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Submanifolds of Symmetric Spaces and Their Time Evolutions

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ABSTRACT. This workshop held March 5-6, 2021 to conduct international research exchanges on "Submanifolds of symmetric spaces and their time evolutions".

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Preface

This volume of OCAMI Reports summarizes the workshop "Submanifolds of Symmetric Spaces and Their Time Evolutions" held from March 5th to March 6th in 2021 online by Zoom because of the COVID-19 pandemic. This workshop was supported by "Osaka city University, Advanced Mathematical Institute MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics." The main focus of this workshop is submanifolds of symmetric spaces (or more general ambient spaces), mean curvature flows in symmetric spaces (or more general ambient spaces), and related geometric flows (line bundle mean curvature flows , which is flows of connections of complex line bundle over Kähler manifold, coupling flows of Ricci flows and heat flows, and so on). This workshop consisted of two 60 minutes keynote lectures on "mean curvature flow for isoparametric submanifolds and polar foliations on symmetric spaces" by Professor Xiaobo Liu (Peking University), two 60 minutes keynote lectures on "deformed Hermitian Yang-Mills connections and line bundle mean curvature flows" by Doctor Hikaru Yamamoto (University of Tsukuba), and seven 50 minutes lectures on "submanifolds of symmetric spaces, submanifolds of generalized s-manifolds, proper Fredholm submanifolds of Hilbert spaces, Lagrangian mean curvature flows, and coupling flows of Ricci flows and heat flows." There were 38 participants in this workshop. This workshop conducted international research exchanges on submanifolds of symmetric spaces, their time evolutions and furthermore related geometric flows.

March 2021

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Table of Contents

Ancient solutions for mean curvature flow of isoparametric submanifolds

Xiaobo Liu

Ancient solutions are important in studying singularities of mean curvature flows (MCF). So far most rigidity results about ancient solutions are modeled on shrinking spheres or spherical caps. In this talk, I will describe the behavior of MCF for a class of submanifolds, called isoparametric submanifolds, which have more complicated topological type. We can show that all such solutions are in fact ancient solutions, i.e. they exist for all time which goes to negative infinity. I will also describe our conjectures proposed together with Terng on rigidity of ancient solutions to MCF for hypersurfaces in spheres. These conjectures are closely related to Chern's conjecture for minimal hypersurfaces in spheres. This talk is based on joint works with Chuu-Lian Terng.

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Ancient solutions for Mean Curvature Flow of Isoparametric Submanifolds

Xiaobo Liu

Peking University

Talk at Workshop on
"Submanifolds of Symmetric Spaces and Their Time Evolution"
March 5-6, 2021

Mean curvature flow

Let M be a submanifold in a Riemannian manifold X. The **mean curvature flow** (abbreviated as **MCF**) of M is a map $f : I \times M \longrightarrow X$ satisfying

$$\frac{\partial f}{\partial t} = H(t, \cdot)$$

where $H(t, \cdot)$ is the mean curvature vector field of $f(t, \cdot)$ and $f(0, \cdot)$ is the immersion of M in X.

If a solution exists on $I = (-\infty, T]$ for some $T > 0$, it is called an **ancient solution**. Such solutions are important in studying singularities of general MCF.

Examples:

- If $f_0 : M \longrightarrow X$ is minimal, then

$$f(t,x) = f_0(x), \qquad x \in M$$

is an ancient solution to MCF. This is the **stationary solution**.

- If $M^n \subset S^{n+1}$ is a subsphere, then the MCF of M is an ancient solution, which shrinks to a point in finite positive time and converges to the equator as $t \to -\infty$. This solution is called the **shrinking spherical cap**.

- If

$$f_0 : M \longrightarrow S^{n+1} \subset \mathbb{R}^{n+2}$$

is a minimal submanifold of the unit sphere S^{n+1} of any codimension (in particular M could be S^{n+1}), then

$$f(t,x) = \sqrt{1-2nt}\, f_0(x), \qquad x \in M$$

is an ancient solution to MCF for M as a submanifold in $\mathbb{R}^{n+2}$.

$M \subset \mathbb{R}^N$. νM: normal bundle of M.

Definition(Terng): M is **isoparametric** if

(1) νM is globally flat, i.e. parallel translations of normal vectors along closed curves are identity.

(2) For any parallel normal vector field η, principal curvatures along η are constant.

An **isoparametric polynomial** is a homogeneous polynomial F on $\mathbb{R}^{n+2}$ such that ΔF and $||\nabla F||$ are constant along level sets of F.

Isoparametric hypersurfaces

- Let F be an isoparametric polynomial which is normalized such that the range of $F\ |_{S^{n+1}}$ is $[-1, 1]$. Then for any $t \in (-1, 1)$, $F^{-1}(t) \cap S^{n+1}$ is an isoparametric submanifold. These are **isoparametric hypersurfaces in sphere**.

  $M_\pm := F^{-1}(\pm 1) \cap S^{n+1}$ are not isoparametric, they are focal submanifolds of isoparametric hypersurfaces.

  Let g be the number of distinct principal curvatures of an isoparametric hypersurface in sphere. Then $g = 1, 2, 3, 4, 6$ (Münzner).

- **Clifford examples**(Ferus-Karcher-Münzner): Assume that $E_1, \ldots, E_{m-1}$ are skew symmetric $l \times l$ matrices such that

$$E_i E_j + E_j E_i = -2\delta_{ij} Id$$

(i.e. these matrices give a representation of the Clifford algebra.)

Then we can construct a homogeneous polynomial of degree 4 on $\mathbb{R}^{2l}$ which is isoparametric.

The corresponding isoparametric hypersurfaces have 4 distinct principal curvatures. Most of them are non-homogeneous.

- **Homogeneous isoparametric submanifolds**: Let G/K be a symmetric space,
$\mathfrak{g}$ the Lie algebra of G,
$\mathfrak{k}$ the Lie algebra of K,
$\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$.
The isotropy representation of K acts on $\mathfrak{p}$. Principal orbits of this representation are isoparametric submanifolds. All these submanifolds are homogeneous.

- Reducible cases: Products of isoparametric submanifolds are isoparametric.

Properties

- If $M \subset \mathbb{R}^N$ is complete isoparametric, then $M = M_1 \times \mathbb{R}^k$ with M_1 compact isoparametric.
- If $M \subset \mathbb{R}^N$ is compact isoparametric, then M is contained in a round sphere. After translation and dilation, we may assume M is contained in the unit sphere centered at origin.
- For any parallel normal vector field η along $M \subset \mathbb{R}^N$, define

$$M_\eta := \{x + \eta(x) \mid x \in M\}.$$

  If M is isoparametric, then M_η is always a smooth submanifold.

Isoparametric foliation:

- If $\dim M_\eta = \dim M$, then M_η is also isoparametric. It is called a **parallel isoparametric submanifold** of M.
- If $\dim M_\eta < \dim M$, then M_η is no longer isoparametric. It is a **focal submanifold** of M.
- $\bigcup_\eta M_\eta$ gives a singular foliation of $\mathbb{R}^N$. If $M \subset S^{N-1}$, then the set of $M_\eta \subset S^{N-1}$ also gives a singular foliation of S^{N-1}. These foliations are called **isoparametric foliations**.

MCF of Isop. Submanifolds

Assume $M \subset S^{N-1} \subset \mathbb{R}^N$ compact isoparametric. MCF of M as a submanifold of $\mathbb{R}^N$ (resp. S^{N-1}) is called the **Euclidean** (resp. **spherical**) MCF of M.
Theorem(Liu-Terng, Duke 2009):

- MCFs of M preserve the isoparametric condition before collapsing.
- Euclidean MCF of M always converge to a focal submanifold in a finite time $T > 0$. Same is true for spherical MCF if M is not minimal in sphere.
- Every focal submanifold is a limit of the MCF of some isoparametric submanifold.

The above results were generalized:

for MCF of equifocal submanifolds of symmetric spaces by N. Koike (2011),

and for MCF of regular leaves of isoparametric foliations on compact non-negativelt curved manifolds by Alexandrino-Radeschi (2016).

Ancient Solutions

Theorem(Liu-Terng, Math. Ann. 2020):
In both Euclidean and spherical cases,

- MCF of M are ancient solutions, i.e. they exist for all $t \in (-\infty, 0]$.
- There is a unique minimal isoparametric submanifold $M_{\min}$ for each isoparametric foliation in S^{N-1}.
- As $t \to -\infty$, MCF of M converges to MCF of $M_{\min}$. (Note that the spherical MCF of $M_{\min}$ is stationary).

More precisely, $\exists$ a unit parallel normal vector field ζ on M in S^{N-1} such that the map $h : M \to S^{N-1}$ defined by

$$h(x) = (\cos r)x + (\sin r)\zeta(x)$$

is the embedding of $M_{\min}$ in S^{N-1}, where r is the spherical distance between M and $M_{\min}$.

Let $f(t, x)$, $F(t, x)$ be the spherical and Euclidean MCF of M. For all $x \in M$,

$$\lim_{t \to -\infty} ||F(t, x) - \sqrt{1 - 2nt}\ h(x)|| = 0,$$
$$\lim_{t \to -\infty} ||f(t, x) - h(x)|| = 0.$$

Hypersurface cases

Let $M^n \subset S^{n+1}$ be an isoparametric hypersurface with g distinct principal curvatures. For any $x_0 \in M$, let $S^1(x_0)$ be the geodesic in S^{n+1} which passes x_0 and is perpendicular to M.

Let $M_\pm$ be the focal submanifolds of M with $\dim M_+ \leq \dim M_-$. Let

$$\begin{aligned} m_1 &:= \dim M - \dim M_-, \\ m_2 &:= \dim M - \dim M_+. \end{aligned}$$

Then m_1 and m_2 are multiplicities of principal curvatures of M.
Note that $m_1 \leq m_2$.
If $g = 1, 3, 6$, then $m_1 = m_2$.

$M_+ \cup M_-$ intersects $S^1(x_0)$ in exactly $2g$ points, evenly distributed along the circle. Let $x_\pm$ be the intersection of $M_\pm$ with $S^1(x_0)$ which are closest to x_0. We may identify the normal space of M in $\mathbb{R}^{n+2}$ at x_0 with $\mathbb{C}$ such that $S^1(x_0)$ is the unit circle in $\mathbb{C}$ and $x_+ = 1$, $x_- = e^{i\pi/g}$, $x_0 = e^{i\theta_0}$ with $0 < \theta_0 < \pi/g$.

Every parallel isoparametric hypersurface intersect the arc $\{e^{i\theta} \mid 0 < \theta < \pi/g\}$ at exactly one point. We can use this point to represent the parallel isoparametric hypersurface.

Let $M_{\min}$ be the unique minimal isoparametric hypersurface which is parallel to M. Then $M_{\min}$ is represented by $e^{i\theta_{\min}}$ where

$$\cos g\theta_{\min} = -\delta$$

with

$$\delta := \frac{m_2 - m_1}{m_2 + m_1} \geq 0.$$

Theorem (Liu-Terng): Let M_t be the spherical mean curvature flow of M. Assume M_t is represented by $e^{i\theta(t)}$. Then $\theta(t)$ is given by

$$\cos g\theta(t) = e^{gnt}\left(\cos g\theta_0 + \delta\right) - \delta.$$

Note that $e^{i\theta_{\min}}$ divides the arc

$$\{e^{i\theta} \mid 0 < \theta < \pi/g\}$$

into two parts. $\theta(t) \to 0$ or π/g as t approaches some positive number. As $t \to -\infty$, $\theta(t) \to \theta_{\min}$.

Remark: For higher codimensional cases, explicit solutions of MCF can also be constructed recursively using Coxeter group structure of the isoparametric submanifolds. In general, it is more difficult to locate the position of minimal leaf of isoparametric foliation in higher codimension.

Rigidity Results

Theorem (Huisken and Sinestrari, JDG 2015):
Let M_t be the spherical MCF of a hypersurface in S^{n+1}, $A(t)$ and $H(t)$ be the shape operator and the mean curvature vector field of M_t as a submanifold of S^{n+1}. M_t is either a shrinking spherical cap or a stationary solution if one of the following conditions is satisfied for all $t < 0$:

- (HS1) For $n \geq 3$ and

$$||A(t)||^2 - \frac{1}{n-1}||H(t)||^2 \leq 2.$$

- (HS2) For some constant $B < 4n$,

$$||A(t)||^2 < e^{-Bt}||H(t)||^2.$$

Huisken and Sinestrari claimed that condition (HS1) is sharp. They justify this claim by considering the MCF of a product of an $(n-1)$-dimensional sphere and a circle in S^{n+1}. This is precisely the MCF of an isoparametric hypersurface with $g = 2$, $m_1 = 1$, $m_2 = n - 1$.

According to our calculations, for this example

$$||A(t)||^2 - \frac{1}{n-1}||H(t)||^2 - 2 = \frac{n-2}{n-1}\tan^2\theta(t)$$

where $e^{i\theta(t)}$ represents M_t. As $t \to -\infty$, $\theta(t) \to \theta_{\min}$, RHS $\to n-2$ which is not arbitrarily small. So the justification for the sharpness of (HS1) is not correct.

Lemma(Liu-Terng): For every $g = 1, 2, 3, 4, 6$, $\exists$ isoparametric hypersurface with g distinct principal curvatures such that its MCF satisfies

$$||A(t)||^2 < e^{-2gnt}||H(t)||^2$$

for all $t < 0$.

Corollary: Condition (HS2) is sharp in the sense that B can not be $\geq 4n$. (Otherwise $g = 2$ case of the above lemma would give counter examples.)

Rigidity conjectures

Conjecture A(Liu-Terng): If $\exists\, c_1, c_2, T > 0$ such that

$$c_1 e^{-2gnt} \leq \frac{||A(t)||^2}{||H(t)||^2} \leq c_2 e^{-2gnt}$$

for all $t < -T$, then the ancient solution of MCF M_t of compact hypersurfaces in sphere is the MCF of an isoparametric hypersurface with g distinct principal curvatures.

Remark: $g = 1$ case of this conjecture is true by Huisken and Sinestrari's result.

The above estimate holds for MCF of isoparametric hypersurfaces.

Conjecture B(Liu-Terng): If $\exists$ some constants $0 < \epsilon < 1$ and $T > 0$ such that either

$$(g-1)n \leq ||A(t)||^2 - \frac{1}{n}||H(t)||^2 \leq (g-1+\epsilon)n$$

or

$$(g-1-\epsilon)n \leq ||A(t)||^2 - \frac{1}{n}||H(t)||^2 \leq (g-1)n$$

for all $t < -T$, then the ancient solution of MCF M_t of compact hypersurfaces in sphere is the MCF of an isoparametric hypersurface with g distinct principal curvatures.

Remark: $g = 1$ case of Conjecture B is true by a result of Lei-Xu-Zhao (2019).

The above estimate holds for MCF of isoparametric hypersurfaces.

The two inequalities in Conjecture B can not be replaced by the following inequality:

$$(g-1-\epsilon)n \leq ||A(t)||^2 - \frac{1}{n}||H(t)||^2 \leq (g-1+\epsilon)n.$$

Otherwise Otsuki's construction would give a counter example (which are minimal of topological type $S^{n-1} \times S^1$).

Chern's Conjecture

Original **Chern's Conjecture**: $M^n \subset S^{n+1}$ compact minimal with constant $||A||$. Then the set of possible values of $||A||^2$ is discrete.

Remark: For minimal isoparametric hypersurface with g distinct principal curvatures, $||A||^2 = n(g-1)$.

Stronger version of Chern's Conjecture: $M^n \subset S^{n+1}$ compact minimal with constant $||A||$. Then M is isoparametric.

Stationary case of Conjecture B

Conjecture C: $M^n \subset S^{n+1}$ compact minimal. If $\exists$ some constants $0 < \epsilon < 1$ such that either

$$(g-1)n \leq ||A||^2 \leq (g-1+\epsilon)n$$

or

$$(g-1-\epsilon)n \leq ||A||^2 \leq (g-1)n,$$

then M is an isoparametric hypersurface with g distinct principal curvatures.

Remark: Conjecture C is stronger than Chern's conjecture.

Theorem (Chern-do Carmo-Kobayashi, 1970): $M^n \subset S^{n+1}$ compact minimal. If

$$0 \leq ||A||^2 \leq n,$$

then M is either an equator (isoparametric with $g = 1$) or a Clifford torus (isoparametric with $g = 2$).

This result implies that Conjecture C is true for $g = 1$ case and half of the $g = 2$ case.

Theorem: Assume $M^n \subset S^{n+1}$ compact minimal with

$$n \leq ||A||^2 \leq (1 + \epsilon)n.$$

M must be a Clifford torus if

- $\epsilon = \frac{1}{12}$ and $||A||$ is constant (Peng-Terng, 1983),
- $\epsilon = \frac{1}{3}$ and $||A||$ is constant (Cheng-Yang, 1998),
- $\epsilon = \frac{3}{7}$ and $||A||$ is constant (Suh-Yang, 2007), or
- $\epsilon = \frac{1}{23}$ (Ding-Xin, 2011), or
- $\epsilon = \frac{1}{22}$ (Xu-Xu, 2017).

These results give some partial answers for second half of the $g = 2$ case of Conjecture C.

Chern's conjecture for isoparametric hypersurface in S^4 was proved by S.P. Chang (1993).

The above results was proved using estimates obtained from elliptic equations for $\Delta \mathrm{II}$ and $\Delta(\nabla \mathrm{II})$. We hope the flow (parabolic) method may provide new insights to Chern's conjecture.

Idea for Proof: Euclidean Case

Assume $M \subset S^{N-1} \subset \mathbb{R}^N$ compact isoparametric. Fix $p \in M$. Let $V = \nu_p M$ be the normal space of M as a submanifold in $\mathbb{R}^N$ at p.

The intersection of V and the union of all focal submanifolds of M is a union of finitely many hyperplanes in V. Let W be the group generated by reflections along these hyperplanes.

W is a finite group, called the **Coxeter group** of M (Terng).

Let C be the interior of the fundamental domain of W acting on V which contains p. C is called the open **Weyl chamber** of W.

C is an open simplicial cone in V.

Every parallel isoparametric submanifold of M intersect C at exactly one point.

Every focal submanifold of M intersect ∂C at exactly one point.

MCF of M is reduced to a flow equation for points in $\overline{C} \subset V$.

W invariant polynomials on V give a new coordinate system on $\overline{C}$. In this coordinates, the Euclidean MCF becomes a flow equation along a polynomial vector field, and can be solved recursively in a rather explicit way. The solution exists as long as it does not hit ∂C.

Since focal submanifolds have lower dimensions, the volume function approaches 0 if the flow approaches ∂C.

As $t \to -\infty$, the volume function of MCF is increasing, hence never hits ∂C. Consequently MCF exists for all $t \in (-\infty, 0]$, i.e. the solution is ancient.

By a result of Palais-Terng, each isoparametric foliation in S^{N-1} contains at least one minimal isoparametric leaf.

To study the limit of MCF as $t \to -\infty$, let $x(t)$ be the Euclidean MCF of M and $\tilde{x}(t)$ the Euclidean MCF of a minimal isoparametric submanifold in S^{N-1} which is parallel to M. Define

$$D(t) := ||x(t) - \tilde{x}(t)||^2.$$

Lemma: $\exists$ constant $b > 0$ such that

$$D'(t) \geq \frac{b}{1-2nt} D(t).$$

Corollary: For all $t < 0$,

$$D(t) \leq D(0)(1-2nt)^{-\frac{b}{2n}}.$$

In particular, $\lim_{t \to -\infty} D(t) = 0$, i.e. MCF of M converges to the MCF of a minimal isoparametric submanifold.

Corollary: Each isoparametric foliation of S^{N-1} contains exactly one minimal isoparametric leaf.

Spherical Case

Let $f(t,x)$ and $F(t,x)$ denote the spherical and Euclidean MCF of M. Then

$$F(t,x) = \sqrt{1-2nt}\, f(-\frac{1}{2n}\ln(1-2nt), x).$$

We can use this formula to prove results about spherical MCF of M.

Geometric quantities

Let α_i be unit normal vectors of walls of Coxeter group (roots), m_i multiplicities of curvature distribution. For $x \in C$,

$$\begin{aligned} H^E(x) &= -\sum_i \frac{m_i \alpha_i}{\langle x, \alpha_i \rangle}, \\ H^S(x) &= -\sum_i \frac{m_i \alpha_i}{\langle x, \alpha_i \rangle} + \frac{nx}{\|x\|^2}, \\ \|A^E(x)\|^2 &= \sum_i \frac{m_i}{\langle x, \alpha_i \rangle^2}, \\ \|A^S(x)\|^2 &= \sum_i \frac{m_i}{\langle x, \alpha_i \rangle^2} - \frac{n}{\|x\|^2}. \end{aligned}$$

If $x(t) \in C$ represents a solution to MCF, then

$$\begin{aligned} \lim_{t \to -\infty} (1 - 2nt) \|H^E(x(t))\|^2 &= n^2, \\ \lim_{t \to -\infty} (1 - 2nt) \|A^E(x(t))\|^2 &= \sum_i \frac{m_i}{\langle x_{\min}, \alpha_i \rangle^2}. \end{aligned}$$

and

$$\begin{aligned} \lim_{t \to -\infty} \|H^S(y(t))\|^2 &= 0, \\ \lim_{t \to -\infty} \|A^S(y(t))\|^2 &= \sum_i \frac{m_i}{\langle x_{\min}, \alpha_i \rangle^2} - n. \end{aligned}$$

Thanks!

Polar foliations in symmetric spaces and their mean curvature flow

Xiaobo Liu

Polar foliations are natural generalizations of orbit foliations of polar actions. In this talk I will describe the relation between polar foliations and isoparametric submanifolds in simply connected symmetric spaces with non-negative curvature. It turns out principal orbits of such foliation are isoparametric submanifolds. If leaves are compact, such foliations must be products of isoparametric foliations in Euclidean spaces and polar foliations in compact symmetric spaces. For polar foliation in compact symmetric spaces, there is a unique regular leaf which is minimal. The mean curvature flow of all regular leaves have ancient solutions and always converge to the minimal regular leaf as time goes to negative infinity. This talk is based on joint works with Marco Radeschi.

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Polar Foliations on Symmetric Spaces and Mean Curvature Flow

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Peking University

Talk at Workshop on
"Submanifolds of Symmetric Spaces and Their Time Evolution"
March 5-6, 2021

Isoparametric Submanifolds

Let N be a Riemannian manifold, $M \subset N$ a submanifold.

Given a parallel normal vector field ξ defined over a small open subset $U \subset M$, let

$$U_\xi := \{\exp(\xi(p)) \mid p \in U\}.$$

If $\|\xi\|$ is small, U_ξ is a smooth submanifold of N. We call U_ξ a **locally parallel submanifold** of M.

Defination(Heintze-Liu-Olmos, 1997): A submanifold $M \subset N$ is **isoparametric** if

(1) Normal bundle νM is flat.

(2) For every $p \in M$, $\exp \nu_p M$ is totally geodesic in a neighbourhood of p (called a **local section**).

(3) Locally parallel submanifolds of M have parallel mean curvature vector fields.

Example: If N is a space form, this coincides with Terng's definition of isoparametric submanifolds.

Unlike the space form case, isoparametric submanifolds in general Riemannian manifold:

(1) may not have constant principal curvatures.

(2) may not produce a global singular foliation of N. If it does, such foliation is called an **isoparametric foliation**.

Example: If N is a compact symmetric space, $M \subset N$ is an **equifocal submanifold** if
(1) νM is flat and abelian (use Lie algebra structure).
(2) Focal distance and multiplicities along locally parallel normal vector fields are constant.

(This definition is due to Terng and Thorbergsson)

Fact: M is equifocal if and only if it is isoparametric with flat sections.

Example: Let G be a Lie group acting isometrically on a Riemannian manifold N. The **action is polar** if there exists a totally geodesic submanifold $\Sigma \subset N$ which intersects all orbits of G and intersects them orthogonally, and the dimension of Σ is complementary to the dimension of principal orbits. Σ is called a **section** of the polar foliation.

The **action is hyperpolar** if it is polar with flat section.

Fact: Principal orbits of polar actions are isoparametric.

Given an isoparametric foliation $(N, \mathcal{F})$, i.e. a singular foliation consists of an isoparametric submanifold L and all its parallel submanifolds. It turns out that all **regular leaves** (i.e. leaves with maximal dimension) of $\mathcal{F}$ are isoparametric submanifolds as well. Passing through every point p, there is a totally geodesic submanifold Σ_p (i.e. a section) which intersects all leaves orthogonally.

A vector $v \in TN$ is **horizontal** if it is tangent to a section. It is **vertical** if it is tangent to a leaf.

Mean curvature flow

Recall **mean curvature flow** (abbreviated as **MCF**) of M over an interval I is a map

$$f : I \times M \longrightarrow N$$

satisfying

$$\frac{\partial f}{\partial t} = H(t, \cdot)$$

where $H(t, \cdot)$ is the mean curvature vector field of $f(t, \cdot)$ and $f(0, \cdot)$ is the immersion of M in N.

If a solution exists on $I = (-\infty, T]$ for some $T > 0$, it is called an **ancient solution**. Such solutions are important in studying singularities of general MCF.

Theorem(Liu-Terng, Duke 2009): If L is an isoparametric submanifold in a Sphere or Euclidean space. Then

(1) MCFs of L flows through leaves of the isoparametric foliation associated to L.

(2) Euclidean MCF of L always converge to a singular leaf in a finite time $T > 0$. Same is true for spherical MCF if L is not minimal in sphere.

The previous result was generalized:

for MCF of equifocal submanifolds in symmetric spaces by N. Koike (2011), and

for MCF of regular leaves of isoparametric foliations on compact non-negatively curved manifolds by Alexandrino-Radeschi (2016).

Theorem(Liu-Terng, Math. Ann. 2020):
If L is an isoparametric submanifold in a Sphere or Euclidean space, then

(1) MCF of L always has ancient solution, i.e. it exists for all $t \in (-\infty, 0]$.

(2) There is a unique minimal isoparametric submanifold $L_{\min}$ for each isoparametric foliation in S^{N-1}.

(3) As $t \to -\infty$, MCF of L converges to MCF of $L_{\min}$. (Note that the spherical MCF of $M_{\min}$ is stationary)

Theorem A(Liu-Radeschi, 2020): Assume $\mathcal{F}$ is an isoparametric foliation on a Riemannian manifold M such that $\mathrm{Ric}_M(v) > \mathrm{Ric}_\Sigma(v)$ for any section Σ and v tangent to Σ, and the leaf space $M/\mathcal{F}$ is compact. Then

(1) There is a unique minimal regular leaf $L_{\min}$ in $\mathcal{F}$.

(2) For any regular leaf L in $\mathcal{F}$, the MCF of L always has ancient solution and it converges to $L_{\min}$ as t goes to $-\infty$.

Remark:This result generalizes the corresponding result of Liu-Terng in spheres.

Polar Foliation

Let N be a complete Riemannian manifold. A singular foliation $\mathcal{F}$ on N is **Riemannian** if every geodesic perpendicular to a leaf at one point must be perpendicular to all leaves which it intersects.

Moreover, $\mathcal{F}$ is **polar** if for $\forall p \in N$, $\exists$ a totally geodesic submanifold $\Sigma \ni p$ with dimension complementary to the dimension of regular leaves such that Σ intersects all leaves of $\mathcal{F}$ orthogonally. Σ is called a **section** of $\mathcal{F}$.

Remark: No restriction for mean curvature of leaves in $\mathcal{F}$.

A **hyperpolar foliation** is a polar foliation with flat sections.

Remark: In general a polar foliation may not be isoparametric.

Starting from an isoparametric submanifold, its parallel submanifolds may not form a foliation.

Example:
Orbits of polar actions form a polar foliation.

Orbits of hyperpolar actions form a hyperpolar foliation.

Isoparametric foliations are polar.

Parallel submanifolds of an equifocal submifold in compact simply connected symmetric spaces form a hyperpolar foliation.

Theorem B(Liu-Radeschi, 2020): Let $\mathcal{F}$ be a polar foliation on a simply connected symmetric space N with non-negative curvature. Then

(1) $\mathcal{F}$ is always isoparametric.

(2) $\mathcal{F}$ is a product of a polar foliation with compact **leaf space** $N/\mathcal{F}$ and an isoparametric foliation in a Euclean space.

(3) If $\mathcal{F}$ has compact leaves, then it is a product of a polar foliation in a compact symmetric space and an isoparametric foliation in a Euclean space.

Corollary (Liu-Radeschi, 2020):
Let N be a simply connected symmetric space with curvature ≥ 0 and $\mathcal{F}$ is a polar foliation on N without trivial or Euclidean factors. Then
(1) There is a unique regular leaf $L_{\min}$ in $\mathcal{F}$ which is minimal in N.
(2) MCF of every regular leaf of $\mathcal{F}$ has ancient solution and it converges to $L_{\min}$ as $t \mapsto -\infty$.

Remark: Together with Liu-Terng's result for isoparametric submanifolds in Euclidean spaces, this result completeluy describes behavior of MCF for polar foliation in nen-negatively curved symmetric spaces in negative time direction.

Idea for proof of Theorem A

Let L be a regular leaf in a polar foliation $\mathcal{F}$ on a Riemannian manifold N. Let $\gamma(t)$ be a geodesic with $\gamma(0) = p \in L$ and $\gamma'(0) \perp T_pL$. Let X be a parallel normal vector field along L with $X(p) = \gamma'(0)$.
Define **end-point map** $\phi_X : L \longrightarrow N$ by

$$\phi_X(q) := \exp_q X(q).$$

For every $v \in T_pL$, we call $J_v(t) := d_p\phi_{tX}(v)$ a **holonomy Jacobi vector field** along γ. Let L_t be the leaf of $\mathcal{F}$ passing $\gamma(t)$. Then $J_v(t) \in T_{\gamma(t)}L_t$ for all t.

Let $\Sigma \ni p$ be a section of $\mathcal{F}$. Then $\gamma(t) \in \Sigma$ for all t.

Let $\mathcal{V}_t := \nu_{\gamma(t)}\Sigma$. We have

$$\mathcal{V}_t \supseteq T_{\gamma(t)}L_t$$

for all t with "=" iff L_t is a regular leaf.

The curvature operator R on N defines a symmetric operator R_t on $\mathcal{V}_t$ by

$$R_t(w) := R(w, \gamma'(t))\gamma'(t)$$

for $w \in \mathcal{V}_t$.

There is a **Raccati operator** S_t on $\mathcal{V}_t$ such that $S_t(J(t)) = J'(t)$ for holonomy Jacobi vector field J at regular times. In fact $-S_t$ is the shape operator of L_t along $\gamma'(t)$ if L_t is a regular leaf. S_t satisfies the **Riccati equation**:

$$S_t' + S_t^2 + R_t = 0.$$

Riccati comparison theorem: Let $n = \dim \mathcal{V}_t$. If $\frac{1}{n}\mathrm{tr}(R_t) > \delta > 0$, then $\frac{1}{n}\mathrm{tr}(S_t)$ is bounded above by the solution of

$$s'(t) + s^2(t) + \delta = 0$$

with $s(0) = \frac{1}{n}\mathrm{tr}(S_0)$.

Given an orthonormal basis $\{e_i \mid i = 1, \cdots, n\}$ of T_pL where $p = \gamma(0)$. Let J_i be the holonomy Jacobi field with $J_i(0) = e_i$. Let ω and ω_t be the volume element of L and L_t respectively. Then at p,

$$\Phi_{tX}^* \omega_t = f_p(t) \cdot \omega$$

where

$$f_p(t) := \det(< J_i(t), E_j(t) >)_{1 \leq i,j \leq n}$$

with $E_j(t)$ the parallel extension of e_j along γ.

Fact:

$$\frac{d}{dt} \ln f_p(t) = \mathrm{tr}(S_t) = - < H_t, X_t > .$$

So we can use Riccati comparison theorem to estimate volume of L_t. In particular, we have

Lemma: If for all regular leaves L and all $v \in \nu_p L$, $\mathrm{tr}_{T_pL} R(\cdot, v)v > 0$, then $V(L) := \mathrm{vol}(L)^{\frac{1}{n}}$ is a strictly concave function on the regular part of $N/\mathcal{F}$.

Corollary: If in addition, $N/\mathcal{F}$ is compact, then there is a unique minimal regular leaf $L_{\min}$.

If $\mathcal{F}$ is isoparametric, MCF of a regular leaf can be reduced to a flow on regular part of $N/\mathcal{F}$ and V is a Lyapunov function for the flow along $-H$ with unique global attractor $L_{\min}$. Hence MCF of L converges to $L_{\min}$ as $t \mapsto -\infty$. This finishes the proof of Theorem A.

Idea for proof of Theorem B

To prove a polar foliation $\mathcal{F}$ on a simply connected manifold N is isoparametric, we need:

Theorem(Alexandrino-Toeben): Let L be a regular leaf of $\mathcal{F}$. Then

(1) L has trivial normal holonomy.

(2) L has constant focal data, i.e., the end point map of a parallel normal vector field has constant rank.

So we only need to show mean curvature vector fields along regular leaves are parallel.

Let N be a simply connected symmetric space with curvature ≥ 0.

Theorem(Lytchak): Every polar foliation in N is a product of a trivial factor, a hyperpolar foliation, and polar foliations with spherical sections (i.e. sections with constant positive curvature).

Theorem(Heintze-Liu-Olmos): Hyperpolar foliations on N are isoparametric.

So we only need to show polar foliations with spherical sections are isoparametric.

We may assume every section Σ is a sphere with curvature $= 1$ after re-scaling metric on N. Fix a regular leaf L of $\mathcal{F}$. Let X be a unit parallel normal vector field along L. For any $p \in L$, let $\gamma(t)$ be the geodesic with $\gamma(0) = p$ and $\gamma'(0) = X(p)$. Then γ is periodic with period 2π. In fact $\exp_q tX(q)$ is periodic for all $q \in L$. This implies that all holonomy Jacobi fields are periodic with period 2π.

We can prove:

(1) Periodicity of holonomy Jacobi fields implies that eigenvalues of R_t along $\gamma(t)$ are squares of integers

$$0 \leq \lambda_1^2 \leq \cdots \leq \lambda_n^2.$$

Let $m = \sum_{i=1}^n \lambda_i$. Then $2m$ is the index of space of holonomy Jacobi fields along $\gamma|_{[0,2\pi)}$. By continuity of index, we see m is independent of $p \in L$.

(2) Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\mathcal{V}_0$ where e_i is an eigenvector of R_0 with eigenvalue λ_i^2. Let $J_i(t)$ be the holonomy Jacobi vector field with $J_i(0) = e_i$. Let $E_i(t)$ be the parallel translation of e_i along $\gamma(t)$. Let

$$f_p(t) := \det(< J_i(t), E_j(t) >).$$

Then

$$f_p(t) = \sum_i a_i \sin(s_i t) + b_i \cos(s_i t)$$

where $s_i = \lambda_1 \pm \lambda_2 \pm \cdots \pm \lambda_n$.

(3) It follows that f_p lies in a space $\mathcal{T}$ of functions with $\dim \mathcal{T} = m + 1$.

(4) Assume γ intersects singular leaves L_j at time $t_j \in (0, \pi)$ for $j = 1, \dots, k$. Let $m_j := \dim(L) - \dim(L_j)$. Then $\sum_{j=1}^{k} m_j = m$ and

$$f_p^{(d)}(t_j) = 0$$

for all $d = 0, \dots, m_j - 1$ and $j = 1, \dots, k$. This gives a system of m linearly independent conditions on space $\mathcal{T}$ which, together with condition $f_p(0) = 1$, uniquely determines f_p.

(5) Note that t_j and m_j are determined by focal data of $\exp tX : L \longrightarrow N$. By constancy of focal data, $f_p(t)$ does not depend on $p \in L$.

(6) Since

$$f_p'(0) = f_p(0)\mathrm{tr}(S_0) = - < H_p, X_p >,$$

it follows that $< H_p, X_p >$ does not depend on $p \in L$ for all parallel normal vector field X. So H is parallel along all regular leaves and $\mathcal{F}$ is isoparametric.

Second part of Theorem B is a splitting theorem for polar foliation. We would like to show that after splits off Euclidean factors, either $N/\mathcal{F}$ is compact, or N is compact.

It follows from a combination of Lytchak's results that if $(N, \mathcal{F})$ is an indecomposable polar foliation with spherical sections, than N has to be compact. So we only need to consider the hyperpolar case.

Assume $\mathcal{F}$ is a hyperpolar foliation on M. For p in the regular part of the foliation, let L_p be the leaf through p. Define

$$D(p) := \{v \in \nu_p L \mid R(\cdot, v)v \equiv 0\}.$$

Since M is a symmetric space with non-netative Ricci curvature, $D(p)$ is always a vector space and defines an integrable totally geodesic distribution when p varies in the regular part of the foliation.

Moreover, it follows that $v \in D(p)$ is equvalent to $\mathrm{tr}R(\cdot, v)v = 0$. It is also equivalent to the geodesic $\exp(tv)$ intersecting singular leaves only finitely many times. The latter fact is proved by estimating index of the space of holonomy Jacobi fields using Riccati comparison theorem.

A section Σ of $\mathcal{F}$ then splits as a product of two submanifolds $\Sigma_1 \times \Sigma_2$, with Σ_1 an integral submanifold of D and Σ_2 perpendicular to D.

Each hyperpolar foliation has a **Weyl group** W which is generated by reflections along affine hyperplanes of Σ called **walls** of W. One can show that the above splitting of Σ induce a splitting of W as well. To prove this, we need to use the fact that for any horizontal geodesic γ with $\gamma(0) \in \Sigma_1 \cap \Sigma_2$, it intersects walls of Weyl group W either finitely times when $\gamma \subset \Sigma_1$ or infinitely many times when $\gamma \nsubseteqq \Sigma_1$. So walls of W is divided into disjoint union of two sets, one has normal vectors tangent to Σ_1, another has normal vectors tangent to Σ_2. The decomposition of walls gives decomposition of W.

Decomposition of the Weyl group implies the splitting of $(N, \mathcal{F})$ into the product of two foliations $(M_1, \mathcal{F}_1) \times (M_2, \mathcal{F}_2)$. (Terng, Heintze-Liu, Ewert, Silva-Speranca)

The first factor $(M_1, \mathcal{F}_1)$, which has Σ_1 as a section, is an isoparametric foliation in Euclidean space (possibly times a trivial factor). The second factor $(M_2, \mathcal{F}_2)$, which has Σ_2 as a section, has the property that its leaf space $M_2/\mathcal{F}_2$ is compact. Every horizontal geodesic $\gamma(t)$ in M_2 meets singular leaves infinitely many times in positive t direction.

If leaves of $(N, \mathcal{F})$ are compact, we can prove M_2 is compact by contradiction:

In fact, if M_2 is not compact, we have a splitting $M_2 = M' \times \mathbb{R}^k$ for some $k > 0$. We then show the projection from a regular leaf L_2 of $\mathcal{F}_2$ to $\mathbb{R}^k$ is a submersion. This is not possible if L_2 is compact.

This completes the proof of Theorem B.

Thanks!

Antipodal sets of generalized s-manifolds

Takashi Sakai

Abstract. We introduce the notion of generalized s-manifold as a generalization of symmetric spaces. Then we study maximal antipodal sets of generalized s-manifolds. This is partly joint work with S. Ohno and Y. Terauchi.

1 Generalized s-manifolds

A Riemannian symmetric space M is a Riemannian manifold which has a geodesic symmetry s_x at each point $x \in M$. The family $\{s_x\}_{x\in M}$ of geodesic symmetries satisfies

$$s_x \circ s_y = s_{s_x(y)} \circ s_x \tag{1}$$

for all $x, y \in M$, that is, a Riemannian manifold M is a quandle. By generalizing (1) to (2), we define generalized s-manifolds as follows.

Definition 1. Let M be a smooth manifold and Γ a group. Let $\{\varphi_x\}_{x\in M}$ be a family of group homomorphisms $\varphi_x : \Gamma \to \mathrm{Diff}(M)$ from Γ to the diffeomorphism group $\mathrm{Diff}(M)$ of M. Then $(\Gamma, \{\varphi_x\}_{x\in M})$ is called a *generalized s-structure* on M if it satisfies the following conditions:

1. For each $\gamma \in \Gamma$, the map $\mu^\gamma : M \times M \to M;\ (x, y) \mapsto \varphi_x(\gamma)(y)$ is smooth. (When Γ is a Lie group, $\mu : \Gamma \times M \times M \to M;\ (\gamma, x, y) \mapsto \varphi_x(\gamma)(y)$ is smooth.)

2. For each $x \in M$, x is an isolated fixed point of the action of $\varphi_x(\Gamma)$ on M, i.e., x is isolated in $F(\varphi_x(\Gamma), M) := \{y \in M \mid \varphi_x(\gamma)(y) = y\ (\forall\gamma \in \Gamma)\}$.

3. For any $x, y \in M$ and $\gamma, \delta \in \Gamma$,

$$\varphi_x(\gamma) \circ \varphi_y(\delta) \circ \varphi_x(\gamma)^{-1} = \varphi_{\varphi_x(\gamma)(y)}(\gamma\delta\gamma^{-1}) \tag{2}$$

holds.

When $\Gamma = \mathbb{Z}_2$, a generalized s-manifold is just a symmetric space in the sense of Loos and Nagano. More generally, a regular s-manifold is a generalized s-manifold with $\Gamma = \mathbb{Z}$, in particular a k-symmetric space is a generalized s-manifold with $\Gamma = \mathbb{Z}_k$ (cf. [2]). A Γ-symmetric space introduced by Lutz [3] is a generalized s-manifold with a finite abelian group Γ.

Let $\mathbb{K}$ be $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. For $n, n_1, \ldots, n_r \in \mathbb{N}$ satisfying $n_1 + \cdots + n_r < n$, we consider a flag manifold

$$F_{n_1,\ldots,n_r}(\mathbb{K}^n) := \{x = (V_1, \ldots, V_r) \mid V_1 \subset \cdots \subset V_r \subset \mathbb{K}^n, \dim V_i = n_1 + \cdots + n_i\}.$$

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The flag manifold $F_{n_1,\dots,n_r}(\mathbb{K}^n)$ has fibrations over Grassmannian manifolds. Symmetries of the Grassmannian manifolds induce a Γ-symmetric structure on $F_{n_1,\dots,n_r}(\mathbb{K}^n)$ with $\Gamma = (\mathbb{Z}_2)^r$. Furthermore, in [4], by using Γ-symmetric triples, we investigated Γ-symmetric structures on R-spaces, which is a natural generalization of symmetric R-spaces. We gave a necessary and sufficient condition for an R-space to admit a natural Γ-symmetric structure in terms of the root system. Then we classified R-spaces that admit natural Γ-symmetric structures, when the root system is irreducible.

2 Antipodal sets of generalized s-manifolds

Chen and Nagano [1] studied antipodal sets of compact symmetric spaces. The definition of antipodal sets of compact symmetric spaces naturally extends to a generalized s-manifold $(M, \Gamma, \{\varphi_x\}_{x\in M})$.

Definition 2. A subset A of M is called an *antipodal set* if $\varphi_x(\gamma)(y) = y$ holds for all $x, y \in A$ and $\gamma \in \Gamma$, i.e., $y \in F(\varphi_x(\Gamma), M) := \{y \in M \mid \varphi_x(\gamma)(y) = y \ (\forall \gamma \in \Gamma)\}$. An antipodal set A of M is said to be *maximal* if $A = A'$ holds for any antipodal set A' of M with $A \subset A'$. The supremum of the cardinalities of antipodal sets of M, denoted by $\#_\Gamma M$, is called the *antipodal number* of M. An antipodal set A is said to be *great* if its cardinality attains $\#_\Gamma M$.

Theorem 1 (Ohno-S.-Terauchi)**.** *Any maximal antipodal set of $F_{n_1,\dots,n_r}(\mathbb{K}^n)$ with respect to the Γ-symmetric structure with $\Gamma \cong (\mathbb{Z}_2)^r$ is congruent to*

$$\begin{aligned} A =\{ & (\langle \boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_{n_1}} \rangle_{\mathbb{K}}, \langle \boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_{n_1+n_2}} \rangle_{\mathbb{K}}, \dots, \langle \boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_{n_1+\cdots+n_r}} \rangle_{\mathbb{K}}) \\ & \mid 1 \le i_1 < \cdots < i_{n_1} \le n,\ 1 \le i_{n_1+1} < \cdots < i_{n_1+n_2} \le n, \dots, \\ & \quad 1 \le i_{n_1+\cdots+n_{r-1}+1} < \cdots < i_{n_1+\cdots+n_r} \le n, \\ & \quad \#\{i_1, \dots, i_{n_1+\cdots+n_r}\} = n_1 + \cdots + n_r\}, \end{aligned}$$

where $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ is the standard basis of $\mathbb{K}^n$.

Furthermore, in [4], we determined maximal antipodal sets of natural Γ-symmetric structures on R-spaces. More precisely, an R-space M can be realized as an orbit of the isotropy representation (s-representation) of a compact symmetric space G/K. Then any maximal antipodal set of the natural Γ-symmetric structure on M is given as an orbit of the Weyl group of G/K. Consequently we obtain that the antipodal number of M is the cardinality of the orbit of the Weyl group, hence it is equal to $\dim H_*(M; \mathbb{Z}_2)$. These are generalizations of results on maximal antipodal sets and the two-numbers of symmetric R-spaces by Tanaka–Tasaki [6] and Takeuchi [5].

References

[1] B.-Y. Chen, T. Nagano, A Riemannian geometric invariant and its applications to a problem of Borel and Serre, Trans. Am. Math. Soc. **308** (1988), 273–297.

[2] O. Kowalski, Generalized symmetric spaces, Lecture Notes in Mathematics **805**, Springer-Verlag, Berlin-New York, 1980.

[3] R. Lutz, Sur la géométrie des espaces Γ-symétriques, C. R. Acad. Sci., Paris, Sér. I **293** (1981), 55–58.

[4] P. Quast and T. Sakai, Natural Γ-symmetric structures on R-spaces, J. Math. Pures Appl. (9) **141** (2020), 371–383.

[5] M. Takeuchi, Two-number of symmetric R-spaces, Nagoya Math. J. **115** (1989), 43–46.

[6] M. S. Tanaka, H. Tasaki, Antipodal sets of symmetric R-spaces, Osaka J. Math. **50** (2013), 161–169.

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Antipodal sets of generalized s-manifolds

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Submanifolds of symmetric spaces and their time evolutions

Introduction

Definition

(M, g) : **Riemannian symmetric space**

$\overset{\text{def}}{\Longleftrightarrow}$ For each $x \in M$, there exists $s_x \in \mathrm{Isom}(M, g)$ s.t.

1. $s_x^2 = \mathrm{id}_M$,
2. x is isolated in $F(s_x, M) := \{y \in M \mid s_x(y) = y\}$.

e.g. $\mathbb{E}^n$, S^n, $\mathbb{R}\mathrm{H}^n$, $\mathrm{G}_r(\mathbb{K}^n)$, compact Lie groups, ...
E. Cartan classified Riemannian symmetric spaces.

Generalizations:

- locally symmetric space $\nabla R = 0$
- pseudo-Riemannian symmetric space, affine symmetric space
- weakly symmetric space (Selberg)
- k-symmetric space, s-manifold (Ledger, Obata, Kowalski, ...)
- Γ-symmetric space (Lutz)

Symmetric space

Definition (Loos, Nagano)

M : **symmetric space**

$\overset{\text{def}}{\Longleftrightarrow}$ For each $x \in M$, there exists $s_x \in \mathrm{Diff}(M)$ s.t.

1. $\mu : M \times M \to M;\ (x, y) \mapsto s_x(y)$ is smooth,
2. $s_x^2 = \mathrm{id}_M$,
3. x is isolated in $F(s_x, M) := \{y \in M \mid s_x(y) = y\}$,
4. $s_x \circ s_y = s_{s_x(y)} \circ s_x \quad (\forall x, y \in M)$.

e.g.

- Euclidean space $\mathbb{R}^n$ $\quad s_x(y) = -y + 2x$
- sphere $S^n \subset \mathbb{R}^{n+1}$ $\quad s_x(y) = -y + 2\langle x, y\rangle x$

Γ-symmetric spaces

Definition (Lutz)

M : C^∞-manifold, $\quad \Gamma$: finite abelian group

$\mu = \{\mu^\gamma : M \times M \to M$ smooth map $\mid \gamma \in \Gamma\}$

Γ-symmetric strucure on M

$\overset{\text{def}}{\Longleftrightarrow}$

1. For each $x \in M$
$$\Gamma \to \mathrm{Diff}(M);\ \gamma \mapsto \gamma_x := \mu^\gamma(x, \cdot)$$
is an injective homomorphism, i.e. $\Gamma_x := \{\gamma_x \mid \gamma \in \Gamma\} \cong \Gamma$.
2. Every x is isolated in
$$F(\Gamma_x, M) := \{y \in M \mid \gamma_x(y) = y\ (\forall \gamma \in \Gamma)\}.$$
3. For all $x \in M$ and $\gamma \in \Gamma$, γ_x is an automorphism of μ, i.e.
$$\mu^\delta\big(\gamma_x(y), \gamma_x(z)\big) = \gamma_x\big(\mu^\delta(y, z)\big) \quad (\forall y, z \in M,\ \delta \in \Gamma).$$

3 $\iff \quad \delta_{\gamma_x(y)} \circ \gamma_x = \gamma_x \circ \delta_y \quad (\forall x, y \in M,\ \gamma, \delta \in \Gamma)$

Generalized s-manifolds

Definition (Ohno-S.)

M : C^∞-manifold, Γ : (Lie) group

$(\Gamma, \{\varphi_x\}_{x\in M})$: **generalized s-structure** on M

$\overset{\text{def}}{\Longleftrightarrow}$

1. For each $x \in M$, $\varphi_x : \Gamma \to \mathrm{Diff}(M)$ is a group homomorphism.
2. For each $\gamma \in \Gamma$, $\mu^\gamma : M \times M \to M$; $(x, y) \mapsto \varphi_x(\gamma)(y)$ is a smooth mapping. (In the case where Γ is a Lie group, $\mu : \Gamma \times M \times M \to M$; $(\gamma, x, y) \mapsto \varphi_x(\gamma)(y)$ is smooth.)
3. x is isolated in
$$F(\varphi_x(\Gamma), M) := \{y \in M \mid \varphi_x(\gamma)(y) = y \ (\forall \gamma \in \Gamma)\}.$$
4. For $\forall x, y \in M$ and $\forall \gamma, \delta \in \Gamma$
$$\varphi_x(\gamma) \circ \varphi_y(\delta) \circ \varphi_x(\gamma)^{-1} = \varphi_{\varphi_x(\gamma)(y)}(\gamma\delta\gamma^{-1}).$$

Generalized s-manifolds

For each $x \in M$, $\Gamma_x := \{\varphi_x(\gamma) \mid \gamma \in \Gamma\}$ is a subgroup of $\mathrm{Diff}(M)$. We call Γ_x the **symmetric transformation group** at $x \in M$.

Remark

- $\Gamma = \mathbb{Z}_2 \implies (M, \Gamma, \{\varphi_x\}_{x\in M})$ is a symmetric space
- $\Gamma = \mathbb{Z}_k \implies (M, \Gamma, \{\varphi_x\}_{x\in M})$ is a k-symmetric space
- $\Gamma = \mathbb{Z} \implies (M, \Gamma, \{\varphi_x\}_{x\in M})$ is a regular s-manifold
- Γ is a finite abelian group
$$\implies (M, \Gamma, \{\varphi_x\}_{x\in M}) \text{ is a } \Gamma\text{-symmetric space}$$

Antipodal sets of compact symmetric spaces

M : compact Riemannian symmetric space

Definition (B.-Y. Chen-Nagano)

For $x \in M$, each connected component of the fixed point set

$$F(s_x, M) := \{y \in M \mid s_x(y) = y\}$$

of s_x is called a **polar**. An isolated polar is called a **pole**.

Definition (B.-Y. Chen-Nagano)

1. $A \subset M$: **antipodal set**
   $\overset{\text{def}}{\iff}$ $s_x(y) = y$ for all $x, y \in A$
2. $A \subset M$: **maximal antipodal set**
   $\overset{\text{def}}{\iff}$ $A' \subset M$: antipodal set, $A \subset A' \Longrightarrow A = A'$
3. $\#_2 M := \sup\{\#A \mid A \subset M : \text{antipodal}\}$ **2-number**
4. $A \subset M$: **great antipodal set** $\overset{\text{def}}{\iff}$ $\#A = \#_2 M$

Antipodal sets of compact symmetric spaces

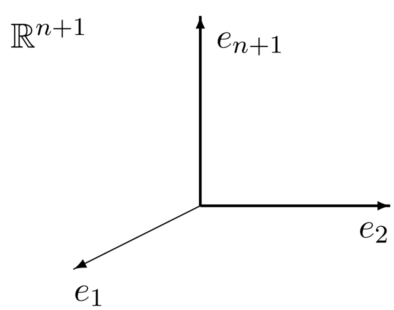
Example:

$o = \langle e_1 \rangle \in \mathbb{R}P^n$

$$F(s_o, \mathbb{R}P^n) = \{o\} \cup \{x \in \mathbb{R}P^n \mid x \perp o\}$$
$$\cong \{o\} \cup \mathbb{R}P^{n-1}$$

$A = \{\langle e_i \rangle \mid i = 1, \dots, n+1\}$

$\#_2 \mathbb{R}P^n = n + 1 = \dim H_*(\mathbb{R}P^n, \mathbb{Z}_2)$



Theorem (Takeuchi)

M : symmetric R-space $\Longrightarrow$ $\#_2 M = \dim H_*(M, \mathbb{Z}_2)$

Antipodal sets of generalized s-manifolds

$(\Gamma, \{\varphi_x\}_{x \in M})$: generalized s-structure on M.

Definition

For $x \in M$, a connected component of the fixed point set

$$F(\varphi_x(\Gamma), M) := \{y \in M \mid \varphi_x(\gamma)(y) = y \ (\forall \gamma \in \Gamma)\}$$

is called a **polar**. An isolated polar is called a **pole**.

Definition

1. $A \subset M$: **antipodal set**
   $\overset{\text{def}}{\Longleftrightarrow}$ $\varphi_x(\gamma)(y) = y$ for all $x, y \in A$ and $\gamma \in \Gamma$
2. $A \subset M$: **maximal antipodal set**
   $\overset{\text{def}}{\Longleftrightarrow}$ $A' \subset M$: antipodal set, $A \subset A' \Longrightarrow A = A'$
3. $\#_\Gamma M := \sup\{\#A \mid A \subset M : \text{antipodal}\}$ **antipodal number**
4. $A \subset M$: **great antipodal set** $\overset{\text{def}}{\Longleftrightarrow}$ $\#A = \#_\Gamma M$

Flag manifolds

$\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$

$n, n_1, \ldots, n_r \in \mathbb{N}$ satisfying $n_1 + \cdots + n_r < n$

$$F_{n_1,\ldots,n_r}(\mathbb{K}^n) := \left\{ x = (V_1, \ldots, V_r) \;\middle|\; \begin{array}{l} V_1 \subset V_2 \subset \cdots \subset V_r \subset \mathbb{K}^n \\ \dim V_i = n_1 + \cdots + n_i \ (\forall i) \end{array} \right\}$$

For $x = (V_1, \ldots, V_r) \in F_{n_1,\ldots,n_r}(\mathbb{K}^n)$, define

$$s_{V_i} = 2P_{V_i} - \mathrm{id}_{\mathbb{K}^n} : \mathbb{K}^n \to \mathbb{K}^n \quad (i = 1, \ldots, r),$$

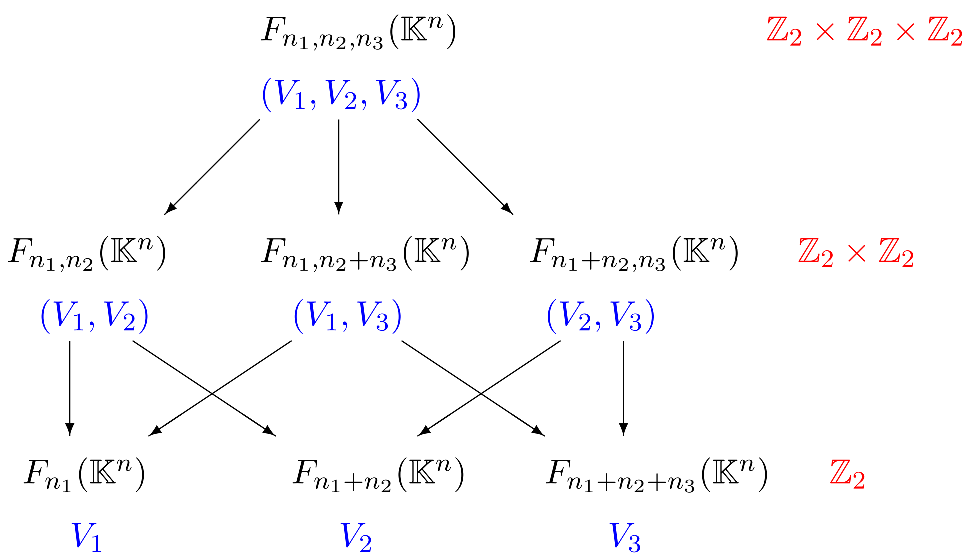
where $P_{V_i} : \mathbb{K}^n \to V_i$ is the orthogonal projection. Then

$$s_{V_i} \in \mathrm{Diff}(M), \quad s_{V_i}^2 = \mathrm{id}_M, \quad s_{V_i} \circ s_{V_j} = s_{V_j} \circ s_{V_i}.$$

Hence $\qquad (\mathbb{Z}_2)^r \cong \langle s_{V_1}, \ldots s_{V_r} \rangle \subset \mathrm{Diff}(M).$

Therefore $F_{n_1,\ldots,n_r}(\mathbb{K}^n)$ is a generalized s-manifold with $\Gamma = (\mathbb{Z}_2)^r$.

Flag manifolds



Maximal antipodal sets of flag manifolds

Theorem (Ohno-S.-Terauchi)

1. *Any maximal antipodal set of $F_{n_1,\dots,n_r}(\mathbb{K}^n)$ with respect to the $\Gamma \cong (\mathbb{Z}_2)^r$-symmetric structure is congruent to*

$$
\begin{aligned}
A = \{ & (\langle \boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_{n_1}} \rangle_{\mathbb{K}}, \langle \boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_{n_1+n_2}} \rangle_{\mathbb{K}}, \dots, \langle \boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_{n_1+\cdots+n_r}} \rangle_{\mathbb{K}}) \\
& \mid 1 \leq i_1 < \cdots < i_{n_1} \leq n,\ 1 \leq i_{n_1+1} < \cdots < i_{n_1+n_2} \leq n, \dots, \\
& \quad 1 \leq i_{n_1+\cdots+n_{r-1}+1} < \cdots < i_{n_1+\cdots+n_r} \leq n, \\
& \quad \#\{i_1, \dots, i_{n_1+\cdots+n_r}\} = n_1 + \cdots + n_r \},
\end{aligned}
$$

   where $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ is the standard basis of $\mathbb{K}^n$.

2. $$
\begin{aligned}
\#_\Gamma(F_{n_1,\dots,n_r}(\mathbb{K}^n)) &= \frac{n!}{n_1! n_2! \cdots n_r! n_{r+1}!} \\
&= \dim H_*(F_{n_1,\dots,n_r}(\mathbb{K}^n); \mathbb{Z}_2),
\end{aligned}
$$

   where $n_{r+1} := n - (n_1 + \cdots n_r)$.

Kähler C-spaces

G : compact connected semisimple Lie group

$x_0(\neq 0) \in \mathfrak{g}$

$$\begin{aligned} M &:= \mathrm{Ad}(G)x_0 \subset \mathfrak{g} \quad : \textbf{Kähler } C\textbf{-space} \\ &\cong G/G_{x_0} \end{aligned}$$

$$\begin{aligned} G_{x_0} &:= \{g \in G \mid \mathrm{Ad}(g)x_0 = x_0\} \\ Z(G_{x_0}) &:= \{g \in G_{x_0} \mid gh = hg \ (\forall h \in G_{x_0})\} \end{aligned}$$

For each $x = \mathrm{Ad}(g_x)x_0 \in M$ and $\gamma \in \Gamma$, define

$$\varphi_x(\gamma) : M \to M;\ y \mapsto \mathrm{Ad}(g_x \gamma g_x^{-1})y,$$

$$\varphi_x : \Gamma \to \mathrm{Diff}(G/K);\ \gamma \to \varphi_x(\gamma).$$

Then $(\Gamma, \{\varphi_x\}_{x\in M})$ is a G-equivariant generalized s-structure.

Moreover M has k-symmetric structures for $\forall k \geq \exists k_0$,
i.e. generalized s-structures with $\Gamma \cong \mathbb{Z}_k$.

Maximal antipodal sets of Kähler C-spaces

$M = \mathrm{Ad}(G)x_0$: a Kähler C-space with $\Gamma = Z(G_{x_0})$

Proposition (Ikawa-Iriyeh-Okuda-S.-Tasaki)

1. For $x, y \in M$

   $$y \text{ is antipodal to } x \iff [x, y] = 0$$

2. $A \subset M$: maximal antipodal set

   $$\Longrightarrow \quad \exists \mathfrak{t} \subset \mathfrak{g} : \text{maximal abelian subalgebra}$$
   $$\text{s.t.} \quad A = M \cap \mathfrak{t}.$$

   Hence A is an orbit of the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{t}$. In particular, any maximal antipodal sets of M are congruent with each other by G.

Generalized s-structures on complex flag manifolds

$F_{n_1,\dots,n_k}(\mathbb{C}^n)$ is realized as an adjoint orbit of $G = \mathrm{SU}(n)$:

$$F_{n_1,\dots,n_r}(\mathbb{C}^n) \cong \mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{r+1})) = G/G_{x_0}$$

Observation

1. $F_{n_1,\dots,n_r}(\mathbb{C}^n)$ admits Γ-symmetric structures with
$$\Gamma = (\mathbb{Z}_2)^r, \qquad \Gamma = \mathbb{Z}_k \ (k \geq k_0), \qquad \Gamma = Z(G_{x_0}).$$
2. Maximal antipodal sets of $F_{n_1,\dots,n_r}(\mathbb{C}^n)$ for these three generalized s-structures coincide, that is an orbit of the Weyl group of $\mathrm{SU}(n)$.

Compact symmetric triads

(G, K_1, K_2) : compact symmetric triad

i.e. $(G, K_1, \sigma_1), (G, K_2, \sigma_2)$: compact symmetric pairs

$\Gamma := \langle \sigma_1, \sigma_2 \rangle \subset \mathrm{Aut}(G)$

$$\begin{tikzcd}
& G \arrow[d, "\pi"] & \\
& G/(K_1 \cap K_2) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
G/K_1 & & G/K_2
\end{tikzcd}$$

Γ-symmetric space

- $\sigma_1 = \sigma_2 \quad \Longrightarrow \quad \Gamma \cong \mathbb{Z}_2$
- $\sigma_1 \neq \sigma_2,\ \sigma_1\sigma_2 = \sigma_2\sigma_1 \quad \Longrightarrow \quad \Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_2$
- $\sigma_1\sigma_2 \neq \sigma_2\sigma_1 \quad \Longrightarrow \quad \Gamma = \langle \sigma_1, \sigma_2 \rangle$ is a non-abelian group

Γ-symmetric triples

Definition (Lutz, Goze-Remm, Ohno-S.-Terauchi)

(G, K, Γ) : **Γ-symmetric triple**

$\overset{\text{def}}{\Longleftrightarrow}$ G : connected Lie group

$\Gamma \subset \mathrm{Aut}(G)$: finite subgroup

$K \subset G$: closed subgroup s.t

$$F(\Gamma, G)_0 \subset K \subset F(\Gamma, G),$$

where $F(\Gamma, G) := \{g \in G \mid \gamma(g) = g \ (\forall \gamma \in \Gamma)\}$.

(G, K, Γ) : Γ-symmetric triple

For each $x = g_x K \in G/K$ and $\gamma \in \Gamma$, define

$$\varphi_x(\gamma) : G/K \to G/K;\ gK \mapsto g_x \gamma(g_x^{-1} g)K,$$

$$\varphi_x : \Gamma \to \mathrm{Diff}(G/K);\ \gamma \to \varphi_x(\gamma).$$

Then $(\Gamma, \{\varphi_x\}_{x \in G/K})$ is a G-equivariant generalized s-structure on G/K.

Natural Γ-symmetric structures on R-spaces

$P = G/K$: simply-connected compact symmetric space

where $G := \mathrm{Isom}(P)_0$

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

$\mathfrak{a} \subset \mathfrak{p}$: maximal abelian subalgebra

$\mathcal{R} \subset \mathfrak{a}^*$: root system of (G, K) with respect to $\mathfrak{a}$

$\Sigma := \{\alpha_1, \ldots, \alpha_r\}$: simple roots of $\mathcal{R}$

$\{\xi_1, \ldots, \xi_r\}$: basis of $\mathfrak{a}$ dual to Σ

For $j \in \{1, \ldots, r\}$, define

$$g_i := \exp(\pi \xi_i) \in G, \quad \text{and} \quad \gamma^i := \mathrm{Int}_G(g_i)\big|_K \in \mathrm{Aut}(K).$$

Note $\gamma^i \gamma^i = \mathrm{id}_K$, $\gamma^i \neq \mathrm{id}_K$, $\gamma^i \gamma^j = \gamma^j \gamma^i$ for all $i, j \in \{1, \ldots, r\}$, hence $(\mathbb{Z}_2)^r \cong \langle \gamma^i : \ i = 1, \ldots, r \rangle \subset \mathrm{Aut}(K)$.

Natural Γ-symmetric structures on R-spaces

For $I \subset I_{\mathrm{reg}} := \{1, \dots, r\}$, define $\xi_I := \sum_{i \in I} \xi_i$ and

$$X_I := \mathrm{Ad}_G(K)\xi_I \subset \mathfrak{p} \quad R\text{-}\mathbf{space}$$

$X_I \cong K/H_I$ where $H_I = \{k \in K \mid \mathrm{Ad}_G(k)\xi_I = \xi_I\}$.

Every R-space in $\mathfrak{p}$ is equivariantly isomorphic to X_I for some $I \subset I_{\mathrm{reg}}$.

Lemma

Let $\emptyset \neq \tilde{I} \subset I \subset I_{\mathrm{reg}} = \{1, \dots, r\}$, then $H_I \subset H_{\tilde{I}}$.

Therefore the choice of $I \subset I_{\mathrm{reg}}$ gives a stratification of orbit types.

Natural Γ-symmetric structures on R-spaces

$$\begin{aligned}\Gamma^I &= \langle \gamma^i : i \in I \rangle \subset \mathrm{Aut}(K) \\ &\cong (\mathbb{Z}_2)^{|I|}\end{aligned}$$

A non-empty subset I of $I_{\mathrm{reg}} = \{1, \dots, r\}$ is called **admissible** if (K, H_I, Γ^I) is a Γ^I-symmetric triple, that is, if

$$F(\Gamma^I, K)_0 \subset H_I \subset F(\Gamma^I, K).$$

Then Γ^I induces a Γ^I-symmetric structure on $X_I \cong K/H_I$.

Theorem (Quast-S.)

$I \subset I_{\mathrm{reg}} = \{1, \dots, r\}$ is admissible

$$\iff \quad \text{for all } \alpha = \sum_{i=1}^{r} c_i^\alpha \alpha_i \in \mathcal{R}^+$$
$$\left(\forall i \in I : c_i^\alpha \text{ even}\right) \implies \left(\forall i \in I : c_i^\alpha = 0\right).$$

Extrinsically Γ-symmetric spaces

Assume that $I \subset I_{\mathrm{reg}}$ is admissible.
Then $K/H_I \cong X_I = \mathrm{Ad}(K)\xi_I$ has a Γ^I-symmetric structure.

$$K/H_I \to X_I := \mathrm{Ad}_G(K)\xi_I;\ kH_I \mapsto \mathrm{Ad}_G(k)\xi_I$$

$$\gamma^i_{\xi_I}(\mathrm{Ad}_G(k)\xi_I) = \mathrm{Ad}_G(g_i)\mathrm{Ad}_G(k)\xi_I.$$

$\gamma^i_{\xi_I}$ extends to the linear endomorphism $\mathrm{Ad}_G(g_i)|_{\mathfrak{p}}$ of $\mathfrak{p}$,

$$\Gamma^I_{\xi_I} = \langle \gamma^i_{\xi_I} : i \in I\rangle \cong \langle \mathrm{Ad}_G(g_i)|_{\mathfrak{p}} : i \in I\rangle \subset \mathrm{End}(\mathfrak{p})$$

$$F(\Gamma^I_{\xi_I}, \mathfrak{p}) = \{Y \in \mathfrak{p} \mid [Y, \xi_I] = 0\} = T^{\perp}_{\xi_i} X_I$$

Hnece X_I can be considered as an **extrinsically Γ-symmetric space**.

Maximal antipodal sets of natural Γ-structures on R-spaces

Let I be an admissible subset of $I_{\mathrm{reg}} = \{1, \ldots, r\}$.

$$K/H_I \to X_I := \mathrm{Ad}_G(K)\xi_I; \quad kH_I \mapsto \mathrm{Ad}_G(k)\xi_I$$

Γ^I-symmetric structure on K/H_I can be transfered to X_I as

$$\gamma^i_{\xi_I}(\mathrm{Ad}_G(k)\xi_I) = \mathrm{Ad}_G(g_i)\mathrm{Ad}_G(k)\xi_I.$$

Hence $\gamma^i_{\xi_I}$ extends to the linear endomorphism $\mathrm{Ad}_G(g_i)|_{\mathfrak{p}}$ of $\mathfrak{p}$.

$$F(\Gamma^I_{\xi_I}, \mathfrak{p}) = \{Y \in \mathfrak{p} \mid [Y, \xi_I] = 0\}$$

Theorem (Quast-S.)

For each antipodal subset A of X_I, there exists some maximal abelian subspace $\mathfrak{a}'$ in $\mathfrak{p}$ such that

$$A = X_I \cap \mathfrak{a}' = W(P, \mathfrak{a}')\xi_I,$$

where $W(P, \mathfrak{a}')$ is the Weyl group of (G, K) with respect to $\mathfrak{a}'$.
Any two maximal antipodal subsets of X_I are conjugate by K.

Maximal antipodal sets of natural Γ-structures on R-spaces

Corollary

Let A be a maximal antipodal subset of X_I. Then

$$\#_\Gamma(X_I) = |A| = \left|W(P, \mathfrak{a}')\xi_I\right| = \dim H_*(X_I, \mathbb{Z}_2).$$

Further problems

1. Classify Γ-symmetric spaces and generalized s-manifolds.
2. Determine maximal antipodal sets and antipodal numbers of generalized s-manifolds.
3. Study geometric meaning of the antipodal numbers of generalized s-manifolds, in particular, a relationship with its topology.

Thank you very much for your attention!

References

- B.-Y. Chen, T. Nagano, *A Riemannian geometric invariant and its applications to a problem of Borel and Serre*, Trans. Am. Math. Soc. **308** (1988), 273–297.
- D. Ferus, *Symmetric submanifolds of Euclidean space*, Math. Ann. 247 (1980), 81–93.
- M. Goze, E. Remm, *Riemannian Γ-symmetric spaces*, Differential geometry, 195–206, World Sci. Publ., Hackensack, NJ, 2009.
- O. Loos, Symmetric spaces **I**, **II**, W. A. Benjamin, New York 1969.

References

- R. Lutz, *Sur la géométrie des espaces Γ-symétriques*, C. R. Acad. Sci., Paris, Sér. I **293** (1981), 55–58.
- P. Quast, T. Sakai, *Natural Γ-symmetric structures on R-spaces*, J. Math. Pures Appl. **141** (2020), 371–383.
- M. Takeuchi, *Two-number of symmetric R-spaces*, Nagoya Math. J. **115** (1989), 43–46.
- M. S. Tanaka, H. Tasaki, *Antipodal sets of symmetric R-spaces*, Osaka J. Math. **50** (2013), 161–169.

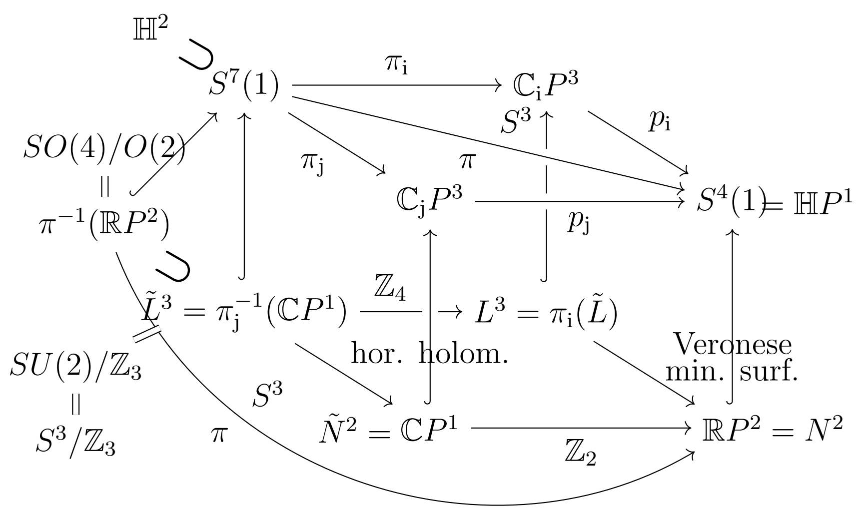
Higher dimensional generalization of the Chiang Lagrangian and totally complex submanifolds

Jong Taek Cho, Kaname Hashimoto*, Yoshihiro Ohnita

Abstract. We will discuss a higher dimensional generalization of Chiang's Lagrangian submanifold in $\mathbb{C}P^3$ from the viewpoint of totally complex submanifolds of $\mathbb{H}P^n$.

1 The Chiang Lagrangian

The *Chiang Lagrangian* is a compact embedded minimal Lagrangian submanifold L^3 of $\mathbb{C}P^3$ given as an $SU(2)$-orbit by the moment map method (R. Chiang [1]). It is known to be a non-symmetric homogeneous space and a strictly Hamiltonian stable with non-parallel second fundamental form ([3], [5]), and so on ([6], [4]). The Chiang Lagrangian L^3 is envolved with several nice structures illustrated in the following diagram:



Here $\mathbb{H} = \mathbb{R}1+\mathbb{R}i+\mathbb{R}j+\mathbb{R}k$ is the quaternion number field with quaternionic imaginary units $\{i, j, k\}$ and $\mathbb{H}^{n+1}$ is considered as an $(n+1)$-dimensional quaternionic vecter space with the rightmultiplication of quaternionic numbers.

2 Totally complex submanifolds and R-spaces

We use the concept of "totally complex submanifolds" of a quaternionic projective space $\mathbb{H}P^n$ (e.g. [7]) in order to generalize the above diagram into the higher dimensional setting. A submanifold of $\mathbb{H}P^n$ is called a *totally complex submanifold* if N can be

*the presenter

locally lifted to a *twistor space* $\mathbb{C}P^{2n+1}$ over $\mathbb{H}P^n$ as a complex Legendrian submanifold. We have $\dim_{\mathbb{R}} N \leq 2n$. If $\dim_{\mathbb{R}} = 2n$, then N is said to be *of maximal dimension*.

Main Theorem 1. *The above diagram can be generalized into higher dimensions as follows:*

$$
\begin{tikzcd}
& \mathbb{H}^{n+1} & & & \\
& S^{4n+3}(1) \arrow[u, phantom, sloped, "\subset"] \arrow[rr, "\pi_{\mathrm{i}}"] \arrow[dr, "\pi_{\mathrm{j}}"] \arrow[drrr, "\pi"] & & \mathbb{C}_{\mathrm{i}}P^{2n+1} \arrow[dr, "p_{\mathrm{i}}"] & \\
\widehat{N} = \pi^{-1}(N) \arrow[ur] \arrow[drrrr, bend right=40, "\pi"'] & & \mathbb{C}_{\mathrm{j}}P^{2n+1} \arrow[rr, "p_{\mathrm{j}}"'] & & \mathbb{H}P^n \\
& \tilde{L}^{2n+1} = \pi_{\mathrm{j}}^{-1}(\tilde{N}) \arrow[ul, phantom, sloped, "\subset"] \arrow[uu, leftrightarrow, "\text{min.}", "\text{Leg.}"'] \arrow[rr, dash] \arrow[dr, "S^3"'] & & L^{2n+1} = \pi_{\mathrm{i}}(\tilde{L}) \arrow[uu, leftrightarrow, "S^3\ \text{min.}", "\text{Lag.}"'] \arrow[dr] & \\
& & \tilde{N}^n \arrow[uu, "\text{cplx.}", "\text{Leg.}"'] \arrow[rr] & & N^{2n} \arrow[uu, "\text{tot.}", "\text{cplx.}"']
\end{tikzcd}
$$

Totally complex submanifolds of $\mathbb{H}P^n$ with parallel second fundamental form have been classified by Kazumi Tsukada [7].

Theorem 1 (Tsukada [7])**.** *Any n-dimensional totally complex submanifold $\tilde{N}$ in $\mathbb{H}P^n$ with parallel second fundamental form is locally congruent to one of canonically immersed totally complex submanifolds:*

(0) $\mathbb{C}P^1 \longrightarrow \mathbb{R}P^2 \subset S^4 = \mathbb{H}P^1$ *(Veronese min surf.)*,

(1) $\mathbb{C}P^n \subset \mathbb{H}P^n$ *(totally geodesic)*,

(2) $Sp(3)/U(3) \longrightarrow \mathbb{H}P^6$,

(3) $SU(6)/S(U(3) \times U(3)) \longrightarrow \mathbb{H}P^9$,

(4) $SO(12)/U(6) \longrightarrow \mathbb{H}P^{15}$,

(5) $E_7/E_6 \cdot T^1 \longrightarrow \mathbb{H}P^{27}$,

(6) $\mathbb{C}P^1(\tilde{c}) \times \mathbb{C}P^1(\tilde{c}/2) \longrightarrow \mathbb{H}P^2$,

(7) $\mathbb{C}P^1(\tilde{c}) \times \mathbb{C}P^1(\tilde{c}) \times \mathbb{C}P^1(\tilde{c}) \longrightarrow \mathbb{H}P^3$,

(8) $\mathbb{C}P^1(\tilde{c}) \times \dfrac{SO(n+1)}{SO(2) \times SO(n-1)} \longrightarrow \mathbb{H}P^n \quad (n \geq 4)$.

An R-space is a compact homogeneous space obtained as an orbit of the isotropy representation of a Riemannian symmetric space, i.e. a so-called s-representaion. Olmos-Sánchez [8] showed the differential geometric characterization of R-spaces by means of the parallelism of the second fundamental form with respect to a *canonical connection*. By constructing explicitly such a canonical connection on the inverse image $\hat{N} = \pi^{-1}(N)$, we can obtain the following theorem.

Main Theorem 2. *Assume that N is a $2n$-dimensional totally complex submanifold of $\mathbb{H}P^n$ with parallel second fundamental form. Then its inverse image $\hat{N} = \pi^{-1}(N) \subset S^{4n+3}(1) \subset \mathbb{H}^{n+1}$ is a standardly embedded R-space associated to a quaternion-Kähler symmetric pair (G, K).*

	totally cplx. imm. $\tilde{N}$	q. K. symm. sp. G/K	$\Pi(G,K)$
$\mathbb{H}P^1$	$\mathbb{C}P^1$	$\dfrac{G_2}{(Sp(1)\times Sp(1))/\mathbb{Z}_2}$	G_2
$\mathbb{H}P^n$	$\mathbb{C}P^n$ (totally geodesic)	$\dfrac{SU(n+3)}{S(U(2)\times U(n+1))}$	B_2
$\mathbb{H}P^6$	$\dfrac{Sp(3)}{U(3)}$	$\dfrac{F_4}{(Sp(3)\times Sp(1))/\mathbb{Z}_2}$	F_4
$\mathbb{H}P^9$	$\dfrac{SU(6)}{S(U(3)\times U(3))}$	$\dfrac{E_6}{(SU(6)\times Sp(1))/\mathbb{Z}_2}$	F_4
$\mathbb{H}P^{15}$	$\dfrac{SO(12)}{U(6)}$	$\dfrac{E_7}{(Spin(12)\times SU(2))/\mathbb{Z}_2}$	F_4
$\mathbb{H}P^{27}$	$\dfrac{E_7}{(U(1)\times E_6)/\mathbb{Z}_3}$	$\dfrac{E_8}{(E_7\times SU(2))/\mathbb{Z}_2}$	F_4
$\mathbb{H}P^2$	$\mathbb{C}P^1(\tilde{c})\times\mathbb{C}P^1(\tilde{c}/2)$	$\dfrac{SO(7)}{SO(4)\times SO(3)}$ $(n=2)$	B_3
$\mathbb{H}P^3$	$\mathbb{C}P^1(\tilde{c})\times\mathbb{C}P^1(\tilde{c})\times\mathbb{C}P^1(\tilde{c})$	$\dfrac{SO(8)}{SO(4)\times SO(4)}$ $(n=3)$	D_4
$\mathbb{H}P^n$	$\mathbb{C}P^1(\tilde{c})\times\dfrac{SO(n+1)}{SO(2)\times SO(n-1)}$	$\dfrac{SO(n+5)}{SO(4)\times SO(n+1)}$ $(n\geq 4)$	B_4

Here $\Pi(G, K)$ denotes the Dynkin diagram of the restricted root systems of symmetric pair (G, K).

It gives a new proof of Theorem 1 different from [7].

In the case when $G/K = \dfrac{G_2}{(Sp(1)\times Sp(1))/\mathbb{Z}_2}$, we have $n = 1$, N^2 is a Veronese minimal surface of $\mathbb{H}P^1 = S^4$ and $L^3 = \pi_i(\tilde{L})$ is nothing but the Chiang Lagrangian in $\mathbb{C}P^3$.

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References

[1] R. Chiang, *New Lagrangian submanifolds of* $\mathbb{C}P^n$. Int. Math. Res. Not. **45** (2004), 2437–2441.

[2] B. Y. Chen, F. Dillen, L. Verstraelen, and L. Vrancken, *An exotic totally real minimal immersions of* S^3 *in* $\mathbb{C}P^3$ *and its characterization*, Proc. Royal Soc. Edinburgh Sect. A, Math. **126** (1996), 153–165.

[3] L. Bedulli and A. Gori, *A Hamiltonian stable minimal Lagrangian submanifold of projective space with nonparallel second fundamental form.* Transform. Groups 12 (2007), no. 4, 611–617.

[4] J. D. Evans and Y. Lekili, Floer cohomology of the Chiang Lagrangian. Selecta Math. (N.S.) 21 (2015), no. 4, 1361–1404.

[5] Y. Ohnita, *Stability and rigidity of special Lagrangian cones over certain minimal Legendrian orbits*, Osaka J. Math. **44** no. 2 (2007), 305-334.

[6] Y. Ohnita, *On deformation of 3-dimensional certain minimal Legendrian submanifolds*, Proc. The 13-th International Workshop on Differential Geometry and Related Fields, **13** (2009), pp.71–87, NIMS, KMS and Grassmann Research Group.

[7] K. Tsukada, *Parallel submanifolds in a quaternion projective space.* Osaka J. Math. **22** (1985), 187–241.

[8] C. Olmos and C. Sánchez, *A geometric characterization of the orbits of s-representations.* J. reine angew. Math. **420** (1991), 195–202.

[9] Y. Ohnita, *Parallel Kähler submanifolds and R-spaces.* (submitted), a preprint, OCAMI Preprint Ser. 20-21.

[10] J.-T. Cho, K. Hashimoto and Y. Ohnita, *Totally complex submanifolds and R-spaces.* in preparation.

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Higher dimensional generalization of the Chiang Lagrangian and totally complex submanifolds

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"Submanifolds of Symmetric Spaces and their Time Evolutions"

R. Chiang Lagrangian submanifolds

Definition (IMRN, 2004)

The Chiang Lagrangian submanifold $L^3 \subset \mathbb{C}P^3$ is defined by

$$L^3 := \left\{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{C}P^3 \;\middle|\; \begin{array}{c} 3|z_0|^2 + |z_1|^2 - |z_2|^2 - 3|z_3|^2 = 0 \\ z_0\bar{z_1} + z_1\bar{z_2} + z_2\bar{z_3} = 0 \end{array} \right\}$$

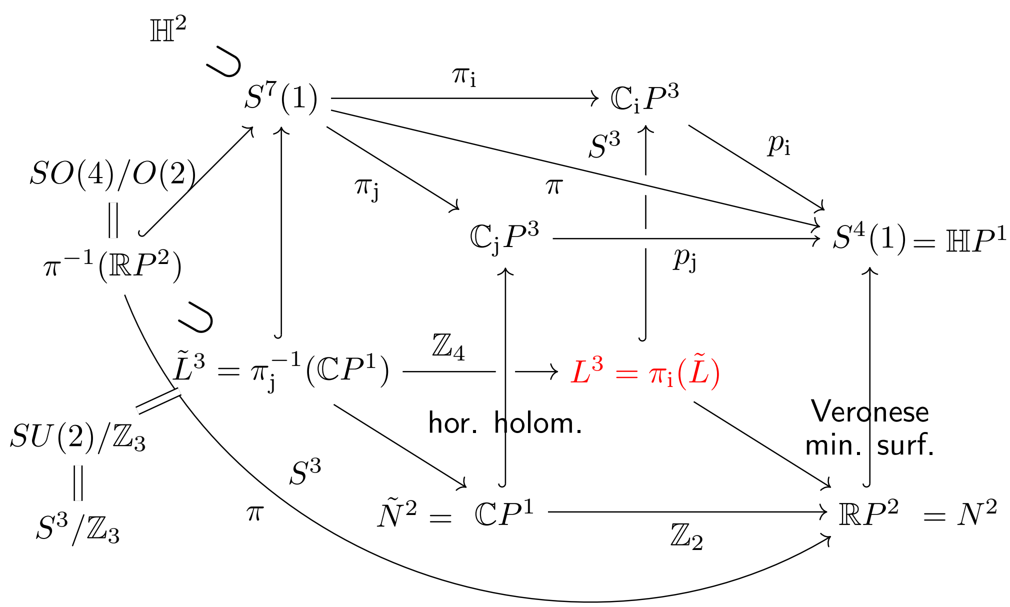
- minimal Lagrangian $SU(2)$ orbit $L^3 = \rho(SU(2))[1:0:0:1]$.
- conn. cpt. embedded minimal Lagranagian submanifold in $\mathbb{C}P^3$.
- L^3 does not possess parallel second fundamental forms. $\nabla^*\alpha^N \neq 0$
- Homogeneous space but not symmetric space.
- Curvature characterization
  (B. Y. Chen, Dillen, Verstraelen, Vrancken, Bolton, 1996)
- Strictly Hamiltonian stable (Ohnita, Bedulli-Gori, 2007)
- Floer homology (Evans-Lekili, 2015), min.Maslov number of $L^3 = 2$

$$\mathbb{H} = \mathbb{R} + \mathbb{R}\mathrm{i} + \mathbb{R}\mathrm{j} + \mathbb{R}\mathrm{k}$$

$$\mathbb{C}_{\mathrm{i}} := \mathbb{R} + \mathrm{i}\mathbb{R} \subset \mathbb{H}, \qquad \mathbb{C}_{\mathrm{j}} := \mathbb{R} + \mathrm{j}\mathbb{R} \subset \mathbb{H}$$

We consider the following diagram:

$$\begin{tikzcd}
 & \mathbb{H}^2 \arrow[d, phantom, "\bigcup"] & \\
 & S^7(1) \arrow[dl, "\pi_{\mathrm{j}}"] \arrow[dr, "\pi_{\mathrm{i}}"'] \arrow[dd, "\pi"', "S^3"] & \\
\mathbb{C}_{\mathrm{j}}P^3 \arrow[dr, "p_{\mathrm{j}}"'] & & \mathbb{C}_{\mathrm{i}}P^3 \arrow[dl, "p_{\mathrm{i}}"] \\
 & S^4(1) &
\end{tikzcd}$$



- $\mathbb{C}P^1 \to \mathbb{C}_{\mathrm{j}}P^3$: Veronese embedding of degree 3.
- $\tilde{L}^3 \subset S^7(1)$: minimal Legendrian submanifold embedded in $S^7(1)$

Higher dimensional generalization

Let $\mathbb{H}^{n+1} \cong \mathbb{R}^{4n+4}$ be an $(n+1)$-dimensional quaternionic vector space with right multiplications by i, j, k. We consider the following standard fibrations:

$$\begin{tikzcd}
 & \mathbb{H}^{n+1} & \\
 & S^{4n+1}(1) \arrow[u, phantom, "\bigcup" description] \arrow[dl, "\pi_{\mathrm{j}}"', "S^1"] \arrow[dd, "\pi"', "S^3"] \arrow[dr, "\pi_{\mathrm{i}}", "S^1"'] & \\
\mathbb{C}_{\mathrm{j}}P^{2n+1} \arrow[dr, "S^2", "p_{\mathrm{j}}"'] & & \mathbb{C}_{\mathrm{i}}P^{2n+1} \arrow[dl, "S^2"', "p_{\mathrm{i}}"] \\
 & \mathbb{H}P^n &
\end{tikzcd}$$

Then $\mathbb{C}_{\mathrm{j}}P^{2n+1}$ has the standard complex contact structure and the holomorphic contact 1-form on $\mathbb{C}_{\mathrm{j}}P^{2n+1}$.
Suppose that $\tilde{N} \to \mathbb{C}_{\mathrm{j}}P^{2n+1}$ is a horizontal holomorphic immersion of an n-dimensional complex manifolds $\tilde{N}$, that is, a complex Legendrian submanifold of $\mathbb{C}_{\mathrm{j}}P^{2n+1}$.

$$\begin{tikzcd}
 & & \mathbb{H}^{n+1} & \\
 & & S^{4n+1}(1) \arrow[u, phantom, "\bigcup" description] \arrow[dl, "\pi_{\mathrm{j}}"', "S^1"] \arrow[dd, "\pi"', "S^3"] \arrow[dr, "\pi_{\mathrm{i}}", "S^1"'] & \\
\tilde{N} \arrow[r, hook, "\text{cplx. Leg.}"'] & \mathbb{C}_{\mathrm{j}}P^{2n+1} \arrow[dr, "S^2", "p_{\mathrm{j}}"'] & & \mathbb{C}_{\mathrm{i}}P^{2n+1} \arrow[dl, "S^2"', "p_{\mathrm{i}}"] \\
 & & \mathbb{H}P^n &
\end{tikzcd}$$

Totally complex submanifold

Let (M^{4n}, g, Q) be a quaternionic Kähler manifold. g is the Riemannian metric on M and Q is a rank 3 subbundle of End TM which satisfies the following conditions:
For each $p \in M$, there is a neighborhood U of p over which there exists a local frame field $\{I, J, K\}$ of Q satisfying

$$I^2 = J^2 = K^2 = -\mathrm{id},$$

$$IJ = -JI = K, JK = -KJ = I, KI = -IK = J.$$

Definition

A submanifold $N^{2k} \subset (M^{4n}, g, Q)$ $(k \leq n)$ is said to be totally complex if, for every $p \in N$, there exists an open neighborhood U of q in N and sections J of $Q|_U$ such that the following properties hold:
(1) $J(TU) = TU$,
(2) $J^2 = -\mathrm{id}$,
(3) $I(T_qN) \subseteq T_q^{\perp}N$ *for every* $q \in U$ *and* $I \in Q_q$ *such that* $J_qI = -IJ_q$,
(4) $\nabla J = 0$.

$\iff$ N has cplx. Leg. submanifolds to twister spase $\mathcal{Z}$.

A horizontal holomorphic map $\tilde{N} \to \mathbb{C}_{\mathrm{j}}P^{2n+1}$ of n-dimensional complex manifold $\tilde{N}$ corresponds to a maximal dimensional totally complex immersion $\tilde{N} \to \mathbb{H}P^n$.

$$\begin{tikzcd}
\mathbb{H}^{n+1} & & & & \\
& S^{4n+3}(1) \arrow[ul, phantom, sloped, "\subset"] \arrow[rr, "\pi_{\mathrm{i}}"] \arrow[dr, "\pi_{\mathrm{j}}"'] \arrow[drrr, "\pi"] & & \mathbb{C}_{\mathrm{i}}P^{2n+1} & \\
& & \mathbb{C}_{\mathrm{j}}P^{2n+1} & & \mathbb{H}P^n \\
& \tilde{L} = \pi_{\mathrm{j}}^{-1}(\tilde{N}) \arrow[uu, hook, "\text{min.}", "\text{Leg}"'] \arrow[rr, "\text{covering}"] \arrow[dr] & & L = \pi_{\mathrm{i}}(\tilde{L}) \arrow[uu, hook, "\text{min.}", "\text{Lag.}"'] & \\
& & \tilde{N}^n \arrow[uu, hook, "\text{cplx.}", "\text{Leg.}"'] \arrow[rr] & & N^{2n} \arrow[uu, hook, "\text{tot.}", "\text{cplx.}"']
\end{tikzcd}$$

Theorem

$$
\begin{tikzcd}
\mathbb{H}^{n+1} \arrow[dr, phantom, sloped, "\supset"] & & & & \\
& S^{4n+3}(1) \arrow[rr, "\pi_{\mathrm{i}}"] \arrow[dr, "\pi_{\mathrm{j}}"'] \arrow[drrr, "\pi"'] & & \mathbb{C}_{\mathrm{i}}P^{2n+1} \arrow[dr, "p_{\mathrm{i}}"] & \\
& & \mathbb{C}_{\mathrm{j}}P^{2n+1} \arrow[rr, "p_{\mathrm{j}}"'] & & \mathbb{H}P^{n} \\
\widehat{N} = \pi^{-1}(N) \arrow[uur, hook] \arrow[dr, phantom, sloped, "\supset"] \arrow[ddrrrr, bend right, "\pi"'] & & & & \\
& \tilde{L}^{2n+1} = \pi_{\mathrm{j}}^{-1}(\tilde{N}) \arrow[uuu, hook, "\text{min.}", "\text{Leg.}"'] \arrow[rr] \arrow[dr, "S^3"'] & & L^{2n+1} = \pi_{\mathrm{i}}(\tilde{L}) \arrow[uuu, hook, "\text{min.}", "\text{Lag.}"', "S^3"' near end] \arrow[dr] & \\
& & \tilde{N}^{n} \arrow[uuu, hook, "\text{cplx.}", "\text{Leg.}"'] \arrow[rr] & & N^{2n} \arrow[uuu, hook, "\text{tot.}", "\text{cplx.}"']
\end{tikzcd}
$$

We use the standard Riemannian fibration π_{i}. Then $\tilde{L}^{2n+1} \to S^{4n+3}(1)$ is a minimal Legendrian immersion relative to I.
Therefore, $\tilde{L}^{2n+1} \to \mathbb{C}_{\mathrm{i}}P^{2n+1}$ is a minimal Lagrangian immersion relative to J, which has the S^1-action induced by the orthogonal complex structure I.

$$
\begin{tikzcd}
\mathbb{H}^{n+1} \arrow[dr, phantom, sloped, "\supset"] & & & \\
& S^{4n+3}(1) \arrow[r, "\pi_{\mathrm{i}}"] \arrow[drr, "\pi"'] & \mathbb{C}_{\mathrm{i}}P^{2n+1} & \\
& & & \mathbb{H}P^{n} \\
& \tilde{L}^{2n+1} = \pi_{\mathrm{j}}^{-1}(\tilde{N}) \arrow[uu, hook, "\text{min.}", "\text{Leg.}"'] \arrow[r, "\text{covering}"] & L^{2n+1} = \pi_{\mathrm{i}}(\tilde{L}) \arrow[uu, hook, "\text{min.}", "\text{Lag.}"', "S^3" near end] &
\end{tikzcd}
$$

Theorem (Tsukada, 1985)

Any n-dimensional totally complex submanifold $\tilde{N}$ in $\mathbb{H}P^n$ with parallel second fundamental form is locally congruent to one of the following immersed totally complex submanifolds:

(0) $\mathbb{C}P^1 \longrightarrow \mathbb{R}P^2 \subset S^4 = \mathbb{H}P^1$ *Veronese min surf.*

(1) $\mathbb{C}P^n \subset \mathbb{H}P^n$ *totally geodesic*

(2) $Sp(3)/U(3) \longrightarrow \mathbb{H}P^6$

(3) $SU(6)/S(U(3) \times U(3)) \longrightarrow \mathbb{H}P^9$

(4) $SO(12)/U(6) \longrightarrow \mathbb{H}P^{15}$

(5) $E_7/((U(1) \times E_6)/\mathbb{Z}_3) \longrightarrow \mathbb{H}P^{27}$

(6) $\mathbb{C}P^1(\tilde{c}) \times \mathbb{C}P^1(\tilde{c}/2) \longrightarrow \mathbb{H}P^2$

(7) $\mathbb{C}P^1(\tilde{c}) \times \mathbb{C}P^1(\tilde{c}) \times \mathbb{C}P^1(\tilde{c}) \longrightarrow \mathbb{H}P^3$

(8) $\mathbb{C}P^1(\tilde{c}) \times \dfrac{SO(n+1)}{SO(2) \times SO(n-1)} \longrightarrow \mathbb{H}P^n \quad (n \geq 4)$

All of those totally complex submanifolds are obtained from the follwing compact homogeneous complex Legendrian submanifolds embedded in $\mathbb{C}P^{2n+1}$.

(1) $\mathbb{C}P^n \subset \mathbb{C}P^{2n+1}$

(2) $Sp(3)/U(3) \longrightarrow \mathbb{C}P^{13}$

(3) $SU(6)/S(U(3) \times U(3)) \longrightarrow \mathbb{C}P^{19}$

(4) $SO(12)/U(6) \longrightarrow \mathbb{C}P^{31}$

(5) $E_7/(U(1) \times E_6)/\mathbb{Z}_3 \longrightarrow \mathbb{C}P^{55}$

(6) $\mathbb{C}P^1(\tilde{c}) \times \mathbb{C}P^1(\tilde{c}/2) \longrightarrow \mathbb{C}P^5$

(7) $\mathbb{C}P^1(\tilde{c}) \times \mathbb{C}P^1(\tilde{c}) \times \mathbb{C}P^1(\tilde{c}) \longrightarrow \mathbb{C}P^7$

(8) $\mathbb{C}P^1(\tilde{c}) \times \dfrac{SO(n+1)}{SO(2) \times SO(n-1)} = \mathbb{C}P^1(\tilde{c}) \times Q_{n-1}(\mathbb{C}) \longrightarrow \mathbb{C}P^{2n+1}$

$(n \geq 4)$

Quaternionic Symmetric space

A Riemannian manifold M is called a quaternionic symmetric space if M satisfies the following conditions:
(i) M is a quaternionic Kähler manifold with quaternionic structure Q.
(ii) M is a symmetric space.
(iii) Q_p is contained in the linear holonomy group for some point p in M.

G	K	$\dim M = G/K$
$Sp(n+1)$	$Sp(n) \times Sp(1)$	$4n$
$SU(n+3)$	$S(U((n+1) \times U(2))$	$4n$
$SO(n+5)$	$SO(n+1) \times SO(4)$	$4n$
G_2	$(Sp(1) \times Sp(1))/\mathbb{Z}_2$	8
F_4	$(Sp(3) \times Sp(1))/\mathbb{Z}_2$	28
E_6	$(SU(6) \times SU(2))/\mathbb{Z}_2$	40
E_7	$(Spin(12) \times SU(2))/\mathbb{Z}_2$	64
E_8	$(E_7 \times SU(2))/\mathbb{Z}_2$	112

Theorem

Suppose that $N^{2n} \subset \mathbb{H}P^n$ *is a totally complex submanifold with* $\nabla^* \alpha^N = 0$.
Then its inverse image $\widehat{N}^{2n+3} = \pi^{-1}(N) \subset S^{4n+3}$ *is an "R-space" associated to a quaternionic symmetric space* G/K *with* $\nabla^* \alpha^{\widehat{N}} \neq 0$.

- $\widehat{N} = \pi^{-1}(N) \subset S^{4n+3}$ is a singular orbit of a quaternionic symmetric space G/K.

$$
\begin{tikzcd}
 & S^{4n+3}(1) \arrow[rr, "\pi_{\mathrm{i}}"] \arrow[dr, "\pi_{\mathrm{j}}"'] \arrow[drrr, "S^3", "\pi"'] & & \mathbb{C}_{\mathrm{i}}P^{2n+1} \arrow[dr, "p_{\mathrm{i}}"] & \\
 & & \mathbb{C}_{\mathrm{j}}P^{2n+1} \arrow[rr, "p_{\mathrm{j}}"'] & & \mathbb{H}P^n \\
\widehat{N} = \pi^{-1}(N) \arrow[uur, hook] \arrow[drrrr, "S^3", "\pi"'] & & & & \\
 & & & & N^{2n} \arrow[uu, hook, "\text{tot. cplx.}"]
\end{tikzcd}
$$

Definition

Let (M, g) be a Riemannian manifold and ∇ Levi-Civita connection of g. A linear connection ∇^c on M is called a **canonical connection** *if it satisfies*:

(1) $\nabla^c g = 0$,

(2) $\nabla^c D = 0$,

where $D := \nabla - \nabla^c$.

Theorem (Olmos-Sánchez, 1991)

Let M be a connected compact Riemannian submanifold of $\mathbb{R}^n$ and let α be its second fundamental form. Then the following three statements are equivalent:

(1) *M admits a canonical connection ∇^c satisfyin $\nabla^c \alpha = 0$.*

(2) *M is a homogeneous submanifold with constant principal curvatures.*

(3) *M is a standard embedding of an R-space.*

On $\widehat{N} = \pi^{-1}(N)$, there exist $\nabla^c := \nabla - D$ such that

$$\nabla^c \alpha^{\widehat{N}} = 0, \qquad \nabla^c D = 0.$$

From Olmos-Sánchez's result we obtain that $\widehat{N}$ is an R-space associated to a quaternionic symmetric space G/K.

We can explicitly construct a tensor field D of type $(1, 2)$ on $\widehat{N}$.
Suppose that N is a totally complex submanifold of $\mathbb{H}P^n$.
For each $\mathrm{x} \in \widehat{N}$, $I^N_{\pi(\mathbf{x})}, J^N_{\pi(\mathbf{x})}, K^N_{\pi(\mathbf{x})} \in \mathbb{C}_{\mathrm{j}}P^{2n+1}$ satisfies

$$I^N_{\pi(\mathbf{x})}(T_{\pi(\mathrm{x})}N) \perp T_{\pi(\mathrm{x})}N, \quad J^N_{\pi(\mathbf{x})}(T_{\pi(\mathrm{x})}N) = T_{\pi(\mathrm{x})}N, \quad K^N_{\pi(\mathbf{x})} \perp T_{\pi(\mathrm{x})}N.$$

There are $\lambda_1(\mathrm{x})$, $\lambda_2(\mathrm{x})$, $\lambda_3(\mathrm{x}) \in Sp(1)$ such that

$$I^N_{\pi(\mathbf{x})}(\mathrm{x}) = \mathrm{x}\lambda_1(\mathrm{x}), \quad J^N_{\pi(\mathbf{x})}(\mathrm{x}) = \mathrm{x}\lambda_2(\mathrm{x}), \quad K^N_{\pi(\mathbf{x})}(\mathrm{x}) = \mathrm{x}\lambda_3(\mathrm{x}).$$

We define a tensor field D of type $(1,2)$ as follow:
At $\mathbf{x} \in \widehat{N}$, for each X, $Y \in T_{\pi(\mathbf{x})}N$, $V \in \mathcal{V}_x\widehat{N}$, v_1, v_2, $v_3 \in \mathbb{R}$,

- $D_{\tilde{X}}(\tilde{Y}) = \langle \widetilde{(J^N_{\pi(\mathbf{x})}X)}, \tilde{Y}\rangle J^N_{\pi(\mathbf{x})}(\mathbf{x}) = g_N(J^N_{\pi(\mathbf{x})}X, Y) J^N_{\pi(\mathbf{x})}(\mathbf{x}) \in \mathcal{V}_{\mathbf{x}}\hat{N}$,
- $D_{\tilde{X}}(V) = v_2 \tilde{X}\lambda_2(\mathbf{x}) = -v_2 \widetilde{(J^N_{\pi(\mathbf{x})}X)} \in \mathcal{H}_{\mathbf{x}}\hat{N} \quad ({}^\forall V = \mathbf{x}\left(\sum_{a=1}^{3} v_a\lambda_a(\mathbf{x})\right))$,
- $D_V(\tilde{X}) = \dfrac{v_2}{2}\tilde{X}\lambda_2(\mathbf{x}) = -\dfrac{v_2}{2}\widetilde{(J^N_{\pi(\mathbf{x})}X)} \in \mathcal{H}_{\mathbf{x}}\hat{N} \quad ({}^\forall V = \mathbf{x}\left(\sum_{a=1}^{3} v_a\lambda_a(\mathbf{x})\right))$,
- $$D_U(V) = \mathbf{x}\left\{(v_2u_3 + \frac{1}{2}v_3u_2)\lambda_1(\mathbf{x}) + (v_3u_1 - v_1u_3)\lambda_2(\mathbf{x}) + (-\frac{1}{2}v_1u_2 - v_2u_1)\lambda_3(\mathbf{x})\right\}$$
$$= \mathbf{x}\begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & -\frac{1}{2}u_2 & u_3 \\ \lambda_1(\mathbf{x}) & \lambda_2(\mathbf{x}) & \lambda_3(\mathbf{x}) \end{vmatrix} \in \mathcal{V}_{\mathbf{x}}\hat{N}.$$

Here note that
$$\widetilde{(J^N_{\pi(\mathbf{x})}X)}_{\mathbf{x}} = (J^N_{\pi(\mathbf{x})}X)(\mathbf{x}) = -(\tilde{X})_{\mathbf{x}}\lambda_2(\mathbf{x}).$$

For each w_1, w_2, $w_3 \in T\widehat{N}$

Lemma

- $g_{\widehat{N}}(D_{w_2}, w_3) + g_{\widehat{N}}(w_2, D_{w_1}w_3) = 0 \quad$ i.e. $\nabla^c g = 0$
- $\nabla^{\widehat{N}} D = D \cdot D \quad$ i.e. $\nabla^c D = 0$

Lemma

Suppose that N is a $2n$-dimensional totally complex submanifold of $\mathbb{H}P^n$ with parallel second fundamental form. Then the second fundamental form $\alpha^{\widehat{N}}$ of $\widehat{N}$ satisfies the equation

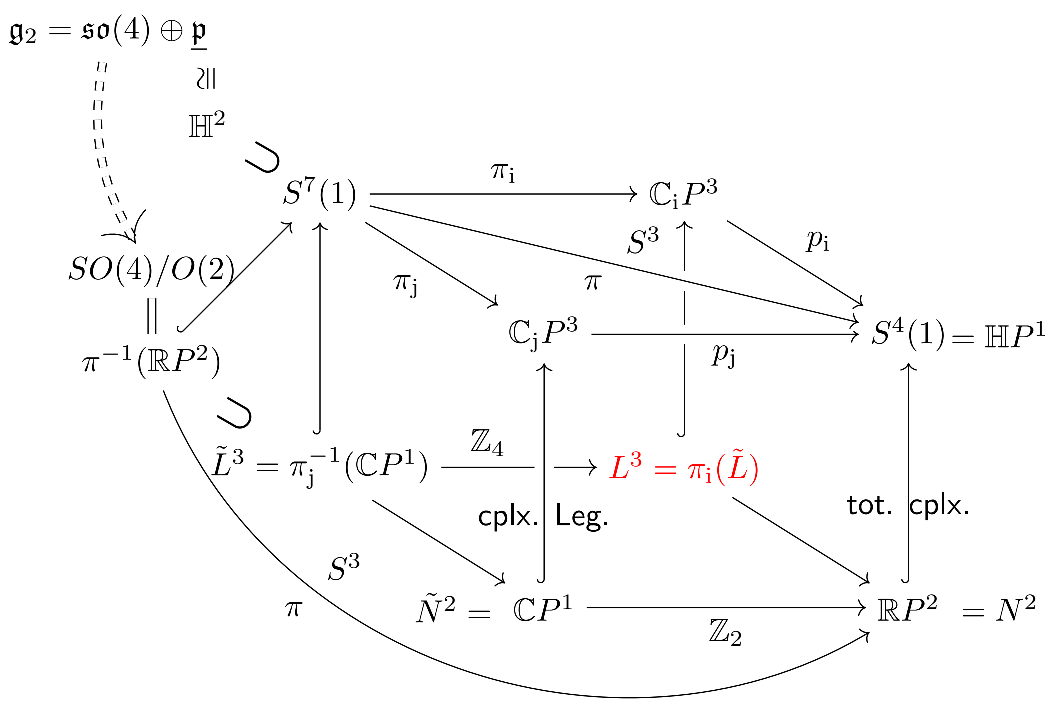
$$\nabla^*_{w_1}\alpha^{\widehat{N}}(w_2, w_3) + \alpha^{\widehat{N}}(D_{w_1}w_2, w_3) + \alpha^{\widehat{N}}(w_2, D_{w_1}w_3) = 0.$$

i.e. $\quad \nabla^c\alpha^{\widehat{N}} = 0$

totally cplx. imm. $\tilde{N}$	q. symm. sp. G/K
$\mathbb{C}P^1$	$\dfrac{G_2}{(Sp(1)\times Sp(1))/\mathbb{Z}_2}$
$\mathbb{C}P^n$	$\dfrac{SU(n+1)}{S(U(2)\times U(n))}$
$\dfrac{Sp(3)}{U(3)}$	$\dfrac{F_4}{(Sp(3)\times Sp(1))/\mathbb{Z}_2}$
$\dfrac{SU(6)}{S(U(3)\times U(3))}$	$\dfrac{E_6}{(SU(6)\times Sp(1))/\mathbb{Z}_2}$
$\dfrac{SO(12)}{U(6)}$	$\dfrac{E_7}{(Spin(12)\times SU(2))/\mathbb{Z}_2}$
$\dfrac{E_7}{(U(1)\times E_6)/\mathbb{Z}_3}$	$\dfrac{E_8}{(E_7\times SU(2))/\mathbb{Z}_2}$
$\mathbb{C}P^1(\tilde{c})\times\mathbb{C}P^1(\tilde{c}/2)$	$\dfrac{SO(7)}{SO(4)\times SO(3)}$ $(n=2)$
$\mathbb{C}P^1(\tilde{c})\times\mathbb{C}P^1(\tilde{c})\times\mathbb{C}P^1(\tilde{c})$	$\dfrac{SO(8)}{SO(4)\times SO(4)}$ $(n=3)$
$\mathbb{C}P^1(\tilde{c})\times\dfrac{SO(n+1)}{SO(2)\times SO(n-1)}$	$\dfrac{SO(n+5)}{SO(4)\times SO(n+1)}$ $(n\geq 4)$

$(G,K)=(G_2,SO(4))$

In this case we obtain the Chiang Lag submfd as $L^3=\pi_{\mathrm{i}}(\pi_{\mathrm{j}}^{-1}(\tilde{N}))$.



References I

[1] R. Chiang, *New Lagrangian submanifolds of* $\mathbb{C}P^n$. Int. Math. Res. Not. **45**, 2437–2441 (2004)

[2] B. Y. Chen, F. Dillen, L. Verstraelen, and L. Vrancken, *An exotic totally real minimal immersions of* S^3 *in* $\mathbb{C}P^3$ *and its characterization*, Proc. Royal Soc. Edinburgh Sect. A, Math. **126** (1996), 153–165.

[3] L. Bedulli and A. Gori, *A Hamiltonian stable minimal Lagrangian submanifold of projective space with nonparallel second fundamental form*. Transform. Groups 12 (2007), no. 4, 611–617.

[4] J. D. Evans and Y. Lekili, *Floer cohomology of the Chiang Lagrangian*, Selecta Math. (N.S.) 21 (2015), no. 4, 1361–1404.

[5] Y. Ohnita, *Stability and rigidity of special Lagrangian cones over certain minimal Legendrian orbits*, Osaka J. Math. **44** no. 2 (2007), 305-334.

References II

[6] Y. Ohnita, *On deformation of 3-dimensional certain minimal Legendrian submanifolds*, Proc. The 13-th International Workshop on Differential Geometry and Related Fields, **13** (2009), pp.71–87, NIMS, KMS and Grassmann Research Group.

[7] K.Tsukada, *Parallel submanifolds in a quaternion projective space*. Osaka J. Math. **22** (1985), 187–241.

[8] C. Olmos and C. Sánchez, *A geometric characterization of the orbits of* s*-representations*. J. reine angew. Math. **420** (1991), 195–202.

[9] Y. Ohnita, *Parallel Kähler submanifolds and* R*-spaces*. (submitted), a preprint, OCAMI Preprint Ser. 20-21.

[10] J.-T. Cho, K. Hashimoto and Y. Ohnita, *Totally complex submanifolds and* R*-spaces*. in preparation.

Thank you very much for your kind attention!!

Minimal PF submanifolds in Hilbert spaces with symmetries

Masahiro Morimoto

There are several kinds of minimal submanifolds which have certain symmetries. A submanifold M of a Riemannian manifold N is called *austere* ([1]) if for each normal vector ξ the eigenvalues with multiplicities of the shape operator A_ξ is invariant under the multiplication by (-1). M is called *reflective* ([3]) if it is a connected component of the fixed point set of an involutive isometry of N. M is called *weakly reflective* ([2]) if for each normal vector ξ at each $p \in M$ there exists an isometry ν_ξ of N satisfying the conditions $\nu_\xi(p) = p$, $d\nu_\xi(\xi) = -\xi$ and $\nu_\xi(M) = M$. From these definitions we have

$$\text{reflective} \quad \Rightarrow \quad \text{weakly reflective} \quad \Rightarrow \quad \text{austere} \quad \Rightarrow \quad \text{minimal.}$$

It is an interesting problem to classify or give examples of these minimal submanifolds.

A fundamental class of submanifolds in Hilbert spaces is given by *proper Fredholm* (PF) submanifolds in Hilbert spaces ([6]). Roughly speaking they are submanifolds in Hilbert spaces where the shape operators are compact operators and the distance functions are compatible with the Palais-Smale condition. Many examples of PF submanifolds are obtained through a Riemannian submersion $\Phi_K : V_\mathfrak{g} \to G/K$ which is called the *parallel transport map* ([7]). Here G/K is a compact normal homogeneous space and $V_\mathfrak{g} := L^2([0,1], \mathfrak{g})$ the Hilbert space of all L^2-paths with values in the Lie algebra $\mathfrak{g}$ of G. It is known that if M is a compact submanifold of G/K then the inverse image $\Phi_K^{-1}(M)$ is a PF submanifold of $V_\mathfrak{g}$.

In my talk I will introduced the concept of reflective submanifolds, weakly reflective submanifolds and austere submanifolds into the class of PF submanifolds in Hilbert spaces and show that under suitable condition if M is a weakly reflective submanifold of G/K then the inverse image $\Phi_K^{-1}(M)$ is a weakly reflective PF submanifold of $V_\mathfrak{g}$.

References

[1] R. Harvey and H. B. Lawson, Jr., *Calibrated geometries*, Acta Math., **148** (1982), 47-157.

[2] O. Ikawa, T. Sakai, H. Tasaki, *Weakly reflective submanifolds and austere submanifolds.* J. Math. Soc. Japan **61** (2009), no. 2, 437-481.

[3] Dominic S. P. Leung, *The reflection principle for minimal submanifolds of Riemannian symmetric spaces*, J. Differential Geom. **8** (1973), 153-160.

[4] M. Morimoto, *On weakly reflective PF submanifolds in Hilbert spaces*, to appear in Tokyo J. Math.

[5] M. Morimoto, *Austere and arid properties for PF submanifolds in Hilbert spaces*, Differential Geom. Appl., Vol. 69 (2020) 101613.

[6] C.-L. Terng, *Proper Fredholm submanifolds of Hilbert space.* J. Differential Geom. **29** (1989), no. 1, 9-47.

[7] C.-L. Terng, G. Thorbergsson, *Submanifold geometry in symmetric spaces.* J. Differential Geom. **42** (1995), no. 3, 665-718.

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Sec. 1 Sec. 2 Sec. 3 Sec. 4 Sec. 5 Sec. 6

Minimal PF submanifolds in Hilbert spaces with symmetries

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Submanifolds of Symmetric Space and Their Time Evolution
March 5 - 6, 2021

Sec. 1 Sec. 2 Sec. 3 Sec. 4 Sec. 5 Sec. 6

Overview

M	$\hookrightarrow$	$\mathbb{E}^n$
submanifold		Euclidean sp.

$\swarrow$ $\searrow$

M	$\hookrightarrow$	N
submanifold		Riemannian mfd (finite dim)

M	$\hookrightarrow$	V
submanifold		Hilbert sp. (infinite dim)

minimal submanifold with symmetry

- reflective submanifold
- weakly reflective submanifold $\longrightarrow$
- austere submanifold

we define and study

- reflective submanifold
- weakly reflective submanifold
- austere submanifold

Plan

1. Minimal submanifolds with symmetries
2. Submanifolds in Hilbert spaces
3. The parallel transport map
4. Submanifold geometries via the parallel transport map
5. Minimal PF submanifolds with symmetries
6. Symmetric properties
   via the parallel transport map (: Main results)

Sec. 1 - Minimal submanifolds with symmetries (1/4)

Definition (Harvey-Lawson 1982)

Let

$$\underset{\text{submanifold}}{M} \hookrightarrow \underset{\text{Riem. manifold.}}{N}$$

M is austere

$\underset{\text{def}}{\Leftrightarrow}$ $\forall p \in M,\ \forall \xi \in T_p^\perp M$, the eigenvalues (with multiplicities) of the shape operator A_ξ^M is invariant under the multiplication by (-1).

Proposition

$$\text{austere} \implies \text{minimal}$$

Problem

Classify (or, find examples of) austere submanifolds.

Sec. 1 - Minimal submanifolds with symmetries (2/4)

Theorem (F. Podestà 1997)

G: Lie group, N: Riemannian manifold.
Suppose $G \curvearrowright N$: isometric action of cohomogeneity one.
(i.e. codim (principal G-orbit) = 1).
$\Rightarrow$ any singular (i.e. non-principal) G-orbit is an austere submanifold of N.

Note

More precisely, Podestà proved that any singular G-orbit M satisfies:
$\forall p \in M$, $\forall \xi \in T_p^\perp M$, there exists $\nu_\xi \in \mathrm{Isom}(N)$ satisfying

$$\nu_\xi(p) = p, \qquad \nu_\xi(M) = M, \qquad d\nu_\xi(\xi) = -\xi.$$

Sec. 1 - Minimal submanifolds with symmetries (3/4)

Definition (Ikawa-Sakai-Tasaki 2009)

Let
$$\underset{\text{submanifold}}{M} \hookrightarrow \underset{\text{Riem. manifold.}}{N}$$

M is weakly reflective
$\underset{\text{def}}{\Leftrightarrow}$ $\forall p \in M$, $\forall \xi \in T_p^\perp M$, there exists $\nu_\xi \in \mathrm{Isom}(N)$ satisfying

$$\nu_\xi(p) = p, \qquad \nu_\xi(M) = M, \qquad d\nu_\xi(\xi) = -\xi.$$

Proposition (Ikawa-Sakai-Tasaki 2009)

reflective $\Rightarrow$ weakly reflective $\Rightarrow$ austere $\Rightarrow$ minimal
(D.S. Leung 1973)

Problem

Classify (or, find examples of) weakly reflective submanifolds.

Sec. 1 - Minimal submanifolds with symmetries (4/4)

Study on weakly reflective submanifolds

- Ikawa-Sakai-Tasaki (2009) classified weakly reflective submanifolds and austere submanifolds which are orbits of s-representations.
- Ohno (2016) gave examples of weakly reflective submanifolds which are orbits of Hermann actions $K_2 \curvearrowright G/K_1$.
- Enoyoshi (2018) showed that there exists unique weakly reflective principal orbits in the cohomogeneity one action $G_2 \curvearrowright \widetilde{Gr}_3(\mathrm{Im}\,\mathbb{O})$

Note

All known examples of weakly reflective submanifolds are homogeneous.

Sec. 2 - Submanifolds in Hilbert spaces (1/6)

Basic Setting

Suppose: $\forall p \in M$, T_pM is a closed subsp. of V. $(\rightsquigarrow T^{\perp}M)$

smooth immersion		
M	$\hookrightarrow$	V
Hilbert manifold		separable Hilbert sp.
g	$\leftrightsquigarrow$	$\langle\cdot,\cdot\rangle$
Riem. metric		inner product
$\wr$		$\wr$
∇		D
Levi-Civita conn.		Levi-Civita conn.

second fundm. form α, shape op. A, normal conn. $\nabla^{\perp}$ is defined:

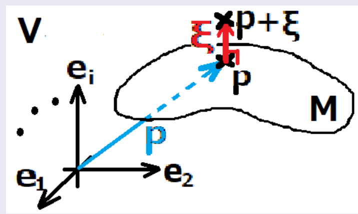
$$\begin{cases} \diamond \quad \text{Gauss formula.} & D_XY = \nabla_XY + \alpha(X,Y), \quad X, Y \in \Gamma(TM). \\ \diamond \quad \text{Weingarten formula} & D_X\xi = -A_\xi X + \nabla^{\perp}_X\xi, \quad \xi \in \Gamma(T^{\perp}M). \end{cases}$$

Difficulty: Spectral theory of shape op. $A_\xi : T_pM \to T_pM$.

Sec. 2 - Submanifolds in Hilbert spaces (2/6)

Suppose (C.-L. Terng 1989)

- M has finite codimension in V.
- The end point map $Y : T^{\perp}M \to V$, $(p, \xi) \mapsto p + \xi$



satisfies: $\forall r > 0$, restriction to D_r (: normal disc bdl of radius r)

$$Y|D_r \ : \ D_r \ \to \ V$$

is proper and Fredholm (i.e. differential is a Fredholm op.).

Then M is called a proper Fredholm submanifold (PF submanifold).

Sec. 2 - Submanifolds in Hilbert spaces (3/6)

Example

- If $V = \mathbb{E}^n$ then
  $M \hookrightarrow \mathbb{E}^n$ is PF $\Leftrightarrow$ immersion $M \hookrightarrow \mathbb{E}^n$ is proper.
- If $\dim V = \infty$, then the unit sphere if V is not PF.
- Every affine subspace of a Hilbert space V is PF.
- Every orbit of the gauge transformations is PF. (page 6/6)

Sec. 2 - Submanifolds in Hilbert spaces (4/6)

Proposition (C.-L. Terng 1989)

If M is a PF submanifold of V, then
(1) the shape operator $A_\xi : T_pM \to T_pM$ is a self-adjoint compact op.
(2) for each $u \in V$, the function $f_u : M \to \mathbb{R}$, $p \mapsto \|p-u\|^2$
satisfies Condition C (Palais-Smale 1960s).

Remark

- $A_\xi : T_pM \to T_pM$ real eigenvalues
$$\underbrace{\mu_1 < \mu_2 < \cdots}_{\text{finite multip.}} < 0 < \underbrace{\cdots < \lambda_2 < \lambda_1}_{\text{finite multip}}$$
These are called the principal curvatures of M in direction ξ.
- the shape op. $A_\xi : T_pM \to T_pM$ is not of trace class in general
(There is no natural definition for the mean curvature.)

Sec. 2 - Submanifolds in Hilbert spaces (5/6)

Definition (King-Terng 1993, Heintze-Liu-Olmos 2006, Koike 2002)

M: PF submanifold, $\xi \in T^\perp M$, $A_\xi : T_pM \to T_pM$: shape operator.
◇ $\mu_1 \le \mu_2 \le \cdots < 0 < \cdots \le \lambda_2 \le \lambda_1$: eigenvalues of A_ξ
◇ $\{\kappa_k\}_{k=1}^\infty$: distinct eigenvalues of A_ξ s.t. $|\kappa_k| > |\kappa_{k+1}|$ or $\kappa_k = -\kappa_{k+1}$.

(1) ζ-regularized mean curvature $\mathrm{tr}_\zeta A_\xi := \lim_{s \searrow 1}(\sum_{k=1}^\infty \lambda_k^s - \sum_{k=1}^\infty |\mu_i|^s)$
(2) regularized mean curvature $\mathrm{tr}_r A_\xi := \sum_{k=1}^\infty (\lambda_k + \mu_k)$
(3) formal mean curvature $\mathrm{tr}_f A_\xi := \sum_{k=1}^\infty \mathrm{multip}(\kappa_k)\kappa_k$

(1) M is ζ-minimal $\underset{\text{def}}{\Leftrightarrow}$ $\forall \xi \in T^\perp M$, $\mathrm{tr}_\zeta A_\xi = 0$
(2) M is r-minimal $\underset{\text{def}}{\Leftrightarrow}$ $\forall \xi \in T^\perp M$, $\mathrm{tr}_r A_\xi = 0$ (and $\mathrm{tr}\, A^2 < \infty$)
(3) M is f-minimal $\underset{\text{def}}{\Leftrightarrow}$ $\forall \xi \in T^\perp M$, $\mathrm{tr}_f A_\xi = 0$

Sec. 2 - Submanifolds in Hilbert spaces (6/6)

Example (Terng 1989, Pinkall-Thorbergsson 1990, Terng 1995)

: orbits of the gauge transformations.

G: conn. compact Lie group with bi-inv. Riem met., $\mathfrak{g}$: its Lie alg.
$P \to [0,1]$: trivial principal G-bundle (i.e. $P := [0,1] \times G$).

(gauge transf. gp) (transitive) gauge transf. (connections of P)

$\mathcal{G} := H^1([0,1], G) \curvearrowright V_{\mathfrak{g}} := L^2([0,1], \mathfrak{g})$

$g * u := gug^{-1} - g'g^{-1}$ $(g \in \mathcal{G},\ u \in V_{\mathfrak{g}})$

path group (Hilbert Lie group) $\cup$ $\quad \| \quad$ path sp. (Hilbert sp.)

subgroup $P(G,H) \curvearrowright V_{\mathfrak{g}} := L^2([0,1], \mathfrak{g})$

$\quad \| \qquad\qquad\qquad \cup$

$\{g \in \mathcal{G} \mid (g(0), g(1)) \in H\}$ $\qquad$ $P(G,H)$-orbit $\uparrow$

($\forall H$: closed subgp of $G \times G$) $\qquad$ PF submanifold !

Sec. 3 - The parallel transport map (1/5)

Setting

G: conn. compact Lie group with bi-invariant Riem met., $\mathfrak{g}$: its Lie algebra.
$\mathcal{G} := H^1([0,1], G)$: path group of all Sobolev H^1-paths from $[0,1]$ to G.
$V_{\mathfrak{g}} := L^2([0,1], \mathfrak{g})$: path space of all L^2-paths from $[0,1]$ to $\mathfrak{g}$.

Definition (Terng 1995, Terng-Thorbergsson 1995)

The parallel transport map is defined by

$$\begin{array}{ccccc} \Phi & : & V_{\mathfrak{g}} & \to & G \\ & & \Psi & & \Psi \\ & & u & \mapsto & \Phi(u) \overset{\text{def}}{:=} g_u(1). \end{array}$$

Here, $g_u \in \mathcal{G}$ is defined by the ODE $\begin{cases} g_u^{-1} g_u' = u, \\ g_u(0) = e \in G. \end{cases}$

Sec. 1 Sec. 2 **Sec. 3** Sec. 4 Sec. 5 Sec. 6

Sec. 3 - The parallel transport map (2/5)

Definition

Define the map $\Psi : \mathcal{G} \to G \times G$ by $\Psi(g) := (g(0), g(1))$ for $g \in \mathcal{G}$.

Proposition (Terng 1995)

The parallel transport map $\Phi : V_{\mathfrak{g}} \to G$ is equivariant with respect to Ψ.
That is, the following diagram commutes:

gauge transformation

$g * u := gug^{-1} - g' g^{-1}$

$$\begin{array}{ccc} \mathcal{G} & \curvearrowright & V_{\mathfrak{g}} \\ \Psi \downarrow & & \Phi \downarrow \\ G \times G & \curvearrowright & G \end{array}$$

$(b_1, b_2) \cdot a := b_1 a b_2^{-1}$

isometric action

That is, $\Phi(g * u) = \Psi(g) \cdot \Phi(u)$.

Sec. 1 Sec. 2 **Sec. 3** Sec. 4 Sec. 5 Sec. 6

Sec. 3 - The parallel transport map (3/5)

Theorem (Terng-Thorbergsson 1995)

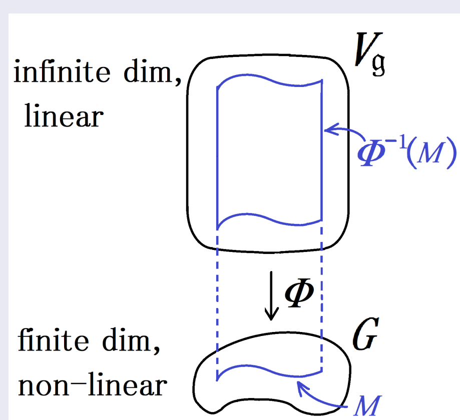
(1) Φ is a Riemannian submersion,

(2) any two fibers of Φ are congruent under the isometry of $V_{\mathfrak{g}}$,

(3) Φ is a principal $P(G, \{e\} \times \{e\})$-bdl,

(4) If M is a closed submfd of G, then $\Phi^{-1}(M)$ is a PF submfd of $V_{\mathfrak{g}}$.

(5) If $M = H \cdot a$ for subgp $H \subset G \times G$ then $\Phi^{-1}(M) = P(G, H) * u$ for $u \in \Phi^{-1}(a)$.



Nice points

(1) We can obtain examples of (homogeneous) PF submanifolds.
(2) We can linearize geometrical problems of submanifold $M \subset G$.

Sec. 1 Sec. 2 **Sec. 3** Sec. 4 Sec. 5 Sec. 6

Sec. 3 - The parallel transport map (4/5)

Generalization (Terng-Thorbergsson1995)

G/K: compact normal homg. sp, $\pi : G \to G/K$: projection.
The parallel transport map over G/K

$$\Phi_{G/K} \overset{\text{def}}{:=} \pi \circ \Phi : V_{\mathfrak{g}} \to G \to G/K.$$

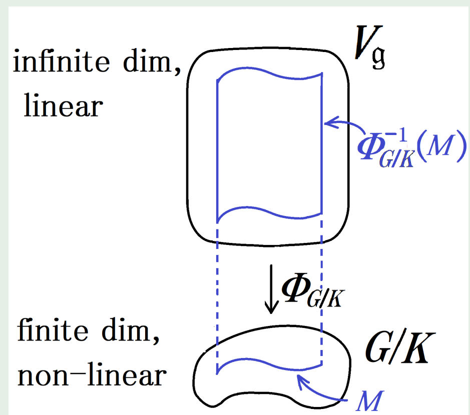
(1) $\Phi_{G/K}$: a Riemannian submersion,

(2) Two fibers of $\Phi_{G/K}$ are congruent under the isometry of $V_{\mathfrak{g}}$,

(3) $\Phi_{G/K}$: a principal $P(G, \{e\} \times K)$-bdl.

(4) If M is a closed submfd of G/K, then $\Phi_{G/K}^{-1}(M)$ is a PF submfd of $V_{\mathfrak{g}}$.

(5) If $M = K' \cdot aK$ for subgp $K' \subset G$ then $\Phi_{G/K}^{-1}(M) = P(G, K' \times K) * u$ for $u \in \Phi_{G/K}^{-1}(aK)$.

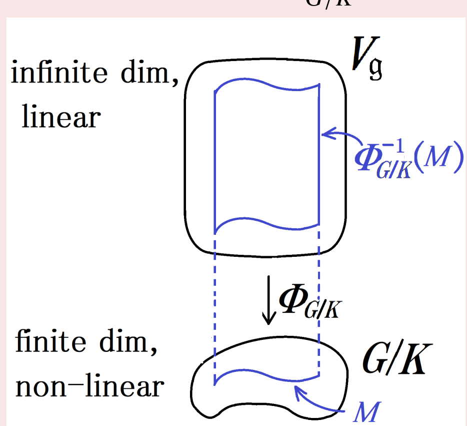


Sec. 1 Sec. 2 **Sec. 3** Sec. 4 Sec. 5 Sec. 6

Sec. 3 - The parallel transport map (5/5)

Fundamental problem

: The geometrical relation between M and $\Phi_{G/K}^{-1}(M)$?



E.g. • M: minimal $\Rightarrow \Phi_{G/K}^{-1}(M)$ is ζ-minimal and r-minimal.
(King-Terng 1993, Heintze-Liu-Olmos 2006).

• Suppose G/K: symmetric space of compact type.
M: equifocal $\Rightarrow \Phi_{G/K}^{-1}(M)$:isoparametric (Terng-Thorbergsson1995).

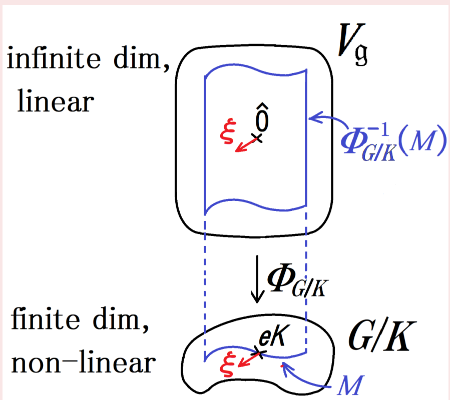
Sec. 1 Sec. 2 Sec. 3 **Sec. 4** Sec. 5 Sec. 6

Sec. 4 - Submanifold geometries via the p.t.m. (1/5)

Problem

G/K : cpt. normal homog. sp., $\mathfrak{g} = \mathfrak{k} + \mathfrak{n}$: orthogonal decomp.

- M : closed submanifold of G/K through $eK \in M$.
- Fix a normal vector $\xi \in T^{\perp}_{eK}M \cong T^{\perp}_{\hat{0}}\Phi^{-1}_{G/K}(M)$.



$\rightsquigarrow \begin{cases} \alpha^{\Phi^{-1}_{G/K}(M)} & : \text{ second f. form} \\ A_{\xi}^{\Phi^{-1}_{G/K}(M)} & : \text{ shape op.} \\ \{\mu\} & : \text{ principal curv.} \end{cases}$

$\rightsquigarrow \begin{cases} \alpha^{M} & : \text{ second f. form} \\ A_{\xi}^{M} & : \text{ shape op.} \\ \{\lambda\} & : \text{ principal curv.} \end{cases}$

The relation ?

Sec. 1 Sec. 2 Sec. 3 **Sec. 4** Sec. 5 Sec. 6

Sec. 4 - Submanifold geometries via the p.t.m. (2/5)

Theorem (M. 2019) : the second fundamental form

$\forall X, Y \in T_{\hat{0}}\Phi^{-1}_{G/K}(M)$,

$$\begin{aligned}\alpha^{\Phi^{-1}_{G/K}(M)}(X, Y) = \alpha^{M}\left(\int_0^1 X(t)_{\mathfrak{n}}dt, \int_0^1 Y(t)_{\mathfrak{n}}dt\right)\\ +\tfrac{1}{2}\left[\int_0^1 X(t)_{\mathfrak{k}}dt, \int_0^1 Y(t)_{\mathfrak{n}}dt\right]^{\perp} - \tfrac{1}{2}\left[\int_0^1 X(t)_{\mathfrak{n}}dt, \int_0^1 Y(t)_{\mathfrak{n}}dt\right]^{\perp}\\ +\tfrac{1}{2}\left[\int_0^1 X(t)dt, \int_0^1 Y(t)dt\right]^{\perp} - \left(\int_0^1 \left[\int_0^t X(s)ds, Y(t)\right]dt\right)^{\perp}.\end{aligned}$$

Theorem (M. 2019) : the shape operator

$\forall X \in T_{\hat{0}}\Phi^{-1}_{G/K}(M)$, $\xi \in T^{\perp}_{\hat{0}}\Phi^{-1}_{G/K}(M)$,

$$\begin{aligned}A_{\xi}^{\Phi^{-1}_{G/K}(M)}(X) = A_{\xi}^{M}\left(\int_0^1 X(t)_{\mathfrak{n}}dt\right) - \tfrac{1}{2}\left[\int_0^1 X(t)_{\mathfrak{n}}dt, \xi\right]_{\mathfrak{k}} + \tfrac{1}{2}\left[\int_0^1 X(t)_{\mathfrak{k}}dt, \xi\right]^{\top}\\ -\tfrac{1}{2}\left[\int_0^1 X(t)dt, \xi\right]^{\top} + \left[\int_0^t X(s)ds, \xi\right] - \left[\int_0^1 \int_0^t X(s)dsdt, \xi\right]^{\perp}.\end{aligned}$$

Sec. 4 - Submanifold geometries via the p.t.m. (3/5)

Theorem (Koike 2002, M. 2019)：Principal curvatures

G/K：compact symmetric sp. $\mathfrak{g} = \mathfrak{k} + \mathfrak{n}$: canonical decomp.
Suppose M：curvature adapted submfd of G/K.
(i.e. $\mathrm{ad}(\xi)^2 : \mathfrak{n} \to \mathfrak{n}$ preserves $T_{eK}M$
and commutes with $A^M_\xi : T_{eK}M \to T_{eK}M$.)
$\{\lambda\}$: eigenvalue of A^M_ξ, $\{\sqrt{-1}\nu\}$: eigenvalue of $\mathrm{ad}(\xi) : \mathfrak{g} \to \mathfrak{g}$.
Then the principal curvatures of $\Phi^{-1}_{G/K}(M)$ in direction ξ is

$$\left\{0,\ \lambda,\ \frac{\nu}{n\pi},\ \frac{\nu}{\arctan\frac{\nu}{\lambda} + m\pi}\right\} \quad \lambda,\ \nu > 0,\ n \in \mathbb{Z}\backslash\{0\},\ m \in \mathbb{Z}.$$

eigenfunctions and multiplicities are given in the next page:

Sec. 4 - Submanifold geometries via the p.t.m. (4/5)

Theorem (Koike 2002, M. 2019)：Principal curvatures

Set $\mu(\nu, \lambda, m) := \dfrac{\nu}{\arctan\frac{\nu}{\lambda} + m\pi}$.

eigenval.	basis of eigenfunctions	multip.
0	$\{x_i^0 \sin n\pi t, y_j^{(0,\lambda)} \cos n\pi t, y_l^{(0,\perp)} \cos n\pi t\}_{n\in\mathbb{Z}_{\geq 1}, \lambda, i, j, l}$	∞
λ	$\{y_j^{(0,\lambda)}\}_j$	$m(0, \lambda)$
$\dfrac{\nu}{n\pi}$	$\{x_r^{(\nu,\perp)} \sin n\pi t - y_r^{(\nu,\perp)} \cos n\pi t\}_r$	$m(\nu, \perp)$
$\mu(\nu, \lambda, m)$	$\left\{\sum_{n\in\mathbb{Z}} \frac{\nu}{n\pi\mu+\nu}(x_k^{(\nu,\lambda)} \sin n\pi t + y_k^{(\nu,\lambda)} \cos n\pi t)\right\}_k$	$m(\nu, \lambda)$

Sec. 4 - Submanifold geometries via the p.t.m. (5/5)

By using the formula of the shape operator, we obtain:

Theorem (M. 2019): The totally geodesic property

The following are equivalent:
(1) $\Phi_{G/K}^{-1}(M)$ is a totally geodesic PF submanifold of $V_{\mathfrak{g}}$
(i.e. an affine subspace of $V_{\mathfrak{g}}$).
(2) M is a totally geodesic submfd of G/K s.t. $T_{eK}^{\perp}M \subset$ (center of $\mathfrak{g}$).

Thus $\Phi_{G/K}^{-1}(M)$ is not totally geodesic, except for rare cases

Sec. 5 - Minimal PF submanifolds with symmetries (1/4)

Definition

Let

$$\underset{\text{PF submfd}}{M} \hookrightarrow \underset{\text{separable Hilbert sp.}}{V}$$

- M is reflective
$\underset{\text{def}}{\Leftrightarrow} \exists\, \sigma : V \to V$: involutive isometry s.t. $M = \mathrm{Fix}(\sigma)_0$.
- M is weakly reflective
$\underset{\text{def}}{\Leftrightarrow} \forall p \in M, \forall \xi \in T_p^{\perp}M, \exists \nu_{\xi} : V \to V$: isometry
s.t. (i) $\nu_{\xi}(p) = p$, (ii) $d\nu_{\xi}(\xi) = -\xi$, (iii) $\nu_{\xi}(M) = M$.
- M is austere
$\underset{\text{def}}{\Leftrightarrow} \forall p \in M, \forall \xi \in T_p^{\perp}M$,
{eigenvalues of A_{ξ} (with multip)} is invariant under $(-1)\times$.

Sec. 5 - Minimal PF submanifolds with symmetries (2/4)

Recall

$$\begin{array}{ccc} M & \hookrightarrow & N \\ \text{submfd} & & \text{Riem mfd} \end{array}$$

reflective $\Rightarrow$ weakly reflective $\Rightarrow$ austere $\Rightarrow$ minimal

Note

$$\begin{array}{ccc} \Phi_{G/K}^{-1}(M) & \hookrightarrow & V_{\mathfrak{g}} \\ \text{PF submfd} & & \text{separable Hilbert sp.} \end{array}$$

reflective $\Rightarrow$ weakly reflective $\Rightarrow$ austere $\Rightarrow$ $\left.\begin{array}{l} \zeta\text{-minimal} \\ \text{r-minimal} \end{array}\right\}$ same

Sec. 5 - Minimal PF submanifolds with symmetries (3/4)

Theorem (M. 2019)

G/K: compact normal homogeneous space.
Then each fiber of the parallel transport map $\Phi_{G/K} : V_{\mathfrak{g}} \to G/K$
is a weakly reflective PF submanifold of $V_{\mathfrak{g}}$.

Proof (outline)

The canonical reflection (M. 2019) of the path space $V_{\mathfrak{g}} = L^2([0,1], \mathfrak{g})$

$$\mathfrak{r} : V_{\mathfrak{g}} \to V_{\mathfrak{g}}\ , \quad u \mapsto \mathfrak{r}(u)\ , \quad \mathfrak{r}(u)(t) := -u(1-t)$$

plays an important role.

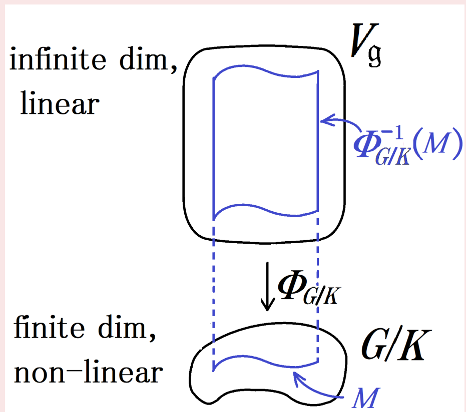
Remark

The fiber is not totally geodesic (except for rare cases).
Therefore, the fiber is not reflective.

Sec. 1 Sec. 2 Sec. 3 Sec. 4 **Sec. 5** Sec. 6

Sec. 5 - Minimal PF submanifolds with symmetries (4/4)

Problem



(1) M : reflective $\Rightarrow$ $\Phi^{-1}_{G/K}(M)$: reflective ?

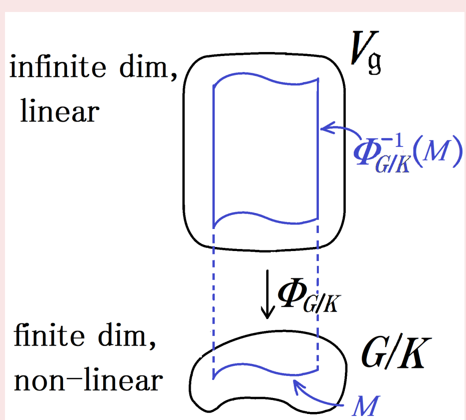
(2) M : weakly reflective $\Rightarrow$ $\Phi^{-1}_{G/K}(M)$: weakly reflective ?

(3) M : austere $\Rightarrow$ $\Phi^{-1}_{G/K}(M)$: austere ?

Sec. 1 Sec. 2 Sec. 3 Sec. 4 Sec. 5 **Sec. 6**

Sec. 6 - Symmetric properties via the p.t.m. (1/7)

Problem



(1) M : reflective $\Rightarrow$ $\Phi^{-1}_{G/K}(M)$: reflective × (e.g. fiber)

(2) M : weakly reflective $\Rightarrow$ $\Phi^{-1}_{G/K}(M)$: weakly reflective ○ (Thm A and B)

(3) M : austere $\Rightarrow$ $\Phi^{-1}_{G/K}(M)$: austere △ (Thm C and D)

Sec. 6 - Symmetric properties via the p.t.m. (2/7)

Theorem A (M. 2019)

G: connected compact semi-simple Lie group
with bi-inv metric induced from negative of the Killing form,
K: symmetric subgroup of G s.t. (G, K): effective.
M: weakly reflective submanifold of G/K.
$\Rightarrow \Phi_{G/K}^{-1}(M)$ is a weakly reflective PF submanifold of $V_{\mathfrak{g}}$.

Theorem B (M. 2020)

N: compact isotropy irreducible Riemannian homogeneous space.
Fix $p \in N$. Set $G := \mathrm{Isom}_0(N)$ and $K := \mathrm{Isom}(N)_p$ so that $N \cong G/K$.
M: weakly reflective submanifold of $N \cong G/K$
$\Rightarrow \Phi_{G/K}^{-1}(M)$ is a weakly reflective PF submanifold of $V_{\mathfrak{g}}$.

Sec. 6 - Symmetric properties via the p.t.m. (3/7)

Theorem C (M. 2020)

Suppose $G/K = SO(n+1)/SO(n)$: sphere.
M: closed submanifold of G/K.
Then the following conditions are equivalent:
(1) M is an austere submanifold of G/K,
(2) $\Phi_{G/K}^{-1}(M)$ is an austere PF submanifold of $V_{\mathfrak{g}}$.

Theorem D (M. 2021)

Let G/K: symmetric of compact type. Suppose G is simple.
M: an austere orbit of a Hermann action with commuting involutions.
$\Rightarrow \Phi_{G/K}^{-1}(M)$ is an austere PF submanifold of $V_{\mathfrak{g}}$.

Sec. 6 - Symmetric properties via the p.t.m. (4/7)

Example

G/K: symmetric sp. of compact type, K': closed subgroup of G.
Suppose the action $K' \curvearrowright G/K$: of cohomogeneity one.
Let $M := K' \cdot aK$: singular orbit through $aK \in G/K$.
($\Rightarrow$ M is weakly reflective submfd of G/K (Podesta 1997, IST2009).
By Thm A, $\Phi_{G/K}^{-1}(M) \, (= P(G, K' \times K) * u)$ is a weakly reflective PF submfd of $V_{\mathfrak{g}}$.

Example

(U, L): compact Riem. symmetric pair. Assume L: connected.
$\mathfrak{u} = \mathfrak{l} + \mathfrak{p}$: canonical decomp. $\mathrm{Ad} : L \to SO(\mathfrak{p})$: isotropy rep.
Suppose $M := \mathrm{Ad}(L)x$: weakly reflective submfd of $S(\|x\|)(\subset \mathfrak{p})$.
(Such M was classified in Ikawa-Sakai-Tasaki 2009)
By Thm A (or Thm B), $\Phi_{G/K}^{-1}(M) \, \left(= P(SO(\mathfrak{p}), \mathrm{Ad}(L) \times SO(\mathfrak{p})_x) * \hat{0}\right)$ is a weakly reflective PF submanifold of $V_{\mathfrak{g}}$.

Sec. 6 - Symmetric properties via the p.t.m. (5/7)

Example

G/K: symmetric space of compact type. K': symmetric subgroup of G.
Suppose $M = K' \cdot aK$: weakly reflective orbit of the Hermann action.
(Such examples were given by Ohno 2016).
Then by Thm A, $\Phi_{G/K}^{-1}(M) \, (= P(G, K' \times K) * u)$ is a weakly reflective PF submfd

Example

G_2 : exceptional Lie grp,
$\tilde{\mathrm{Gr}}_3(\mathrm{Im}\,\mathbb{O}) \; (\cong SO(7)/(SO(3) \times SO(4)))$: Grassmann manifold
$G_2 \curvearrowright \tilde{\mathrm{Gr}}_3(\mathrm{Im}\,\mathbb{O})$ has unique weakly reflective orbit M [Enoyoshi 2018].
Then by Thm A, $\Phi_{G/K}^{-1}(M) \left(= P(SO(7), G_2 \times SO(3) \times SO(4)) * \hat{0}\right)$ is a weakly reflective PF submfd of $V_{\mathfrak{o}(7)}$.

Sec. 6 - Symmetric properties via the p.t.m. (6/7)

Example

(U, L): compact Riem. symmetric pair. Assume L: connected.
$\mathfrak{u} = \mathfrak{l} + \mathfrak{p}$: canonical decomp. $\mathrm{Ad} : L \to SO(\mathfrak{p})$: isotropy rep.
Suppose $M := \mathrm{Ad}(L)x$: austere submfd of $S(\|x\|)(\subset \mathfrak{p})$.
(Such M was classified by Ikawa-Sakai-Tasaki 2009)
Then by Thm C , $\Phi_{G/K}^{-1}(M)$ $\left(= P(SO(\mathfrak{p}), \mathrm{Ad}(L) \times SO(\mathfrak{p})_x) * \hat{0}\right)$ is an austere PF submanifold of $V_{\mathfrak{g}}$.

Example

G/K: symmetric space of compact type. Suppose G is simple.
Suppose M: austere orbit of a Hermann action with commuting involutions.
(Such examples classified by Ikawa 2011).
Then by Thm D, $\Phi_{G/K}^{-1}(M)$ $(= P(G, K' \times K) * u)$ is an austere PF submfd of $V_{\mathfrak{g}}$.

Sec. 6 - Symmetric properties via the p.t.m. (7/7)

Note

Recently, Taketomi (2018) introduced a generalized concept of weakly reflective submanifolds, namely arid submanifolds:

$$\text{reflective} \Rightarrow \text{weakly reflective} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \text{austere} \\ \\ \text{arid} \end{array} \begin{array}{c} \searrow \\ \nearrow \end{array} \text{minimal}$$

A submanifold M of a Riemannian manifold N is arid
$\underset{\text{def}}{\Leftrightarrow}$ $\forall p \in M$, $\forall \xi \in T_p^{\perp} M$ there exists $\varphi_\xi \in \mathrm{Isom}(N)$ satisfying

$$\varphi_\xi(p) = p, \qquad \varphi_\xi(M) = M, \qquad d\varphi_\xi(\xi) \neq \xi.$$

Theorems A and B can be formulated to the arid case.

On homogeneous minimal submanifolds

Note

In infinite dimensional Hilbert spaces,
there exist many examples of homogeneous minimal PF submanifolds which are not totally geodesic (by Theorem in Sec. 4.)

Theorem (Di Scala 2002)

In finite dimensional Euclidean spaces,
any homogeneous minimal submanifolds must be totally geodesic.

Reference I

[1] J. Berndt, L. Vanhecke, *Curvature-adapted submanifolds*, Nihonkai Math. J. 3 (1992), no. 2, 177–185.

[2] K. Enoyoshi, *Principal curvatures of homogeneous hypersurfaces in a Grassmann manifold* $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\,\mathbb{O})$ *by the* G_2*-action*, to appear in Tokyo J. Math.

[3] R. Harvey and H. B. Lawson, Jr., *Calibrated geometries*, Acta Math., **148** (1982), 47-157.

[4] E. Heintze, X. Liu, C. Olmos, *Isoparametric submanifolds and a Chevalley-type restriction theorem*, Integrable systems, geometry, and topology, 151-190, AMS/IP Stud. Adv. Math., **36**, Amer. Math. Soc., Providence, RI, 2006.

Reference II

[5] E. Heintze, R. Palais, C.-L. Terng, G. Thorbergsson, *Hyperpolar actions on symmetric spaces*, Geometry, topology, & physics, 214-245, Conf. Proc. Lecture Notes Geom. Topology, IV, Int. Press, Cambridge, MA, 1995.

[6] O. Ikawa, T. Sakai, H. Tasaki, *Weakly reflective submanifolds and austere submanifolds*. J. Math. Soc. Japan **61** (2009), no. 2, 437-481.

[7] O. Ikawa, *The geometry of symmetric triad and orbit spaces of Hermann actions*, J. Math. Soc. Japan **63** (2011), no. 1, 79-136.

[8] C. King, C.-L. Terng, *Minimal submanifolds in path space*, Global analysis in modern mathematics, 253-281, Publish or Perish, Houston, TX, 1993.

[9] N. Koike, *On proper Fredholm submanifolds in a Hilbert space arising from submanifolds in a symmetric space*, Japan. J. Math. (N.S.) **28** (2002), no. 1, 61-80.

Reference III

[10] M. Morimoto, *On weakly reflective PF submanifolds in Hilbert spaces*, to appear in Tokyo J. Math.

[11] M. Morimoto, *Austere and arid properties for PF submanifolds in Hilbert spaces*, Differential Geom. Appl., Vol. 69 (2020) 101613.

[12] M. Morimoto, *On weakly reflective submanifolds in compact isotropy irreducible Riemannian homogeneous spaces*, arXiv:2003.04674.

[13] M. Morimoto, *Minimal PF submanifolds in Hilbert spaces with symmetries*, Ph.D. Thesis. doi/10.24544/ocu.20200615-005

[14] M. Morimoto, *Curvatures and austere property of orbits of path group actions induced by Hermann actions*, in preparation.

[15] S. Ohno, *A sufficient condition for orbits of Hermann actions to be weakly reflective*, Tokyo J. Math. **39** (2016), no. 2, 537-564.

Reference IV

[16] R. S. Palais, *Morse theory on Hilbert manifolds*, Topology **2** (1963), 299-340.

[17] R. S. Palais, C.-L. Terng, *Critical Point Theory and Submanifold Geometry*, Lecture Notes in Math., vol 1353, Springer-Verlag, Berlin and New York, 1988.

[18] U. Pinkall, G. Thorbergsson, *Examples of infinite dimensional isoparametric submanifolds*, Math. Z. **205** (1990), no. 2, 279-286.

[19] F. Podestà, *Some remarks on austere submanifolds*, Boll. Un. Mat. Ital. B (7) **11** (1997), no. 2, suppl., 157-160.

[20] A. J. D. Scala, *Minimal homogeneous submanifolds in Euclidean spaces*, Ann. Global Anal. Geom. **21** (2002), no. 1, 15-18.

[21] S. Smale, *Morse theory and a non-linear generalization of the Dirichlet problem*, Ann. of Math. (2) **80** (1964), 382-396.

Reference V

[22] S. Smale, *An infinite dimensional version of Sard's theorem*, Amer. J. Math. **87** (1965), 861-866.

[23] Y. Taketomi, *On a Riemannian submanifold whose slice representation has no nonzero fixed points*, Hiroshima Math. J. **48** (2018), no. 1, 1-20.

[24] C.-L. Terng, *Proper Fredholm submanifolds of Hilbert space.* J. Differential Geom. **29** (1989), no. 1, 9-47.

[25] C.-L. Terng, *Polar actions on Hilbert space*. J. Geom. Anal. **5** (1995), no. 1, 129-150.

[26] C.-L. Terng, G. Thorbergsson, *Submanifold geometry in symmetric spaces.* J. Differential Geom. **42** (1995), no. 3, 665-718.

Thank you very much for your attention.

Lagrangian submanifolds in complex projective space and quaternionic Kähler geometry

Makoto Kimura

Abstract. We discuss a relationship of certain Lagrangian submanifolds in complex projective space and submanifolds in complex 2-plane Grassmannian.

1. Let $\mathbb{CP}^n$ be an n-dimensional complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 4. For a Lagrangian submanifold M^n in $\mathbb{CP}^n$ and a unit normal vector field N on M, we find the condition such that for $r \in \mathbb{R}$ with sufficiently small $|r| > 0$, 1-parameter family of 'parallel' submanifolds

$$M_r := \{\exp_p(rN_p) |\ p \in M\}$$

are Lagrangian submanifolds in $\mathbb{CP}^n$, by computing the differential of the normal exponential map (cf. [1]).

2. For a Lagrangian submanifold M^n in $\mathbb{CP}^n$ and a unit normal vector field N on M, we define a 'Gauss map' γ to complex 2-plane Grassmannian $G_2(\mathbb{C}^{n+1})$ (cf. [2], [3], [4]). If M and N admits parallel Lagrangian submanifolds M_r as 1., and moreover if JN is an eigenvector of the shape operator A_N and the eigenvalue is constant, then the rank of γ is $n-1$ and $\gamma(M)$ is a quarter dimensional totally real submanifold with respect to both complex and quaternionic Kähler structure of $G_2(\mathbb{C}^{n+1})$.

3. Conversely, let Σ^{n-1} be a quarter dimensional submanifold in $G_2(\mathbb{C}^{n+1})$ which is totally real with respect to both complex and quaternionic Kähler structure of $G_2(\mathbb{C}^{n+1})$. Then we have locally a horizontal lift of Σ in the twistor space of $G_2(\mathbb{C}^{n+1})$, and we can construct ruled (i.e., foliated by geodesics in $\mathbb{CP}^n$) Lagrangian immersion φ from a circle bundle M^n over Σ to $\mathbb{CP}^n$, provided φ is regular, and M admits parallel family of Lagrangian submanifolds M_r in $\mathbb{CP}^n$.

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References

[1] T. E. Cecil and P. J. Ryan, *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc. **269** (1982), no. 2, 481–499.

[2] J. T. Cho and M. Kimura, *Hopf hypersurfaces in complex hyperbolic space and submanifolds in indefinite complex 2-plane Grassmannian I*, Topology Appl., **196** (2015), part B, 594–607.

[3] M. Kimura, *Hopf hypersurfaces in complex projective space and half-dimensional totally complex submanifolds in complex 2-plane Grassmannian I*, Diff. Geom. Appl. **35**, suppl., 266–273, *II*, ibid. **54**, part A, 44–52,

[4] B. Palmer, *Hamiltonian minimality and Hamiltonian stability of Gauss maps*, Diff. Geom. Appl. **7** (1997), no. 1, 51–58.

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Lagrangian submanifolds in complex projective space and quaternionic Kähler geometry

Makoto Kimura(Ibaraki University)

March 5, 2021

Submanifolds of Symmetric Spaces and Their Time evolutions

Contents

- 'Parallel family' of Lagrangian submanifolds in $\mathbb{CP}^n$,

Contents

- 'Parallel family' of Lagrangian submanifolds in $\mathbb{CP}^n$,
- 'Gauss map' from Lagrangian submanifolds in $\mathbb{CP}^n$ to complex 2-plane Grassmannian $\mathbb{G}_2(\mathbb{C}^{n+1})$,

Contents

- 'Parallel family' of Lagrangian submanifolds in $\mathbb{CP}^n$,
- 'Gauss map' from Lagrangian submanifolds in $\mathbb{CP}^n$ to complex 2-plane Grassmannian $\mathbb{G}_2(\mathbb{C}^{n+1})$,
- 'Totally real' submanifolds in $\mathbb{G}_2(\mathbb{C}^{n+1})$,

Contents

- 'Parallel family' of Lagrangian submanifolds in $\mathbb{CP}^n$,
- 'Gauss map' from Lagrangian submanifolds in $\mathbb{CP}^n$ to complex 2-plane Grassmannian $\mathbb{G}_2(\mathbb{C}^{n+1})$,
- 'Totally real' submanifolds in $\mathbb{G}_2(\mathbb{C}^{n+1})$,
- Examples.

Parallel hypersurfaces

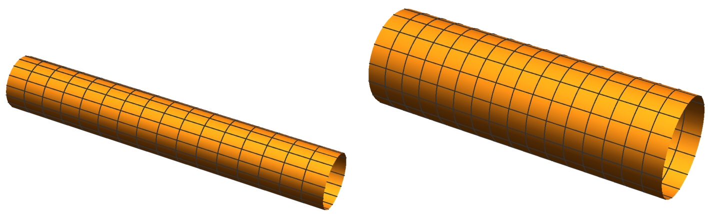
Parallel hypersurfaces

$\widetilde{M}$: Riemann manifold,
M: An oriented hypersurface of $\widetilde{M}$,
N_x: Unit normal vector field M in $\widetilde{M}$,

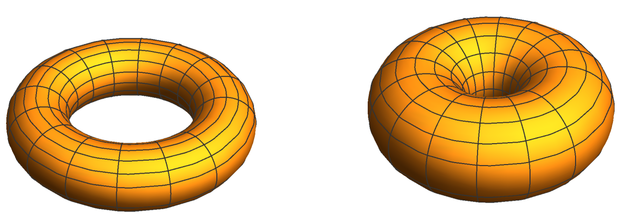
$$\phi_r(M) := \{\exp_x(rN_x) \mid x \in M\} \quad (0 < r)$$

is called a parallel hypersurface of M, provided $\phi_r(M)$ is a smooth hypersurface of $\widetilde{M}$.

Examples of parallel hypersurfaces in $\mathbb{R}^3$: Circular cylinders



Examples of parallel hypersurfaces in $\mathbb{R}^3$: tori of revolution



Isoparametric hypersurfaces in space forms

Isoparametric hypersurfaces in space forms

If M_0^n: is an isoparametric hypersurface (i.e., principal curvatures are constant) in $\widetilde{M}^{n+1}(c)$ of constant sectional curvature c, then its parallel hypersurfaces M_r^n in $\widetilde{M}^{n+1}(c)$ is also isoparametric.

Hopf hypersurfaces in Kähler manifold

Structure vector of real hypersurface

M^{2n-1}: Oriented real hypersurface in Kähler manifold $\widetilde{M}^n$,
N: unit normal vector field of M in $\widetilde{M}$,
$\xi := -JN$: structure vector field of M.

Hopf hypersurface

M^{2n-1}: real hypersurface in Kähler manifold $\widetilde{M}^n$,
M: Hopf hypersurface in $\widetilde{M}$
$\Leftrightarrow$ $A\xi = \mu\xi$ (μ: Hopf principal curvature) .

A: shape operator of M in $\widetilde{M}$.

Hopf hypersurfaces in complex space forms

Hopf principal curvature

μ is constant when $\widetilde{M} = \widetilde{M}^n(c)$ with constant holomorphic sectional curvature $c \neq 0$.

Hopf hypersurface

M^{2n-1} is a Hopf hypersurface in $\widetilde{M}^n(c)$ $(c \neq 0)$ $\Rightarrow$

- Parallel hypersurfaces M_r in $\widetilde{M}^n(c)$ are also Hopf,
- Each integral curve of ξ is a geodesic on M and a 'circle' in $\widetilde{M}^1(c) \subset \widetilde{M}^n(c)$.

Parallel family of Lagrangian submanifolds

Parallel family of Lagrangian submanifolds in $\mathbb{CP}^n$

In this talk, we treat with 1-parameter family of 'parallel' Lagrangian submanifolds M_r^n in $\mathbb{CP}^n$ with Fubini-Study metric of constant holomorphic sectional curvature 4.

Problem

Problem

Let M_0 be a Lagrangian submanifold in $\mathbb{CP}^n$ and let N be a unit normal vector field on M. For $r \in \mathbb{R}$ with sufficient small $|r|$, when each of the parallel family

$$M_r := \{\exp_p(rN_p) \in \mathbb{CP}^n |\ p \in M_0\} \qquad (1)$$

is a Lagrangian submanifold in $\mathbb{CP}^n$?

Answer

Answer

For $r \in \mathbb{R}$ (with small $|r|$), each n-dimensional submanifold M_r in $\mathbb{CP}^n$ given by (1) is Lagrangian if and only if the following 4 equations are satisfied:

1. $\nabla_{JN}(JN) = 0$, i.e., JN is a geodesic vector field on M_0, where J and ∇ denote the complex structure and the induced Levi-Civita connection on M_0, respectively,
2. $\langle A_N(JN), \nabla_X(JN)\rangle = 0$, where X, A and $\langle\ ,\ \rangle$ denote a tangent vector field orthogonal to JN, the shape operator and the induced metric of M_0, respectively,

Answer

Answer

3. $\langle \nabla_X(JN), Y\rangle = \langle \nabla_Y(JN), X\rangle$, where X and Y are tangent vector fields on M_0 which are orthogonal to JN, i.e., $\{JN\}^\perp$ is an integrable distribution on M_0,
4. $\langle \nabla_X(JN), A_N Y\rangle = \langle \nabla_Y(JN), A_N X\rangle$, where X and Y are tangent vector fields on M_0 which are orthogonal to JN.

Integral curve of JN

Integral curve of JN

Furthermore we consider the case:

5. JN is an eigenvector of the shape operator A_N, i.e., $A_N(JN) = \mu JN$ and the eigenvalue μ is constant on M_0.

Then each integral curve of JN in M_0 is a circle in a complex projective line $\mathbb{CP}^1$ in $\mathbb{CP}^n$. Moreover for each r, with respect to a unit normal vector field $(N_r)_{\exp(rN_p)} := (d\exp)_{rN_p}(N_p)$ on M_r, each integral curve of JN_r in M_r is either a circle or a geodesic (ruled Lagrangian submanifold) in $\mathbb{CP}^n$.

Complex Stiefel manifold

Euclidean inner product on $\mathbb{C}^{n+1}$

For $z, w \in \mathbb{C}^{n+1}$, we define $\mathbb{R}$-valued Euclidean inner product by $\langle z, w\rangle = \mathrm{Re}(z^* w)$.

Complex Stiefel manifold

$$V_2(\mathbb{C}^{n+1}) = \{(u_1, u_2) \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} |$$
$$\|u_1\| = \|u_2\| = 1,\ \langle u_1, u_2\rangle = \langle u_1, iu_2\rangle = 0\}$$

:Complex Stiefel manifold,
$M_V := V_2(\mathbb{C}^{n+1}) = U(n+1)/U(n-1)$.

Complex 2-plane Grassmann manifold

Complex 2-plane Grassmann manifold

Complex 2-plane Grassmann manifold $\mathbb{G}_2(\mathbb{C}^{n+1})$ is realized as a quotient space of an action of $U(2)$ on $V_2(\mathbb{C}^{n+1})$ as $U(2) \ni g$, $(u_1, u_2) \mapsto (u_1, u_2)g$.

$\mathbb{G}_2(\mathbb{C}^{n+1}) = U(n+1)/U(n-1) \times U(2)$.

Circle actions on $V_2(\mathbb{C}^{n+1})$

Circle actions on complex Stiefel manifold

We consider circle actions on M_V:
$S^1 = \mathbb{R}/2\pi\mathbb{Z} \ni \theta$, $(u_1, u_2) \mapsto (e^{i\theta}u_1, e^{i\theta}u_2)$,

$SO(2) \ni \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$,
$(u_1, u_2) \mapsto (u_1, u_2) \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$.

Sequence of projections

Sequence of projections

$$M_V \xrightarrow{\pi_{S^1}} M_S \xrightarrow{\pi_{SO(2)}} M_{\mathcal{Z}} \xrightarrow[S^2]{\pi_{\mathcal{Z}}} M_{\mathbb{G}},$$

$M_{\mathbb{G}}^{4n-4}$: $\mathbb{G}_2(\mathbb{C}^{n+1})$, Complex 2-plane Grassmannian,

$M_{\mathcal{Z}}^{4n-2}$: twistor space of $M_{\mathbb{G}}$, complex contact manifold,

M_S^{4n-1}: S^1-bundle over $M_{\mathcal{Z}}$ (3-Sasakian).

Tangent spaces

Tangent space of M_V

$$\begin{gathered} T_{(u_1,u_2)}M_V = (\{u_1,u_2\}^{\perp} \times \{u_1,u_2\}^{\perp}) \\ \oplus\mathbb{R}(iu_1, iu_2) \oplus \mathbb{R}(-u_2, u_1) \\ \oplus\mathbb{R}(-iu_1, iu_2) \oplus \mathbb{R}(iu_2, iu_1), \end{gathered}$$

Tangent space of M_S

$$\begin{gathered} T_{[u_1,u_2]}M_S \cong (\{u_1,u_2\}^{\perp} \times \{u_1,u_2\}^{\perp}) \\ \oplus\mathbb{R}(-u_2, u_1) \\ \oplus\mathbb{R}(-iu_1, iu_2) \oplus \mathbb{R}(iu_2, iu_1), \end{gathered}$$

Tangent spaces

Tangent space of $M_{\mathcal{Z}}$

$$\begin{gathered} T_{[u_1,u_2]}M_{\mathcal{Z}} \cong (\{u_1,u_2\}^{\perp} \times \{u_1,u_2\}^{\perp}) \\ \oplus\mathbb{R}(-iu_1, iu_2) \oplus \mathbb{R}(iu_2, iu_1), \end{gathered}$$

Tangent space of $M_{\mathbb{G}}$

$$T_{[u_1,u_2]}M_{\mathbb{G}} \cong (\{u_1,u_2\}^{\perp} \times \{u_1,u_2\}^{\perp}).$$

Q.K. structure of $M_{\mathbb{G}}$

Q.K. structure of $M_{\mathbb{G}}$

Then for $(u_1, u_2) \in V_2(\mathbb{C}^{n+1})$, a basis I_1, I_2, I_3 of quaternionic Kähler structure of $M_{\mathbb{G}}$ at $[u_1, u_2]$ is give as follows:

$$(x_1, x_2) \in \{u_1, u_2\}^\perp \times \{u_1, u_2\}^\perp,$$
$$I_1 : (x_1, x_2) \mapsto (-x_2, x_1),$$
$$I_2 : (x_1, x_2) \mapsto (-ix_1, ix_2),$$
$$I_3 : (x_1, x_2) \mapsto (ix_2, ix_1).$$

Complex structure is $J : (x_1, x_2) \mapsto (ix_1, ix_2)$.

'Gauss map' to $\mathbb{G}_2(\mathbb{C}^{n+1})$

'Gauss map' to $\mathbb{G}_2(\mathbb{C}^{n+1})$

Let M_0^n be a Lagrangian submanifold in $\mathbb{CP}^n$ and let N be a unit normal vector field on M_0. Then we have a 'Gauss map' γ_N from M_0 to the complex 2-plane Grassmannian $\mathbb{G}_2(\mathbb{C}^{n+1})$, where $\gamma_N(p)$ is the complex 2-plane spanned by the position vector p and N_p in $\mathbb{C}^{n+1}$ for $p \in M^n$.

Theorem 1

Theorem 1

Let M^n be a Lagrangian submanifold in $\mathbb{CP}^n$ and let N be a unit normal vector field on M. Suppose that for each $r \in \mathbb{R}$ with sufficiently small $|r|$, each of the parallel family M_r of M_0 is a Lagrangian submanifold in $\mathbb{CP}^n$ and JN is an eigenvector of the shape operator A_N of M_0 with constant eigenvalue μ.

Theorem 1

Theorem 1

Then

1. The image $\gamma_N(M_0)$ is a quarter-dimensional totally real submanifold with respect to both complex structure and Quaternionic Kähler structure of $\mathbb{G}_2(\mathbb{C}^{n+1})$, and
2. Each fiber of γ_N is an integral curve of JN on M_0.
3. Hence M_0 is a total space of circle bundle over $\gamma_N(M_0)$ with projection γ.

'Gauss map' of real hypersurfaces

'Gauss map' of real hypersurfaces

Similar result was obtained for real hypersurfaces, in particular Hopf hypersurfaces in $\mathbb{CP}^n$ and half-dimensional totally complex submanufolds in $\mathbb{G}_2(\mathbb{C}^{n+1})$, [K, 2014].

Gauss map of hypersurface in sphere

Gauss map of hypersurface in $\mathbb{S}^{n+1}$

$x : M^n \to \mathbb{S}^{n+1}$: an immersion,
$N : M^n \to \mathbb{R}^{n+2}$: unit normal vector field,
$\rightsquigarrow$
$\gamma : M^n \to \widetilde{\mathbb{G}}_2(\mathbb{R}^{n+2}) \cong \mathbb{Q}^n$: Gauss map, defined by $\gamma(p) = x(p) \wedge N(p)$.

Here M^n is an oriented hypersurface in $\mathbb{S}^{n+1}$,
$\widetilde{\mathbb{G}}_2(\mathbb{R}^{n+2})$: oriented real 2-plane Grassmannian,
$\mathbb{Q}^n$: complex hyperquadric in $\mathbb{CP}^{n+1}$.

Gauss map of hypersurface in sphere

Gauss map of hypersurface in $\mathbb{S}^{n+1}$

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$\widetilde{\mathbb{G}}_2(\mathbb{R}^{n+2})$: oriented real 2-plane Grassmannian,
$\mathbb{Q}^n$: complex hyperquadric in $\mathbb{CP}^{n+1}$.

Gauss map of hypersurface in sphere

Theorem (Palmer, 1997)

M^n; oriented hypersurface in $\mathbb{S}^{n+1}$
$\Rightarrow$
$\gamma(M)$: Lagrangian submanifold in $\mathbb{Q}^n$, and
M^n is either isoparametric or austere in $\mathbb{S}^{n+1}$
$\Rightarrow$
$\gamma(M)$: Minimal Lagrangian in $\mathbb{Q}^n$.

For $t \in \mathbb{R}$, $M_t := \{\cos t x(p) + \sin t N(p)\}$: parallel hypersurface of M, the Gauss image is not changed:

$$\gamma(M_t) = \gamma(M).$$

Proposition 2

Proposition 2

Let $\varphi : \Sigma^{n-1} \to M_{\mathbb{G}}$ be a totally real immersion concerning both complex structure and quaternionic Kähler structure of $M_{\mathbb{G}}$, $\widetilde{\Sigma}$ the universal covering of Σ, and $\pi_\Sigma : \widetilde{\Sigma} \to \Sigma$ the covering projection. Then there exists a horizontal immersion $\widetilde{\varphi} : \widetilde{\Sigma} \to M_{\mathcal{Z}}$ such that $\pi_{\mathcal{Z}} \circ \widetilde{\varphi} = \varphi \circ \pi_\Sigma$ ($\widetilde{\varphi}$: horizontal lift of φ).

Converse construction

Converse construction

Let $\psi : \Sigma^{n-1} \to M_{\mathcal{Z}}$ be a horizontal immersion such that $\pi_{\mathcal{Z}} \circ \psi : \Sigma \to M_{\mathbb{G}}$ is a totally real immersion concerning both complex structure and quaternionic Kähler structure of $M_{\mathbb{G}}$.
Concerning a circle bundle $M_S \to M_{\mathcal{Z}}$, let $\pi_\psi : \psi^* M_S \to \Sigma$ be the pullback bundle over Σ for ψ.

Converse construction

Converse construction

Then we have the following commutative diagram:

$$\begin{tikzcd} \psi^* M_S \arrow[r, "\eta"'] \arrow[d, "\pi_\psi"'] & M_S \arrow[d, "\pi_{\mathcal{Z}}"] \\ \Sigma^{n-1} \arrow[r, "\psi"'] & M_{\mathcal{Z}}. \end{tikzcd}$$

Converse construction

Converse construction

For a projection $\mathrm{pr} : M_S \to \mathbb{CP}^n$, $[u_1, u_2] \mapsto [u_1]$, We define a map $\Phi_0 := \mathrm{pr} \circ \eta : \psi^* M_S \to \mathbb{CP}^n$. Then the image of each fiber $\pi^{-1}(p)$ for $p \in \Sigma$ is a geodesic c_p in $\mathbb{CP}^n$ and on the open subset $O \subset \psi^* M_S$ of regular points for Φ_0, $\Phi_0(O)$ is a ruled Lagrangian submanifold in $\mathbb{CP}^n$.

Converse construction

Converse construction

For a ruled Lagrangian submanifold $M_0 = \Phi_0(O)$, along geodesics normal to M_0 with initial vector $J\dot{c}_p$, each of parallel family of n-dimensional submanifold M_r in $\mathbb{CP}^n$ is also Lagrangian for sufficiently small $|r|$.

Special case

Special case

Complex quadric $\mathbb{Q}^{n-1}$ in $\mathbb{CP}^n$ which is identified with real oriented 2-plane Grassmannian $\widetilde{\mathbb{G}}_2(\mathbb{R}^{n+1})$ is half-dimensional totally geodesic, totally complex (concerning Q.K. structure) and totally real (concerning complex structure) submanifold in $\mathbb{G}_2(\mathbb{C}^{n+1})$.

Gauss image of hypersurfaces in S^n

Special case

Hence Gauss image of oriented hypersurface Σ^{n-1} in S^n is considered as a totally real submanifold concerning both complex structure and Q.K. structure in $\mathbb{G}_2(\mathbb{C}^{n+1})$, so it satisfies conditions to construct parallel family of Lagrangian submanifolds M_r in $\mathbb{CP}^n$.

Examples

Examples

Let Σ^{n-1} be an oriented hypersurface, p a position vector of Σ in $S^n \subset \mathbb{R}^{n+1}$ and N_p a unit normal of Σ^{n-1} in S^n at $p \in \Sigma$.
For $r \in \mathbb{R}$ with sufficiently small $|r|$, we define a map, $\Phi_r : M^n = S^1 \times \Sigma^{n-1} \to \mathbb{CP}^n$,

$$\begin{aligned}\Phi_r(t, p) := \pi((&\cos r \cos t - i \sin r \sin t)p \\ &+(-\sin r \cos t + i \cos r \sin t)N_p),\end{aligned}$$

where $\pi : S^{2n+1} \to \mathbb{CP}^n$ is the Hopf fibration.

Examples

Examples

Then $\{\Phi_r(M^n)\}$ gives a parallel family of Lagrangian submanifold in $\mathbb{CP}^n$, provided Φ_r is an immersion. Here t is a parameter of integral curve (a circle lies in $\mathbb{CP}^1$) of JN.
In particular when $r = 0$, $\Phi_0(M)$ is a ruled Lagrangian submanifold in $\mathbb{CP}^n$, provided the shape operator of Σ^{n-1} in S^n is non-degenerate.

References

References

1. D. V. Alekseevsky and S. Marchiafava, *A twistor construction of Kähler submanifolds of a quaternionic Kähler manifold*, Ann. Mat. Pura Appl. **184** (2005), no. 1, 53–74.
2. J. T. Cho and M. Kimura, *Hopf hypersurfaces in complex hyperbolic space and submanifolds in indefinite complex 2-plane Grassmannian I*, Topology Appl., **196** (2015), part B, 594–607.

References

References

3. M. Kimura, *Hopf hypersurfaces in complex projective space and half-dimensional totally complex submanifolds in complex $\mathbf{2}$-plane Grassmannian I*, Diff. Geom. Appl. **35**, suppl., 266–273, *II*, ibid. **54**, part A, 44–52,
4. B. Palmer, *Hamiltonian minimality and Hamiltonian stability of Gauss maps*, Diff. Geom. Appl. **7** (1997), no. 1, 51–58.

An introduction to the deformed Hermitian Yang-Mills (dHYM) connections

Hikaru Yamamoto

Abstract. In this talk, I gave an introduction to the deformed Hermitian Yang–Mills (dHYM) connections. The talk was started with the introduction of some basic notions of special Lagrangian submanifolds. After that, the easy version of the real Fourier–Mukai transform was explained. Finally, I gave a list of recent results obtained in a joint works with K. Kawai.

1 Special Lagrangian submanifold

Roughly speaking, a deformed Hermitian Yang–Mills connection is a mirror object of a special Lagrangian submanifold in the sense of mirror symmetry. So, it's better to start with the introduction of special Lagrangian submanifolds.

Let $\mathbb{C}^n \cong \mathbb{R}^{2n}$ be the standard complex plane with coordinates $x = x + iy$. Denote the standard Kähler form and the holomorphic volume form on $\mathbb{C}^n$ by $\omega = \sum_i dx^i \wedge dy^i$ and $\Omega := dz^1 \wedge \cdots \wedge dz^n$, respectively.

Definition 1. An n-dimensional real submanifold L in $\mathbb{C}^n$ is called a special Lagrangian submanifold with phase $e^{i\theta}$ if it satisfies $\omega|_L = 0$ and $\mathrm{Im}(e^{-i\theta}\Omega)|_L = 0$.

It is well-known that a special Lagrangian submanifold is volume minimizing in its homology class. One can easily see that if a Lagrangian submanifold $L \in \mathbb{C}^n$ is written by the graph of the gradient of some function f on $\mathbb{R}^n$ the special Lagrangian condition is equivalent to

$$\arg\det\left(I + i\,\mathrm{Hess}\, f\right) = \theta. \tag{1}$$

It is also known (as a result of McLean) that the moduli space of special Lagrangian submanifolds (around L) is a smooth $b^1(L)$-dimensional manifold.

2 The real Fourier–Mukai transform

Historically, the definition of deformed Hermitian Yang–Mills connections is introduced by the real Fourier–Mukai transform in the paper in the paper of Leung, Yau and Zaslow. To explain that, let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Then, as explained above, we obtain the graph of $Y := \nabla f$ denoted by S_Y. On the other hand, we also get a connection by

$$\nabla^Y := d + \sum_{i=1}^{n} Y^i dy^i.$$

This is an Hermitian connection of the trivial $\mathbb{C}$-bundle over $\mathbb{C}^n$. Then, by Leung, Yau and Zaslow, it was found that f satisfies (1) if and only if the curvature 2-form F_{∇^Y} satisfies

$$\mathrm{Im}\left(e^{-i\theta}(\omega + F_{\nabla^Y})^n\right) = 0.$$

We remark that the condition that Y is written as ∇f is equivalent to $F^{0,2}_{\nabla^Y} = 0$. Thus, we have reached the definition of deformed Hermitian Yang–Mills connections.

Definition 2. Let (X, ω) be a Kähler manifold with $\dim_{\mathbb{C}} X = n$ and $L \to X$ be a smooth $\mathbb{C}$-bundle with an Hermitian metric h. Then, an Hermitian connection ∇ of (L, h) is called a deformed Hermitian Yang–Mills connection with phase $e^{i\theta}$ if it satisfies $F^{0,2}_{\nabla} = 0$ and

$$\operatorname{Im}\left(e^{-i\theta}(\omega + F_{\nabla^Y})^n\right) = 0,$$

where F_∇ is the curvature 2-form of ∇.

The correspondence between S_Y and ∇^Y is (roughly) called the real Fourier–Mukai transform.

3 Some results

We are wondering whether some properties of special Lagrangian submanifolds also hold for deformed Hermitian Yang–Mills connections or not. In other words, we are wondering whether deformed Hermitian Yang–Mills connections are similar to special Lagrangian submanifolds or not. This is a motivation of a joint work with K. Kawai. We answered the following questions affirmatively.

- Is a deformed Hermitian Yang–Mills connection a minimizer of some functional?
- Is there a flow similar to mean curvature flows?
- Is the moduli space of deformed Hermitian Yang–Mills connections smooth and finite dimensional?

We give some more detail on the first and second question. First, the functional V on the set of all Hermitian connections of (L, h) is given by

$$V(\nabla) := \int_X \sqrt{\det\left(\mathrm{id}_{TX} - iF^\sharp_\nabla\right)} \frac{\omega^n}{n!}.$$

We call V the volume functional for Hermitian connections. As the case of submanifolds, we can define a 1-form depends on ∇, denoted by $H(\nabla)$, which satisfies the first variation formula:

$$\delta_\nabla V = -\int_X \langle \cdot\,, H(\nabla)\rangle.$$

Then, the first main theorem of a joint work with Kawai is as follows .

Theorem 1. *Assume that (X, ω) is a Kähler manifold with $\dim_{\mathbb{C}} X = 3$ or 4. Then, the functional V has a topologically determined lower bound and it is attained at ∇ if and only if it is a deformed Hermitian Yang–Mills connection.*

As the case of mean curvature flows, we can consider the negative gradient flow of V and it written as

$$\frac{\partial}{\partial t}\nabla = H(\nabla). \tag{2}$$

Of course, the volume functional V is decreasing along this flow. So, we can expect that the flow converges a deformed Hermitian Yang–Mills connection if the flow has a long time solution. As the first step to realize this expectation, we have proved the following, the short time existence and uniqueness.

Theorem 2. *Assume that X is compact. Then, the flow* (2) *satisfies the short time existence and uniqueness for any initial connection ∇.*

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1

~~Intro~~

dHYM $\overset{\text{mirror}}{\longleftrightarrow}$ sLag (in mirror sym)

§ special Lag submfd

$\mathbb{C}^n \simeq \mathbb{R}^{2n}$ $(z = x + iy)$.

$\omega = dx^i \wedge dy^i$, $\Omega = dz^1 \wedge \cdots \wedge dz^n$

Def $\theta \in \mathbb{R}$. $L^n \subset \mathbb{C}^n$ is sLag with phase $e^{i\theta}$

$\Leftrightarrow$ $\omega|_L \equiv 0$, $\mathrm{Im}(e^{-i\theta}\Omega)|_L \equiv 0$

Thm sLag L is volume minimizing in $[L]$.

(skip the proof)

graphical case

$Y := (Y^1, \cdots, Y^n) : B \to \mathbb{R}^n$: smooth (B: open set in $\mathbb{R}^n$)

$S_Y := \{(x, Y(x)) \mid x \in B\} \subset \mathbb{C}^n$ (n-dim sub)

Fact 1 S_Y is Lag

$\Leftrightarrow \dfrac{\partial Y^i}{\partial x^j} = \dfrac{\partial Y^j}{\partial x^i} \Leftrightarrow \exists f \in C^\infty(B)$ s.t. $Y = \nabla f$

Fact 2

$\Omega|_{S_Y} = \det\left(I + \sqrt{-1}\,\mathrm{Hess} f\right) dx^1 \wedge \cdots \wedge dx^n$

$\rightsquigarrow S_Y$ is sLag with phas $e^{i\theta}$

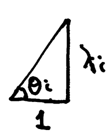
$\Leftrightarrow \arg\left(\det(I + \sqrt{-1}\,\mathrm{Hess} f)\right) = \theta$

2

Let $\lambda_1(x), \cdots, \lambda_n(x)$ be eigenvalues of $\mathrm{Hess} f(x)$

Define $r_i(x)$, $\theta_i(x)$ by

$$1 + \sqrt{-1}\,\lambda_i(x) = r_i(x)\, e^{i\theta_i(x)}$$

$\rightsquigarrow$  $\rightsquigarrow \theta_i(x) = \arctan \lambda_i(x)$

Def the Lag angle of S_Y

$\Theta(x) := \theta_1(x) + \cdots + \theta_n(x)$

Prop S_Y is sLag with phas $e^{i\theta}$

$\Leftrightarrow \Theta \equiv \theta$

Fact 3 (McLean)

Around a sLag L, there is a $b^1(L)$-dim family of sLags.

($\rightsquigarrow$ The moduli space of sLag (around L) is smooth $b^1(L)$-dim mfd)

(very rough proof)

$\mathcal{F} : \{\text{Lag } L' \text{ (close to } L)\} \longrightarrow C^\infty(L')$

$\mathcal{F}(L') := $ "Θ of L'"

$\underbrace{\mathcal{F}^{-1}(\theta)}_{=\text{ Moduli sp}}$ (θ: const)

Apply implicit funct. thm to $\mathcal{F}$.

$\rightsquigarrow \delta_L \mathcal{F}$ is surjective

and $\mathrm{Ker}\,\delta_L\mathcal{F} \cong \underbrace{\{\text{harmonic 1-form on } L\}}_{b^1(L)\text{-dim}}$

1st Betti number of L

3

§ The real Fourier-Mukai transform

• Toy model by Leung-Yau-Zaslow

$Y = (Y^1, \cdots, Y^n) : B^n \longrightarrow \mathbb{R}^n$: smooth

Then, we get two objects. $X = T^*B$

① $S_Y := \{(x, Y(x)) \mid x \in B\} \subset \mathbb{C}^n$

$\updownarrow$ ← The FM-transform

② $\nabla^Y := d + \sqrt{-1}\,(\underbrace{Y^1(x)dx^1 + \cdots + Y^n(x)dx^n}_{= A_Y \in \Gamma(B, T^*B)})$

$(\nabla^Y = d + \sqrt{-1}\underbrace{A_Y}_{\text{connection form}})$

Fact 1 S_Y is Lag $\Leftrightarrow \dfrac{\partial Y^i}{\partial x^j} = \dfrac{\partial Y^j}{\partial x^i}$

$\Leftrightarrow F^{0,2}_{\nabla^Y} = 0$ ← integrability condi.

Fact 2 $(Y = \nabla^3 f)$

S_Y is sLag $\Leftrightarrow \arg(\det(I + \sqrt{-1}\,\mathrm{Hess} f)) = \theta$

$\Leftrightarrow \mathrm{Im}(e^{-i\theta}(\omega + F_{\nabla^Y})^n) = 0$ (☆)

↑ sLag cond. for connections.

Def (X, ω) : Kähler with $\dim_{\mathbb{C}} X = n$

$L \to X$: smooth $\mathbb{C}$-bdl with Herm metric h.

Then, a Hermitian connection ∇ of L is called a deformed Hermitian Yang-Mills connection with phase $e^{i\theta}$ if

$F^{0,2}_{\nabla} \equiv 0$ and

$\mathrm{Im}(e^{-i\theta}(\omega + F_{\nabla})^n) \equiv 0$

4

§ In low dimension

Fix $\theta = 0$.

① $\dim_{\mathbb{C}} X = 1$

$\rightsquigarrow \mathrm{Im}(\omega + F_{\nabla}) = 0$

$\Leftrightarrow F_{\nabla} = 0$ ← flat conn.

② $\dim_{\mathbb{C}} X = 2$

$\mathrm{Im}(\omega + F_{\nabla})^2 = 0$

$\Leftrightarrow \omega \wedge F_{\nabla} = 0$ ← Hermitian Yang-Mills connection

③ If $\dim_{\mathbb{C}} X \geq 3$,

dHYM $\neq$ HYM

§ Some variants

• metric version of dHYM

(X, ω) : Kähler

$L \to X$: hol $\mathbb{C}$-bdl

A Hermitian metric h of L is dHYM metric (with phase $e^{i\theta}$)

$\Leftrightarrow \mathrm{Im}(e^{-i\theta}(\omega + F_h)^n) = 0$

where $F_h := -\partial\bar{\partial}\log h$

5

§ Joint work with Kawai

Motivation

"Does a dHYM connection have a property ~~with~~ wich a sLag has?"

☑ dHYM is a minimizer of some functional?

☑ Is there a flow similar to "mean curvature flow"?

☑ Is the moduli space of dHYM smooth? finite dim? orientable? ← Part II

Yes

Line bundle mean curvature flows and the moduli space of dHYM connections

Hikaru Yamamoto

Abstract. In this talk, I introduced two recent results. One is given in a joint work with X. Han. It is about an ε-regularity theorem for line bundle mean curvature flow. The other is given in a joint work with K. Kawai. It proves that the moduli space of deformed Hermitian Yang–Mills connections is a smooth finite dimensional manifold.

1 Line bundle mean curvature flow

In the former talk, I introduced a flow for Hermitian connections. There is a metric version of the flow, the so-called line bundle mean curvature flow. It was introduced by Jacob and Yau.

Definition 1. Let (X, ω) be a Kähler manifold with $\dim_{\mathbb{C}} = n$ and $L \to X$ be a holomorphic line bundle. Then, a one-parameter family of Hermitian metrics $\{ h_t \}_{t \in [0,T)}$ of L is called a line bundle mean curvature flow if it satisfies

$$\frac{\partial}{\partial}(-\log h_t) = \Theta(h_t) - \theta,$$

for some fixed constant $\theta \in \mathbb{R}$.

In the above definition, the angle function $\Theta(h)$ is defined to satisfy

$$(\omega - \partial\bar{\partial} \log h)^n = R(h) e^{i\Theta(h)} \omega^n.$$

The function $R(h)$ is called the radius function.

The line bundle mean curvature flow is considered as a mirror object of a mean curvature flow. So, it could be singular in finite time as the mean curvature flow case. Thus, I considered that establishing an ε-regularity theorem for line bundle mean curvature flows will be useful.

To introduce our main result, we should put some assumptions. Assume that X is diffeomorphic to the product of balls $B(r_1)$ and $B(r_2)$ in $\mathbb{R}^n$ and L is the trivial $\mathbb{C}$-bundle. Also assume that the Kähler form and h_t are y-invariant. Then, we define a kind of "the Gaussian density" for Hermitian connections by

$$D(h_t) := (\mathrm{Vol}(B(r_2)))^{-1} \frac{1}{(4\pi(T-t))^{n/2}} \int_X \exp\left(-\frac{|x|^2 + |\partial \log h_t|^2}{4(T-t)} \right) R(h_t) \frac{\omega^n}{n!}.$$

Then, the ε-regularity theorem given in a joint work with Han is expressed as follows.

Theorem 1. *There exist $C > 0$ and $\varepsilon > 0$ satisfying the following condition. If a line bundle mean curvature flow $\{ h_t \}_{t \in [0,T)}$ satisfies*

$$|F(h_t)| \leq C \quad \text{and} \quad \lim_{t \to T} D(h_t) < 1 + \epsilon,$$

then h_t can be extended beyond T.

2 The moduli space of dHYM connections

Next, we switch the subject to connections from metrics. So, we consider deformed Hermitian Yang–Mills connections, not metrics. This part is based on a joint work with K. Kawai. We studied the moduli space of deformed Hermitian Yang–Mills connections.

The setting is as follows. Let (X, ω) be a compact Kähler manifold and let $L \to X$ be a smooth $\mathbb{C}$-bundle with an Hermitian metric h. Fix $\theta \in \mathbb{R}$. Put

$$\mathcal{M} := \{ \text{dHYM connections of } (L, h) \}/U(1)\text{-gauge}$$

Then, we proved the following.

Theorem 2. *If $\mathcal{M} \neq \emptyset$, then $\mathcal{M}$ is a smooth $b^1(X)$-dimensional manifold. Moreover, it has an affine structure and it is orientable.*

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Line bundle mcfs
and the moduli space of dHYM conn.

§1 Line bundle mcf for metrics

(joint work with X. Han in Tsinghua Univ.)

This is the metric version of dHYM (not the connection ver.)

Recall

(X, ω) : Kähler $\dim_{\mathbb{C}} X = n$

$L \to X$: hol. $\mathbb{C}$-bdl

the line bdl mcf is introduced by Jacob-Yau.

To find dHYM metric it's nice to study the lbmcf

1

Def A 1-parameter family of Hermitian metrics $\{h_t\}_{t \in [0,T)}$ of L is called a line bundle mcf

$$\Leftrightarrow \frac{\partial}{\partial t}(-\log h_t) = \underbrace{\Theta(h_t)}_{\text{angle func.}} - {}^{\exists}\theta$$

where

$$\Theta(h) := \arg\left(\det(\omega - \partial\bar{\partial}\log h)^n\right)$$

Fact

- The ~~volume~~ lbmcf V is decreasing along lbmcf. exist!
- The stationary solution of lbmcf is dHYM metric with phase $e^{i\theta}$.

2

Motivation of study

- A difficulty and its resolution in Lag mcf side.

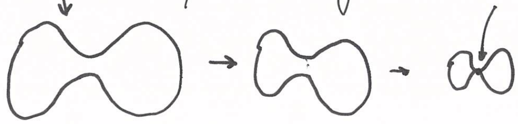
In general mcf develops singularities in finite time.

type I sing

type II sing

L_0

sing. pt.



But, Wang and Neves proved that "if L_0 is graded Lag then type I sing does not occur."

A key to prove this was White's ε-regularity theorem.

White's ε-reg (very rough ver)

$\exists$ very nice $\varepsilon > 0$ satisfying the following property.

If a mcf $\{S_t\}_{t \in [0,T)}$ satisfies

$$\lim_{t \to T} (GD)(S_t) < 1 + \varepsilon$$

then S_t does not develop singularity at $t = T$.

Gauss Density

Q. Can we make similar reg. thm for lbmcf?

A. Yes

3

§ The result

Assumptions

$X \cong B(r_1) \times B(r_2) \ni (x, y)$

$L = \underline{\mathbb{C}}$

ω is y-inv, h_t is y-inv

Def Some density is defined by

$$D(h_t) := \mathrm{Vol}(B(r_2))^{-1} \times \frac{1}{(4\pi(T-t))^{n/2}} \times \int_X \exp\left(-\frac{|x|^2 + |\partial \log h_t|^2}{4(T-t)}\right) R(h_t) \frac{\omega^n}{n!}$$

Prop

① $\frac{d}{dt} D(h_t) \leq -\int_X |$[redacted]$|^2 \frac{\omega^n}{n!} + O\left(\frac{c}{r_1^{n+1}}\right)$

$\rightsquigarrow$ $D(h_t)$ is decreasing along lbmcf

② [redacted] $= 0$

$\Leftrightarrow \Theta(h_t) = \frac{1}{T-t}\left(\phi - \frac{1}{2} x^k \frac{\partial \phi}{\partial x^k}\right)$

(where $\phi := -\log h_t$).

called a self-shrinker in our paper.

4

Prop If T is nonsingular time then $\lim_{t \to T} D(h_t) = 1$.

Our ε-reg thm ensures that its converse is also true. (in some sense)

Thm (Han - Y.)

$\exists C > 0$, $\exists \varepsilon > 0$ satisfying the following properties.

If lbmcf $\{h_t\}_{t \in [0,T)}$ satisfies

$|F(h_t)| \leq C$ and $\lim_{t \to T} D(h_t) < 1 + \varepsilon$

then h_t can be extended beyond T.

Future work

We want to apply this thm to prove " lbmcf does not develop type I sing if initial is graded " (as Wang and Neves.)

5

§2 The moduli space of dHYM connections.

(joint work with Kawai Gakushuin)

I switch the subject from dHYM metric to dHYM connections.

$\mathcal{M}_{SLag}$ is smooth, finite. dim.

(☆) The same property holds for dHYM.

Setting

(X, ω) : compact Kähler with $\dim_{\mathbb{C}} X = n$.

$L \to X$: smooth $\mathbb{C}$-bdl with Herm metric h

Fix $\theta \in \mathbb{R}$.

$\mathcal{M} := \{$ dHYM conn. with $e^{i\theta}$ $\} / U(1)$-gauge

↑ moduli dHYM

Thm (Kawai-Y.)

If $\mathcal{M} \neq \phi$, then $\mathcal{M}$ is a smooth mfd with

$\dim_{\mathbb{R}} \mathcal{M} = b^1(X)$ ← 1st Betti number of X

In addition,

- $\mathcal{M}$ admits an affine structure.
- $\mathcal{M}$ is orientable.

6

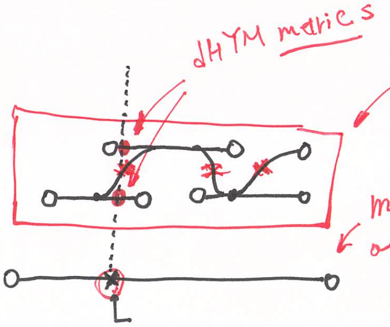
Implication

[1] For any $\nabla \in \mathcal{M}$, all $\nabla' \in \mathcal{M}$ (sufficiently close to ∇) is written as

$\nabla + \alpha$

(where $\alpha \in \mathcal{H}^1(X)$) ← harmonic 1-form on X.

[2]



(Proof) of Main thm

$\theta = 0$ for simplicity.

Put $\mathcal{A} := \{ \nabla \mid$ Herm conn. of $(L, h) \}$

$\mathcal{F} : \mathcal{A} \to \Omega^{0,2} \oplus \Omega^{2n}$ by

$\mathcal{F}(A) := \left(F_\nabla^{0,2}, \ \mathrm{Im}(\omega + F_\nabla)^n \right)$

$\mathcal{M} = \mathcal{F}^{-1}(0,0) / U(1)$-gauge.

Proof is the implicit funct. thm.

$\delta_\nabla \mathcal{F} : T_\nabla \mathcal{A} \to \Omega^{0,2} \oplus \Omega^{2n}$

is very complicated.

7

Define

$$\tilde{\omega}_\nabla := \left(\frac{1}{R(\nabla)}\right)^{\frac{1}{n-1}} \times \left(\mathrm{id}_{TX} + \frac{F_\nabla^\#}{\sqrt{-1}}\right)^* \omega$$

Hermitian metric on X.

This is not Kähler

but is balanced ($d\tilde{\omega}_\nabla^{n-1} = 0$)

By using $\tilde{\omega}_\nabla$, $\delta_\nabla \mathcal{F}$ can be

expressed simply.

$\rightsquigarrow$ $\dim \operatorname{Ker} \delta_\nabla \mathcal{F} = b^1(X)$.

$h \in \Gamma(X, L^* \otimes L)$

$h : X \to L^* \otimes L$

Lagrangian mean curvature flows with generalized perpendicular symmetries

Akifumi Ochiai

Abstract. We show a method of constructing an invariant Lagrangian mean curvature flow in a Calabi-Yau manifold with the use of generalized perpendicular symmetries. We use moment maps of the action of Lie groups, which are not necessarily abelian. We also show a general way to construct an invariant mean curvature flow in a Riemannian manifold.

1 Preliminaries

Definition 1. Let $\phi : \Sigma \to M$ be an immersion from a manifold Σ to a manifold M. For a smooth map

$$\begin{cases} F : \Sigma \times [0, T) \to M; \quad (p, t) \mapsto F_t(p) \\ F_0 = \phi \end{cases},$$

if $F_t(\cdot) : \Sigma \to M$ is an immersion for any $t \in [0, T)$, then we call F a *deformation* of ϕ.

Definition 2. Let $\phi : \Sigma \to (M, g)$ be an immersion from a manifold Σ to a Riemannian manifold (M, g). The *mean curvature flow* $F = (F_t)_{t\in[0,T)}$ of ϕ is the deformation of ϕ such that it is a smooth solution of the following partial differential equation for the mean curvature vector field $\mathcal{H}^t$ of F_t: $\frac{\partial}{\partial t}F(p, t) = \mathcal{H}^t(p)$.

2 Constructions of invariant mean curvature flows

Let (M, g) be a Riemannian manifold, H a Lie group which acts on M, K a closed subgroup of H, $Z(\mathfrak{h}^*)$ the center of the Lie coalgebra $\mathfrak{h}^*$, $L_h : M \to M$ the translation by an element $h \in H$, $\xi^\#$ a fundamental vector field generated by $\xi \in \mathfrak{h} =:$ Lie (H), L^K a subset in L defined by $L^K = \{p \in L \mid H_p = K\}$ for any submanifold L of M, where H_p is the isotropy subgroup of H at $p \in L$, V a submanifold of M such that $V \subset M^K$, $\phi_V : (H/K) \times V \to M$ a map defined by $(hK, p) \mapsto hp$. We also use these notations in the following section.

Definition 3. If the map ϕ_V is an immersion with the mean curvature vector field $\mathcal{H}$ and it holds that

$$\mathcal{H}(hK, p) = (L_h)_{*p}\mathcal{H}(K, p) \quad ((hK, p) \in (H/K) \times V),$$

then we say that V has the *property* $(*)$. Moreover, if there exists a deformation $f : V \times [0, T) \to M^K$ of V such that the immersed submanifold $V_t := f_t(V)$ also has the property $(*)$, we say that f *preserves* the property $(*)$ of V.

Definition 4. Suppose there exists a deformation $f : V \times [0, T) \to M^K$. If ϕ_{V_t} is an immersion for any $t \in [0, T)$, the following map F defines a deformation of ϕ_{V_0} and we say F the *expansion* of f:

$$F : (H/K) \times V \times [0, T) \to M; \quad (hK, p, t) \mapsto hf_t(p).$$

Theorem 1. *Suppose a submanifold $V \subset M^K$ has the property $(*)$ and there exists a deformation $f : V \times [0, T) \to M^K$ of V with the expansion F satisfying the followings.*

(i) For any $t \in [0, T)$ and any $p \in V$, it holds that $\frac{\partial}{\partial t} F_t(K, p) = \mathcal{H}^t(K, p)$, and

(ii) the deformation f preserves the property $(*)$,

where, $\mathcal{H}^t$ is the mean curvature vector field of the immersion F_t. Then, the family of maps $(F_t)_{t\in[0,T)}$ is the mean curvature flow of the map ϕ_V.

Corollary 1. *Suppose that a submanifold $V \subset M^K$ has the property $(*)$ and there exists a vector field A satisfying the followings.*

(i.a) The vector field A generates a deformation $f : V \times [0, T) \to M^K$ of V with the expansion F,

(i.b) for any $t \in [0, T)$ and any $p \in V$, it holds that $\mathcal{H}^t(K, p) = A_{f_t(p)}$, and

(ii) the deformation f preserves the property $(*)$.

Then, the family of maps $(F_t)_{t\in[0,T)}$ is the mean curvature flow of the map ϕ_V.

We note that for a given submanifold V, if we find a vector field A satisfying the condition (i.b), then the condition (i.a) is an ordinary differential equation.

3 Constructions of invariant Lagrangian mean curvature flows

We show a method of constructing a Lagrangian mean curvature flow in a Calabi-Yau manifold by the corollary above, using generalized perpendicular symmetries.

Theorem 2. *Let (M, I, g, Ω) be a connected Calabi-Yau manifold with a complex structure I, Kähler metric g and a Calabi-Yau structure Ω, H a connected Lie group which acts on M preserving I and g with a moment map $\mu : M \to \mathfrak{h}^*$, K a closed subgroup of H such that H/K is orientable and the K-action preserves Ω, L a special Lagrangian submanifold, $c \in Z(\mathfrak{h}^*)$, V_c an $\big(n - \dim(H/K)\big)$-dimensional submanifold of M such that $V_c \subset \mu^{-1}(c) \cap L^K$ and ϕ_{V_c} is an immersion. Then, it holds that*

(1) the map ϕ_{V_c} is a Lagrangian immersion, and

(2) there exists a vector field A_H along L^K such that

$$\mathcal{H}^c(hK, p) = (A_H)_{hp} + (L_h)_{*p} I_p \big\{ (\mathrm{grad}_{\phi^*_{V_c} g} \theta_c(K, \cdot))_p \big\} \quad ((hK, p) \in (H/K) \times V_c)$$

holds, where $\mathcal{H}^c$ is the mean curvature vector field of ϕ_{V_c} and $\theta_c : (H/K) \times V_c \to \mathbb{R}/2\pi\mathbb{Z}$ is the Lagrangian angle of ϕ_{V_c}.

Moreover, suppose that A_H generates a deformation $f : V_c \times [0, T) \to L^K$ with the expansion F and for any $t \in [0, T)$ and $V_t := f_t(V_c)$, the following condition holds.

$$\xi^\#_p \in \mathrm{T}^\perp_p L \oplus \mathrm{T}_p V_c \text{ and } \xi^\#_p \notin \mathrm{T}_p V_c \setminus \{0\} \quad (\text{“generalized perpendicular condition”}).$$

Then, the family of maps $(F_t)_{t\in[0,T)}$ is the Lagrangian mean curvature flow of ϕ_{V_c}.

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Lagrangian mean curvature flows with generalized perpendicular symmetries

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Submanifolds of Symmetric Spaces and Their Time Evolutions

§1. **Goals**

•Goal（general cases）:
To construct mean curvature flows by symmetries of Lie groups in Riemannian mfds.

•Goal（special cases）:
To construct Lagrangian mean curvature flows by generalized perpendicular symmetries of Lie groups in Calabi-Yau mfds.

§2. Previous Researches

•Previous Researches:

Yamamoto(2016)	
construct	generalized Lag MCF
in	toric almost Calabi-Yau mfds
using	moment map & toric symm.
Konno(2018)	
construct	Lag MCF
in	Calabi-Yau mfds
using	moment map & perp. symm. of abelian actions

•Our Researches:

Ours (general cases)	
construct	MCF
in	Riem. mfds
using	symm. of general actions
Ours (special cases)	
construct	Lag MCF
in	Calabi-Yau mfds
using	moment map & generalized perp. symm. of general actions

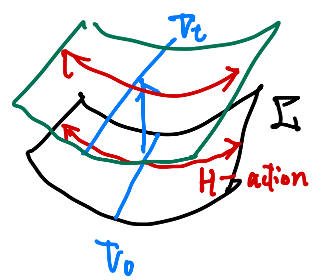
§3. Overview

$H \curvearrowright M$

How to construct MCF by symm. of Lie groups

M : Riem. mfd, H: Lie grp s.t. $H \curvearrowright M$,
Σ: H-invariant submfd of M.

Step.1 Find a nice sumfd $V_0 \subset \Sigma$ s.t. $H \cdot V_0 = \Sigma$.



Step.2 Study how V_0 is deformed by the MCF of Σ.

Step.3 Have the MCF of Σ by $\Sigma_t := H \cdot V_t$.

§4. Preliminaries

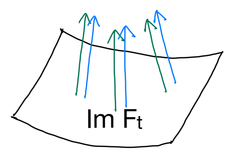
Def. 1 Let $\phi : \overset{\text{mfd}}{\Sigma} \to \overset{\text{mfd}}{M}$ be an immersion. For a smooth map

$$\begin{cases} F : \Sigma \times [0, T) \to M; \quad (p, t) \mapsto F_t(p) \\ F_0 = \phi \end{cases},$$

if $F_t(\cdot) : \Sigma \to M$ is an immersion for $\forall t \in [0, T)$, then we call F the **deformation** of ϕ (or Σ).

Let $\phi : \overset{\text{mfd}}{\Sigma} \to \overset{\text{Riem.mfd}}{(M, g)}$ be an immersion. The **mean curvature flow** $F = (F_t)_{t\in[0,T)}$ of ϕ is the deformation of ϕ s.t. it is a smooth solution of the following PDE:

$$\frac{\partial}{\partial t}F(p, t) = \mathcal{H}^t(p) \quad \text{w/ } \mathcal{H}^t : \text{mcv of } F_t.$$

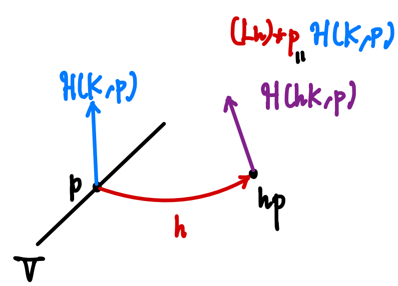


Fact. 2 *MCF preserves the "Lagrangeness" in Kähler-Einstein mfds.*

§5. Constructions of MCFs

•Setting $(*1)$:

- (M, g): Riem. mfd,
- H: Lie grp s.t. $H \curvearrowright M$,
- K: closed subgrp of H,
- V: submfd of M s.t. $V \subset M^K$.



Def. 3 Under $(*1)$,

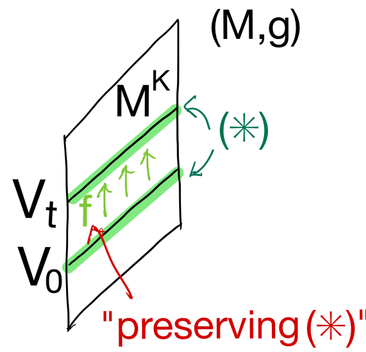
$$Z(\mathfrak{h}^*) : \text{the center of the Lie coalgebra } \mathfrak{h}^*,$$
$$L^K := \{p \in L \mid H_p = K\}, \quad \text{w/}L : \text{any submfd of } M,$$
$$\phi_V : (H/K) \times V \to M; \quad (hK, p) \mapsto hp.$$

Def. 4 (property $(*)$) Under $(*1)$, if ϕ_V is an immersion & its mean curvature vectors are H-invariant, i.e., it holds that

$$\mathcal{H}(hK, p) = (L_h)_{*p}\mathcal{H}(K, p), \tag{$*$}$$

then we say that V has the **property $(*)$** wrt the H-actions.

Def. 5 (preserve the property (∗)) Let V_0 is a submfd of M s.t. $V_0 \subset M^K$ & has the property (∗). Under (∗1), if ∃ a deformation of V_0 in M^K & $V_t := f_t(V)$ also has the property (∗), we say that f **preserves** the property (∗) of V_0.



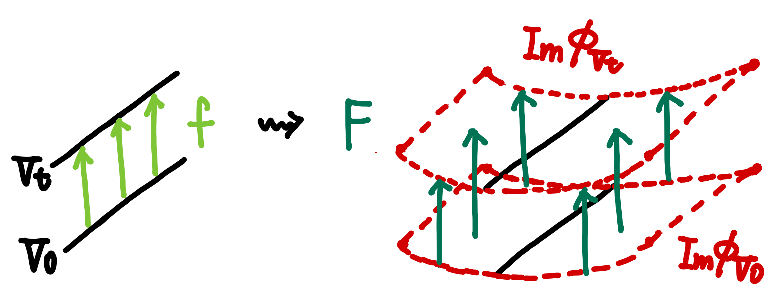
Under (∗1), suppose that ∃ a deformation $f : V_0 \times [0, T) \to M^K$.

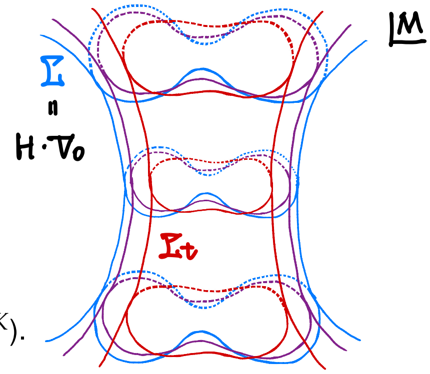
Def. 6 (expansion of deformation) If ϕ_{V_t} is an immersion for $\forall t \in [0, T)$, we can define a deformation F of ϕ_{V_0} by

$$F : (H/K) \times V_0 \times [0, T) \to M; \quad (hK, p, t) \mapsto hf_t(p) =: F_t(hK, p).$$

We call F the **expansion** of f.

We denote the mean curvature vector of F_t by $\mathcal{H}^t$.





•Setting $(*2)$:

- (M, g): Riem. mfd,
- H: Lie grp s.t. $H \curvearrowright M$,
- K: closed subgrp of H,
- V_0: submfd with $(*)$ of M (s.t. $V_0 \subset M^K$).

Thm. 7 *Under $(*2)$, suppose that $\exists$ a deformation f of V_0 with its expansion F satisfying (i) & (ii):*

(i) *For $\forall t \in [0, T)$, $\forall p \in V_0$,*

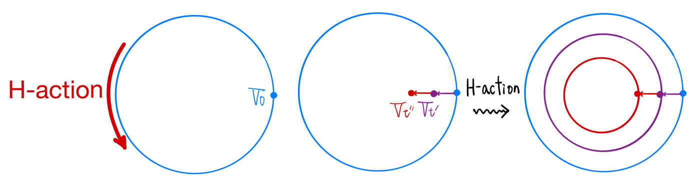
$$\frac{\partial}{\partial t} F_t(K, p) = \mathcal{H}^t(K, p) \quad (\text{``restricted MCF condition''}),$$

(ii) *f preserves the property $(*)$.*

Then, $(F_t)_{t\in[0,T)}$ is the MCF of ϕ_{V_0}.

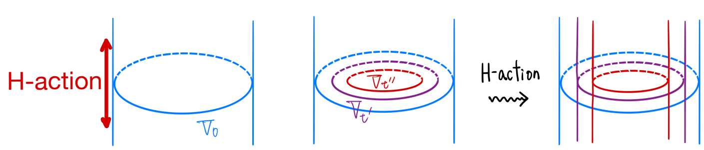
e.g. 8 (circle, sphere)

$\phi : S^n \to \mathbb{R}^{n+1}$, $V_0 :=$ single point, $H := SO(n+1)$.



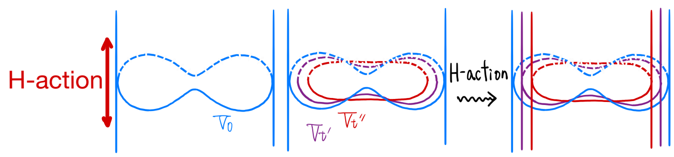
e.g. 9 (cylinder)

$\phi : S^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n+1}, V_0 := S^m,\ H := \mathbb{R}^{n-m}.$



e.g. 10 (generalized cylinder)

$\phi : M^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n+1},\ V_0 := M,\ H := \mathbb{R}^{n-m}.$



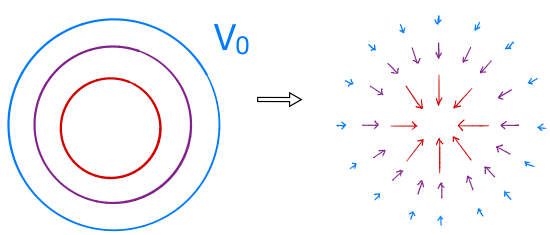
Question: How to reduce the restricted MCF eq to an ODE ?

► Additional assumption:

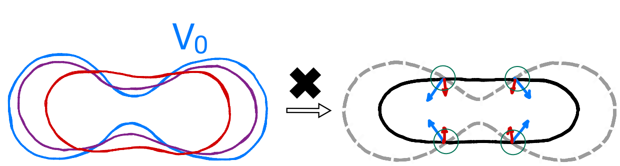
The evolution of the restricted MCF forms a vector field of the mean curvature vectors.

e.g. 11

(1) The MCF of S^n forms a vector filed of their mcv.



(2) The MCF of Dumbbell-like surfaces do not.



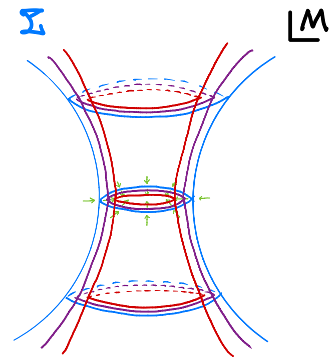
Cor. 12 *Under* $(*2)$, *suppose that the restricted MCF of* (V_0, ϕ_{V_0}) *forms a vector filed* A, *i.e.,* $\exists$ *a vector field* A *satisfying* (i.a) & (i.b)*:*

(i.a) A *generates a deformation* f *of* V_0 *in* M^K *with* F, *i.e.,*

$$\frac{d}{dt}F_t(K,p) = A_{f_t(p)} \quad (\forall p \in V_0, \forall t \in [0,T]) \quad \leftarrow \text{ODE}$$

(i.b) *For* $\forall t \in [0,T)$ & $p \in V_0$,

$$\mathcal{H}^t(K,p) = A_{f_t(p)}.$$



Moerover, suppose that

(ii) f *preserves the property* $(*)$.

Then, $(F_t)_{t\in[0,T)}$ *is the MCF of* ϕ_{V_0}.

$\rightsquigarrow$ How to find V_0 with A satisfying (i.b) for constructing Lag MCFs in CY mfds ?

§6. Constructions of Lag MCFs

•Setting $(*3)$:

- (M,ω): $2n$-$\dim_{\mathbb{R}}$ symp. mfd,
- H: Lie grp s.t. $H \curvearrowright (M,\omega)$ with moment map $\mu : M \to \mathfrak{h}^*$,
- K: closed subgrp of H,
- V_c: submfd of M s.t. $V_c \subset M^K$,
- ϕ_{V_c}: immersion.

Prop. 13 *Under* $(*3)$, *suppose*

(i) V_c *is isotropic,*

(ii) *("moment map condition")* $V_c \subset \mu^{-1}(c)$ *for* $c \in Z(\mathfrak{h}^*)$.

(iii) $\dim H/K + \dim V_c = n$

Then ϕ_{V_c} *is Lagrangian. Conversely, if* ϕ_{V_c} *is connected & Lagrangian, then* (i), (ii) *and* (iii) *hold.*

Def. 14 (Lagrangian angle) (M, I, g, Ω): Calabi-Yau mfd, L: oriented Lag submafd of M,

$$\theta : L \to \mathbb{R}/2\pi\mathbb{Z} : \textbf{Lagrangian angle} :\Leftrightarrow \iota^*\Omega = e^{\sqrt{-1}\theta}\mathrm{vol}_{\iota^*g}$$

w/ $\iota : L \to M$: inclusion map.

$$L : \textbf{special Lagrangian submfd} :\Leftrightarrow \theta \equiv \mathrm{const.}$$

Prop. 15 *$\mathcal{H}(p)$: mean curvature vector of L at $p \in L$. Then,*

$$\mathcal{H}(p) = I_{\iota(p)}\left\{\iota_{*p}(\mathrm{grad}_{\iota^*g}\theta)_p\right\}.$$

•<u>Setting $(*4)$</u>:

- (M, I, ω, Ω): connected Calabi-Yau mfd,
- H: connected Lie grp s.t. $H \curvearrowright (M, I, \omega)$ with moment map $\mu : M \to \mathfrak{h}^*$,
- K: closed subgrp of H s.t. H/K: orientable & $K \curvearrowright \Omega$,
- V_c: orientable submfd of M s.t. $V_c \subset \mu^{-1}(c) \cap M^K$,
- ϕ_{V_c}: Lag immersion.

Prop. 16 *Under* $(*4)$,

(1) $\theta_c(hK, p) = \exists\, \theta_H(hK) + \exists\, \theta_{V_c}(p)$, *w/ θ_c: Lag angle of ϕ_{V_c},*
$\uparrow$ *defined only by* (M, H, K)

(2) $\mathcal{H}^c(hK, p) = \exists\, (A_H)_{hp} + (L_h)_{*p} I_p\left\{(\mathrm{grad}_{\phi^*_{V_c}g}\theta_{V_c})_p\right\}$,
$\uparrow$ *defined only by* (M, H, K)
w/ $\mathcal{H}^c$:MCV of ϕ_{V_c}.

If $\theta_{V_c} \equiv \mathrm{const.} \rightsquigarrow \mathcal{H}^c = A_H$ holds and V_c accomodates to Cor.12

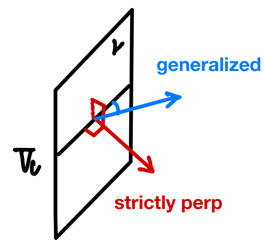
•Setting ($*$5):

- (M, I, ω, Ω): connected Calabi-Yau mfd,
- H: connected Lie grp s.t. $H \curvearrowright (M, I, \omega)$
  with moment map $\mu : M \to \mathfrak{h}^*$,
- K: closed subgrp of H s.t. H/K: orientable & $K \curvearrowright \Omega$,
- A_H: vector field along M^K as in Prop.16
- L: special Lag submfd with Lag angle $\theta(p) \equiv \theta$,
- $c \in Z(\mathfrak{h}^*)$,
- V_c: $\big(n - \dim(H/K)\big)$-dim submfd of M s.t. $V_c \subset \mu^{-1}(c) \cap L^K$.

Prop. 17 *Under* ($*$5), *suppose*

$$\forall p \in V_c, \forall \xi \in \mathfrak{h}, \quad \xi^\#_p \in \mathrm{T}^\perp_p L \oplus \mathrm{T}_p V_c \ \&\ \xi^\#_p \notin \mathrm{T}_p V_c \setminus \{0\}.$$

("generalized perp. condition")

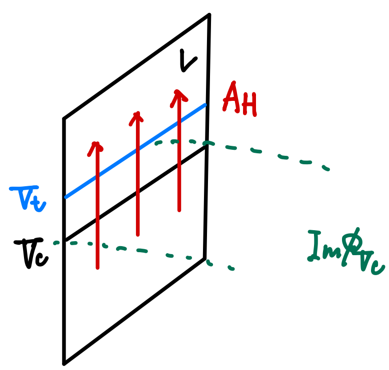


Then,

(1) $\theta_{V_c}(p) = \theta - \frac{\pi}{2} \dim(H/K), \quad \leftarrow \text{const.}$

(2) $\mathcal{H}^c(hK, p) = (A_H)_{hp}$.

Thm. 18 *Under* ($*$5), *suppose that* A_H *generates a deformation* $f : V_c \times [0, T) \to L^K$ *with its expansion* F, *and for* $\forall t \in [0, T)$ *and* $V_t := f_t(V_c)$, *the generalized perpendicular condition holds.*
Then, A_H *and* V_c *satisfies the condition of Cor.12 and* $(F_t)_{t \in [0,T)}$ *is a Lag MCF of* ϕ_{V_c}.



T^*S^n.

$A_H \equiv 0$

§7. **Examples**

	construct	*Lag self-similar solution*
e.g. 19	*in*	$\mathbb{C}^4$
	using	*strictly perp. symm.* of $U(1) \times SO(3)$

	construct	*Lag MCF*
e.g. 20	*in*	$\mathbb{C}^5$
	using	*gen. perp. symm.* of $\mathbb{R} \times SO(2)$

	construct	*Lag translating soliton*
e.g. 21	*in*	$\mathbb{C}^5$
	using	*strictly perp. symm.* of $U(1) \times SO(3)$

	construct	*Lag translating soliton*
e.g. 22	*in*	$\mathbb{C}^6$
	using	*gen. perp. symm.* of $\mathbb{R} \times SO(2)$

Thank you very much for your attention.

Ricci flow, heat equation, Liouville type theorem

Keita Kunikawa

Abstract. In this talk, we will see a Liouville type theorem for heat equation along ancient (super) Ricci flow using Perelman's reduced distance.

1. Liouville theorems

Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Ric} \geq -K$ $(K \geq 0)$ and $B_R(x_0)$ be a geodesic ball on M centered at $x_0 \in M$ with radius $R > 0$. In 1975, Cheng-Yau established a gradient estimate for positive harmonic function $u : B_R(x_0) \to \mathbb{R}$;

$$\frac{|\nabla u|}{u} \leq C_n \left(\frac{1}{R} + \sqrt{K} \right) \quad \text{on} \quad B_{\frac{R}{2}}(x_0),$$

where C_n is a constant depending only on n. When $K = 0$, Cheng-Yau's gradient estimate implies the celebrated Yau's Liouville theorem, that is, positive harmonic function on a complete Riemannian manifold with $\mathrm{Ric} \geq 0$ must be constant.

A parabolic analogue of this theory is considered by Souplet-Zhang [2]. They obtained a space-only gradient estimate for a positive solution to the heat equation on a Riemannian manifold with $\mathrm{Ric} \geq -K$. More precisely, they showed that for a positive solution to $\partial_t u = \Delta u$, $0 < u \leq A$ on a parabolic cylinder $Q_{R,T}(x_0, t_0) = B_R(x_0) \times [t_0 - T, t_0]$,

$$\frac{|\nabla u|}{u} \leq C_n \left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{K} \right) \left(1 + \log \frac{A}{u} \right) \quad \text{on} \quad Q_{\frac{R}{2}, \frac{T}{4}}(x_0, t_0).$$

As a corollary, they showed Liouville type results for ancient solutions (i.e., solutions defined for all the negative time $t \in (-\infty, 0]$) to heat equation on a complete Riemannian manifold with $\mathrm{Ric} \geq 0$;

(a1) **positive** ancient solution u with $u(x,t) = \exp\left[o(d_g(x) + \sqrt{|t|})\right]$ near infinity must be constant,

(b1) ancient solution u with $u(x,t) = o(d_g(x) + \sqrt{|t|})$ near infinity must be constant.

Without the growth conditions, there exist nontrivial ancient solutions. For (a1), we know a positive ancient solution $u(x,t) = e^{x+t}$ on $M = \mathbb{R}$ which does not satisy the growth condition. Likewise, we have another example for (b1): $u(x,t) = x$. This is a static ancient solution on $M = \mathbb{R}$ which does not satisfy the growth condition. In this sense, Souplet-Zhang's growth conditions are essential and sharp in the space direction.

2. Main results

In this talk, we will see that the analogous results to Souplet-Zhang hold under Ricci flow background. Althohg it is natural to consider the time-dependent Riemannian distance $d_{g(t)}(x)$, one finds that it does not work well. The problem is that the condition $\mathrm{Ric} \geq -K$ is not sufficient to estimate the time derivative of $d_{g(t)}(x)$. If we additionally impose $\mathrm{Ric} \geq -K$, then it is possible to estimate the time derivative of $d_{g(t)}(x)$ and we can obtain a gradient estimate as above. However, in order to derive Liouville type results, this situation can deal with only trivial situation, i.e., Ricci flat ($K = 0$).

To overcome this difficulty, we adobt Perelman's reduced geometry. In reduced geometry, it is natural to use reverse time parameter $\tau := -t$. Instead of $d_{g(t)}(x)$, we adopt the so-called reduced distance $\ell(x, \tau)$. This makes it possible for us to avoid the estimate of time-derivative of $d_{g(t)}(x)$.

Here, we will consider more general situation than the Ricci flow. Let $(M, g(\tau))$ be a time-dependent Riemannian manifold, $\tau \in [0, \infty)$. Set $h := \frac{1}{2}\partial_\tau g$ and $H := \mathrm{tr}_g h$. For $(x, \tau) \in M \times (0, \infty)$, let $L(x, \tau)$ stand for the L-distance from a space-time base point $(x_0, 0)$, i.e., the infimum of the so-called $\mathcal{L}$-length over all curves $\gamma : [0, \tau] \to M$ with $\gamma(0) = x_0$ and $\gamma(\tau) = x$. We only consider the case that the infimum is achieved by a minimal $\mathcal{L}$-geodesic. Then the reduced distance from $(x_0, 0)$ and its squred root is defined by

$$\ell(x, \tau) = \frac{1}{2\sqrt{\tau}} L(x, \tau), \quad \mathfrak{d}(x, \tau) = \sqrt{4\tau\ell(x, \tau)}.$$

In the static case of $g(\tau) \equiv g$, it holds that $\ell(x, \tau) = d_g(x)^2/4\tau$. Moreover, we introduce the Müller quantity $\mathcal{D}(V)$ and the trace Harnack quantity $\mathcal{H}(V)$ for (time-dependent) vector field V on M by

$$\begin{aligned}
\mathcal{D}(V) &:= -\partial_\tau H - \Delta H - 2||h||^2 + \mathrm{div}h(V) - 2g(\nabla H, V) + 2\mathrm{Ric} - 2h(V, V), \\
\mathcal{H}(V) &:= -\partial\tau H - \frac{H}{\tau} - 2g(\nabla H, V) + 2h(V, V).
\end{aligned}$$

Main Theorem. (K.-Sakurai [1]) *Let $(M, g(\tau))_{\tau \in [0,\infty)}$ be a complete ancient backward super Ricci flow* $\mathrm{Ric} \geq h$ *with $\mathcal{D}(V) \geq 0$, $\mathcal{H}(V) \geq -H/\tau$ and $H \geq 0$ for any V. Then,*

(a2) ***positive*** *ancient solution to $\partial_\tau u = -\Delta_{g(\tau)} u$ with $u(x, \tau) = \exp[o(\mathfrak{d}(x, \tau) + \sqrt{\tau})]$ near infinity must be constant;*

(b2) *ancient solution to $\partial_\tau u = -\Delta_{g(\tau)} u$ with $u(x, \tau) = o(\mathfrak{d}(x, \tau) + \sqrt{\tau})$ near infinity must be constant.*

Remark. For a static manifold with $\mathrm{Ric} \geq 0$, all the assumptions automatically hold and $\mathfrak{d}(x, \tau) = d_g(x)$. So, our theorem includes Souplet-Zhang's Liouville type result. As for the Ricci flow $\mathrm{Ric} = h$ with bounded nonnegative curvature operator Rm, again all the assumptions are satisfied and we obtain a Liouville type result.

References

[1] K. Kunikawa and Y. Sakurai, *Liouville theorem for heat equation along ancient super Ricci flow via reduced geometry*, preprint, arXiv:2005.04882.

[2] P. Souplet and Q.S. Zhang, *Sharp gradient estimate and Yau's Liouville theorem for the heat equation on noncompact manifolds*, Bull. London Math. Soc. **38** (2006), no. 6, 1045–1053.

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Ricci flow, heat equation, Liouville type theorem

Keita Kunikawa

(Utsunomiya University)

Joint work with Y. Sakurai

1

History of Liouville theorems

- Bounded hol. fct. on $\mathbb{C}$
- " harmonic " on $\mathbb{R}^n$

$\Big\} \Rightarrow$ const.

Yau '75 (M^n, g) : cplt $\mathrm{Ric} \geq 0$

$u > 0$: harmonic fct.

$\Rightarrow u \equiv$ const.

Rem u : b'dd $\rightsquigarrow u > 0$

L2

Cheng-Yau '75 : Local grad. estimate

$$\mathrm{Ric} \geq -K \quad (K \geq 0)$$

$u : B(x_0, R) \longrightarrow \mathbb{R}$

positive harmonic fct.

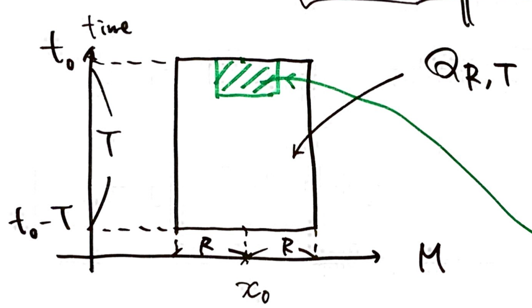
Then, $\dfrac{|\nabla u|}{u} \leq C_n \left(\dfrac{1}{R} + \sqrt{K} \right)$ in $B(x_0, \frac{R}{2})$

$K = 0$, $R \to \infty$ $\Rightarrow$ Liouville thm.

L3

Heat eq. on (M, g) : cplt, $\mathrm{Ric} \geq -K$

$$\partial_t u = \Delta u \quad \text{in} \quad \underbrace{B(x_0, R) \times [t_0 - T, t_0]}_{\parallel}$$



Li-Yau '86 : $\forall a > 0$

$$\frac{|\nabla u|^2}{u^2} - a \frac{\partial_t u}{u} \leq C_{n,a} \left(\frac{1}{R^2} + \frac{1}{T} + K \right) \quad \text{in } Q_{\frac{R}{2}, \frac{T}{4}}$$

$\rightsquigarrow$ does not imply Liouville result.

14

Souplet - Zhang '06 : Same setting as LY.

$$\partial_t u = \Delta u, \quad 0 < u \leq A \quad \text{in } Q_{R,T}$$

Then, $\dfrac{|\nabla u|}{u} \leq C_n\left(\dfrac{1}{R} + \dfrac{1}{\sqrt{T}} + \sqrt{K}\right)\left(1 + \log\dfrac{A}{u}\right)$ in $Q_{\frac{R}{2},\frac{T}{4}}$

Cor. $\mathrm{Ric} \geq 0$ $(K = 0)$

(a) $u > 0$ ancient sol, $t \in (-\infty, 0]$

with $u(x,t) = \exp[o(d_g(x) + \sqrt{|t|})]$ near ∞

$\Rightarrow u \equiv$ const. $\quad u(x,t) = e^{x+t} > 0$ on $\mathbb{R}$

(b) u : ancient.

with $u(x,t) = o(d(x) + \sqrt{|t|})$ near ∞

$\Rightarrow u \equiv$ const. $\quad u(x,t) = x$ on $\mathbb{R}$

15

Today : Ricci flow ver. of SZ.

$$\begin{cases} (M, g(t)) : \partial_t g = -2\mathrm{Ric}(g(t)), \text{ ancient.} \\ \partial_t u = \Delta_{g(t)} u \end{cases}$$

In the following $\tau := -t$, $\tau \in [0, \infty)$

	SZ		KS
curv.	$\mathrm{Ric} \geq 0$	$\longrightarrow$	$\mathrm{Rm} \geq 0$, bounded
dist.	$d_g(x)$	$\longrightarrow$	$\bar{d}(x,\tau) = \sqrt{4\tau \ell(x,\tau)}$

$x_0 \in M$: fixed

reduced dist. from $(x_0, 0)$

$g(\tau) \equiv g$

$\rightsquigarrow \ell = \dfrac{d^2}{4\tau}$

Thm A (K. - Sakurai '20) 16

$(M, g(\tau))_{\tau\in[0,\infty)}$: $\partial_\tau g = 2\mathrm{Ric}$ backward RF.

$Rm \geq 0$, bounded

Then,

(a) $\partial_\tau u = -\Delta u$, $u > 0$ in $M\times[0,\infty)$

with $u(x,\tau) = \exp[o(\bar{d}(x,\tau)+\sqrt{\tau})]$

$\Rightarrow$ $u \equiv$ const.

(b) $\partial_\tau u = -\Delta u$ in $M\times[0,\infty)$

with $u(x,\tau) = o(\bar{d}(x,\tau)+\sqrt{\tau})$

$\Rightarrow$ $u \equiv$ const.

Rem. Bailesteanu - Cao - Pulemotov '10 : grad. est. using $d_{g(\tau)}$ .. $|\mathrm{Ric}| \leq K$ same as SZ.

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proof of SZ '06. $\mathrm{Ric} \geq -K$ 17

w.l.o.g. $0 < u \leq 1$ in $Q_{R,T}$.

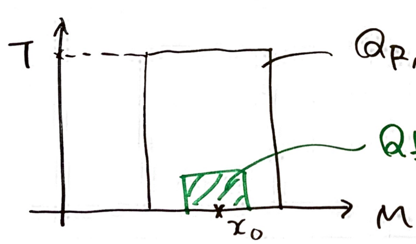
$f := \log u$, $w := \dfrac{|\nabla f|^2}{(1-f)^2}$

Bochner formula : $\frac{1}{2}\Delta|\nabla u|^2 = \langle \nabla \Delta u, \nabla u\rangle + |\mathrm{Hess}\, u|^2 + \underbrace{\mathrm{Ric}(\nabla u,\nabla u)}_{\geq -K|\nabla u|^2}$; $\Delta u = -\partial_\tau u$

$\Downarrow$

Some estimate $(\partial_\tau + \Delta) w \geq$ ▨ $\cdots (*)$

Cutoff fct. $\psi = \psi(d(x),\tau)$ supported on ~~$M\times[0,\infty)$~~ $Q_{R,T}$



$Q_{R,T}$: ~~$0\leq\psi\leq 1$~~ $0 \leq \psi \leq 1$.

$Q_{\frac{R}{2},\frac{T}{4}}$: $\psi \equiv 1$.

18

ψw attains max in $Q_{R,T}$

$$\underbrace{\Delta(\psi w)}_{\Delta d \,\to\, Ric \geq -K.} \leq 0, \quad \underbrace{\partial_\tau(\psi w)}_{\partial_\tau d = 0 \text{ in SZ.}} \leq 0, \quad \nabla(\psi w) = 0.$$

Combining this with $(*)$, we have

$$\frac{|\nabla u|}{u} \leq C_n\left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{K}\right)\left(1 + \log\frac{1}{u}\right) \quad \text{in } Q_{\frac{R}{2}, \frac{T}{4}}$$

- $K = 0$, $R \to \infty$, $T \to \infty$

$\Rightarrow$ Liouville type result (a), (b) in SZ.

$\square$

19

BCP '10 $\quad \begin{cases} \partial_\tau g = 2\,Ric \\ \partial_\tau u = -\Delta_{g(\tau)} u \end{cases}$

Note: $d(x, \tau) \leftarrow g(\tau)$: RF

cutoff. $\psi = \psi(d(x,\tau), \tau)$ upper bound.

$\partial_\tau \psi \rightsquigarrow \partial_\tau d_{g(\tau)} \rightsquigarrow |Ric| \leq K$ $\quad Ric \leq K$

~~(bound from above)~~

Same argument ~~gives~~ implies

$$\frac{|\nabla u|}{u} \leq C_n\left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{K}\right).$$

$\square$.

K-S '20. Idea: use reduced dist. $l(x,\tau)$ 10

$\bar{d}(x,\tau) := \sqrt{4\tau l(x,\tau)}$, $l(x,\tau) \geq 0$ if $Scal \geq 0$

$\bar{L} := \bar{d}(x,\tau)^2$

$Rm \geq 0$, bounded $\rightarrow$ Hamilton's trace Harnack ineq.

$\downarrow$ $\downarrow$

$(\partial_\tau + \Delta)\bar{L} \leq 2n$ $|\nabla \bar{d}|^2 \leq 3$

Cutoff fct. $\psi = \psi(\bar{d}(x,\tau), \tau)$.

Same arg. as SZ

$\rightsquigarrow \dfrac{|\nabla u|}{u} \leq C_n\left(\dfrac{1}{R} + \dfrac{1}{\sqrt{T}}\right)\left(1 + \log\dfrac{A}{u}\right)$ in $Q_{\frac{R}{2}, \frac{T}{4}}$. $\square$

Thm B (K.-S '20) 11

$$\begin{cases} (M, g(\tau))_{\tau \in [0,\infty)} : \text{backward } (-K)\text{-RF.} \\ \qquad \left(Ric - \frac{1}{2}\partial_\tau g = -K \right) \\ Rm \geq 0, \text{ bounded} \\ \partial_\tau u = -\Delta u \\ 0 < u \leq A \text{ in } Q_{R,T} = \{(x,\tau) \in M \times [0,T] \mid \bar{d} \leq R\} \end{cases}$$

Then,

$$\frac{|\nabla u|}{u} \leq C_n\left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{K}\right)\left(1 + \log\frac{A}{u}\right) \text{ in } Q_{\frac{R}{2}, \frac{T}{4}}.$$

Antipodal sets of compact symmetric spaces and polars of compact Lie groups

Makiko Sumi Tanaka

Abstract. This presentation is based on the author's collaboration with Hiroyuki Tasaki. In their former research, Tasaki and the author classified maximal antipodal sets of some classical compact Riemannian symmetric spaces by using their embeddings into connected compact Lie groups as polars. In order to continue the classification of maximal antipodal sets of other compact Riemannian symmetric spaces, their realization as polars of disconnected compact Lie groups is needed. In this presentation the author explain the recent research relating to them.

Let M be a compact Riemannian symmetric space and s_x denote the point symmetry at a point x in M. A subset A of M is called an *antipodal set* if $s_x(y) = y$ holds for any points x, y in A. The *2-number* of M is the maximum of the cardinalities of antipodal sets. In the 1980's Chen and Nagano introduced these notions and gave detailed studies of the 2-numbers. In the past ten years there was progress on the research of antipodal sets. Our interest has been shifted to maximal antipodal sets themselves rather than their cardinalities. Tasaki and the author classified maximal antipodal subgroups of some classical compact Lie groups G and gave their explicit descriptions (J. Lie Theory, 2017), and after that, they classified maximal antipodal sets of some classical compact Riemannian symmetric spaces M (Differential Geom. Appl., 2020). The basic principle is to make use of an embedding of M into G as a polar with respect to the identity element and apply the classification of maximal antipodal subgroups of G. In order to continue the classification of maximal antipodal sets for some other classical compact Riemannian symmetric spaces M, the realization of M as a polar of a disconnected compact Lie group is needed. Chen-Nagano (Duke Math. J., 1978) and Nagano (Tokyo J. Math., 1988) gave detailed studies of polars of connected compact Riemannian symmetric spaces. Tasaki and the author studied polars of disconnected compact Lie groups in their submitting paper.

Let G be a compact Lie group and we denote by e the identity element of G. There exists a biinvariant Riemannian metric on G and G is a Riemannian symmetric space with respect to the metric. For any $x \in G$, the point symmetry at x, denoted by s_x, is given by $s_x(y) = xy^{-1}x$ ($y \in G$). Let G be a connected compact Lie group and σ be an involutive automorphism of G. We denote by $\langle\sigma\rangle$ the subgroup of the group of automorphisms of G generated by σ. For the semidirect product $G \rtimes \langle\sigma\rangle$, $G \rtimes \langle\sigma\rangle = (G, e') \cup (G, \sigma)$ is the decomposition into a disjoint union of the connected components, where $e' = \mathrm{id}_G$ denotes the identity element of $\langle\sigma\rangle$. We denote by $\hat{e}$ the identity element of the semidirect product $G \rtimes \langle\sigma\rangle$. We define the action ρ_σ of G on G by $\rho_\sigma(g)(h) = gh\sigma(g)^{-1}$ ($g, h \in G$), which is called the twisted conjugate action by σ. For an isometry f of G, we denote by $F(f, G)$ the set of fixed points of f. The following is the main theorem:

Theorem (Tanaka-Tasaki, submitted). *Let G be a connected compact Lie group and σ be an involutive automorphism of G. Then we obtain*

$$F(s_{\hat{e}}, G \rtimes \langle \sigma \rangle) = (F(s_e, G), e') \cup (F(s_e \circ \sigma, G), \sigma).$$

In particular, each connected component of $(F(s_e \circ \sigma, G), \sigma)$ is a polar of $G \rtimes \langle \sigma \rangle$. Moreover, the connected component of $(F(s_e \circ \sigma, G), \sigma)$ containing (e, σ) coincides with $(\rho_\sigma(G) \cdot e, \sigma)$, where $\rho_\sigma(G) \cdot e$ is a symmetric space defined by a symmetric pair $(G, F(\sigma, G))$, which is realized by the imbedding $G/F(\sigma, G) \to G\ ;\ gF(\sigma, G) \mapsto g\sigma(g)^{-1}$.

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Antipodal sets of compact symmetric spaces and polars of compact Lie groups

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Contents

1. Introduction

M: a Riemannian manifold

M is called a Riemannian symmetric space if for $\forall x \in M$, the point symmetry s_x at x is given, i.e., (i) s_x is an isometry of M, (ii) $s_x \circ s_x = \mathrm{id}_M$, (iii) x is an isolated fixed point of s_x.

- The differential $(ds_x)_x$ is $-\mathrm{id}_{T_x M}$.
- When M is connected, s_x is uniquely determined by (i)-(iii) and s_x is the geodesic symmetry.

$F(s_x, M) := \{y \in M | s_x(y) = y\}$

A connected component of $F(s_x, M)$ is called a polar w.r.t. x.

By (iii), $\{x\}$ is a polar w.r.t. x, called the trivial polar.

- A polar M^+ of positive dimension is a totally geodesic submanifold and hence M^+ is a Riemannian symmetric space. The point symmetry at $y \in M^+$ is given by $s_y|_{M^+}$.

- $\mathbb{R}^n$: Euclidean space, $F(s_x, \mathbb{R}^n) = \{x\}$
- S^n: a sphere, $F(s_x, S^n) = \{x, -x\}$

• P^n: **the projective space,** $F(s_x, P^n) = \{x\} \cup P^{n-1}$

(∵) **Set** $\mathbb{K} = \mathbb{R}, \mathbb{C}$, **or** $\mathbb{H}$ **and denote** P^n **by** $\mathbb{K}P^n$. **Since** s_x **is induced by the reflection along** x **in** $\mathbb{K}^{n+1}$, $F(s_x, \mathbb{K}P^n) =$
$\{x\} \cup \{1\text{–dim. subspaces in } x^\perp\} (= \mathbb{K}P^{n-1})$.

• **If** M **is of noncompact type,** $F(s_x, M) = \{x\}$.

Hereafter we consider the case where M **is compact.**

• **A compact connected Riem. sym. sp.** M **is** (i) **of compact type (**$I(M)$ **is compact and semisimple),** (ii) **a torus, or a product of** (i) **and** (ii) **locally.**

A**: a subset of** M

A **is called an antipodal set if for** $\forall x, y \in A$, $s_x(y) = y$ **holds.**

For $\forall x \in A$, $A \subset F(s_x, M)$. x **is an isolated point in** $F(s_x, M)$ **as well as in** A. **Thus** A **is discrete. Hence an antipodal set is finite.**

The 2-number of M is $\#_2 M := \max\{|A| \mid A \subset M : \text{an antipodal set}\}$.
If A satisfies $|A| = \#_2 M$, A is called great.
If $A \subset A'$ implies $A = A'$, we say A is maximal.

- **A great antipodal set is maximal but the converse is not true.**
- **$\#_2 S^n = 2$ and $\{x, -x\}$ is a great antipodal set.**

Bang-Yen Chen and Tadashi Nagano gave detailed studies of the 2-numbers (Chen-Nagano, 1988).

In the past ten years there was progress in the research of antipodal sets. Our interest is in maximal antipodal sets themselves rather than their cardinalities. We are working on the classification of maximal antipodal sets.

- **In (T.-Tasaki, 2017) we classified max. antip. subgr. of some classical cpt. Lie groups G.**
- **In (T.-Tasaki, 2020) we classified max. antip. sets of some classical cpt. Riem. sym. sp. M.**

The basic principle is to make use of an

embedding of M into G as a polar w.r.t. the identity element and apply the classification of max. antip. subgr. of G.

• In order to continue the classification of max. antip. sets for some other classical cpt. Riem. sym. sp. M, we need a realization of M as a polar of a disconnected cpt. Lie gr.

• Chen-Nagano and Nagano gave detailed studies of polars of connected cpt. Riem. symmetric spaces.

• We studied polars of disconnected cpt. Lie groups (T.-Tasaki, submitted).

2. Relations between antipodal sets and polars

G: a compact Lie group

e: the identity element of G

G_0: the identity component of G

$\exists$ a biinvariant Riemannian metric on G

G is a compact Riem. symmetric space.

$\forall x \in G$, $s_x(y) = xy^{-1}x \ (y \in G)$

• $s_e(y) = y^{-1}, \ s_x(y) = L_x \circ s_e \circ L_{x^{-1}}(y)$

$F(s_e, G) = \{x \in G \mid x^2 = e\}$

$F(s_e, G) = \bigcup_{j=0}^{r} G_j^+$, G_j^+: a polar, $G_0^+ = \{e\}$
In general, when a polar consists of a single point x, we call x a pole.

Proposition 1

$Z_G(G_0)$: the centralizer of G_0 in G
$\tilde{Z}_2(G) := Z_G(G_0) \cap F(s_e, G)$
- The set of poles coincides with $\tilde{Z}_2(G)$.
- For a point x in G_j^+, $G_j^+ = \{I_g(x) \mid g \in G_0\}$, where $I_g(x) = gxg^{-1}$.

Hence each polar is a G_0-conjugacy class of involutive elements.

A: an antipodal set of G
We can assume $e \in A$ by left (or right) translations. Then,
- $x^2 = e\ (x \in A),\ xy = yx\ (x, y \in A)$.
- If A is maximal, A is a subgroup $\cong$ $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.

We call such A a maximal antipodal subgroup.

Example. $G = O(n)$: **the orthogonal group**

$G_0 = SO(n)$

1_n: **the identity matrix**

$I_j = \text{daig}(\underbrace{-1, \ldots, -1}_{j}, 1, \ldots, 1) \in O(n)$

$G_j^+ = \{gI_jg^{-1} \mid g \in SO(n)\}$

$\cong SO(n)/S(O(j) \times O(n-j))$

$= G_j(\mathbb{R}^n)$: **the real Grassmann mfd.**

$A_0 = \{\text{diag}(\epsilon_1, \ldots, \epsilon_n) \mid \epsilon_i = \pm 1\}$ **is a maximal antipodal subgroup of** $O(n)$**.**

$\tilde{Z}_2(O(n)) = \{\pm 1_n\}$

• A_0 **is a unique max. antip. subgr. of** $O(n)$ **up to conjugation, while a max. antip. subgr. of** $O(n)/\{\pm 1_n\}$ **is not unique up to conjugation when** n **is even and** $n \geq 4$**.**

$M = G_j^+$: **a polar of positive dim.**

M **is a connected cpt. Riem. sym. sp.**

$x_0 \in M$, $M = \{I_g(x_0) \mid g \in G_0\}$

• $I_0(M) = \{I_g|_M \mid g \in G_0\}$

• **If** A **is an antip. set of** M**, then** $A \cup \{e\}$ **is an antip. set of** G**.**

• $\exists \tilde{A}$: a max. antip. subgr. $A \cup \{e\} \subset \tilde{A}$

• If A is maximal in M, then $A = M \cap \tilde{A}$.

$C_1, \ldots, C_k$: G_0-conjugacy classes of maxi. antip. subgr. of G

B_s: a representative of C_s $(1 \leq s \leq k)$

(We gave their explicit descriptions for some classical G.)

$\exists g \in G_0,\ 1 \leq \exists s \leq k, \quad \tilde{A} = I_g(B_s)$

$A = M \cap \tilde{A} = M \cap I_g(B_s) = I_g(M \cap B_s)$

Hence A is $I_0(M)$-congruent to $M \cap B_s$.

Therefore, a representative of an $I_0(M)$-congruence class of maximal antipodal sets of M is one of $M \cap B_1, \ldots, M \cap B_k$.

• Using this principle, for some classical cpt. Riem. sym. sp. M, we determined $I_0(M)$-cong. classes of max. antip. sets of M and gave explicit descriptions of their representatives.

• $\exists M$, realized as a polar not of a connected G but of a disconnected G.

e.g., $U(n)/O(n), U(2n)/Sp(n)$

3. Polars of disconnected compact Lie groups

G: a compact Lie group

G_0: the identity component of G

$G = G_0 \cup \bigcup_{\lambda \in \Lambda} G_\lambda$, G_λ: a conn. component

$F(s_e, G) = (F(s_e, G) \cap G_0) \cup \bigcup_{\lambda \in \Lambda} (F(s_e, G) \cap G_\lambda)$

We know $F(s_e, G) \cap G_0$ by Chen-Nagano.

We study $F(s_e, G) \cap G_\lambda$.

If $F(s_e, G) \cap G_\lambda \neq \emptyset$, for $\forall x_\lambda \in G_\lambda \cap F(s_e, G)$ we have $G_\lambda = G_0 x_\lambda = x_\lambda G_0$.

I_{x_λ} ($I_{x_\lambda}(y) = x_\lambda y x_\lambda^{-1}$) is an involutive automorphism of G_0.

The action defined by $g.h = ghI_{x_\lambda}(g)^{-1}$ ($g, h \in G_0$) is called the twisted conjugate action by I_{x_λ}. (It is a Hermann action.)

T_λ: a maximal torus of the identity comp. of $F(I_{x_\lambda}, G_0)$.

By a property of Hermann actions we have:

<u>Proposition 2</u> $G_\lambda = \bigcup_{g \in G_0} g(x_\lambda T_\lambda)g^{-1}$

(It is well-known $G_0 = \bigcup_{g \in G_0} gTg^{-1}$ for a maximal torus T of G_0.)

$$F(s_e, G) \cap G_\lambda = \bigcup_{g \in G_0} g\{x \in x_\lambda T_\lambda \mid x^2 = e\} g^{-1}$$

In order to determine $F(s_e, G) \cap G_\lambda$, it is enough to determine $\{x \in x_\lambda T_\lambda \mid x^2 = e\}$ and G_0-conjugacy classes of each element of the set.
We can carry out them for each G on a case-by-case argument.

On the other hand, we have the following:
Proposition 3 **Assume $G_\lambda \cap F(s_e, G) \neq \emptyset$.**
(1) $G_0 \cup G_\lambda$ **is a subgroup.**

(2) **For $x_\lambda \in G_\lambda \cap F(s_e, G)$, $G_0 \cup G_\lambda$ is isomorphic to $G_0 \rtimes \langle I_{x_\lambda} \rangle$, where $\langle I_{x_\lambda} \rangle$ is the subgroup of** $\mathrm{Aut}(G_0)$ **generated by I_{x_λ}.**

Hence, the determination of polars of G is reduced to the determination of polars of $G_0 \rtimes \langle I_{x_\lambda} \rangle$.

$G_0 \rtimes \langle I_{x_\lambda} \rangle$ **consists of two connected components:**
$G_0 \rtimes \langle I_{x_\lambda} \rangle = \{(g, \mathrm{id}) \mid g \in G_0\} \cup \{(g, I_{x_\lambda}) \mid g \in G_0\}$

The group operation of $G_0 \rtimes \langle I_{x_\lambda}\rangle$:
For $g, h \in G_0,\ e' := \mathrm{id},\ \tau := I_{x_\lambda}$,
$(g, e')(h, e') = (gh, e'),\ (g, e')(h, \tau) = (gh, \tau)$,
$(g, \tau)(h, e') = (g\tau(h), \tau),\ (g, \tau)(h, \tau) = (g\tau(h), e')$.

Proof of Prop. 3: (1) **is easily seen by the group operation.** (2) $\varphi : G_0 \rtimes \langle I_{x_\lambda}\rangle \to G_0 \cup G_\lambda$ **defined by** $\varphi(g, \mathrm{id}) = g, \varphi(g, I_{x_\lambda}) = gx_\lambda$ **gives a Lie group isomorphism.**

G**: a connected cpt. Lie group**
σ**: an involutive atumorphism of** G
$\widehat{e} = (e, \mathrm{id})$**: the identity element of** $G \rtimes \langle\sigma\rangle$

<u>Theorem 4</u>

$F(s_{\widehat{e}}, G \rtimes \langle\sigma\rangle) = (F(s_e, G), \mathrm{id}) \cup (F(s_e \circ \sigma, G), \sigma)$
In particular, each connected component of $(F(s_e \circ \sigma, G), \sigma)$ **is a polar of** $G \rtimes \langle\sigma\rangle$**. Moreover, the conn. comp. of** $(F(s_e \circ \sigma, G), \sigma)$ **containing** (e, σ) **coincides with** $(\rho_\sigma(G) \cdot e, \sigma)$**, where** ρ_σ **is the twisted conjugate action by** σ**, and** $\rho_\sigma(G) \cdot e \cong G/F(\sigma, G)$**.**

Proof of Thm. 4 :
$F(s_{\widehat{e}}, G \rtimes \langle\sigma\rangle) = F(s_{\widehat{e}}, (G, \mathrm{id})) \cup F(s_{\widehat{e}}, (G, \sigma))$

$F(s_{\widehat{e}}, (G, \mathrm{id})) = (F(s_e, G), \mathrm{id})$
$F(s_{\widehat{e}}, (G, \sigma)) = (F(s_e \circ \sigma, G), \sigma)$
$(\because)\ \forall g \in G,$
$s_{\widehat{e}}(g, \sigma) = (g, \sigma)$
$\Leftrightarrow\ (g, \sigma) = (g, \sigma)^{-1} = (\sigma(g^{-1}), \sigma)$
$\Leftrightarrow\ g = \sigma(g^{-1})$
$\Leftrightarrow s_e \circ \sigma(g) = g$

As stated before, if we obtain the classification of max.antip. sugr. of $G \rtimes \langle \sigma \rangle$, we can determine max. antip. sets of $G/F(\sigma, G)$.

4. Examples

$U(n)$: **the unitary group**
$F(s_{1_n}, U(n)) =$
$\{x \in U(n) \mid x^2 = 1_n\} = \bigcup_{j=0}^{n} \{g\, I_j\, g^{-1} \mid g \in U(n)\}$
$I_j = \mathrm{diag}(\underbrace{-1, \ldots, -1}_{j}, 1, \ldots, 1) \in U(n)$
The polars of $U(n)$ **w.r.t.** 1_n **is:**
$\{1_n\}$, $\{-1_n\}$,
$U(n)/(U(j) \times U(n-j)) = G_j(\mathbb{C}^n) \quad (1 \leq j \leq n-1)$ **the complex Grassmann mfd.**
$\tau(g) := \bar{g}\ \ (g \in U(n))$

τ **is an involutive autom. of** $U(n)$

$G = U(n) \rtimes \langle\tau\rangle,\ \ \langle\tau\rangle = \{e', \tau\}$

$G = \{(g, e') \mid g \in U(n)\} \cup \{(g, \tau) \mid g \in U(n)\} \cdots (*)$

We write (g, e') **by** g, **and** (g, τ) **by** $g\tau$.

$(*) \rightsquigarrow G = U(n) \cup U(n)\tau$

$F(s_{\hat{e}}, G) = (F(s_{\hat{e}}, G) \cap U(n)) \cup (F(s_{\hat{e}}, G) \cap U(n)\tau)$

$F(s_{\hat{e}}, G) \cap U(n) = F(s_{1_n}, U(n)) = \bigcup_{j=0}^{n} G_j(\mathbb{C}^n)$

We study $F(s_{\hat{e}}, G) \cap U(n)\tau$ **by using Thm. 4.**

T**: a maximal torus of** $F(\tau, U(n)) = O(n)$

$U(n)\tau = \bigcup_{g \in U(n)} g(\tau T)g^{-1}$ **(by Prop. 2)**

$$F(s_{\hat{e}}, G) \cap U(n)\tau = \bigcup_{g \in U(n)} g\{x \in \tau T \mid x^2 = 1_n\}g^{-1}$$

So we study $\{x \in \tau T \mid x^2 = 1_n\}$. **We can take** $T \subset O(n)$ **as**

$$T = \left\{ \begin{bmatrix} R(\theta_1) & & & \\ & \ddots & & \\ & & R(\theta_k) & \\ & & & (1) \end{bmatrix} \middle|\ \theta_1, \ldots, \theta_k \in \mathbb{R} \right\},$$

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \qquad k = \lfloor \tfrac{n}{2} \rfloor$$

$\forall t \in T,\ \ \tau t = (1_n, \tau)(t, e') = (\tau(t), \tau) = t\tau,$

$(\tau t)^2 = \tau^2 t^2 = t^2$

Hence, $\{x \in \tau T \mid x^2 = 1_n\} = \tau\{t \in T \mid t^2 = 1_n\}$

$$= \tau\left\{\left.\begin{bmatrix} \epsilon_1 1_2 & & & \\ & \ddots & & \\ & & \epsilon_k 1_2 & \\ & & & (1) \end{bmatrix} \right| \epsilon_1, \ldots, \epsilon_k = \pm 1\right\}.$$

$F(s_{\hat{e}}, G) \cap U(n)\tau = \bigcup_{g \in U(n)} g\tau\{t \in T \mid t^2 = 1_n\}g^{-1}$

- $\forall t \in T, \forall g \in U(n),\ g(\tau t)g^{-1} = g\, t\, {}^t\! g\, \tau$
- **Since** $(i1_2)(-1_2)(i1_2) = 1_2$,

$\forall t \in T, t^2 = 1_n,\ \exists h \in U(n)$ s.t. $h\, t\, {}^t\! h = 1_n$.

Hence, if $t \in T, t^2 = 1_n,\ \{g(\tau t)g^{-1} \mid g \in U(n)\} =$ $\{g\, t\, {}^t\! g \mid g \in U(n)\}\tau = \{g\, 1_n\, {}^t\! g \mid g \in U(n)\}\tau.$

So $F(s_{\hat{e}}, G) \cap U(n)\tau = \{g\, 1_n\, {}^t\! g \mid g \in U(n)\}\tau$

Here $g 1_n\, {}^t\! g = g 1_n \bar{g}^{-1} = g 1_n \tau(g)^{-1} = \rho_\tau(g)(1_n)$.

ρ_τ**: the twisted conjugate action by** τ**.**

Hence $\{g\, 1_n\, {}^t\! g \mid g \in U(n)\}$ **is an orbit of** $\rho_\tau(G)$ **through** 1_n**.**

$g\, 1_n\, {}^t\! g = 1_n \Leftrightarrow {}^t\! g = g^{-1} = {}^t\bar{g} \Leftrightarrow g \in O(n)$

$F(s_{\hat{e}}, G) \cap U(n)\tau \cong U(n)/O(n)$ **(connected)**

$U(n)/O(n)$ **is realized as a polar of** $U(n) \rtimes \langle \tau \rangle$**.**

($U(n)/O(n)$ **is not realized as a polar of a connected compact Lie group.)**

Thank you for your kind attention.