L^{∞} decay estimate and asymptotic behavior of solutions to 1D Schrodinger equations with long range dissipative nonlinearity

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Abstract

This manuscript presents the decay estimate of the solutions to the initial value problem of 1D Schrödinger equations including a subcritical dissipative nonlinearity $\lambda|u|^{p-1}u$ with $2.367 \approx p_0 \le p < 3$, $\mathrm{Im}\lambda < 0$ and $(p-1)|\mathrm{Re}\lambda| \le 2\sqrt{p}|\mathrm{Im}\lambda|$. Our aim is to determine the decay-rate of the solutions in L^∞, without size restriction on the initial data, for smaller nonlinear power p than those treated in the former known results. If the nonlinear power rises up to $2.686 \approx (5+\sqrt{33})/4 < p < 3$, the asymptotic leading term of the solution is also obtained.

1 Introduction

We consider the Cauchy problem of nonlinear Schrödinger equations :

$$\begin{cases} i\partial_t u = -\frac{1}{2}\partial_x^2 u + \lambda\,\mathcal{N}(u) \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $(t,x) \in \mathbf{R}\times\mathbf{R}$ and $u(t,x)$ is a complex-valued unknown function. For the gauge-invariant nonlinearity, we assume that

$$\begin{aligned} \lambda &= \lambda_1 + i\lambda_2 \quad \text{with} \quad \lambda_1, \lambda_2 \in \mathbf{R}, \\ \mathcal{N}(u) &= |u|^{p-1}u. \end{aligned}$$

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If $p = 3$, the equation (1.1) arises in the nonlinear fiber engineering. According to Chapter 11 in [1] or physical manuscript [2], there is a complex Ginzburg-Landau equation:

$$i\frac{\partial U}{\partial \xi} - \frac{1}{2}(s+id)\frac{\partial^2 U}{\partial \tau^2} + N^2|U|^2U = \frac{i}{2}(\mu - \mu_2|U|^2)U, \tag{1.2}$$

where U stands for the dimension-less amplitude of electric field propagating through an optical fiber, ξ the position along the fiber, τ the time-variable expressing the oscillation of the electric field, s, d, N, μ and μ_2 denotes the scale of dispersion, diffusion, nonlinear Kerr effect, amplification and nonlinear dissipation, respectively. Therefore the equation (1.1) corresponds to the special and artificial case of (1.2) in which $s < 0$, $d = 0$ and $\mu = 0$ together with the power of nonlinearity generalized from 3 to p. We want to treat (1.2) as an evolution equation and so replace the variables ξ and τ by t and x respectively following the mathematical convention.

There are large amount of articles concerning with the decay estimate and asymptotic behavior of solutions to nonlinear Schrödinger equations. In the case of small data, it is often said that the smaller the nonlinear power p is, the more difficult the problem seeking for asymptotic behavior of the solution becomes. In detail, the difficulty depends on whether the improper integral $\int_1^\infty \mathcal{N}(u(t))/u(t)dt$ is finite or not when $u(t)$ is expected to be like a free solution : $u(t) = O(t^{-1/2})$ as $t \to \infty$ in L^∞. From this point of view, the asymptotic behavior of solutions dramatically changes at $p = 3$. In fact, for $p > 3$ (supercritical case), the nonlinearity decays rapidly enough as $t \to \infty$, and so it is easy to show that there exists some function ϕ such that $u(t)$ asymptotically tends to $(it)^{-1/2}e^{ix^2/2t}\hat{\phi}(x/t)$ as $t \to \infty$ without depending on the sign of λ_2. On the other hand, for $p = 3$ (critical case) and $\lambda_2 = 0$, Hayashi-Naumkin [8] obtained $\|u(t)\|_{L^\infty} = O(t^{-1/2})$ and

$$u(t,x) \sim (it)^{-1/2}e^{ix^2/2t}e^{-i\Theta(t,x/t)}\hat{\phi}(x/t) \quad \text{as} \quad t \to \infty, \tag{1.3}$$

where the real-valued phase modification $\Theta(t,x)$ arises in the asymptotic leading term of the solution. For $p = 3$ and the dissipative nonlinear case, i.e., $\lambda_2 < 0$, Shimomura [19] proved that $\|u(t)\|_{L^\infty} = O(t^{-1/2}(\log t)^{-1/2})$ and (1.3) with $\Theta(t,x)$ complex-valued. It is noticeable that, in this critical dissipative case, the nonlinearity explicitly affects the decay rate of $u(t)$. In the similar situation, but the initial data belongs to the homogeneous weighted L^2 space so that the scale-invariance is valid, Hayashi-Li [6] also obtained the same decay property as in [19]. For $1 < p < 3$ (sub-critical case) and $\lambda_2 = 0$, Hayashi-Kaikina-Naumkin [5] obtained $\|u(t)\|_{L^\infty} = O(t^{-1/2})$ and specified the asymptotic leading term like (1.3) by imposing the high regularity and non-zero condition on the initial data. Their work also showed that the expression of real-valued $\Theta(t,x)$ changes at $p=2$ and lower. For $\lambda_2<0$ and $3-\delta<p<3$ with $\delta>0$ sufficiently small, Kita-Shimomura [14] obtained $\|u(t)\|_{L^\infty}=O(t^{-1/(p-1)})$ and (1.3) with $\Theta(t,x)$ complex-valued, where the small initial data satisfies $u_0\in H^1$ and $xu_0\in L^2$ without imposing the high regularity and non-zero condition. Surveying these known results and referring to the decay-rate of $\|u(t)\|_{L^\infty}$, we know that, in the dissipative nonlinear case, the dispersion caused by the Lapracian survives for large time if $p>3$, but the dissipation caused by $\lambda\mathcal{N}(u)$ survives if $p<3$. We note here that, in the sub-critical dissipative case, the decay rate of $\|u(t)\|_{L^\infty}$ is just similar to that of a solution to the corresponding ordinary differential equation : $i\partial_t u=\lambda\mathcal{N}(u)$.

We have so far summarized the progress for small solutions to (1.1) in particular. One may then ask how about a large initial data. As far as the authors know, it is possible to obtain the decay estimate and asymptotic behavior of $u(t)$ without size restriction on the initial data, but so-called the strong dissipative condition : $\lambda_2<0$ and $(p-1)|\lambda_1|\le 2\sqrt{p}|\lambda_2|$ is required. Under this strong dissipative condition, it is well-known that the nonlinear operator $F(u)\equiv-\frac{1}{2}\partial_x^2u+\lambda\mathcal{N}(u)$ becomes monotone, i.e., $\mathrm{Im}(F(u_1)-F(u_2),u_1-u_2)<0$ holds, where $(\,\cdot\,,\,\cdot\,)$ stands for the L^2-inner product (refer to [17] for detail). Furthermore the monotone property helps $\|Ju(t)\|_{L^2}$ bounded by $\|xu_0\|_{L^2}$ for all time, where $J=\exp(it\partial_x^2/2)x\exp(-it\partial_x^2/2)=x+it\partial_x$ is the infinitesimal generator of Galilean transform. The global estimate of $\|Ju(t)\|_{L^2}$ always lies at the core of our interest because the remainder terms arising in the proof is more or less bounded by $\|Ju(t)\|_{L^2}$. The strong dissipative condition therefore reduces the complexity of the estimates of remainder terms for large initial data. Following this idea, Kita-Shimomura [15] obtained the decay-estimate and asymptotic behavior of $u(t)$, which is similar to those of small data solutions as mentioned above. In [15], the range of p is assumed to be $2.686\approx(5+\sqrt{33})/4<p\le3$. Recently Jin-Jin-Li [12] have succeeded in extending the range of p to $2.586\approx(19+\sqrt{145})/12<p\le3$. Their approach is based on the estimate : $\|J^2u(t)\|_{L^2}\le Ct^{3-p}$ which is deduced from $u_0\in H^1$, $x^2u_0\in L^2$ and the strong dissipative condition. The asymptotic analysis for critical nonlinear Schrödinger equations has been extensively studied by several mathematicians. For example, refer to [5, 7, 10, 11] for derivative nonlinear case, and [9, 18] for final value problem.

Our aim in this manuscript is to research decay property of the solution for the sub-critical nonlinearity without size restriction on the initial data, but under the strong dissipative condition. We note here that the problem will be solved for the nonlinear power p smaller than those treated in [15, 12]. More concretely speaking, the proof will be performed for $p_0\le p<3$, where the lower bound p_0 is the smallest positive root of the quartic equation:

$$(7p^2+22p-93)^2-48(3p-7)^2(p+3)=0. \tag{1.4}$$

The numerical value of p_0 is $2.367\cdots$ due to Excel of Microsoft Office 2013. Our goals are the following.

Theorem 1.1. *Let* $2.367 \approx p_0 \le p < 3$*, where* p_0 *is the smallest positive root of* (1.4)*. Also let* $\lambda_2 < 0$ *and*

$$(p-1)|\lambda_1| \le 2\sqrt{p}|\lambda_2|. \tag{1.5}$$

If $u_0 \in H^1$ *and* $xu_0 \in L^2$*, then there exists a unique global solution* u *to* (1.1) *such that* $u \in C([0,\infty);H^1) \cap C^1([0,\infty);H^{-1})$ *and* $xu \in C([0,\infty);L^2)$*. Furthermore, for some positive constants* K_0 *and* K_1*, the solution satisfies*

$$\|u(t)\|_{L^\infty} \le K_0(1+t)^{-1/(p-1)}, \tag{1.6}$$

$$\|u(t)\|_{L^2} \le K_1(1+t)^{-(2/3)(1/(p-1)-1/2)}. \tag{1.7}$$

Enhancing the constraint of p, we are able to specify the asymptotic leading term of $u(t)$ in L^∞.

Theorem 1.2. *Let* $2.686 \approx (5+\sqrt{33})/4 < p < 3$ *and* $\lambda_2 < 0$ *together with* (1.5)*. If* $u_0 \in H^1$ *and* $xu_0 \in L^2$*, then, for the solution in Theorem 1.1, there exists some function* $\hat{\phi} \in L^\infty \cap L^2$ *such that*

$$\begin{aligned} u(t,x) &= (it)^{-1/2}e^{ix^2/2t}e^{-i\Theta(t,x/t)}\hat{\phi}(x/t) \\ &\quad + o(t^{-1/(p-1)}) \quad \text{as } t\to\infty \text{ in } L^\infty, \end{aligned} \tag{1.8}$$

where the complex-valued function $\Theta(t,x)$ *is described as*

$$\Theta(t,x) = \frac{\lambda}{(p-1)|\lambda_2|}\log\left\{1+\frac{2(p-1)|\lambda_2|}{3-p}(t^{(3-p)/2}-1)|\hat{\phi}(x)|^{p-1}\right\}.$$

In Theorem 1.2 (1.8), the function $\hat{\phi}$ is not always equal to 0 if $u_0 \ne 0$ and $\|u_0\|_{H^1} + \|xu_0\|_{L^2}$ is sufficiently small. The detail of this statement will be given as a remark on the final stage of this manuscript. Therefore the decay rate of $\|u(t)\|_{L^\infty}$ in (1.6) is sharp. In Theorem 1.1 (1.7), one might think whether the decay rate of $\|u(t)\|_{L^2}$ is sharp or not. The authors believe that it depends on the feature of $\hat{\phi}(x)$ as $|x| \to \infty$. Roughly speaking, if we assume that the top term on the right hand side of (1.8) is dominant as $t \to \infty$ in L^2 as well, we see that

$$\begin{aligned}\|u(t)\|_{L^2} &\sim \|e^{-i\Theta(t)}\hat{\phi}\|_{L^2} \\ &= \left\| \left\{ 1 + \frac{2(p-1)|\lambda_2|}{3-p}(t^{(3-p)/2}-1)|\hat{\phi}|^{p-1} \right\}^{-1/(p-1)} \hat{\phi} \right\|_{L^2}\end{aligned}$$

as $t \to \infty$. If $\hat{\phi}$ is compactly supported on $\mathbf{R}$ (this is not so general case), then $\|u(t)\|_{L^2} = O(t^{-1/(p-1)+1/2})$ as $t \to \infty$. On the other hand, if $\hat{\phi}(x) = |x|^{-1/2-\sigma}$ for $|x| >> 1$ with $\sigma > 0$, then we can deduce $\|u(t)\|_{L^2} = O(t^{\{-1/(p-1)+1/2\}\times\{2\sigma/(1+2\sigma)\}})$ as $t \to \infty$. This example suggests that the smaller $\sigma > 0$ one takes, the worse decay rate the $\|u(t)\|_{L^2}$ comes to hold.

Throughout this manuscript, we employ the following notation. For $q \in [1,\infty)$, L^q denotes the set of the measurable function f satisfying $\|f\|_{L^q} \equiv \left(\int_{\mathbf{R}} |f(x)|^q dx\right)^{1/q} < \infty$. L^∞ denotes the set of the measurable function f satisfying $\|f\|_{L^\infty} \equiv \operatorname{ess.sup}_{x\in\mathbf{R}} |f(x)| < \infty$. The Sobolev space H^1 stands for the set of the integrable function f such that $f \in L^2$ and $\partial_x f \in L^2$. The norm of H^1 is given by $\|f\|_{H^1} = \|f\|_{L^2} + \|\partial_x f\|_{L^2}$. The dual space of H^1 is denoted by H^{-1}. Let $U(t) = \exp(it\partial_x^2/2)$ be the Schrödinger group. It is also expressed by the integral kernel, i.e.,

$$U(t)f(x) = \frac{1}{\sqrt{2\pi i t}} \int e^{i|x-y|^2/2t} f(y)dy.$$

By expanding $|x-y|^2$, we see that $U(t)$ is factorized like

$$U(t)f = MD\mathcal{F}Mf,$$

where M is the multiplication of $e^{ix^2/2t}$, $Df(x) = (it)^{-1/2}f(x/t)$, $\mathcal{F}$ the Fourier transform given by $\mathcal{F}f(\xi) = (2\pi)^{-1/2}\int e^{-i\xi x}f(x)dx$. We will often use the operator $J = U(t)xU(-t) = x + it\partial_x$, which is the infinitesimal generator of Galilean transform. The operator J is convenient to obtain the weighted estimate of the solution since it commutes with $i\partial_t + \frac{1}{2}\partial_x^2$.

In section 2, the transformed function $v(t) = U(-t)u(t)$ will be induced, which plays an important role to estimate $\|u(t)\|_{L^\infty}$, since

$$u(t) = U(t)v(t) = MD\mathcal{F}Mv(t) = MD\mathcal{F}v(t) + (\text{a remainder term}).$$

In section 3, we will present the proof of Theorem 1.1 and 1.2 after providing Proposition 3.2. Our idea of the proof is based on the iteration technique by which the decay estimate of $\|\mathcal{F}v(t)\|_{L^{q_{n+1}}}$ is obtained successively from that of $\|\mathcal{F}v(t)\|_{L^{q_n}}$. The detail of such iteration argument will be stated in

Proposition 3.2. It also helps the relaxation of the range of the nonlinear power, since the former known results employed a rough estimate where $\|\mathcal{F}v(t)\|_{L^\infty}$ arising in the nonlinear estimate was simply bounded by $\|u(t)\|_{L^2}$ and $\|Ju(t)\|_{L^2}$. The delicate treatment of $\|\mathcal{F}v(t)\|_{L^\infty}$ through the Gagliardo-Nirenberg inequality : $L^{q_n} \cap H^1 \subset L^\infty$ brings us the refinement. In the final part of section 3, we will discuss the property that $\hat{\phi} \neq 0$ for small u_0.

2 Basic Estimate and Deformation

In this section, some deformations of (1.1) are heuristically described. These deformations will play an important role to prove Theorem 1.1 in the subsequent sections. The argument for the deformations looks somewhat formal because the derivatives of several quantities with respect to t are going to be performed without rigorous justification like the cut-off and regularization technique.

By the contraction mapping principle applied to the integral equation associated with (1.1), the local existence of the solution such that $u \in C([0,T_0];H^1) \cap C^1([0,T_0];H^{-1})$ and $xu \in C([0,T_0];L^2)$ follows for some $T_0 > 0$ (refer to [3, 4, 13]). It is easy to see the decreasing property of $\|u(t)\|_{L^2}$ since

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 = \lambda_2 \|u(t)\|_{L^{p+1}}^{p+1} < 0.$$

To observe the decay property of u, the global estimate of $\|Ju(t)\|_{L^2}$ is required where $J = \exp(it\partial_x^2/2)x\exp(-it\partial_x^2/2) = x + it\partial_x$, since it will often arise in the estimates of reminder terms. Note that $[J, i\partial_t + \frac{1}{2}\partial_x^2] = 0$, where $[A,B] = AB - BA$. Then we have

$$\begin{aligned}(i\partial_t + \frac{1}{2}\partial_x^2)Ju &= \lambda J\mathcal{N}(u) \\ &= \lambda\frac{p+1}{2}|u|^{p-1}Ju - \lambda\frac{p-1}{2}|u|^{p-3}u^2\overline{Ju}\end{aligned}$$

and hence it follows that

$$\begin{aligned}\frac{d}{dt}\|Ju\|_{L^2}^2 &= 2\mathrm{Im}\left(\lambda\int \overline{Ju}J\mathcal{N}(u)dx\right) \\ &= (p+1)\lambda_2\int |u|^{p-1}|Ju|^2dx \\ &\qquad -(p-1)\mathrm{Im}\left(\lambda\int |u|^{p-3}u^2(\overline{Ju})^2dx\right) \\ &\leq C_{p,\lambda}\|u\|_{L^\infty}^{p-1}\|Ju\|_{L^2}^2, \qquad (2.1)\end{aligned}$$

where $C_{p,\lambda} = (p+1)\lambda_2 + (p-1)|\lambda| \le 0$ due to (1.5) in Theorem 1.1. This implies that $\|Ju(t)\|_{L^2} \le \|xu_0\|_{L^2}$ for any $t \ge 0$. The analogy holds for $\|\partial_x u(t)\|_{L^2}$, and the local solution can be continued to the global one. We have the following basic estimates of the solution.

Lemma 2.1. *Let* $\lambda_2 < 0$ *and* (1.5) *hold. Then it follows that there exists a unique global solution* u *to* (1.1) *such that* $u \in C(0,\infty; H^1) \cap C^1(0,\infty; H^{-1})$ *and* $xu \in C(0,\infty; L^2)$*. Furthermore, the solution satisfies*

$$\begin{aligned} \|u(t)\|_{L^2} &\le \|u_0\|_{L^2}, && (2.2)\\ \|\partial_x u(t)\|_{L^2} &\le \|\partial_x u_0\|_{L^2}, && (2.3)\\ \|Ju(t)\|_{L^2} &\le \|xu_0\|_{L^2}. && (2.4) \end{aligned}$$

To obtain both Theorem 1.1 and 1.2, we write $u(t) = U(t)v(t)$, where $v(t) = U(-t)u(t)$. Note that $U(t) = MD\mathcal{F}M$. Making use of $v(t)$, we have

$$\begin{aligned} \|u(t)\|_{L^\infty} &= \|U(t)v(t)\|_{L^\infty} \\ &= \|MD\mathcal{F}Mv(t)\|_{L^\infty} \\ &\le t^{-1/2}\|\mathcal{F}v(t)\|_{L^\infty} + t^{-1/2}\|\mathcal{F}(M-1)v(t)\|_{L^\infty} \qquad (2.5) \end{aligned}$$

The remainder term of (2.5) is estimated by the Sobolev inequality and Plancherel identity, i.e.,

$$\begin{aligned} \|\mathcal{F}(M-1)v(t)\|_{L^\infty} &\le C\|\mathcal{F}(M-1)v(t)\|_{L^2}^{1/2}\|\partial_\xi \mathcal{F}(M-1)v(t)\|_{L^2}^{1/2} \\ &\le Ct^{-1/4}\|Ju(t)\|_{L^2}. \qquad (2.6) \end{aligned}$$

Note here that, to obtain the last inequality of (2.6), we used $|M-1| \le |x|/\sqrt{t}$ and $\|xv\|_{L^2} = \|U(t)xU(-t)u\|_{L^2} = \|Ju\|_{L^2}$. Plugging (2.6) into (2.5) and applying (2.4), we have

$$\begin{aligned} \|u(t)\|_{L^\infty} &\le t^{-1/2}\|\mathcal{F}v(t)\|_{L^\infty} + Ct^{-3/4}\|Ju(t)\|_{L^2} \\ &\le t^{-1/2}\|\mathcal{F}v(t)\|_{L^\infty} + Ct^{-3/4}\|xu_0\|_{L^2}. \qquad (2.7) \end{aligned}$$

Our interest now turns into the estimate of $\|\mathcal{F}v(t)\|_{L^\infty}$. Via the same derivation as Duhamel's principle from the nonlinear Schrödinger equation (1.1), it follows that

$$\partial_t(\mathcal{F}v) = -i\lambda \mathcal{F}U(-t)\mathcal{N}(U(t)v).$$

Note that $U(-t) = M^{-1}\mathcal{F}^{-1}D^{-1}M^{-1}$ and $\mathcal{N}(e^{i\theta}u) = e^{i\theta}\mathcal{N}(u)$ for $\theta \in \mathbf{R}$ (the gauge invariance of $\mathcal{N}(u)$). Then we have

$$\begin{aligned} \partial_t(\mathcal{F}v) &= -i\lambda t^{-(p-1)/2}\mathcal{F}M^{-1}\mathcal{F}^{-1}\mathcal{N}(\mathcal{F}Mv) \\ &= -i\lambda t^{-(p-1)/2}\mathcal{N}(\mathcal{F}v) + R(t), \qquad (2.8) \end{aligned}$$

where

$$\begin{aligned} R(t) &\equiv -i\lambda t^{-(p-1)/2}\left(\mathcal{F}M^{-1}\mathcal{F}\mathcal{N}(\mathcal{F}Mv)-\mathcal{N}(\mathcal{F}v)\right) \\ &= -i\lambda t^{-(p-1)/2}(\mathcal{F}(M^{-1}-1)\mathcal{F}\mathcal{N}(\mathcal{F}Mv)) \\ &\quad -i\lambda t^{-(p-1)/2}(\mathcal{N}(\mathcal{F}Mv)-\mathcal{N}(\mathcal{F}v)). \end{aligned} \tag{2.9}$$

To control the first term on the right hand side of (2.8) which looks slowly decaying, we transform $\mathcal{F}v$ like $w(t,\xi)=e^{i\lambda\Phi(t,\xi)}\mathcal{F}v(t,\xi)$, where

$$\Phi(t,\xi)=\int_1^t \tau^{-(p-1)/2}|\mathcal{F}v(\tau,\xi)|^{p-1}d\tau.$$

By (2.8), we know that the new function w satisfies

$$\partial_t w = e^{i\lambda\Phi}R(t). \tag{2.10}$$

In addition, the phase function Φ satisfies

$$\begin{aligned} \partial_t\Phi &= t^{-(p-1)/2}|\mathcal{F}v|^{p-1} \\ &= t^{-(p-1)/2}|w|^{p-1}e^{(p-1)\lambda_2\Phi}, \end{aligned}$$

which is equivalent to

$$e^{(p-1)|\lambda_2|\Phi} = 1+(p-1)|\lambda_2|\int_1^t \tau^{-(p-1)/2}|w(\tau)|^{p-1}d\tau. \tag{2.11}$$

For the estimate of$\|u(t)\|_{L^\infty}$ and in particular $\|\mathcal{F}v(t)\|_{L^\infty}$, the inequalities and identities (2.7), (2.8), (2.10) and (2.11) will be often taken into account.

3 Proof of Theorems

We are going to prove Theorem 1.1 and 1.2 in this section. The refinement of the estimate of the remainder term (2.9) significantly contributes to the extension of the range of p.

Lemma 3.1. *Let $2\le q\le r\le\infty$ and $v(t)=U(-t)u(t)$. Then, for some positive constant C, it follows that*

$$\begin{aligned} \|R(t)\|_{L^r} &\le Ct^{-(p-1)/2-1/2r-1/4} \\ &\quad \times(\|\mathcal{F}v\|_{L^q}^{(p-1)q/(q+2)}\|Ju\|_{L^2}^{(2p+q)/(q+2)}+t^{-(p-1)/4}\|Ju\|_{L^2}^p) \end{aligned} \tag{3.1}$$

Proof. Let $R(t) = R_1(t) + R_2(t)$, where

$$R_1(t) = -i\lambda t^{-(p-1)/2}(\mathcal{F}(M^{-1}-1)\mathcal{F}\mathcal{N}(\mathcal{F}Mv)), \tag{3.2}$$
$$R_2(t) = -i\lambda t^{-(p-1)/2}(\mathcal{N}(\mathcal{F}Mv) - \mathcal{N}(\mathcal{F}v)). \tag{3.3}$$

By the Gagliardo-Nirenberg inequality : $\|f\|_{L^r} \le C\|f\|_{L^2}^{(r+2)/2r}\|\partial_\xi f\|_{L^2}^{(r-2)/2r}$,

$$\begin{aligned}\|R_1(t)\|_{L^r} &\le Ct^{-(p-1)/2}\|\mathcal{F}(M^{-1}-1)\mathcal{F}\mathcal{N}(\mathcal{F}Mv)\|_{L^2}^{(r+2)/2r} \\ &\qquad \times\|\partial_\xi\mathcal{F}(M^{-1}-1)\mathcal{F}\mathcal{N}(\mathcal{F}Mv)\|_{L^2}^{(r-2)/2r}.\end{aligned}$$

Also making use of $|M^{-1}-1| \le |x|/\sqrt{t}$ and the Plancherel identity, we have

$$\begin{aligned}\|R_1(t)\|_{L^r} &\le Ct^{-(p-1)/2-1/2r-1/4}\|\partial_\xi\mathcal{N}(\mathcal{F}Mv)\|_{L^2} \\ &\le Ct^{-(p-1)/2-1/2r-1/4}\|\mathcal{F}Mv\|_{L^\infty}^{p-1}\|\partial_\xi\mathcal{F}Mv\|_{L^2} \\ &\le Ct^{-(p-1)/2-1/2r-1/4}\|\mathcal{F}Mv\|_{L^\infty}^{p-1}\|Ju\|_{L^2}.\end{aligned} \tag{3.4}$$

To estimate $\|\mathcal{F}Mv\|_{L^\infty}$ in (3.4), we apply the simple trigonometric inequality $\|\mathcal{F}Mv\|_{L^\infty} \le \|\mathcal{F}v\|_{L^\infty} + \|\mathcal{F}(M-1)v\|_{L^\infty}$ and the Gagliardo-Nirenberg inequality $L^q \cap H^1 \subset L^\infty$, i.e.,

$$\begin{aligned}\|\mathcal{F}Mv\|_{L^\infty} &\le \|\mathcal{F}v\|_{L^\infty} + \|\mathcal{F}(M-1)v\|_{L^\infty} \\ &\le C\|\mathcal{F}v\|_{L^q}^{q/(q+2)}\|\partial_\xi\mathcal{F}v\|_{L^2}^{2/(q+2)} + Ct^{-1/4}\|Ju\|_{L^2}. \\ &\le C\|\mathcal{F}v\|_{L^q}^{q/(q+2)}\|Ju\|_{L^2}^{2/(q+2)} + Ct^{-1/4}\|Ju\|_{L^2}\end{aligned} \tag{3.5}$$

Plugging (3.5) into (3.4), we see that

$$\begin{aligned}&\|R_1(t)\|_{L^r} \\ &\quad\le Ct^{-(p-1)/2-1/2r-1/4} \\ &\qquad\times(\|\mathcal{F}v\|_{L^q}^{q/(q+2)}\|Ju\|_{L^2}^{2/(q+2)} + Ct^{-1/4}\|Ju\|_{L^2})^{p-1}\|Ju\|_{L^2} \\ &\quad\le Ct^{-(p-1)/2-1/2r-1/4} \\ &\qquad\times(\|\mathcal{F}v\|_{L^q}^{(p-1)q/(q+2)}\|Ju\|_{L^2}^{(2p+q)/(q+2)} + t^{-(p-1)/4}\|Ju\|_{L^2}^p)\end{aligned} \tag{3.6}$$

As for $R_2(t)$, we see that

$$\|R_2(t)\|_{L^r} \le Ct^{-(p-1)/2}(\|\mathcal{F}Mv\|_{L^\infty} + \|\mathcal{F}v\|_{L^\infty})^{p-1}\|\mathcal{F}(M-1)v\|_{L^r}$$

Applying the Sobolev inequality to $\|\mathcal{F}(M-1)v\|_{L^r}$ and using $|M-1| \le |x|/\sqrt{t}$, we have

$$\|R_2(t)\|_{L^r} \le Ct^{-(p-1)/2-1/2r-1/4}(\|\mathcal{F}Mv\|_{L^\infty} + \|\mathcal{F}v\|_{L^\infty})^{p-1}\|Ju\|_{L^2}.$$

According to the analogy of the estimate (3.5),

$$\begin{aligned}\|R_2(t)\|_{L^r} \\ \leq\ & Ct^{-(p-1)/2-1/2r-1/4} \\ & \times(\|\mathcal{F}v\|_{L^q}^{(p-1)q/(q+2)}\|Ju\|_{L^2}^{(2p+q)/(q+2)} + t^{-(p-1)/4}\|Ju\|_{L^2}^p). \qquad (3.7)\end{aligned}$$

Combining (3.6) and (3.7), we have

$$\begin{aligned}\|R(t)\|_{L^r} \\ \leq\ & Ct^{-(p-1)/2-1/2r-1/4} \\ & \times(\|\mathcal{F}v\|_{L^q}^{(p-1)q/(q+2)}\|Ju\|_{L^2}^{(2p+q)/(q+2)} + t^{-(p-1)/4}\|Ju\|_{L^2}^p). \qquad (3.8)\end{aligned}$$

This completes the proof of (3.1). □

In the proof of Lemma 3.1, $\|\mathcal{F}Mv\|_{L^\infty}$ and $\|\mathcal{F}v\|_{L^\infty}$ appeared. In the idea of [15], these two quantities were roughly estimated like $\|\mathcal{F}Mv\|_{L^\infty} \leq C\|\mathcal{F}Mv\|_{H^1} \leq C(\|u_0\|_{L^2} + \|xu_0\|_{L^2})$ etc. due to the Sobolev inequality and Lemma 2.1. However we are going to take much care of $\|\mathcal{F}Mv\|_{L^\infty}$ and $\|\mathcal{F}v\|_{L^\infty}$ so that $R(t)$ gains more decay rate. This is the key to succeed in the extension of the range of p. By the iteration technique together with Lemma 3.1, we obtain (rough) decay estimate of $\|\mathcal{F}v(t)\|_{L^q}$ for any large $q < \infty$.

Proposition 3.2. *Let u be the solution to* (1.1) *and $v = U(-t)u$. For $n = 1, 2, \cdots$, let $\{q_n\}$ and $\{d_n\}$ be the sequences determined by the recurrent relation :*

$$\begin{aligned} q_1 &= 2, & q_{n+1} &= q_n + (p-1), \\ d_1 &= \frac{3-p}{3(p-1)}, & d_{n+1} &= \frac{2q_nd_n + (3-p)}{2q_{n+1}} \end{aligned} \qquad (3.9)$$

Furthermore let p_0 be the smallest positive root of (1.4)*. Then, for $p \in [p_0, 3)$, there exist some positive constants C_n's such that*

$$\|\mathcal{F}v(t)\|_{L^{q_n}} \leq C_n(1+t)^{-d_n}. \qquad (3.10)$$

Remark here that q_n and d_n satisfying the recurrent relation (3.9) are able to be explicitly described. In fact, we have

$$q_n = 2 + (p-1)(n-1), \quad d_n = \frac{(3-p)(4+3(p-1)(n-1))}{6(p-1)(2+(p-1)(n-1))}. \qquad (3.11)$$

Then it is easy to see that $\{d_n\}$ is monotone increasing and $\lim_{n\to\infty} d_n = 1/(p-1) - 1/2$, but C_n in (3.10) possibly diverges as $n \to \infty$.

Proof. We are going to apply the mathematical induction.
(The case $n = 1$) Taking account of (2.8), we see that

$$\begin{aligned} \frac{d}{dt}\|\mathcal{F}v\|_{L^2}^2 &\leq 2\mathrm{Re}\int \overline{\mathcal{F}v}\partial_t\mathcal{F}v d\xi \\ &= -2|\lambda_2|t^{-(p-1)/2}\|\mathcal{F}v\|_{L^{p+1}}^{p+1} + 2\mathrm{Re}\int \overline{\mathcal{F}v}R(t)d\xi. \end{aligned} \quad (3.12)$$

By Hölder's inequality and Placherel's identity, we have

$$\begin{aligned} \|\mathcal{F}v\|_{L^2}^{2p} &\leq \|\mathcal{F}v\|_{L^1}^{p-1}\|\mathcal{F}v\|_{L^{p+1}}^{p+1} \\ &\leq C\|\mathcal{F}v\|_{L^2}^{(p-1)/2}\|\xi\mathcal{F}v\|_{L^2}^{(p-1)/2}\|\mathcal{F}v\|_{L^{p+1}}^{p+1} \\ &\leq C\|\mathcal{F}v\|_{L^2}^{(p-1)/2}\|\partial_x u\|_{L^2}^{(p-1)/2}\|\mathcal{F}v\|_{L^{p+1}}^{p+1}, \end{aligned}$$

which turns out to be

$$\|\mathcal{F}v\|_{L^\infty}^{(3p+1)/2} \leq C\|\partial_x u\|_{L^2}^{(p-1)/2}\|\mathcal{F}v\|_{L^{p+1}}^{p+1}. \quad (3.13)$$

Note that $\|\partial_x u\|_{L^2} \leq \|\partial_x u_0\|_{L^2}$ due to Lemma 2.1. Then, plugging (3.13) into (3.12) and applying Lemma 3.1 with $r = q = 2$ and $\|Ju\|_{L^2} \leq \|xu_0\|_{L^2}$, we have

$$\frac{d}{dt}\|\mathcal{F}v\|_{L^2}^2 \leq -C|\lambda_2|t^{-(p-1)/2}\|\mathcal{F}v\|^{(3p+1)/2} + Ct^{-(p-1)/2-1/2}\|\mathcal{F}v\|_{L^2},$$

which is equivalent to

$$\frac{d}{dt}\|\mathcal{F}v\|_{L^2} \leq -C|\lambda_2|t^{-(p-1)/2}\|\mathcal{F}v\|_{L^2}^{(3p-1)/2} + Ct^{-(p-1)/2-1/2}. \quad (3.14)$$

We are going to solve this ordinary differential inequality by following the idea of Li-Sunagawa [16]. For readers' convenience, we introduce how to solve it here. Let $\gamma > 0$ be sufficiently large. Then, by the simple computation and (3.14),

$$\begin{aligned} &\frac{d}{dt}(t^\gamma\|\mathcal{F}v\|_{L^2}) \\ &= \gamma t^{\gamma-1}\|\mathcal{F}v\|_{L^2} + t^\gamma\frac{d}{dt}\|\mathcal{F}v\|_{L^2} \\ &\leq \gamma t^\gamma\|\mathcal{F}v\|_{L^2} - C|\lambda_2|t^{\gamma-(p-1)/2}\|\mathcal{F}v\|_{L^2}^{(3p-1)/2} + Ct^{\gamma-p/2}. \end{aligned} \quad (3.15)$$

Applying Young's inequality to the first term on the RHS of (3.15), we see that

$$\gamma t^{\gamma}\|\mathcal{F}v\|_{L^2} \leq C_\varepsilon t^{\gamma-2p/(3p-3)} + \varepsilon t^{\gamma-(p-1)/2}\|\mathcal{F}v\|_{L^2}^{(3p-1)/2}, \qquad (3.16)$$

where the positive constant C_ε diverges as $\varepsilon \downarrow 0$. Plugging (3.16) into (3.15) and taking $\varepsilon > 0$ so small that $\varepsilon \leq C|\lambda_2|$ holds, we see that

$$\frac{d}{dt}(t^{\gamma}\|\mathcal{F}v\|_{L^2}) \leq \gamma t^{\gamma-2p/(3p-3)} + Ct^{\gamma-p/2}.$$

Since $7/3 \leq p$ as in the assumption of Proposition 3.2, $t^{\gamma-2p/(3p-3)} \geq t^{\gamma-p/2}$ for $t \geq 1$. Then it follows that

$$\frac{d}{dt}(t^{\gamma}\|\mathcal{F}v\|_{L^2}) \leq Ct^{\gamma-2p/(3p-3)}. \qquad (3.17)$$

Integrating (3.17) from 1 to t and dividing with t^{γ}, we have

$$\|\mathcal{F}v(t)\|_{L^2} \leq t^{-\gamma}\|\mathcal{F}v(1)\|_{L^2} + C(t^{-(3-p)/(3p-3)} - t^{-\gamma}). \qquad (3.18)$$

Taking $\gamma > -(3-p)/(3p-3)$, we obtain the case $n = 1$ of Proposition 3.2.

(The case $n = k+1$) Assume that

$$\|\mathcal{F}v(t)\|_{L^{q_k}} \leq C_k t^{-d_k} \qquad (3.19)$$

holds. By (2.8),

$$\begin{aligned}
&\frac{1}{q_{k+1}}\frac{d}{dt}\|\mathcal{F}v\|_{L^{q_{k+1}}}^{q_{k+1}} \\
&= \mathrm{Re}\int |\mathcal{F}v|^{q_{k+1}-2}\overline{\mathcal{F}v}\partial_t\mathcal{F}v d\xi \\
&\leq -|\lambda_2|t^{-(p-1)/2}\|\mathcal{F}v\|_{L^{q_{k+1}+p-1}}^{q_{k+1}+p-1} + \|\mathcal{F}v\|_{L^{q_{k+1}}}^{q_{k+1}-1}\|R(t)\|_{L^{q_{k+1}}}. \qquad (3.20)
\end{aligned}$$

Apply Hölder's inequality : $\|\mathcal{F}v\|_{L^{q_{k+1}}}^{2q_{k+1}} \leq \|\mathcal{F}v\|_{L^{q_k}}^{a_k}\|\mathcal{F}v\|_{L^{q_{k+1}+p-1}}^{q_{k+1}+p-1}$ to the first term of (3.20) and Lemma 3.1 with $r = q_{k+1}$, $q = q_k$ to the second. Then we have

$$\begin{aligned}
\frac{d}{dt}\|\mathcal{F}v\|_{L^{q_{k+1}}} \leq\ & -|\lambda_2|t^{-(p-1)/2}\frac{\|\mathcal{F}v\|_{L^{q_{k+1}}}^{q_{k+1}+1}}{\|\mathcal{F}v\|_{L^{q_k}}^{q_k}} \\
& +Ct^{-(p-1)/2-1/2q_{k+1}-1/4}\|\mathcal{F}v\|_{L^{q_k}}^{(p-1)q_k/(q_k+2)} \\
& +Ct^{-(p-1)/2-1/2q_{k+1}-p/4}.
\end{aligned}$$

Plugging the assumption (3.19) into the above inequality, we see that

$$\begin{aligned} \frac{d}{dt}\|\mathcal{F}v\|_{L^{q_{k+1}}} &\leq -\frac{|\lambda_2|}{C_k}t^{-\sigma_1}\|\mathcal{F}v\|_{L^{q_{k+1}}}^{q_{k+1}+1} \\ &\quad +CC_k^{(p-1)q_k/(q_k+2)}t^{-\sigma_2}+Ct^{-\sigma_3}, \end{aligned} \tag{3.21}$$

where

$$\begin{aligned} \sigma_1 &= (p-1)/2 - q_k d_k, \\ \sigma_2 &= (p-1)/2 + 1/2q_{k+1} + 1/4 + (p-1)q_k d_k/(q_k+2), \\ \sigma_3 &= (p-1)/2 + 1/2q_{k+1} + p/4. \end{aligned} \tag{3.22}$$

Analogously in the argument (3.15)–(3.18), we have

$$\|\mathcal{F}v(t)\|_{L^{q_{k+1}}} \leq Ct^{-(1-\sigma_1)/q_{k+1}} + Ct^{-(\sigma_2-1)} + Ct^{-(\sigma_3-1)}. \tag{3.23}$$

Firstly it is immediately shown that $\sigma_2 < \sigma_3$ if $7/3 < p < 3$, due to $d_k < 1/(p-1) - 1/2$ and $2 \leq q_k$. Secondly it can be shown that $(1-\sigma_1)/q_{k+1} \leq \sigma_2 - 1$ for $2.367 \approx p_0 \leq p < 3$. This is because $(1-\sigma_1)/q_{k+1} \leq \sigma_2 - 1$ is equivalent to

$$\frac{(3-p)/2 + q_k d_k}{q_{k+1}} \leq \frac{2p-5}{4} + \frac{1}{2q_{k+1}} + \frac{(p-1)q_k d_k}{q_k+2}, \tag{3.24}$$

and, substituting (3.11) into (3.24) together with the tough computation, we have the quadratic inequality of k:

$$\begin{aligned} 0 &\leq 3(p-1)^2(3p-7)k^2 + (p-1)(7p^2+22p-93)k \\ &\qquad + 4(3p-7)(p+3). \end{aligned} \tag{3.25}$$

In order to verify (3.25) for all $k = 1, 2, \cdots$, we require $3p - 7 > 0$ and alternative conditions below :

(i) $7p^2 + 22p - 93 \geq 0$,

(ii) $7p^2 + 22p - 93 < 0$ and the discriminant is non-positive, i.e.,

$$(7p^2+22p-93)^2 - 48(3p-7)^2(p+3) \leq 0. \tag{3.26}$$

Let $p_0 \approx 2.367$ be the smallest positive real root of the LHS of (3.26). Also note that we are considering the proof under the original constraint $p < 3$. Then $(1-\sigma_1)/q_{k+1} \leq \sigma_2 - 1$ holds for $p_0 \leq p < 3$, which implies, due to (3.23), that

$$\|\mathcal{F}v(t)\|_{L^{q_{k+1}}} \leq C't^{-(1-\sigma_1)/q_{k+1}}. \tag{3.27}$$

By putting $C_{k+1} = C'$ and $d_{k+1} = (1-\sigma_1)/q_{k+1}$, the proof is complete. □

Now we are ready for the proof of our main theorems.

Proof of Theorem 1.1. The global existence and uniqueness of the solution directly follows from Lemma 2.1. The L^2-decay estimate (1.7) is obtained by Proposition 3.2 with $n = 1$. It remains to prove the L^∞-decay estimate of u. Differentiating $t^\gamma|\mathcal{F}v|^2$ with respect to t and applying (2.8), we have

$$\begin{aligned} &\partial_t(t^\gamma|\mathcal{F}v|) \\ &= \gamma t^{\gamma-1}|\mathcal{F}v| + t^\gamma \mathrm{Re}\left(\frac{\overline{\mathcal{F}v}}{|\mathcal{F}v|}\partial_t\mathcal{F}v\right) \\ &\le \gamma t^{\gamma-1}|\mathcal{F}v| - |\lambda_2|t^{\gamma-(p-1)/2}|\mathcal{F}v|^p + t^\gamma\|R(t)\|_{L^\infty}. \end{aligned} \tag{3.28}$$

According to Young's inequality, the first term of (3.28) is bounded by $C_\varepsilon t^{\gamma-1/(p-1)-1/2} + \varepsilon t^{\gamma-(p-1)/2}|\mathcal{F}v|^p$, where $\varepsilon > 0$ is taken so small that $\varepsilon < |\lambda_2|$. Then we have

$$\partial_t(t^\gamma|\mathcal{F}v|) \le C_\varepsilon t^{\gamma-1/(p-1)-1/2} + t^\gamma\|R(t)\|_{L^\infty}. \tag{3.29}$$

Applying Lemma 3.1 and Proposition 3.2 with $q = q_n$ for sufficiently large n to $\|R(t)\|_{L^\infty}$ in (3.29), we see that

$$\partial_t(t^\gamma|\mathcal{F}v|) \le C_\varepsilon t^{\gamma-1/(p-1)-1/2} + Ct^{\gamma-5/4+\delta_n}, \tag{3.30}$$

where $\delta_n > 0$ is sufficiently small and we have used $q_n \to \infty$ and $d_n \to 1/(p-1) - 1/2$ as $n \to \infty$. Integrating (3.30) from 1 to t and dividing both hand sides with t^γ, we have

$$\|\mathcal{F}v\|_{L^\infty} \le Ct^{-1/(p-1)+1/2}. \tag{3.31}$$

Recall (2.7). Then $\|u(t)\|_{L^\infty} \le Ct^{-1/(p-1)}$ follows. □

Proof of Theorem 1.2. By (2.10) and (2.11),

$$w(t) = w(T) + \int_T^t e^{i\lambda\Phi}R(\tau)d\tau, \tag{3.32}$$

$$|e^{i\lambda\Phi}| \le \left\{1 + (p-1)|\lambda_2|\int_1^t \tau^{-(p-1)/2}|w(\tau)|^{p-1}d\tau\right\}^{1/(p-1)}. \tag{3.33}$$

Let $C_T = \max\{1, \sup_{0\le t\le T}\|w(t)\|_{L^\infty}\}$ for $T > 1$, and let

$$T^* = \sup\{T'; \sup_{T\le t<T'}\|w(t)\|_{L^\infty} < 2C_T\}. \tag{3.34}$$

We first show that $T^* = \infty$ by the contradiction argument. Note that, for $t \in [T, T^*)$, (3.33) yield

$$|e^{i\lambda\Phi}| \le Ct^{(3-p)/2(p-1)}C_T, \tag{3.35}$$

where the positive constant C does not depend on T – we will employ this convention for the positive constant C in what follows. Then, applying Lemma 3.1 and (3.31) to (3.32), we have

$$\begin{aligned} \|w(t)\|_{L^\infty} &\le C_T + CC_T \int_T^\infty \tau^{(3-p)/2(p-1)} \cdot \tau^{-5/4} d\tau \\ &\le (1 + CT^{(3-p)/2(p-1)-1/4})C_T. \end{aligned} \tag{3.36}$$

Note that $(3-p)/2(p-1) - 1/4 < 0$. Then, taking T sufficiently large in (3.36), we see that

$$\|w(t)\|_{L^\infty} \le \frac{3}{2}C_T. \tag{3.37}$$

If $T^* > 0$ was finite, it would follow that $2C_T \le (3/2)C_T$ by taking $t \uparrow T^*$ in (3.37). This is the contradiction. Therefore $\|w(t)\|_{L^\infty}$ is bounded from above. So is $\|w(t)\|_{L^2}$ due to the similar argument. Let $\hat{\psi} \in L^\infty \cap L^2$ be

$$\hat{\psi} = w(1) + \int_1^\infty e^{i\lambda\Phi(\tau)}R(\tau)d\tau, \tag{3.38}$$

where the improper integral is taken in $L^\infty \cap L^2$ and

$$\Phi(\tau) = \frac{1}{(p-1)|\lambda_2|} \log\left(1 + (p-1)|\lambda_2| \int_1^\tau s^{-(p-1)/2}|w(s)|^{p-1}ds\right). \tag{3.39}$$

By Lemma 2.1, we have

$$\begin{aligned} \|w(\tau) - \hat{\psi}\|_{L^\infty} &\le \int_\tau^\infty \|e^{i\lambda\Phi}R(s)\|_{L^\infty}ds \\ &\le C\tau^{-\beta} \end{aligned} \tag{3.40}$$

with $\beta = 1/4 - (3-p)/2(p-1) > 0$. From (3.40), it follows that

$$\begin{aligned} \||w(\tau)|^{p-1} - |\hat{\psi}|^{p-1}\|_{L^\infty} &\le C(\|w(\tau)\|_{L^\infty}^{p-2} + \|\hat{\psi}\|_{L^\infty}^{p-2})\|w(\tau) - \hat{\psi}\|_{L^\infty} \\ &\le C\tau^{-\beta}. \end{aligned} \tag{3.41}$$

Hence, for large $T > 0$, there exists some real-valued function $\eta_T \in L^\infty$ such that $\|\eta_T\|_{L^\infty} \le 1/2$ and furthermore

$$\begin{aligned} &(p-1)|\lambda_2| \int_T^t \tau^{-(p-1)/2}(|w(\tau)|^{p-1} - |\hat{\psi}|^{p-1})d\tau \\ &= \eta_T + O(t^{-\beta+(3-p)/2}) \end{aligned} \tag{3.42}$$

as $t \to \infty$ in L^∞. We here note that $-\beta+(3-p)/2 < 0$ if $(5+\sqrt{33})/4 < p$, and this is the main reason why the constraint of p is required in the assumption of Theorem 1.2. Applying (3.42) to (3.39), we see that

$$\Phi(t) = \frac{1}{(p-1)|\lambda_2|} \log \left(A(t,\xi) + \rho(t,\xi) \right) \tag{3.43}$$

as $t \to \infty$ in L^∞, where

$$A(t,\xi) = 1 + \eta_T + \frac{2(p-1)|\lambda_2|}{3-p} (t^{(3-p)/2} - T^{(3-p)/2}) |\hat{\psi}(\xi)|^{p-1},$$

and $\rho(t,\cdot)$ belongs to L^∞ with $\|\rho(t,\cdot)\|_{L^\infty} = O(t^{-\beta+(3-p)/2})$ as $t \to \infty$. Applying (3.40) and (3.43) to $\mathcal{F}v = e^{-i\lambda\Phi}w$, we have

$$\begin{aligned} &\mathcal{F}v(t) \\ &= \hat{\psi} \left\{ \exp\left(\frac{-i\lambda}{(p-1)|\lambda_2|} \log A(t,\cdot) \right) \right\} (1 + O(t^{-\beta+(3-p)/2})) \end{aligned} \tag{3.44}$$

as $t \to \infty$ in L^∞. Furthermore let $\hat{\phi} = \hat{\psi} \exp\left(\frac{-i\lambda}{(p-1)|\lambda_2|} \log(1+\eta_T) \right)$. Then (3.44) yields

$$\mathcal{F}v(t) = e^{-i\Theta(t)} \hat{\phi} \, (1 + O(t^{-\beta+(3-p)/2})) \tag{3.45}$$

as $t \to \infty$ in L^∞, where

$$\Theta(t) = \frac{\lambda}{(p-1)|\lambda_2|} \log \left(1 + \frac{2(p-1)|\lambda_2|}{3-p} (t^{(3-p)/2} - T^{(3-p)/2}) |\hat{\phi}|^{p-1} \right).$$

Noting that $u(t) = MD\mathcal{F}v + O(t^{-3/4+\alpha})$ in L^∞ by (2.7), we obtain (1.8). $\square$

Remark. In Theorem 1.2, the final state $\hat{\phi}$ is not equal to 0 if $u_0 \neq 0$ and $\|u_0\|_{H^1} + \|xu_0\|_{L^2}$ is sufficiently small. We want to prove this statement here. Let $u_0 = \varepsilon f_0$ with $\|f_0\|_{H^1} + \|xf_0\|_{L^2} < 1$ and $\varepsilon > 0$ sufficiently small. We denote, by $u_\varepsilon(t)$, the solution to (1.1) with εf_0 as the initial data. We also write $v_\varepsilon(t) = U(-t)u_\varepsilon(t)$ and $w_\varepsilon(t) = e^{i\lambda\Phi_\varepsilon(t)}\mathcal{F}v_\varepsilon(t)$ where $\Phi_\varepsilon(t) = \int_1^t \tau^{-(p-1)/2} |\mathcal{F}v_\varepsilon(\tau)|^{p-1} d\tau$. By (3.38), we have

$$\begin{aligned} \hat{\psi}_\varepsilon &= L^2\text{-}\lim_{t\to\infty} w_\varepsilon(t) \\ &= w_\varepsilon(1) + \int_1^\infty e^{i\lambda\Phi_\varepsilon} R(\tau) d\tau. \end{aligned} \tag{3.46}$$

We note that, by the integral equation, the first term of (3.46) is written as

$$\begin{aligned} w_\varepsilon(1) &= \mathcal{F}v_\varepsilon(1) \\ &= \varepsilon\mathcal{F}f_0 - i\lambda\int_0^1 \mathcal{F}U(-s)\mathcal{N}(u_\varepsilon(s))ds. \end{aligned} \tag{3.47}$$

Since $\sup_{0\le t\le 1}\|u_\varepsilon(t)\|_{H^1} \le C\varepsilon$ by the contraction mapping principle, (3.47) yields

$$w_\varepsilon(1) = \varepsilon\mathcal{F}f_0 + O(\varepsilon^p) \quad \text{in } L^2. \tag{3.48}$$

We next note that the second term of (3.46) is estimated as

$$\left\|\int_1^t e^{i\lambda\Phi_\varepsilon}R(\tau)d\tau\right\|_{L^2} \le \int_1^\infty \|e^{i\lambda\Phi_\varepsilon}\|_{L^\infty}\|R(\tau)\|_{L^2}d\tau.$$

Applying (3.35) to $\|e^{i\lambda\Phi_\varepsilon}\|_{L^\infty}$ and Lemma 3.2 (3.14) to $\|R(\tau)\|_{L^2}$, we see that

$$\begin{aligned} &\left\|\int_1^t e^{i\lambda\Phi_\varepsilon}R(\tau)d\tau\right\|_{L^2} \\ &\le C\varepsilon\int_1^\infty \tau^{(3-p)/2(p-1)} \times \tau^{-(p-1)/2-1/2+\alpha}(\|\mathcal{F}v_\varepsilon(\tau)\|_{L^\infty} + \varepsilon t^{-1/4+\alpha})^{p-1}d\tau. \end{aligned}$$

Write $\|\mathcal{F}v_\varepsilon\|_{L^\infty} = \|\mathcal{F}v_\varepsilon\|_{L^\infty}^{1-\theta}\cdot\|\mathcal{F}v_\varepsilon\|_{L^\infty}^{\theta} \le C\|\mathcal{F}v_\varepsilon\|_{L^\infty}^{1-\theta}\cdot\|u_\varepsilon\|_{L^2}^{\theta/2}\|Ju_\varepsilon\|_{L^2}^{\theta/2}$ by the Sobolev inequality with $\theta > 0$ sufficiently small. Then, applying $\|\mathcal{F}v_\varepsilon\|_{L^\infty} \le Kt^{-1/(p-1)+1/2}$ and Proposition 3.1, we have

$$\begin{aligned} &\left\|\int_1^t e^{i\lambda\Phi_\varepsilon}R(\tau)d\tau\right\|_{L^2} \\ &\le C\varepsilon^{1+\theta(p-1)}\int_1^\infty \tau^{(3-p)/2(p-1)} \times \tau^{-(p-1)/2-1/2+\alpha} \\ &\qquad \times\tau^{(1-\theta)(-1+(p-1)/2)+\theta\alpha(p-1)/2}d\tau \\ &\le C\varepsilon^{1+\theta(p-1)}. \end{aligned} \tag{3.49}$$

Combining (3.46), (3.48) and (3.49), we see that

$$\hat{\psi}_\varepsilon = \varepsilon\mathcal{F}f_0 + O(\varepsilon^p) + O(\varepsilon^{1+\theta(p-1)}) \quad \text{in } L^2.$$

Hence, taking $\varepsilon > 0$ sufficiently small , we have $\|\hat{\psi}_\varepsilon\|_{L^2} \ne 0$ if $f_0 \ne 0$. Recall that $\hat{\phi} = \hat{\psi}\exp\left\{\frac{-i\lambda}{(p-1)|\lambda_2|}\log(1+\eta)\right\}$ and $\|\eta\|_{L^\infty} \le C\varepsilon^{p-2}$. Then $\hat{\phi} \ne 0$ if $u_0 \ne 0$ and the size of u_0 is sufficiently small.

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References

[1] G. P. Agrawal, *Nonlinear fiber optics*, Academic Press, Inc. (1995).

[2] G. P. Agrawal, *Effect of two-photon absorption on the amplification of ultrashort optical pulses*, Physical Review E **48** (1993), 2316–2318.

[3] T. Cazenave, *Semilinear Schrödinger Equations*, Courant Lecture Notes in Mathematics **10**, American Mathematical Society (2003).

[4] J. Ginibre, *An introduction to nonlinear Schrödinger equations*, in "Nonlinear Waves," (R. Agemi, Y. Giga and T. Ozawa, Eds.), GAKUTO International Series, Mathematical Sciences and Applications **10** (1997), 85–133.

[5] N. Hayashi, E.I. Kaikina and P.I. Naumkin, *Large time behavior of solutions to the generalized derivative nonlinear Schrödinger equation*, Discrete and Continuous Dynamical Systems **5** (1999), 93–106.

[6] N. Hayashi and C. Li, *Critical nonlinear Schrödinger equations with data in homogeneous weighted L^2 spaces*, J. Math. Anal. Appl. **419** (2014), 1214–1234.

[7] N. Hayashi and P.I. Naumkin, *Asymptotic behavior in time of solutions to the derivative nonlinear Schrödinger equation revisited*, Discrete Contin. Dynam. Systems **3** (1997), 383–400.

[8] N. Hayashi and P.I. Naumkin, *Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations*, Amer. J. Math. **120** (1998), 369–645.

[9] N. Hayashi and P.I. Naumkin, *Modified wave operator for Schrödinger type equations with sub-critical dissipative nonlinearities*, Inverse Problems and Imaging **1** (2007), 391-398.

[10] N. Hayashi, P.I. Naumkin and H. Sunagawa, *On the Schrödinger equation with dissipative nonlinearities of derivative type*, SIAM J. Math. Anal. **40** (2008), 278–291.

[11] N. Hayashi, P.I. Naumkin and H. Uchida, *Large time behavior of solutions for derivative cubic nonlinear Schrödinger equations*, Publ. Res. Inst. Math. Sci. **35** (1999), 501–513.

[12] G. Jin, Y. Jin and C. Li, *The initial value problem for nonlinear Schrödinger equations with a dissipative nonlinearity in one space dimension*, preprint (2016).

[13] T. Kato, *Nonlinear Schrödinger equations*, in "Schrödinger Operators," (H. Holden and A. Jensen Eds.), Lecture Notes in Phys. **345**, Springer-Verlag (1989), 218–263.

[14] N. Kita and A. Shimomura, *Asymptotic behavior of solutions to Schrödinger equations with a sub-critical dissipative nonlinearity*, J. Differential Equations **242** (2007), 192–210.

[15] N. Kita and A. Shimomura, *Large time behavior of solutions to Schrödinger equations with a dissipative nonlinearity for arbitrarily large initial data*, J. Math. Soc. Japan **61** (2009), 39–64.

[16] C. Li and H. Sunagawa, *On Schrödinger systems with cubic dissipative nonlinearities of derivative type*, Nonlinearity **29** (2016), 1537–1563.

[17] N. Okazawa and T. Yokota, *Global existence and smoothing effect for the complex Ginzburg-Landau equation with p-Laplacian*, J. Differential Equations **182** (2002), 541–576.

[18] T. Ozawa, *Long range scattering for nonlinear Schrödinger equations in one space dimension*, Comm. Math. Phys. **139** (1991), 479–493.

[19] A. Shimomura, *Asymptotic behavior of solutions for Schrödinger equations with dissipative nonlinearities*, Comm. Partial Differential Equations **31** (2006), 1407–1423.

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