Decay estimate of solutions to dissipative nonlinear Schrodinger equations

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DECAY ESTIMATE OF SOLUTIONS TO DISSIPATIVE NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract

This paper presents the decay estimate of solutions to the initial value problem of 1D Schrödinger equations containing a sub-critical dissipative nonlinearity $\lambda|u|^{p-1}u$, where $2.468 \approx p_0 \leq p < 3$, $\mathrm{Im}\lambda < 0$ and $(p-1)|\mathrm{Re}\lambda| \leq 2\sqrt{p}|\mathrm{Im}\lambda|$. Our aim is to obtain the decay estimate of the solutions, without assuming size restriction on the initial data and under the extended lower bound of nonlinear power.

1 Introduction and main results

We consider the Cauchy problem of 1D-nonlinear Schrödinger equation:

$$\begin{cases} i\partial_t v + \frac{1}{2}\partial_x^2 v = \lambda|v|^{p-1}v, \\ v(0,x) = \phi(x), \end{cases} \tag{1.1}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}$, $1 < p < 3$ and $\lambda = \lambda_1 + i\lambda_2$ $(\lambda_1, \lambda_2 \in \mathbb{R})$ satisfying

$$\lambda_2 < 0, \quad |\lambda_2| \geq \frac{p-1}{2\sqrt{p}}|\lambda_1|. \tag{1.2}$$

It is well known that the asymptotic behavior of solutions to (1.1) with $1 < p \leq 3$ is different from that of solutions to the corresponding free equation (see [1]). There is some research on the small initial problem (1.1)(see e.g., [2] and [6]). The large initial problem (1.1) with (1.2) was first considered in [5], where the $\mathbf{L}^\infty$-decay estimates of the global solutions for $2.686\cdots \approx \frac{5+\sqrt{33}}{4} < p \leq 3$ was investigated. Jin-Jin-Li also considered the large initial problem (1.1) with (1.2) and improved the previous result in [5]. They proved the

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$\mathbf{L}^\infty$-decay estimate of the global solutions to (1.1) with(1.2), i.e., $\|v(t)\|_{\mathbf{L}^\infty} \le C(1+t)^{-1/(p-1)}$ for $t>0$, if $2.586\cdots \approx \frac{19+\sqrt{145}}{12} < p < 3$ holds; $\|v(t)\|_{\mathbf{L}^\infty} \le C(1+t)^{-1/2}\left(\log(1+t)\right)^{-1/2}$ for $t>0$, if $p=3$. Since $\frac{19+\sqrt{145}}{12} < \frac{5+\sqrt{33}}{4}$, Jin-Jin-Li extended the lower bound of p by using the operator $J^2(t) = U(t)\, x^2 U(-t)$. Our purpose is to investigate $\mathbf{L}^2$- decay estimates of solutions and achieve a better $\mathbf{L}^\infty$-decay estimates of solutions to (1.1) with (1.2) for arbitrarily large initial data.

Let $\mathbf{L}^q(\mathbb{R})$ denote the usual Lebesgue space with the norm

$$\|\phi\|_{\mathbf{L}^q(\mathbb{R})} = \left(\int_{\mathbb{R}} |\phi(x)|^q \, dx\right)^{\frac{1}{q}}$$

if $1 \le q < \infty$ and

$$\|\phi\|_{\mathbf{L}^\infty(\mathbb{R})} = \underset{x\in\mathbb{R}}{\text{ess.sup}} \, |\phi(x)| \, .$$

For $m, s \in \mathbb{R}$ and $1 \le q \le \infty$, weighted Sobolev space $\mathbf{H}_q^{m,s}(\mathbb{R})$ is defined by

$$\mathbf{H}_q^{m,s}(\mathbb{R}) = \left\{ f \in \mathbf{L}^q(\mathbb{R})\, ; \|f\|_{\mathbf{H}_q^{m,s}(\mathbb{R})} = \left\|(1-\partial_x^2)^{\frac{m}{2}}(1+|x|^2)^{\frac{s}{2}} f\right\|_{\mathbf{L}^q(\mathbb{R})} < \infty \right\}.$$

We write $\mathbf{H}_2^{m,s}(\mathbb{R}) = \mathbf{H}^{m,s}$ and $\mathbf{H}^{m,0}(\mathbb{R}) = \mathbf{H}^m$ for simplicity.

Let us introduce some more notations. We define the dilation operator by

$$(D_t\phi)(x) = \frac{1}{(it)^{\frac{1}{2}}}\phi\left(\frac{x}{t}\right)$$

and define $M = e^{\frac{i}{2t}x^2}$ for $t \ne 0$. Evolution operator $U(t)$ is written as

$$U(t) = M D_t \mathcal{F} M,$$

where the Fourier transform of f is

$$(\mathcal{F}f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x) dx.$$

We also have

$$U(-t) = M^{-1}\mathcal{F}^{-1}D_t^{-1}M^{-1},$$

where the inverse Fourier transform of f is

$$(\mathcal{F}^{-1}f)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} f(\xi) d\xi.$$

We denote by the same letter C various positive constants.

The standard generator of the Galilei transformations is given as

$$J(t) = U(t)\, xU(-t) = x + it\partial_x.$$

We have

$$J^2(t) = U(t)\, x^2 U(-t) = M(-t^2\partial_x^2)M^{-1}.$$

We also have commutation relations with J^α and $L = i\partial_t + \frac{1}{2}\partial_x^2$ such that

$$[L, J^\alpha] = 0,$$

where $\alpha = 1, 2$.

Before stating our main theorem, we introduce the function space

$$X_{2,T} = \{y \; ; \; U(-t)y \in C([0,T); \mathbf{H}^1 \cap \mathbf{H}^{0,2}), \|y\|_{X_{2,T}} < \infty\},$$

where $\|y\|_{X_{2,T}} = \sup_{0\le t<T} \|U(-t)y\|_{\mathbf{H}^1\cap\mathbf{H}^{0,2}}$. Our main result is

Theorem 1.1. *Let $p_0 < p \le \frac{19+\sqrt{145}}{12}$, where $p_0 \approx 2.486\cdots$ is a unique real root of*

$$12p^3 - 41p^2 + 35p - 18 = 0.$$

We assume that $\phi \in \mathbf{H}^{0,2} \cap \mathbf{H}^1$ (without size restriction) and the strong dissipative condition (1.2) *holds. Then* (1.1) *has a unique global solution $v \in X_{2,\infty}$ satisfying the following time-decay estimates :*

$$\|v(t)\|_{\mathbf{L}^2} \le C(1+t)^{-(2/3)d}, \tag{1.3}$$

$$\|v(t)\|_{\mathbf{L}^\infty} \le C(1+t)^{-1/(p-1)}, \tag{1.4}$$

for $t > 0$, where $d = 1/(p-1) - 1/2$.

Combining Theorem 1.1 and theorems in [3], we have

Corollary 1.2. *Let $p_0 < p < 3$, where $p_0 \approx 2.486\cdots$ is a unique real root of*

$$12p^3 - 41p^2 + 35p - 18 = 0.$$

We assume that $\phi \in \mathbf{H}^{0,2} \cap \mathbf{H}^1$ (without size restriction) and the strong dissipative condition (1.2) *holds. Then* (1.1) *has a unique global solution $v \in X_{2,\infty}$ satisfying the following time-decay estimates :*

$$\|v(t)\|_{\mathbf{L}^\infty} \le C(1+t)^{-1/(p-1)}, \tag{1.5}$$

for $t > 0$.

Remark 1.1. In fact, we have L^2- decay (1.3) under the same assumptions as in Corollary 1.2. This will be proved in Section 2 (see Proposition 2.1).

Remark 1.2. The decay rate of $\|v(t)\|_{\mathbf{L}^\infty}$ is similar to that of the solution to the ODE: $i\partial_t v = \lambda |v|^{p-1}v$. This suggests that the nonlinear effect is dominant to the dispersion emerging from $\frac{1}{2}\partial_x^2$ for large $t > 0$.

In our strategy for the proof, we are going to be engaged in the refinement of the estimate of $\|J^2 v(t)\|_{\mathbf{L}^2}$, where $J = x + it\partial_x$. In [3], they derived, from (1.1) and (1.2),

$$\begin{aligned} \frac{d}{dt}\|J^2 v\|_{\mathbf{L}^2} &\leq C\|v\|_{\mathbf{L}^\infty}^{p-2}\|Jv\|_{\mathbf{L}^\infty}\|Jv\|_{\mathbf{L}^2} \\ &\leq Ct^{-1/2}\|v\|_{\mathbf{L}^\infty}^{p-2}\|J^2 v\|_{\mathbf{L}^2}^{1/2}\|Jv\|_{\mathbf{L}^2}^{3/2}, \end{aligned}$$

and used $\|v\|_{\mathbf{L}^\infty} \leq Ct^{-1/2}\|v\|_{\mathbf{L}^2}^{1/2}\|Jv\|_{\mathbf{L}^2}^{1/2}$ together with $\|v\|_{\mathbf{L}^2} + \|Jv\|_{\mathbf{L}^2} \leq C$ and $\phi \in \mathbf{H}^{0,2}$. Let $2 < p < 3$. Solving the above differential inequality, they obtained

$$\|J^2 v\|_{\mathbf{L}^2} \leq C(1+t)^{3-p} \tag{1.6}$$

for $t > 0$, which played an important role to control the remainder terms appearing in the process to determine the decay estimate of $v(t)$. On the other hand, we will, more carefully than Jin-Jin-Li did, treat $\|v\|_{\mathbf{L}^\infty}$ (refer to Proposition 2.3) after deriving some decay estimate of $\|v(t)\|_{\mathbf{L}^2}$ (refer to Proposition 2.1), and obtain

$$\|J^2 v\|_{\mathbf{L}^2} \leq C(1+t)^{p(11-4p)/6} \tag{1.7}$$

for $t > 0$, if $p_0 < p \leq \frac{19+\sqrt{145}}{12}$. We remark here that the growth order in (1.7) is better than that of (1.6), since $p(11-4p)/6 < 3-p$ holds sufficiently if $p_0 < p \leq \frac{19+\sqrt{145}}{12}$. Finally, making use of (1.7), we will obtain the sharp decay rate of $\|v(t)\|_{\mathbf{L}^\infty}$. The number p_0 will arise in the estimate of (3.5) below. In (3.5), we will regard $t^{-1/(p-1)+1/2}$ as a dominant to $t^{-(12p^2-29p)/12}$ for $t > 1$. This observation is valid if $p_0 < p$.

2 Preliminaries

We first consider the $\mathbf{L}^2$-decay estimate of $v(t)$. This will be used later for a rough $\mathbf{L}^\infty$-decay estimate (see Proposition 2.3).

Proposition 2.1 (L^2-Decay). *Assume the same assumptions as in Corollary 1.2. Let $v \in X_{2,\infty}$ be the global solution to* (1.1). *Then it satisfies*

$$\|v(t)\|_{\mathbf{L}^2} \le C(1+t)^{-(2/3)d}, \tag{2.1}$$

for $t > 0$, where $d = 1/(p-1) - 1/2$.

When we prove Proposition 2.1, the lemma below will be often taken into account, the proof of which follows from [3] and [5]. We note that, to derive Lemma 2.2 without size restriction on ϕ, the strong dissipative condition (1.2) is useful.

Lemma 2.2. *Let $v \in X_{2,\infty}$ be the global solution to* (1.1). *Then it satisfies*

$$\|\partial_x v(t)\|_{\mathbf{L}^2} \le \|\partial_x \phi\|_{\mathbf{L}^2}, \tag{2.2}$$

$$\|Jv(t)\|_{\mathbf{L}^2} \le \|x\phi\|_{\mathbf{L}^2}. \tag{2.3}$$

Proof of Proposition 2.1. We first note that $\mathcal{F}U(-t)v$ satisfies

$$\partial_t(\mathcal{F}U(-t)v) = -i\lambda t^{-(p-1)/2}|\mathcal{F}U(-t)v|^{p-1}\mathcal{F}U(-t)v + R(t), \tag{2.4}$$

where $U(t) = \exp(it\partial_x^2/2)$ and

$$\begin{aligned} R(t) = -i\lambda t^{-(p-1)/2}\big(&\mathcal{F}M^{-1}\mathcal{F}^{-1}|\mathcal{F}MU(-t)v|^{p-1}\mathcal{F}MU(-t)v \\ &-|\mathcal{F}U(-t)v|^{p-1}\mathcal{F}U(-t)v\big) \end{aligned} \tag{2.5}$$

with $M = \exp(ix^2/2t)$. By (2.4), we have

$$\begin{aligned} &\partial_t\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^2 \\ &= -2|\lambda_2|t^{-(p-1)/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^{p+1}}^{p+1} + 2\mathrm{Re}(\mathcal{F}U(-t)v, R(t)). \end{aligned} \tag{2.6}$$

To estimate $\|\mathcal{F}U(-t)v\|_{\mathbf{L}^{p+1}}^{p+1}$ on the RHS of (2.6) from the below, we firstly use Hölder's inequality : $\|f\|_{\mathbf{L}^2} \le \|f\|_{\mathbf{L}^1}^{(p-1)/2p}\|f\|_{\mathbf{L}^{p+1}}^{(p+1)/2p}$ and secondly use $\|f\|_{\mathbf{L}^1} \le C\|f\|_{\mathbf{L}^2}^{1/2}\|\xi f\|_{\mathbf{L}^2}^{1/2}$. Considering also

$$\begin{aligned} \|R(t)\|_{\mathbf{L}^2} &\le Ct^{-(p-1)/2-1/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty}^{p-1}\|Jv\|_{\mathbf{L}^2} \\ &\le Ct^{-p/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{(p-1)/2}\|\partial_\xi\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{(p-1)/2}\|Jv\|_{\mathbf{L}^2} \\ &\le Ct^{-p/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{(p-1)/2}\|Jv\|_{\mathbf{L}^2}^{(p+1)/2}, \end{aligned}$$

in which the following Sobolev embedding

$$\|f\|_{\mathbf{L}^\infty} \le C\|f\|_{\mathbf{L}^2}^{\frac{1}{2}}\|\partial_x f\|_{\mathbf{L}^2}^{\frac{1}{2}}$$

is used, we see that

$$\begin{aligned}\partial_t\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2} &\le -Ct^{-(p-1)/2}\frac{\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{(3p-1)/2}}{\|\partial_x v\|_{\mathbf{L}^2}^{(p-1)/2}} \\ &\qquad +Ct^{-p/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{(p-1)/2}\|Jv\|_{\mathbf{L}^2}^{(p+1)/2}.\end{aligned}$$

By (2.2) and (2.3),

$$\begin{aligned}\partial_t\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2} &\le -Ct^{-(p-1)/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{(3p-1)/2} \\ &\qquad +Ct^{-p/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{(p-1)/2}. \qquad (2.7)\end{aligned}$$

Young's inequality : $ab \le h\theta a^{1/\theta} + h^{-\theta/(1-\theta)}(1-\theta)b^{1/(1-\theta)}$, where $0<\theta<1$ and $h>0$, gives

$$\begin{aligned}&t^{-p/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{(p-1)/2} \\ &\quad \le \varepsilon t^{-(p-1)/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{(3p-1)/2} + C_\varepsilon t^{-(2p-1)(p+1)/4p}\end{aligned}$$

for any $\varepsilon = \frac{h(p-1)}{3p-1} > 0$. Applying this inequality to (2.7) and using $\|v\|_{\mathbf{L}^2} \le C$, we have

$$\begin{aligned}\partial_t\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2} &\le -Ct^{-(p-1)/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{(3p-1)/2} \\ &\quad + Ct^{-(2p-1)(p+1)/4p}. \qquad (2.8)\end{aligned}$$

To solve (2.8), we invoke Sunagawa's idea [4]. The quantity $t^\gamma\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}$ with $\gamma>0$ sufficiently large satisfies the following differential inequality:

$$\begin{aligned}&\partial_t(t^\gamma\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}) \\ &\quad = \gamma t^{\gamma-1}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2} + t^\gamma\partial_t\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2} \\ &\quad \le \gamma t^{\gamma-1}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2} - Ct^{\gamma-(p-1)/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{(3p-1)/2} \\ &\qquad\quad +Ct^{\gamma-(2p-1)(p+1)/4p}. \qquad (2.9)\end{aligned}$$

Young's inequality gives

$$t^{\gamma-1}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2} \le \varepsilon t^{\gamma-(p-1)/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{(3p-1)/2} + C_\varepsilon t^{\gamma-2p/3(p-1)},$$

where $\varepsilon>0$. Let $\varepsilon>0$ be so small that $\gamma\varepsilon < C$. Then, applying this inequality to (2.9), we see that

$$\begin{aligned}\partial_t(t^\gamma\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}) &\le \gamma\varepsilon t^{\gamma-(p-1)/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{(3p-1)/2} \\ &\qquad -Ct^{\gamma-(p-1)/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{(3p-1)/2} \\ &\qquad +\gamma C_\varepsilon t^{\gamma-2p/3(p-1)} + Ct^{\gamma-(2p-1)(p+1)/4p} \\ &\le \gamma C_\varepsilon t^{\gamma-2p/3(p-1)} + Ct^{\gamma-(2p-1)(p+1)/4p} \qquad (2.10)\end{aligned}$$

Integrating (2.10) over $[1, t]$, we have

$$\begin{aligned} &\|\mathcal{F}U(-t)v(t)\|_{\mathbf{L}^2} \\ &\leq \; t^{-\gamma}\|\mathcal{F}U(-1)v(1)\|_{\mathbf{L}^2} + Ct^{1-2p/3(p-1)} + Ct^{1-(2p-1)(p+1)/4p}. \end{aligned} \tag{2.11}$$

Since the second term of (2.11) is dominant if $p > p_0$, taking $\gamma > 0$ sufficiently large and noting that $1 - 2p/3(p-1) = -(2/3)d$ yield

$$\|\mathcal{F}U(-t)v(t)\|_{\mathbf{L}^2} \;\leq\; Ct^{-(2/3)d}$$

for $t > 1$. By Plancherell's identity, $\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2} = \|v\|_{\mathbf{L}^2}$. This completes the proof of Proposition 2.1. □

We next derive a rough decay estimate of $\|v(t)\|_{\mathbf{L}^\infty}$ by making use of Proposition 2.1.

Proposition 2.3 (Rough L^∞-Decay)**.** *Assume the same assumption as in Theorem 1.1. Let $v \in X_{2,\infty}$ be the global solution to* (1.1)*. Then it satisfies*

$$\|v(t)\|_{\mathbf{L}^\infty} \leq C(1+t)^{-(4p-3)/12}, \tag{2.12}$$

for $t > 0$.

Proof of Proposition 2.3. Since

$$\begin{aligned} \|v\|_{\mathbf{L}^\infty} &= t^{-1/2}\|\mathcal{F}MU(-t)v\|_{\mathbf{L}^\infty} \\ &\leq t^{-1/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty} + t^{-1/2}\|\mathcal{F}(M-1)U(-t)v\|_{\mathbf{L}^\infty} \\ &\leq t^{-1/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty} \\ &\quad + Ct^{-1/2}\|\mathcal{F}(M-1)U(-t)v\|_{\mathbf{L}^2}^{1/2}\|\partial_\xi\mathcal{F}(M-1)U(-t)v\|_{\mathbf{L}^2}^{1/2} \\ &\leq t^{-1/2}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty} + Ct^{-3/4}\|Jv\|_{\mathbf{L}^2} \end{aligned} \tag{2.13}$$

in which the Sobolev embedding

$$\|f\|_{\mathbf{L}^\infty} \leq C\|f\|_{\mathbf{L}^2}^{\frac{1}{2}}\|\partial_x f\|_{\mathbf{L}^2}^{\frac{1}{2}}$$

is used, it is enough to observe the decay of $\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty}$. From (2.4) and

$$\|R(t)\|_{\mathbf{L}^\infty} \leq Ct^{-(p-1)/2-1/4}(\|\mathcal{F}MU(-t)v\|_{\mathbf{L}^\infty}^{p-1} + \|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty}^{p-1})\|Jv\|_{\mathbf{L}^2},$$

it follows that

$$\begin{aligned} &\partial_t|\mathcal{F}U(-t)v| \\ &\quad\leq -|\lambda_2|t^{-(p-1)/2}|\mathcal{F}U(-t)v|^p \\ &\qquad + Ct^{-(p-1)/2-1/4}(\|\mathcal{F}MU(-t)v\|_{\mathbf{L}^\infty}^{p-1} + \|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty}^{p-1})\|Jv\|_{\mathbf{L}^2} \end{aligned} \tag{2.14}$$

Since

$$\|\mathcal{F}MU(-t)v\|_{\mathbf{L}^\infty} \leq \|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty} + Ct^{-1/4}\|Jv\|_{\mathbf{L}^2}$$

holds from (2.13) and $\|Jv\|_{\mathbf{L}^2} \leq C$ due to Lemma 2.2, we have

$$\begin{aligned}\partial_t|\mathcal{F}U(-t)v| &\leq -|\lambda_2|t^{-(p-1)/2}|\mathcal{F}U(-t)v|^p \\ &\quad +Ct^{-(p-1)/2-1/4}(\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty}^{p-1} + t^{-(p-1)/4}). \quad (2.15)\end{aligned}$$

To estimate $\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty}^{p-1}$ in (2.15), we use the Sobolev embedding :

$$\begin{aligned}\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty} &\leq C\|\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{1/2}\|\partial_\xi\mathcal{F}U(-t)v\|_{\mathbf{L}^2}^{1/2} \\ &= C\|v\|_{\mathbf{L}^2}^{1/2}\|Jv\|_{\mathbf{L}^2}^{1/2}.\end{aligned}$$

Apply Proposition 2.1 to $\|v\|_{\mathbf{L}^2}^{1/2}$ and Lemma 2.2 to $\|Jv\|_{\mathbf{L}^2}^{1/2}$. Then we see that

$$\begin{aligned}\partial_t|\mathcal{F}U(-t)v| &\leq -|\lambda_2|t^{-(p-1)/2}|\mathcal{F}U(-t)v|^p \\ &\quad +Ct^{-(p-1)/2-1/4}(t^{-(p-1)d/3} + t^{-(p-1)/4}). \quad (2.16)\end{aligned}$$

Since $(p-1)d/3 < (p-1)/4$ and $-(p-1)/2-1/4-(p-1)d/3 = -(4p+3)/12$, (2.16) yields

$$\partial_t|\mathcal{F}U(-t)v| \leq -|\lambda_2|t^{-(p-1)/2}|\mathcal{F}U(-t)v|^p + Ct^{-(4p+3)/12}. \quad (2.17)$$

We now invoke Sunagawa's idea again. By (2.17), we see that

$$\begin{aligned}&\partial_t(t^\gamma|\mathcal{F}U(-t)v|) \\ &\leq \gamma t^{\gamma-1}|\mathcal{F}U(-t)v| - |\lambda_2|t^{\gamma-(p-1)/2}|\mathcal{F}U(-t)v|^p + Ct^{\gamma-(4p+3)/12}. \quad (2.18)\end{aligned}$$

Young's inequality : $t^{\gamma-1}f \leq \varepsilon t^{\gamma-(p-1)/2}f^p + C_\varepsilon t^{\gamma-1/(p-1)-1/2}$ with $\varepsilon > 0$ and $\gamma\varepsilon < |\lambda_2|$ gives

$$\partial_t(t^\gamma|\mathcal{F}U(-t)v|) \leq Ct^{\gamma-1/(p-1)-1/2} + Ct^{\gamma-(4p+3)/12}. \quad (2.19)$$

Since $p_0 < p \leq \frac{19+\sqrt{145}}{12}$, then the second term on the RHS of (2.19) is dominant for large $t > 0$. Integrating (2.19) over $[1,t]$, we have

$$\begin{aligned}\|\mathcal{F}U(-t)v(t)\|_{\mathbf{L}^\infty} &\leq Ct^{-\gamma} + Ct^{-(4p-9)/12} \\ &\leq Ct^{-(4p-9)/12}, \quad (2.20)\end{aligned}$$

if $\gamma > 0$ is taken large enough. Combining (2.13) and (2.20), we obtain Proposition 2.3. □

In virtue of Proposition 2.3, we achieve the estimate of $\|J^2v\|_{\mathbf{L}^2}$, which is a refinement of Jin-Jin-Li's estimate.

Proposition 2.4 (Estimate of $J^2 v$)**.** *Assume that* $p_0 < p \le \frac{19+\sqrt{145}}{12}$*. Let* $v \in X_{2,\infty}$ *be the global solution to* (1.1)*. Then it satisfies*

$$\|J^2 v(t)\|_{\mathbf{L}^2} \le C(1+t)^{p(11-4p)/6}, \tag{2.21}$$

for $t > 0$.

Proof of Proposition 2.4. By (1.2), we see that

$$\begin{aligned} \partial_t \|J^2 v\|_{\mathbf{L}^2} &\le C\|v\|_{\mathbf{L}^\infty}^{p-2} \|JvJv\|_{\mathbf{L}^2} \\ &\le C\|v\|_{\mathbf{L}^\infty}^{p-2} \|Jv\|_{\mathbf{L}^\infty} \|Jv\|_{\mathbf{L}^2} \\ &\le Ct^{-1/2}\|v\|_{\mathbf{L}^\infty}^{p-2} \|J^2 v\|_{\mathbf{L}^2}^{1/2} \|Jv\|_{\mathbf{L}^2}^{3/2}. \end{aligned} \tag{2.22}$$

Apply Proposition 2.3 to $\|v\|_{\mathbf{L}^\infty}^{p-2}$ and Lemma 2.2 to $\|Jv\|_{\mathbf{L}^2}^{3/2}$. Then we have

$$\partial_t \|J^2 v\|_{\mathbf{L}^2}^{1/2} \le Ct^{-1/2-(p-2)(4p-3)/12}. \tag{2.23}$$

Integrating (2.23) over $[1, t]$, we obtain Proposition 2.4. □

3 Proof of Theorem 1.1

Since Proposition 2.1 proves (1.3), we are going to concentrate ourselves into the proof of (1.4).

Proof of Theorem 1.1. By (2.4) and

$$\|R(t)\|_\infty \le Ct^{-(p-1)/2-3/4}(\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty}^{p-1} + t^{-(p-1)/4})\|J^2 v\|_{\mathbf{L}^2}, \tag{3.1}$$

we have

$$\begin{aligned} &\partial_t |\mathcal{F}U(-t)v| \\ &\quad \le -|\lambda_2| t^{-(p-1)/2} |\mathcal{F}U(-t)v|^p \\ &\qquad + Ct^{-(p-1)/2-3/4}(\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty}^{p-1} + t^{-(p-1)/4})\|J^2 v\|_{\mathbf{L}^2}. \end{aligned} \tag{3.2}$$

Apply (2.20) to $\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty}^{p-1}$ and Proposition 2.4 to $\|J^2 v\|_{\mathbf{L}^2}$. Also notice that $\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty}^{p-1}$ is dominant to $t^{-(p-1)/4}$ in (3.2). Then, after some careful computation to determine the decay order of the remainder term, we have

$$\partial_t |\mathcal{F}U(-t)v| \le -|\lambda_2| t^{-(p-1)/2} |\mathcal{F}U(-t)v|^p + Ct^{-(12p^2-29p+12)/12}. \tag{3.3}$$

Similarly to the derivation of (2.19),

$$\partial_t(t^\gamma|\mathcal{F}U(-t)v|) \ \leq \ Ct^{\gamma-1/(p-1)-1/2}+Ct^{\gamma-(12p^2-29p+12)/12}. \qquad (3.4)$$

Integrating (3.4) over $[1,t]$, we see that

$$|\mathcal{F}U(-t)v| \ \leq \ Ct^{-\gamma}+Ct^{-1/(p-1)+1/2}+Ct^{-(12p^2-29p)/12}. \qquad (3.5)$$

We here want to regard $t^{-1/(p-1)+1/2}$ as a dominant term and $t^{-(12p^2-29p)/12}$ as a remainder one. Such observation becomes true if

$$-\frac{1}{p-1}+\frac{1}{2}>-\frac{12p^2-29p}{12},$$

which is equivalent to

$$12p^3-41p^2+35p-18>0.$$

There is only one real root of the polynomial $12p^3-41p^2+35p-18$, which is numerically described as $p=p_0\approx 2.486\cdots$. This is the main reason why the weird number p_0 is included in the assumption of Theorem 1.1. Anyway, if $p_0<p$, (3.5) yields

$$\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty} \ \leq \ Ct^{-1/(p-1)+1/2}$$

for $t>1$. Applying the above inequality to $\|\mathcal{F}U(-t)v\|_{\mathbf{L}^\infty}$, we get

$$\begin{aligned}\|v\,(t)\|_{\mathbf{L}^\infty} &\leq Ct^{-\frac{1}{2}}\left\|\mathcal{F}U\,(-t)\,v\right\|_{\mathbf{L}^\infty}+Ct^{-\frac{1}{2}}\left\|\mathcal{F}\,(M-1)\,U\,(-t)\,v\,(t)\right\|_{\mathbf{L}^\infty}\\ &\leq Ct^{-\frac{1}{p-1}}+Ct^{-\frac{3}{4}}\left\|xU\,(-t)\,v\,(t)\right\|\\ &\leq Ct^{-\frac{1}{p-1}}\end{aligned}$$

for $t>1$. □

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