal{H}^1(\Gamma(t))^2,\]
or equivalently,
\[\frac{d}{dt} \left\{\lambda^1_1(\Sigma) g(t)^2 - \sqrt{2\pi} h(t) - \int_0^t 2\mathcal{H}^1(\Gamma(\tau))^2 d\tau\right\} \leq 0. \hfill (2.6)\]
for a.e \(t > 0\). Note that the function \(l(t) = 2\mathcal{H}^1(\Gamma(t))^2\) is integrable on the interval \(t \in (0, \|\psi_\beta\|_{L_\infty(\partial \Sigma)})\), and thus \(l(t) = \frac{d}{dt} \int_0^t l(\tau) d\tau\).

Fix \(\varepsilon > 0\) so small such that \(\varepsilon < m_\ast\). Integrating (2.6) from \(m_\varepsilon = m_\ast - \varepsilon\) to \(t\), we have
\[\lambda^1_1(\Sigma) g(t)^2 - \sqrt{2\pi} h(t) - \int_0^t 2\mathcal{H}^1(\Gamma(\tau))^2 d\tau \leq \lambda^1_1(\Sigma) g(m_\varepsilon)^2 - \sqrt{2\pi} h(m_\varepsilon) - \int_{m_\varepsilon}^t 2\mathcal{H}^1(\Gamma(\tau))^2 d\tau,\]
which implies
\[\sqrt{2\pi} h(m_\varepsilon) \leq \lambda^1_1(\Sigma) g(m_\varepsilon)^2 - \lambda^1_1(\Sigma) g(t)^2 + \sqrt{2\pi} h(t) + \int_{m_\varepsilon}^t 2\mathcal{H}^1(\Gamma(\tau))^2 d\tau.\]
We easily see that
\[
\int_{m_c}^t 2\tau \mathcal{H}^1(\Gamma(\tau))^2 d\tau \leq \mathcal{H}^1(\partial \Sigma)^2 \int_{m_c}^t 2\tau d\tau = \mathcal{H}^1(\partial \Sigma)^2 \left( M^2 - m_c^2 \right)
\]
for any \( t > m_c \). Letting \( t \to +\infty \), and noting that \( U(t) \) is empty for sufficiently large \( t \), we obtain
\[
h(m_c) \leq \frac{\lambda_1^\beta(\Sigma)}{\sqrt{2\pi}} g^2(m_c) + \frac{1}{\sqrt{2\pi}} \mathcal{H}^1(\partial \Sigma)^2 \left( M^2 - m_c^2 \right).
\]
\( g(m_c) \) and \( h(m_c) \) are given by
\[
g(m_c) = \int_{\Sigma} \psi_\beta \, d\mathcal{H}^2,
\]
\[
h(m_c) = -\int_{\Sigma} \left< \nabla_{\Sigma} \left( \frac{1}{2} \psi_\beta \right), \nabla_{\Sigma} f \right> \, d\mathcal{H}^2
\]
\[
= \int_{\Sigma} \frac{1}{2} \psi_\beta^2 \Delta f \, d\mathcal{H}^2 - \frac{1}{2} \int_{\partial \Sigma} \psi_\beta \frac{\partial f}{\partial \nu} \, d\mathcal{H}^1.
\]
Since \( \Delta_{\Sigma} f = 2 \) by (1.2), we have
\[
\int_{\Sigma} \psi_\beta^2 \, d\mathcal{H}^2 - \frac{1}{2} \int_{\partial \Sigma} \psi_\beta \frac{\partial f}{\partial \nu} \, d\mathcal{H}^1 \leq \frac{\lambda_1^\beta(\Sigma)}{\sqrt{2\pi}} \left( \int_{\Sigma} \psi_\beta \, d\mathcal{H}^2 \right)^2 + \frac{1}{\sqrt{2\pi}} \mathcal{H}^1(\partial \Sigma)^2 \left( M^2 - m_c^2 \right).
\]
Finally letting \( \varepsilon \to 0 \), we obtain the result. \( \Box \)

**Remark 2.1.** In the case that \( \Omega = B_R \subset \mathbb{R}^2 \) is a disc of radius \( R \), then the inequality in Theorem 1.3 becomes the equality
\[
\int_{B_R} \psi_\beta^2 \, dx = \frac{\lambda_1^\beta(\Omega)}{4\pi} \left( \int_{B_R} \psi_\beta \, dx \right)^2 + \frac{R}{2} \int_{\partial \Omega} \psi_\beta^2 \, d\mathcal{H}^1. \tag{2.7}
\]

This is because, first, \( \psi_\beta \) is positive, radial and decreasing in the radial direction on \( B_R \) ([3]:Proposition 2.6). Therefore \( \psi_\beta \equiv c > 0 \) on \( \partial B_R \) and \( U(c) = B_R \), \( \partial U(t) = S(t) \) for any \( t > c \). Also \( |\nabla \psi_\beta| \) is constant on \( S(t) \). Secondly, we can use the sharp isoperimetric inequality as the equality \( 4\pi \mathcal{H}^1(U(t)) = \mathcal{H}^1(S(t))^2 \) in (2.5) in this case. Finally, the unique solution \( f_{B_R} \) of (1.2) is \( f_{B_R} = \frac{1}{2}[x]^2 - \frac{1}{4}R^2 \). By these reasons, we see all inequalities in the proof of Theorem 1.1 are equalities and we obtain
\[
\frac{d}{dt} \left\{ \lambda_1^\beta(B_R) g(t)^2 - 4\pi h(t) \right\} = 0
\]
for a.e. \( t > c \), instead of (2.6). Integrating this from \( t = c \) to \( t \), and letting \( t \to \infty \), we obtain (2.7).

**References**

A Payne-Rayner type inequality for the Robin problem 7


Futoshi Takahashi
Department of Mathematics, Osaka City University,
3-3-138 Sugimoto, Sumiyoshi-ku, Osaka
558-8585, Japan
e-mail: futoshi@sci.osaka-cu.ac.jp

Akinobu Uegaki
Department of Mathematics, Osaka City University,
3-3-138 Sugimoto, Sumiyoshi-ku, Osaka
558-8585, Japan
e-mail: u.akinobu@gmail.com