# Classical Poincaré conjecture via 4D topology

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## Classical Poincaré conjecture via 4D topology

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ABSTRACT

The classical Poincaré conjecture that every homotopy 3-sphere is diffeomorphic to the 3-sphere is proved by G. Perelman by solving Thurston's program on geometrizations of 3-manifolds. A new confirmation of this conjecture is given by combining R. H. Bing's result on this conjecture with Smooth Unknotting Conjecture for an $S^2$-link and Smooth 4D Poincaré Conjecture.

*Keywords:* Homotopy 3-sphere, Smooth unknotting, Smooth homotopy 4-sphere.
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### 1. Introduction

A *homotopy 3-sphere* is a smooth 3-manifold $M$ homotopy equivalent to the 3-sphere $S^3$. It is well-known that a simply connected closed connected 3-manifold is a smooth homotopy 3-sphere. The following theorem, called the classical Poincaré Conjecture coming from [22, 23] is positively shown by Perelman [20, 21] solving positively Thurston's program [24] on geometrizations of 3-manifolds (see [19] for detailed historical notes).

**Theorem 1.1.** Every homotopy 3-sphere $M$ is diffeomorphic to the 3-sphere $S^3$.

The purpose of this paper is to give an alternative proof to Theorem 1.1 by combining R. H. Bing's result in [2, 3] on the classical Poincaré conjecture with Smooth Unknotting Conjecture and Smooth 4D Poincaré Conjecture to be explained from now on. Let $F$ be a smooth surface-link with a component system $F_i$, ($i =$

$1, 2, \ldots, n$) in the 4-sphere $S^4$. The fundamental group $\pi_1(S^4 \setminus F, v)$ (with $v$ a base point) is a *meridian-based free group* if the group $\pi_1(S^4 \setminus F, v)$ is a free group with a basis represented by a meridian system $m_i$ $(i = 1, 2, \ldots, n)$ of $F_i$, $(i = 1, 2, \ldots, n)$ with a base point $v$. The smooth surface-link $F$ is a *trivial surface-link* if the components $F_i$, $(i = 1, 2, \ldots, n)$ bound a disjoint handlebody system smoothly embedded in $S^4$. Smooth Unknotting Conjecture for a surface-link is the following conjecture.

**Smooth Unknotting Conjecture.** Every smooth surface-link $F$ in $S^4$ with a meridian-based free fundamental group $\pi_1(S^4 \setminus F, v)$ is a trivial surface-link.

The positive proof of this conjecture is claimed by [13, 15] with supplement [14]. The result when $F$ is an $S^2$-link (i.e., a surface-link with only $S^2$-components) is applied in this paper. A *homotopy 4-sphere* is a smooth 4-manifold $X$ homotopy equivalent to the 4-sphere $S^4$. Smooth 4D Poincaré Conjecture is the following conjecture.

**Smooth 4D Poincaré Conjecture.** Every 4D smooth homotopy 4-sphere $X$ is diffeomorphic to the 4-sphere $S^4$.

The positive proof of this conjecture is claimed by [16, 17]. For the proof of Theorem 1.1, the following result of R. H. Bing in [2, 3] is used:

**Bing's Theorem.** A homotopy 3-sphere $M$ is diffeomorphic to $S^3$ if, for every knot $k$ in $M$, there is a 3-ball in $M$ containing the knot $k$.

Thus, the main result of this paper is to prove the following lemma.

**Lemma 1.2.** For every knot $k$ in $M$, there is a 3-ball in $M$ containing the knot $k$.

For the proof of Lemma 1.2, Artin's spinning construction of a knot in $S^3$ in [1] is generalized into a connected graph in a homotopy 3-sphere $M$ to produce a spun $S^2$-link in $S^4$ with free fundamental group (not always meridian-based free group). This explanation is done in Section 2. In Section 3, it is shown that every $S^2$-link in $S^4$ with free fundamental group is a ribbon $S^2$-link by using Smooth Unknotting Conjecture for an $S^2$-link and Smooth 4D Poincaré Conjecture. In Section 4, the proof of Lemma 1.2 is done. To do this, it is shown that the spun torus-knot of a knot in $M$ is a ribbon-torus knot in $S^4$ which is a sum of the spun $S^2$-link of a proper arc system $a_*$ in a boundary collar of a compact once-punctured manifold $M^{(o)}$ of $M$ and the spun $S^2$-link of a proper arc system $e_*$ in $M^{(o)}$ with meridian-based free fundamental group $\pi_1(M^{(o)} \setminus e_*, v)$. To see this, an argument of a chord diagram of the spun $S^2$-link of a proper arc system $a_*$ in a boundary collar of $M^{(o)}$ in [12] is used. In this way, it is shown that the knot $k$ is in a 3-ball of $M$ completing the proof of Lemma 1.2 and the proof of Theorem 1.1 is completed.

*Conventions.* The unit $n$-disk is denoted by $D^n$ with the origin $\mathbf{0}$ as a standard notation, but the unit 2-disk $D^2$ is fixed in the complex plane $\mathbb{C}$. A smooth $n$-manifold diffeomorphic to the unit $n$-disk $D^n$ is called an *$n$-ball* for $n \geq 3$ or *$n$-disk* for $n = 2$. A point $\mathbf{1}$ is fixed in the $n$-sphere $S^n = \partial D^{n+1}$.

## 2. Artin's spinning construction of a connected graph in a homotopy 3-sphere

For a homotopy 3-sphere $M$, let $M^{(o)}$ be the compact once-punctured manifold $\mathrm{cl}(M \setminus B)$ of $M$ for a 3-ball $B$ in $M$. Let

$$S = \partial B = \partial M^{(o)}$$

be the boundary 2-sphere of $M^{(o)}$. The closed smooth 4-manifold $X(M)$ defined by

$$X(M) = M^{(o)} \times S^1 \cup S \times D^2$$

is called the *spun manifold* of $M$ with *axis* 4-submanifold $S \times D^2$. As a convention, the 3-submanifold $M^{(o)} \times 1$ of the product $M^{(o)} \times S^1$ is identified with $M^{(o)}$. In particular, a point $(q, 1) \in M^{(o)} \times 1$ is identified with the point $q \in M^{(o)}$. This 4-manifold $X(M)$ is a smooth homotopy 4-sphere by the van Kampen theorem and a homological argument and hence $X(M)$ is diffeomorphic to the 4-sphere $S^4$ by Smooth 4D Poincaré Conjecture. A *legged loop* with *base point* $v$ is the union $k \cup \omega$ of a loop $k$ and an arc $\omega$ joining the base point $v$ with a point of $k$. The arc $\omega$ is called the *leg*. A *legged loop system* with base point $v$ is the union

$$\gamma = \cup_{i=1}^{n} k_i \cup \omega_i$$

of $n$ legged loops $k_i \cup \omega_i \, (i = 1, 2, \ldots, n)$ meeting only at the same base point $v$. Let $k(\gamma) = \cup_{i=1}^{n} k_i = k_*$ denote the loop system of the legged loop system of $\gamma$. Let $\omega_* = \cup_{i=1}^{n} \omega_i$ and $v_* = k_* \cap \omega_*$. For a maximal tree $\tau$ of $\gamma$ containing the base point $v$, a regular neighborhood $B$ of $\tau$ in $M$ with $\gamma \cap B$ a regular neighborhood of $\tau$ in $\gamma$ is taken as 3-ball $B$ used for the compact once-punctured manifold $M^{(o)} = \mathrm{cl}(M \setminus B)$ of $M$. Deform the subgraph $\gamma \cap B$ of $\gamma$ so that

$$\omega_* \subset B, \quad \omega_* \cap S = \partial \omega_* \quad \text{and} \quad k_* \cap B = k_* \cap S = a'_*$$

for an arc system $a'_*$ in $k_*$, where note that the base point $v$ is moved into $S$. Let

$$a(\gamma) = \cup_{i=1}^{n} a_i = a_*$$

for a proper arc $a_i = \mathrm{cl}(k_i \setminus a'_i)$ $(i = 1, 2, \ldots, n)$ in $M^{(o)}$. Let

$$\dot{a}(\gamma) = \partial a_* = \partial a'_*$$

be the set of $2n$ points in the boundary 2-sphere $S$ of $M^{(o)}$. The *spun* $S^2$-link of the graph $\gamma$ is the $S^2$-link $S(\gamma)$ in the 4-sphere $X(M)$ defined by

$$S(\gamma) = a(\gamma) \times S^1 \cup \dot{a}(\gamma) \times D^2.$$

**Lemma 2.1.** The inclusion $M^{(o)} \setminus a(\gamma) \subset X(M) \setminus S(\gamma)$ induces an isomorphism

$$\sigma : \pi_1(M \setminus \gamma, v) \to \pi_1(X(M) \setminus S(\gamma), v)$$

sending a meridian system of the proper arc system $a(\gamma)$ in $M^{(o)}$ to a meridian system of $S(\gamma)$.

**Proof of Lemma 2.1.** Note that there is a canonical isomorphism

$$\pi_1(M^{(o)} \setminus a(\gamma), v) \cong \pi_1(M \setminus \gamma, v).$$

Then the desired isomorphism $\sigma$ is obtained by applying the van Kampen theorem between $(M^{(o)} \setminus a(\gamma)) \times S^1$ and $(S \setminus \dot{a}(\gamma)) \times D^2$. This completes the proof of Lemma 2.1. $\square$

Here is a note on Lemma 2.1.

**Note 2.2.** A general connected graph $\gamma$ with Euler characteristic $\chi(\gamma) = 1 - n$ in $M$ is deformed into a legged loop system $\gamma$ in $M$ by choosing a maximal tree to shrink to a base point $v$. Note that there are only finitely many maximal trees of $\gamma$ such that the loop systems $k(\gamma)$ of the resulting legged loop systems $\gamma$ are distinct as links. By Lemma 2.1, we can obtain finitely many distinct spun $S^2$-links in $S^4$ with isomorphic fundamental groups obtained by taking different maximal trees of the connected graph $\gamma$. This is a detailed explanation on the spun $S^2$-link of a connected graph associated with a maximal tree in [7, p.204] when $M = S^3$.

An argument on Lemma 2.1 is further developed when the homotopy 3-sphere $M$ is given by a Heegaard spitting $V \cup V'$ pasting along a Heegaard surface $F = \partial V = \partial V'$ of genus $n$. A *spine* of a handlebody $V$ of genus $n$ is a legged loop system $\gamma$ with base point $v$ in $F = \partial V$ such that the inclusion map $\gamma \to V$ induces an isomorphism $\pi_1(\gamma, v) \to \pi_1(V, v)$. A regular neighborhood $\dot{V}$ of $\gamma$ in $F$ is a planar surface in $F$.

By [5, Theorem 10.2], there is a diffeomorphism $(\dot{V} \times [0,1], \dot{V} \times 0) \to (V, \dot{V})$ sending every point $(x, 0) \in \dot{V} \times 0$ to $x \in \dot{V}$. The surface $\dot{V}$ is called a *spine surface* of $V$. Let $\gamma$ and $\gamma'$ be spines of the handlebodies $V$ and $V'$ with the same base point $v \in F$, respectively. A *legged Heegaard loop system* in $M$ is the legged loop system $\gamma\gamma'$ in $M$ with base point $v$ obtained by pushing $\gamma \backslash v$ and $\gamma' \backslash v$ into the interiors Int$V$ and Int$V'$, respectively. The fundamental groups of the spun $S^2$-links $S(\gamma\gamma') = S(\gamma) \cup S(\gamma)$, $S(\gamma)$ and $S(\gamma)$ in the 4-sphere $X(M)$ given by Lemma 2.1 are free groups, as shown in the following lemma:

**Lemma 2.3.** The fundamental groups $\pi_1(X(M) \setminus S(\gamma), v)$ and $\pi_1(X(M) \setminus S(\gamma'), v)$ are free groups of rank $n$ and the fundamental group $\pi_1(X(M) \setminus S(\gamma\gamma'), v)$ is a free group of rank $2n$.

**Proof of Lemma 2.3.** The closed complements $\mathrm{cl}(M \setminus N(\gamma))$, $\mathrm{cl}(M \setminus N(\gamma'))$ and $\mathrm{cl}(M \setminus N(\gamma))$ are diffeomorphic to the handlebodies $V'$, $V$ and $F^{(o)} \times [0,1]$ for the once-punctured surface $F^{(o)}$ of $F$ , respectively. Since the fundamental groups $\pi_1(V', v)$, $\pi_1(V, v)$ and $\pi_1(F^{(o)} \times [0,1], v)$ are free groups of ranks $n$, $n$ and $2n$, respectively, the desired result is obtained from Lemma 2.1. □

It should be noted that these free groups in Lemma 2.3 are not necessarily meridian-based free groups. Here is an example.

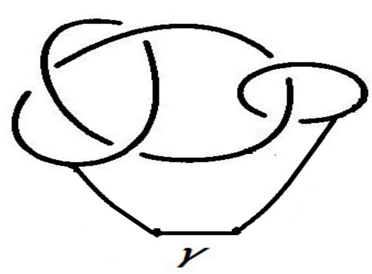


Figure 1: A legged loop system $\gamma$ in $S^3$ with free fundamental group of rank 2

**Example 2.4.** Let $\gamma$ be a legged loop system with base point $v$ in $S^3$ illustrated in Fig. 1 with free fundamental group $\pi_1(S^3 \backslash \gamma, v)$ of rank 2. In fact, a trivial legged loop system is obtained by sliding an edge along another edge, so that the fundamental group $\pi_1(S^3 \setminus k(\gamma), v)$ is a free group of rank 2. A regular neighborhood $V$ of $\gamma$ in $S^3$ and the closed complement $V' = \mathrm{cl}(S^3 \setminus V)$ constitute a genus 2 Heegaard splitting

$V \cup V'$ of $S^3$ by noting that the 3-manifold $V'$ is a handlebody of genus 2 by the loop system theorem and the Alexander theorem (cf. e.g., [7]). Thus, the union $V \cup V'$ is a genus 2 Heegaard splitting of $S^3$. The legged loop system $\gamma$ with vertex $v$ is a spine of $V$ by sliding the base point $v$ into $\partial V$. By Lemma 2.3, the spun $S^2$-link $S(\gamma)$ in the 4-sphere $X(S^3) = S^4$ has the free fundamental group $\pi_1(X(S^3) \setminus S(\gamma), v)$ of rank 2, which does not admit any meridian basis because the $S^2$-link $S(\gamma)$ contains a component of the spun trefoil $S^2$-knot in $S^4$ whose fundamental group is known to be not infinite cyclic.

Given a proper arc system $a_*$ in $M^{(o)}$, there is a legged loop system $\gamma$ in $M$ with the proper arc system $a(\gamma) = a_*$ in $M^{(o)}$. The $S^2$-link $S(\gamma)$ in $X(M)$ is uniquely determined by the arc system $a_*$ and thus denoted by $S(a_*)$. The following lemma is directly used for the proof of Lemma 1.2.

**Lemma 2.5.** Let $a_*$ be a proper arc system in a compact once-punctured manifold $M^{(o)} = \mathrm{cl}(M \setminus B)$ of a homotopy 3-sphere $M$. If the $S^2$-link $S(a_*)$ in the 4-sphere $X(M)$ is a trivial $S^2$-link, then the proper arc system $a_*$ is in a boundary-collar $S \times [0,1]$ of $M^{(o)}$.

**Proof of Lemma 2.5.** By Lemma 2.1, the fundamental group $\pi_1(M^{(o)} \setminus a(\gamma), v)$ is a meridian-based free group. Consider the 2-sphere $S$ is the boundary of the product $d \times [0,1]$ for a disk $d$ so that $d \times 0$ contains one end of the proper arc system $a_*$ and $d \times 1$ contains the other end of the proper arc system $a_*$. Let $(E; E_0, E_1)$ be the triplet obtained from $(M^{(o)}, d \times 0, d \times 1)$ by removing a tubular neighborhood of $a_*$ in $M^{(o)}$. Then the inclusion $E_0 \subset E$ induces an isomorphism

$$\pi_1(E_0, v) \to \pi_1(E, v).$$

By [5, Theorem 10.2], $E$ is diffeomorphic to the connected sum of the product $E_0 \times [0,1]$ and a homotopy 3-sphere. This means that the proper arc system $a_*$ is in a boundary-collar $S \times [0,1]$. This completes the proof of Lemma 2.5. □

**3. Ribbonness of an $S^2$-link with free fundamental group** The *4D handlebody* of *genus* $n$ is the boundary 3-disk sum

$$Y^D = D^4 \natural_{i=1}^n S^1 \times D_i^3$$

obtained from $n$ copies $S^1 \times D_i^3$ $(i = 1, 2, \ldots, n)$ of the 4D solid torus $S^1 \times D^3$ and the 4-disk $D^4$ by pasting a 3-disk system consisting of a boundary 3-disk in $(S^1 \setminus \{1\}) \times D_i^3$ for every $i$ to a system of disjoint $n$ boundary 3-disks of $D^4$. A legged loop system

$\gamma^D$ in the 4D handlebody $Y^D$ of genus $n$ is *standard* if the legged loop system $\gamma^D$ has the following two conditions:

- The loop system $k(\gamma^D)$ is consistent with the system $S^1 \times \mathbf{1}_i$ $(i = 1, 2, \ldots, n)$, and
- The base point $v$ is in the 4-disk $D^4$ and the legs $\omega_i$ $(i = 1, 2, \ldots, n)$ of $\gamma^D$ do not meet the 3-disks $1 \times D^3_i$ $(i = 1, 2, \ldots, n)$.

Note that the legs $(i = 1, 2, \ldots, n)$ of $\gamma^D$ are $\partial$-relatively unique up to isotopies in $Y^D$. The *4D closed handlebody* of *genus* $n$ is the double of the 4D handlebody $Y^D$ of genus $n$, that is the 4-manifold

$$\partial(Y^D \times [0,1]) = Y^D \times 0 \cup (\partial Y^D) \times [0,1] \cup Y^D \times 1$$

which is canonically identified with the following 4-manifold

$$Y^S = S^4 \#_{i=1}^n S^1 \times S^3_i,$$

where the connected summands $S^3$ and $S^1 \times S^3_i$ correspond to the doubles of the 3-disk summands $D^4$ and $S^1 \times D^3_i$, respectively. The 4D handlebody $Y^D \times 0$ in $Y^S$ is identified with $Y^D$. A legged loop system $\gamma$ with vertex $v$ of the 4D closed handlebody $Y^S$ of genus $n$ is *standard* if it is $v$-relatively isotopic to a standard legged loop system $\gamma^D$ of $Y^D \subset Y^S$. A standard legged loop system of $Y^S$ is denoted by $\gamma^S$. A homology 4-sphere is a smooth 4-manifold $X$ with an isomorphism $H_*(X; \mathbf{Z}) \cong H_*(S^4; \mathbf{Z})$. A *4D closed homology handlebody* of *genus* $n$ is a smooth 4-manifold $Y$ with an isomorphism $H_*(Y; \mathbf{Z}) \cong H_*(Y^S; \mathbf{Z})$ for the 4D closed handlebody $Y^S$ of genus $n$. For an $S^2$-link $L$ in $X$, take a normal disk bundle $L \times D^2$ in $X$ and a 3-disk system $D^3_L$ with $\partial D^3_L = L$. This transformation from $X$ into the 4-manifold

$$Y = \mathrm{cl}(X \setminus L \times D^2) \cup D^3_L \times S^1$$

is called the *surgery* of $X$ along the $S^2$-link $L$. Conversely, the transformation from $Y$ into $X$ is called the *surgery* of $Y$ along the loop system $\mathbf{0}_* \times S^1$ by observing that $D^3_L \times S^1$ is a regular neighborhood of $\mathbf{0}_* \times S^1$ in $Y$. The following lemma is a more or less known fact.

**Lemma 3.1.** Let $Y$ be the 4-manifold obtained from a homology 4-sphere $X$ by surgery along any $n$-component $S^2$-link $L$. Then the 4-manifold $Y$ is a 4D closed homology handlebody of genus $n$ such that the inclusion $X \setminus L \times D^2 \subset Y$ induces an isomorphism

$$\pi_1(X \setminus L \times D^2, v) \to \pi_1(Y, v).$$

**Proof of Lemma 3.1.** To see that $H_2(Y;\mathbf{Z}) = 0$, use the Euler characteristic $\chi(Y) = 2n$. Since $H_1(Y;\mathbf{Z}) \cong \mathbf{Z}^n$, we have $H_2(Y;\mathbf{Z}) = 0$ by Poincaé duality, which shows that $Y$ is a 4D closed homology handlebody of genus $n$. The isomorphism $i_* : \pi_1(X \setminus L \times D^2, v) \to \pi_1(Y, v)$ is obtained by a general position argument. □

A *meridian system* of an $S^2$-link $L$ in $X$ is a legged loop system $\gamma_L$ in the closed complement $\mathrm{cl}(X \setminus L \times D^2)$ for a normal disk bundle $L \times D^2$ in $X$ such that the loop system $k(\gamma_L)$ is the loop system $p_* \times S^1$ for a point system $p_*$ in $L$ with one point for every component of $L$. By Lemma 3.1, note that the meridian system $\gamma_L$ induces a legged loop system $\gamma$ in $Y$ such that the loop system $k(\gamma)$ represents a homological basis of the homology group $H_1(Y;\mathbf{Z})$. Conversely, given any legged loop system $\gamma$ in $Y$ such that the loop system $k(\gamma)$ represents a homological basis of $H_1(Y;\mathbf{Z})$, then the 4-manifold $X$ obtained from $Y$ along the loop system $k(\gamma)$ is a homology 4-sphere and the legged loop system $\gamma$ induces a meridian system $\gamma_L$ of an $S^2$-link $L$ in $X$. A *4D closed homotopy handlebody* of *genus* $n$ is a 4D closed homology handlebody $Y$ of genus $n$ such that the fundamental group $\pi_1(Y, p)$ is a free group of rank $n$. A legged loop system $\gamma$ with base point $v$ in a 4D closed homotopy handlebody $Y$ of genus $n$ is a *basis system* if the inclusion $\gamma \subset Y$ induces an isomorphism

$$\pi_1(\gamma, v) \to \pi_1(Y, v).$$

For example, a standard legged loop system $\gamma^S$ of the 4D closed handlebody $Y^S$ is a basis system. The following classification lemma is a result of Smooth Unknotting Conjecture for an $S^2$-link and Smooth 4D Poincaré Conjecture.

**Lemma 3.2.** Let $Y^S$ be the 4D closed handlebody of genus $n$, and $\gamma^S$ a standard legged loop system with base point $v^S$ of $Y^S$. For every 4D closed homotopy handlebody $Y$ of genus $n$ and every basis system $\gamma$ in $Y$, there is an orientation-preserving diffeomorphism

$$f : Y \to Y^S$$

such that $f(\gamma) = \gamma^S$. Given any spin structures on $Y$ and $Y^S$, the diffeomorphism $f$ can be taken spin-structure-preserving.

**Proof of Lemma 3.2.** Let $X$ be the 4-manifold obtained from $Y$ by surgery along the loop system $k_* = k(\gamma)$. This 4-manifold $X$ is diffeomorphic to the 4-sphere $S^4$ by Smooth 4D Poincaré Conjecture since it is a smooth homotopy 4-sphere by the van Kampen theorem and a homological argument. Since $X$ is obtained from $Y$ by replacing a normal disk bundle $k_* \times D^3$ of $k_*$ in $Y$ with $D^2_* \times S^2$ for the disk system $D^2_*$ bounded by $k_*$. Then there is an $S^2$-link $L = 0_* \times S^2$ in $X$. Since the basis system $\gamma$

of $Y$ induces a meridian system of $L$ in $X$, Lemma 3.1 implies that the fundamental group $\pi_1(X \setminus L, v)$ is a meridian based free group. By Smooth Unknotting Conjecture for an $S^2$-link, the $S^2$-link $L$ is a trivial $S^2$-link in the 4-sphere $X$. By the back surgery replacing $D^2_* \times S^2$ in $X$ with $k(\gamma) \times D^3$ in $Y$, there is an orientation-preserving diffeomorphism $f : Y \to Y^S$ with $f(k_*) = k(\gamma^S_*)$. Since a regular neighborhood $N(f(\gamma))$ of $f(\gamma)$ in $Y^S$ is isotopic to $Y^D$ in $Y^S$, the diffeomorphism $f : Y \to Y^S$ is modified to have $f(\gamma) = \gamma^S$. Given any spin structures on $Y$ and $Y^S$, note that there is an orientation-preserving spin-structure-changing diffeomorphism : $S^1 \times S^3 \to S^1 \times S^3$ (see [4] for a similar diffeomorphism on $S^1 \times S^2$). Thus, by composing $f$ with the orientation-preserving spin-structure-changing diffeomorphisms on some connected summands of $Y^S$ which are copies of $S^1 \times S^3$, the diffeomorphism $f : Y \to Y'$ is modified into an orientation-preserving spin-structure-preserving diffeomorphism. This completes the proof of Lemma 3.2. □

The following corollary is directly obtained from Lemmas 2.3, 3.1 and 3.2.

**Corollary 3.3.** Let $\gamma\gamma'$ be a legged Heegaard loop system of a homotopy 3-sphere $M$ associated with a Heegaard.splitting $V \cup V'$ of genus $n$, and $Y(M; \gamma\gamma')$ the 4D closed homology handlebody obtained from the 4-sphere $X(M)$ by surgery along the spun $S^2$-link $L(\gamma\gamma')$ of $\gamma\gamma'$. Then the 4D closed homology handlebody $Y(M; \gamma\gamma')$ is diffeomorphic to the 4D closed handlebody $Y^S$ of genus $2n$.

A surface-link $L$ in $S^4$ is a *ribbon* surface-link if $L$ is equivalent to a surface-link obtained from a trivial $S^2$-link $L^S$ in $S^4$ by surgery along embedded 1-handles on $L^S$ (see [18]). The following lemma is obtained.

**Lemma 3.4.** Any $S^2$-link $L$ in $S^4$ with free fundamental group $\pi_1(S^4 \setminus L, v)$ is a ribbon $S^2$-link.

**Proof of Lemma 3.4.** Let $K_i\,(i = 1, 2, \ldots, n)$ be the components of $L$. Let $Y$ be the 4-manifold obtained from $S^4$ by surgery along $L$. Let $\gamma$ be a legged loop system in $Y$ induced from a meridian system $\gamma_L$ of $L$ in $S^4$. Let $k(\gamma) = k_*$ be the loop system of $\gamma$ in $Y$. The surgery manifold $X$ of $Y$ along $k_*$ is identified with the 4-sphere $S^4$. In precise, let $X = \mathrm{cl}(Y \setminus N(k_*)) \cup D_* \times S^2$ for a regular neighborhood $N(k_*) = k_* \times D^3$ of $k_*$ in $Y$ and the disk system $D_*$ with $\partial D_* = k_*$, where the 2-sphere system $0_* \times S^2$ is identified with $L$. By Lemma 3.2, $Y$ is identified with the closed 4D handlebody $Y^S$ of genus $n$. Let $\gamma^S$ be a standard legged loop system of $Y = Y^S$ with the same vertex $v$ as $\gamma$. Let $k(\gamma^S) = k^S_*$ be the loop system of $\gamma^S$ in $Y$, which is disjoint from $k_*$. Let $x_i\,(i = 1, 2, \ldots, n)$ be a basis of the free group $\pi_1(Y, v)$ of rank $n$ represented by $\gamma^S$. Let $y_i\,(i=1,2,\dots,n)$ be an element system in $\pi_1(Y,v)$ represented by $\gamma$. By a basis change of the basis $x_i\,(i=1,2,\dots,n)$, assume that the product $x_i^{-1}y_i$ is in the commutator subgroup $[\pi_1(Y,v),\pi_1(Y,v)]$ of $\pi_1(Y,v)$ for every $i$. Let

$$Y^0 = \mathrm{cl}(Y \setminus N(k_*^S))$$

for a regular neighborhood $N(k_*^S) = k_*^S \times D^3$ of $k_*^S$ in $Y$. Also, let

$$X^0 = \mathrm{cl}(X \setminus N(k_*^S))$$

by considering $N(k_*^S)$ in $X$. Since the loop system $k_*^S$ is a trivial loop system in the 4-sphere $X$, there is a disjoint disk system $\Omega_*$ with $\partial\Omega_* = k_*^S$ smoothly embedded in $X$. Note that the intersection $N(k_*^S) \cap \Omega_*$ is a boundary collar of $\Omega_*$. Let

$$\Omega'_* = \mathrm{cl}(\Omega_* \setminus (N(k_*^S) \cap \Omega_*)$$

which is a proper disk system in $X^0$. Let $S^1 \times S_i^3 = k_i^S \times S^3\,(i=1,2,\dots,n)$ be the connected summands of the closed 4D handlebody $Y = Y^S$. For every $i$, let $S_i^3 = p_i \times S_i^3$ for a point $p_i \in k_i^S$. Let $V_i = S_i^3 \cap Y^0$ be a 3-ball obtained from $S_i^3$ by removing the interior of a 3-ball neighborhood of the point $p_i = p_i \times \mathbf{1}$ with $\partial V_i \subset \partial Y^0$. Let

$$Y^+ = Y^0 \cup_{i=1}^n \widetilde{\Omega}_i \times d$$

be the 4-manifold obtained from $Y^0$ by attaching 2-handles $\widetilde{\Omega}_i \times d\,(i=1,2,\dots,n)$ to the boundary $\partial Y^0 = \cup_{i=1}^n k_i^S \times S^2$ of $Y^0$ where $\widetilde{\Omega}_i$ is a disk with $\partial\widetilde{\Omega}_i = \partial\Omega'_i$ and a disk $d$ in the 2-sphere $S^2$. Similarly, let

$$X^+ = X^0 \cup_{i=1}^n \widetilde{\Omega}_i \times d$$

be the 4-manifold obtained from $X^0$ by attaching 2-handles $\widetilde{\Omega}_i \times d\,(i=1,2,\dots,n)$ to the boundary $\partial X^0$ identical to $\partial Y^0$. Let $(k_*^{S+}, p_*^+)$ be a moving of the pair $(k_*^S, p_*)$ into the boundary pair $(\partial Y^0, \partial V_*)$. Let $k_i^{S+} \times [0,1]$ be an annulus in $k_i^{S+} \times S^2 \subset \partial Y^0$ for an arc $[0,1]$ in $S^2$. Consider that the element $x_i^{-1}$ is represented by the loop $k_i^{S+} \times 0$ in $Y^0$. Since $y_i$ is a word of the letters $x_j\,(j=1,2,\dots,n)$ in the fundamental group $\pi_1(Y,v)$, the element $y_i$ is represented in $Y^0$ by a band sum $k_i$ of the loop $k_i^{S+} \times 1$ and the boundary loop system $\partial P_i$ of a disk system $P_i$ consisting of suitably oriented parallel disks of $\widetilde{\Omega}_j$ in $\widetilde{\Omega}_j \times d\,(j=1,2,\dots,n)$ along a band system $\mu_i$. Let $b_i$ be a band in the anulus $k_i^{S+} \times [0,1]$ spanning the loop $k_i^{S+}$ and the loop $k_i$ with the centerline $\dot{b}_i = p_i^+ \times [0,1]$. Let $k'_i$ be the loop in $Y^0$ obtained by a band sum of $k_i^{S+} \times 0$ and $k_i$ along the band $b_i$. The union

$$\Delta_i = \mathrm{cl}(k_i^{S+} \times [0,1] \setminus b_i) \cup_{i=1}^n P_i \cup \mu_i$$

is considered as a disk smoothly embedded in $Y^+$ whose boundary loop $\partial\Delta_i$ represents the element $x_i^{-1}y_i$ in $Y^0$. Further, the disk system $\Delta_i\,(i=1,2,\dots,n)$ is made disjoint. By construction, the disk $\Delta_i$ meets the 3-ball system $V_*$ only with the isolated finite point set $P_i\cap\partial V_*$ and with simple proper arcs $\beta_{i,j}\,(j=1,2,\dots,n_i)$ in $\Delta_i$ coming from the transverse intersection of the band system $\mu_i$ and the interior $\mathrm{Int}V_*$ of the 3-ball system $V_*$. Let $B_{i,j}\,(j=1,2,\dots,n_i)$ be disjoint 3-ball neighborhoods of the arcs $\beta_{i,j}\,(j=1,2,\dots,n_i)$ in $\mathrm{Int}V_i$, and $S_{i,j}\,(j=1,2,\dots,n_i)$ the boundary 2-spheres of $B_{i,j}\,(j=1,2,\dots,n_i)$. Then the following claim (#) is obtained.

**(#)** The $S^2$-link $\cup_{i=1}^n\cup_{j=1}^{n_i}S_{i,j}$ in $Y$ becomes a trivial $S^2$-link in the 4-sphere $X$ after the surgery of $Y$ along the loop system $k_*$.

By assuming the proof of the claim (#), the proof of Lemma 3.4 is completed as follows. Let $(S^3)_i^{(*)}$ be a multi-punctured 3-ball obtained from $S_i^3$ by removing the interiors of the 3-balls $B_{i,j}\,(j=1,2,\dots,n_i)$ and a 3-ball neighborhood $N(q_i)=q_i\times D^3$ of the point $q_i=p_i^+\times 1\in k_i$ in $V_i$. Note that the $S^2$-link $\cup_{i=1}^n\partial N(q_i)$ in $Y$ changes into the $S^2$-link $L=\cup_{i=1}^n K_i$ in $X$ after the surgery of $Y$ along $k_*$. Since $K_i$ is equivalent to a 2-sphere in $(S^3)_i^{(*)}$ obtained from the trivial $S^2$-link $\partial V_i\cup_{i=1}^n\cup_{j=1}^{n_i}S_{i,j}$ in $X$ by surgery along disjoint embedded 1-handles in $(S^3)_i^{(*)}$, it is shown that the $S^2$-link $L$ is a ribbon $S^2$-link in the 4-sphere $X$. This completes the proof of Lemma 3.4 assuming the claim (#).

**Proof of (#).** Let $V'_*$ be the 3-ball system obtained from the 3-ball system $V_*$ by removing an open boundary collar which remains containing all the arcs $\beta_{i,j}$, so that $V'_*\cap\widetilde{\Omega}_j=\emptyset$. Since every arc $\beta_{i,j}$ splits the disk $\Delta_h$ containing the arc $\beta_{i,j}$ into two regions, there is an arc $\beta_{i',j'}$ such that a region $\Delta'_h$ of the disk $\Delta_h$ splitted by the $\beta_{i',j'}$ does not contain any other arc $\beta_{i'',j''}$ and does not meet the arc system $b_*\cap k_*$. The boundary of a regular neighborhood relative to $V'_*$ of the region $\Delta'_h$ in $Y^+$ is a 3-sphere containing the 3-ball $B_{i',j'}$ whose complementary 3-ball is denoted by $\widetilde{B}_{i',j'}$. Let $V''_*$ be the 3-ball system obtained from $V'_*$ by replacing the 3-ball $B_{i',j'}$ with the 3-ball $\widetilde{B}_{i',j'}$. Then $V''_*\cap\Delta'_h=\emptyset$. Continue this process on $V''_*$ instead of $V'_*$. Finally, a system of disjoint 3-balls $\widetilde{B}_{i,j}\,(i=1,2,\dots,n;j=1,2,\dots,n_i)$ bounded by the 2-spheres $S_{i,j}\,(i=1,2,\dots,n;j=1,2,\dots,n_i)$ and a 3-ball system $V'''_*$ disjoint from the union $\Delta_*\cup b_*$ are obtained in $Y^+$. Consider that $X^+$ is obtained from $Y^+$ by a surgery along a loop system $k_*^+$ disjointedly parallel to the loop system $k_*$ in $Y^+$ so that $k_*^+$ is in the interior $\mathrm{Int}(Y^0)$ of $Y^0$ and disjoint from the disk system $\Delta_*$. The disk system $\Delta_*$ is now embedded into $X^+$ and the 3-ball $\widetilde{B}_{i,j}$ for any $i,j$ is embedded into a regular neighborhood of $\Delta_*$ in the 4-manifold $\mathrm{cl}(Y^+\setminus N(k_*^+))=\mathrm{cl}(X^+\setminus N(L))$. Since the band system $\mu_i$ except for the attaching part is made disjoint from the disk system $\Omega'_*$, the loop system $k^+_*$ is made disjoint from the disk system $\Omega'_*$. For a normal disk bundle $\Omega'_* \times d$ of $\Omega'_*$ in $\mathrm{cl}(Y^0 \setminus N(k^+_*)) = \mathrm{cl}(X^0 \setminus N(L))$, the union $U = \Omega'_* \times d \cup \widetilde{\Omega}_* \times d = (\Omega'_* \cup \widetilde{\Omega}_*) \times d$ in $\mathrm{cl}(Y^+ \setminus N(k^+_*)) = \mathrm{cl}(X^+ \setminus N(L))$ is diffeomorphic to the product $S^2 \times d$ and the intersection $U \cap \Delta_*$ coincides with the disk system $P_*$. By an isotopy of $X^+$ keeping $U$ setwise fixed and keeping the outside of a neighborhood of $U$ in $X^+$ fixed, the disk system $P_*$ is deformed into a disk system $P^X_*$ in $\Omega'_* \times d \subset X^0$, so that the disk system $\Delta_*$ is deformed into a disk system $\Delta^X_*$ in $\Omega'_* \times d \subset X^0$. Since the 3-ball $\widetilde{B}_{i,j}$ for any $i, j$ is embedded in a regular neighborhood of $\Delta_*$ in the 4-manifold $X^+$, the 3-ball system $\widetilde{B}_{i,j}$ is isotopically deformed into a 3-ball system$\widetilde{B}^X_{i,j}$ in $X^0$ while the 2-spheres $S_{i,j}\,(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n_i)$ are fixed. This means that the 2-spheres $S_{i,j}\,(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n_i)$ are a trivial $S^2$-link in the surgery manifold $X$. This completes the proof of (#). □

This completes the proof of Lemma 3.4. □

A group presentation $(y_1, y_2, \ldots, y_{n+s}| \, r_1, r_2, \ldots, r_s)$ of deficiency $n$ is a *Wirtinger presentation* if every relator $r_i$ is written as a form $y^{-1}_{j_i} w_j y_{j'_i} w^{-1}_i$ for two generators $y_j j_i, y_{j'_i}$ with distinct indexes $j_i, j'_i$ and a word $w_i$ in the letters $y_j\,(j = 1, 2, \ldots, n+s)$. It is known that the fundamental group of an $n$-component ribbon $S^2$-link has a Wirtinger presentation of deficiency $n$ for some $s$ (cf. [7, p. 193], [18, pp. 56-60]). An algebraic version of Lemma 3.4 means the following result in combinatorial group theory.

**Corollary 3.5.** Let $\mathbf{F}_n$ be the free group of rank $n$ with a basis $x_i\,(i = 1, 2, \ldots, n)$. Let $x'_i\,(i = 1, 2, \ldots, n)$ be a set of elements normally generating the free group $\mathbf{F}_n$ written as words in the letters $x_i\,(i = 1, 2, \ldots, n)$ such that the products $x'_i x^{-1}_i\,(i = 1, 2, \ldots, n)$ belong to the commutator subgroup $[\mathbf{F}_n, \mathbf{F}_n]$ of $\mathbf{F}_n$. Then the free group $\mathbf{F}_n$ admits a Wirtinger presentation

$$(y_1, y_2, \ldots, y_{n+s}| \, r_1, r_2, \ldots, r_s)$$

of deficiency $n$ for some $s$ such that the elements $y_i\,(i = 1, 2, \ldots, n + s)$ are written as words in the letters $x_i\,(i = 1, 2, \ldots, n)$ containing the elements $x'_i\,(i = 1, 2, \ldots, n)$ as the given words.

## 4. Main result: Proof of Lemma 1.2

The following observation relates a knot to a Heegaard splitting of a closed connected orientable 3-manifold.

**Lemma 4.1.** For any knot $k$ in any closed connected orientable 3-manifold $M$, there is a Heegaard splitting $V \cup V'$ of $M$ such that the knot $k$ is equivalent to a component of the loop system $k(\gamma)$ of a spine $\gamma$ of $V$ in $M$.

**Proof of Lemma 4.1.** By considering $k$ as a polygonal loop in $M$, there is a triangulation $\mathcal{T}$ of $M$ whose 1-skeleton $\mathcal{T}^{(1)}$ contains the knot $k$. The graph $\mathcal{T}^{(1)}$ is deformed into a legged loop system $\gamma$ in $M$ so that $k$ is a component of the loop system $k(\gamma)$. Let $V$ be a regular neighborhood of $\gamma$ in $M$ which is a handlebody. The closed complement $V' = \mathrm{cl}(M \setminus V)$ is also a handlebody, so that we have a Heegaard splitting $V \cup V'$ of $M$. The legged loop system $\gamma$ is deformed into a spine of the handlebody $V$. □

By combining Lemmas 2.3, 3.4 with Lemma 4.1, the following corollary is obtained, because any component of a ribbon $S^2$-link in $S^4$ is a ribbon $S^2$-knot in $S^4$.

**Corollary 4.2.** For any knot $k$ in any homotopy 3-sphere $M$, the spun-$S^2$-knot $S(k)$ of $k$ in $X(M) = S^4$ is a ribbon $S^2$-knot in $S^4$.

A chord diagram is a diagram $C$ in $S^2$ consisting of a based loop system $o$ (i.e., a trivial oriented link diagram ) and a chord system $\alpha$ joining the based loops where intersections among the chords are permitted (see [8, 9, 10, 11, 12] for the detailed arguments). For a disk $\delta$ in $S^2$, a *chord diagram* in the delta $\delta$ is the intersection $C \cap \delta$ for a chord diagram $C = C(o, \alpha)$ in $S^2$ such that the circle $\partial\delta$ does not meet the based loop system $o$ and meets the chord system $\alpha$ transversely. From a chord diagram $C = C(o, \alpha)$ in $S^2$, a ribbon surface-link $R(C)$ in the 4-sphere $S^4$ is constructed in a unique way. In fact, the ribbon surface-link $R(C)$ is obtained from a trivial oriented $S^2$-link $L^0$ in $S^4$ constructed from the based loop system $o$ by surgery along an embedded 1-handle system $h(\alpha)$ on $L^0$ thickening the chord system $\alpha$. The ribbon surface-link $R(C)$ in $S^4$ is uniquely constructed from the chord diagram $C$ by using the Horibe-Yanagawa's lemma in [18] for uniqueness of the trivial $S^2$-link $L^0$ constructed from the based loop system $o$ and an argument in [6] for uniqueness of the embedded 1-handle system $h(\alpha)$ constructed from the chord system $\alpha$.

**Lemma 4.3.** Let $a_*$ be a proper oriented arc system in a compact once-punctured manifold $M^{(o)} = \mathrm{cl}(M \setminus B)$ of a homotopy 3-sphere $M$ which is obtained from an oriented proper arc diagram $D$ in a disk $\delta$ contained in the boundary 2-sphere $S$ of $M^{(o)}$ by pushing the interior of an upper-arc around every crossing point of $D$ into the interior of $M^{(o)}$. Then the $S^2$-link $S(a_*)$ in $X(M)$ is a ribbon $S^2$-link in $X(M)$ with a chord diagram $C$ in $\delta$ obtained from the arc diagram $D$ by changing every crossing point as in Fig. 2.

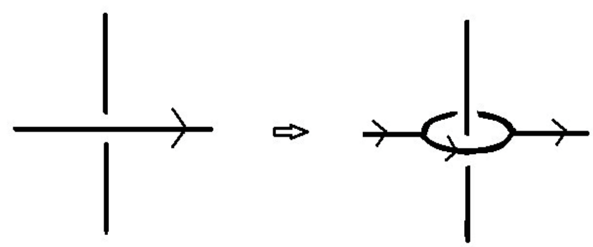

Figure 2: Changing a crossing point into a based loop with chords

**Proof of Lemma 4.3.** This fact is observed in [12, Theorem 2.3 (3)] for an inbound arc diagram whose closure is a knot chord diagram. The present claim is similarly shown for any oriented arc diagram. □

In Lemma 4.3, note that the arc diagram $D$ is recovered from the chord diagram $C$ by taking the upper-arc of every based loop. The proof of Lemma 1.2 is given as follows.

**4.4: Proof of Lemma 1.2.** Let $k$ be a non-trivial knot in a homotopy 3-sphere $M$. By Corollary 4.2, the spun $S^2$-knot $S(k)$ in the 4-sphere $X(M) = S^4$ is a ribbon $S^2$-knot. The *spun torus-knot* of $k$ in the 4-sphere $X(M)$ is given by the inclusion

$$T(k) = k \times S^1 \subset M^{(o)} \times S^1 \subset M^{(o)} \times S^1 \cup S \times D^2 = X(M).$$

The spun $S^2$-knot $S(k)$ in $X(M)$ is obtained from $T(k)$ by a 2-handle surgery and conversely the spun torus-knot $T(k)$ is obtained from the spun $S^2$-knot $S(k)$ by 1-handle surgery. By definition, the spun torus-knot $T(k)$ is a ribbon torus-knot and hence bounds a ribbon solid torus $V_R$ in $X(M)$. Let

$$V_R = \cup_{i=1}^{n} B_i \cup h_i$$

for a disjoint 3-ball system $B_i\,(i = 1, 2, \ldots, n)$ in $X(M)$ and an embedded disjoint 1-handle system $h_i\,(i = 1, 2, \ldots, n)$ on the 2-sphere system $\partial B_i\,(i = 1, 2, \ldots, n)$ in $X(M)$ so that the 1-handle $h_i$ spans $\partial B_i$ and $\partial B_{i+1}$ for every $i$ with $B_{n+1} = B_1$ and every 3-ball $B_i$ meets just one 1-handle $h_{j_i}$ for some $j_i\,(1 \leq j_i \leq n)$ with a transverse disk $d_{j_i}$ in the interior of $B_i$. Since the knot $k$ is non-trivial in $M^{(o)}$ and there is a canonical isomorphism

$$\pi_1(M^{(o)} \setminus k, v) \to \pi_1(X(M) \setminus T(k), v)$$

by the van Kampen theorem, the longitude of $k$ in $M^{(o)}$ represents an infinite order element in the fundamental group $\pi_1(X(M)\setminus T(k), v)$, which implies that the meridian loop of $V_R$ (i.e., the simple loop of $T(k)$ bounding a meridian disk of $V_R$) is a uniquely specified loop in $T(k)$ up to isotopies of $T(k)$. Fix an orientation of knot $k$. Then by the construction of $T(k)$, the meridian disk orientation of the ribbon solid torus $V_R$ is uniquely specified and the ribbon solid torus $V_R$ specifies uniquely a disjoint oriented deformed meridian disk system $d_i\,(i = 1, 2, \ldots, n)$ in $V_R$ so that the knot $k$ meets the disk $d_i$ with just one boundary arc orientation-coherently and just one interior point transversely and the union $k \cup_{i=1}^n d_i$ (called a *chord-disk system*) recovers $V_R$ uniquely by thickening $k$ and $d_i\,(i = 1, 2, \ldots, n)$ (see the left figure of Fig. 3). The disk system $d_i\,(i = 1, 2, \ldots, n)$ is isotopically deformed into $M^{(o)}$ by an isotopy of $X(M)$ keeping $k$ fixed, so that the chord-disk system $k \cup_{i=1}^n d_i$ is in $M^{(o)}$. To show this claim, let $\alpha_i$ be a simple arc in $d_i$ joining the point $k \cap \mathrm{Int} d_i$ with a point in the arc $k \cap \partial d_i$ for all $i$. The arc system $\alpha_i\,(i = 1, 2, \ldots, n)$ is deformed into a bi-collar neighborhood $M^{o)} \times [-1, 1]$ of $M^{(o)}$ with $M^{(o)} \times 0 = M^{(o)}$ in $X(M)$ by an isotopy keeping $M^{(o)}$ fixed. Then the arc system $\alpha_i\,(i = 1, 2, \ldots, n)$ is projected into $M^{(o)}$ by a general position argument. A deformed disk system $d_i\,(i = 1, 2, \ldots, n)$ in $M^{(o)}$ is obtained from the arc system $\alpha_i\,(i = 1, 2, \ldots, n)$ in $M^{(o)}$ by extending them as a small disk system, completing the proof of the claim. Let $k^\times$ be the graph in $M^{(o)}$ obtained from the chord-disk system $k \cup_{i=1}^n d_i$ by shrinking every disk $d_i$ into a 4-degree vertex for every $i$. By taking a maximal tree $\tau(k^\times)$ of $k^\times$, one finds a disk $\delta$ in $M^{(o)}$ containing the maximal tree $\tau(k^\times)$. Let $e_i\,(i = 1, 2, \ldots, n+1)$ be the arc system $\mathrm{cl}(k^\times \setminus \tau(k^\times))$ where the number $n + 1$ is uniquely determined by the Euler characteristic $\chi(K^\times) = -n$. Then the chord-disk system

$$k^{\times\times} = \mathrm{cl}((k \cup_{i=1}^n d_i) \setminus (\cup_{i=1}^{n+1} e_i))$$

can be drawn as a chord diagram $C$ in the disk $\delta$ with the based loop system $o_i = \partial d_i\,(i = 1, 2, \ldots, n)$ so that the chord diagram of the two arcs of $k$ on the disk $d_i$ for every $i$ are drawn with the two arcs as bold lines transversely meeting as in the right figure of Fig. 3. Let $a_i\,(i = 1, 2, \ldots, n+1)$ be the arc system $\mathrm{cl}(k \setminus \cup_{i=1}^{n+1} e_i)$. By replacing the chord diagram of the two arcs of $k$ on the disk $d_i$ for every $i$ with an arc diagram, that is, by replacing the right diagram of Fig. 2 with the left diagram of Fig. 2, the diagram $C$ changes into an arc diagram $D$ of the arc system $a_i\,(i = 1, 2, \ldots, n)$ in the disk $\delta$. Deform the disk $\delta$ into the 2-sphere $S = \partial M^{(o)}$ so that a collar $\delta \times [0, 1]$ of $\delta$ in $M^{(o)}$ with $\delta \times 0 = \delta$ belongs to a boundary collar $S \times [0, 1]$ of $S$ in $M^{(o)}$ with $S \times 0 = S$. The arc system $a_i\,(i = 1, 2, \ldots, n)$ is realized in the collar $\delta \times [0, 1]$

from the arc diagram $D$ by pushing the interiors of the upper-arcs of $D$ into the interior of $\delta \times [0,1]$. By Lemma 4.3, the spun $S^2$-link $\cup_{i=1}^n S(a_i)$ in $X(M)$ with the chord system $C$ in $\delta$ is obtained as in Fig. 2. This means that the spun $S^2$-link $\cup_{i=1}^n S(a_i)$ bounds a part $V'_R$ of the ribbon solid torus $V_R$ belonging to the 4-ball $A = (\delta \times [0,1]) \times S^1 \cup \delta \times D^2$ in $X(M)$. Since the spun torus-knot $T(k)$ is the union of the spun $S^2$-link $\cup_{i=1}^n S(e_i)$ and the spun $S^2$-link $\cup_{i=1}^n S(a_i)$ by deleting the common disk interiors, the spun $S^2$-link $\cup_{i=1}^n S(e_i)$ in $X(M)$ bounds disjoint 3-balls $\mathrm{cl}(V_R \setminus V'_R)$ in the 4-ball $A' = \mathrm{cl}(X(M) \setminus A)$. Let $X'(M)$ be the spun 4-sphere of $M$ on the once-punctured manifold $M_\delta^{(o)} = \mathrm{cl}(M^{(o)} \setminus \delta \times [0,1])$ of $M$, and $S' = \partial M_\delta^{(o)}$ the boundary 2-sphere. The spun $S^2$-link $\cup_{i=1}^n S(e_i)$ is a trivial $S^2$-link in the 4-sphere $X'(M)$. By Lemma 2.5, the proper arc system $e_i\,(i = 1, 2, \dots, n)$ is in a boundary-collar $S' \times [0,1]$ of the once-punctured manifold $M_\delta^{(o)}$. This means that there is a 3-ball in $M^{(o)}$ containing the knot $k$. This completes the proof of Lemma 1.2. □

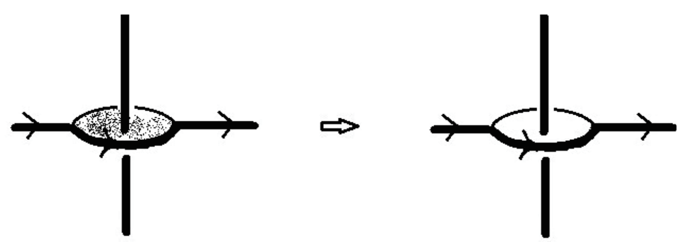

Figure 3: A diagram of the two arcs of $k$ on the disk $d_i$

This completes the proof of Theorem 1.1.

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