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Quantum Field Theory and Related Mathematical Aspects

Organized by
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March 13 – 15, 2022

Abstract

We report the international workshop "Quantum Field Theory and Related Mathematical Aspects" held in Shuzenji Sogo Kaikan, Shizuoka Prefecture.

2020 *Mathematics Subject Classification.*
20C32, 70H06, 81T30, 81T32, 81T35, 81T40

Key words and Phrases.
Integrability, Geometry, Representation theory, Supersymmetry, Quantum field theory, String theory

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Preface

Quantum field theory (QFT) is one of the most successful theoretical frameworks in Physics, which has been widely used in particle physics, condensed matter and cosmology. To understand the behavior of QFT, in particular the non-perturbative effect of QFT, geometry, representation theory etc in Mathematics play a crucial role. Although this fact is not new at all, the idea in Mathematics are continuosly contributing to the development of QFT, and the new discovery in QFT could lead to the progress in geometry and representation theory.

We focus in this international workshop “Quantum Field Theory and Related Mathematical Aspects” on this relation between Physics and Mathematics from the view point of QFT, so as to strengthen the relationship between Physics and Mathematics communities, and furthermore to discover new connections. The workshop covers the topics such as exact results in supersymmetric QFT, nonperturbative methods, generalized symmetry of QFT, and holography. We invited the experts of these fields to discuss the recent developments. At the same time, young researchers and graduate students are welcome to interact them to stimulate their researches.

The workshop was held in the in-person format in Shuzenji Sogo Kaikan, Izu City, Shizuoka Prefecture. 36 people have participated. We had 10 talks by the invited speakers including young researchers. They gave talks focusing on recent developments, which promoted the various discussions. The venue is in a remote area, but this makes participants easy to communicate each other and produce a new project.

In this report, we record the abstracts and the slides of the talks.

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Hiroshi Itoyama

Abstract.

(H. Itoyama) NITEP

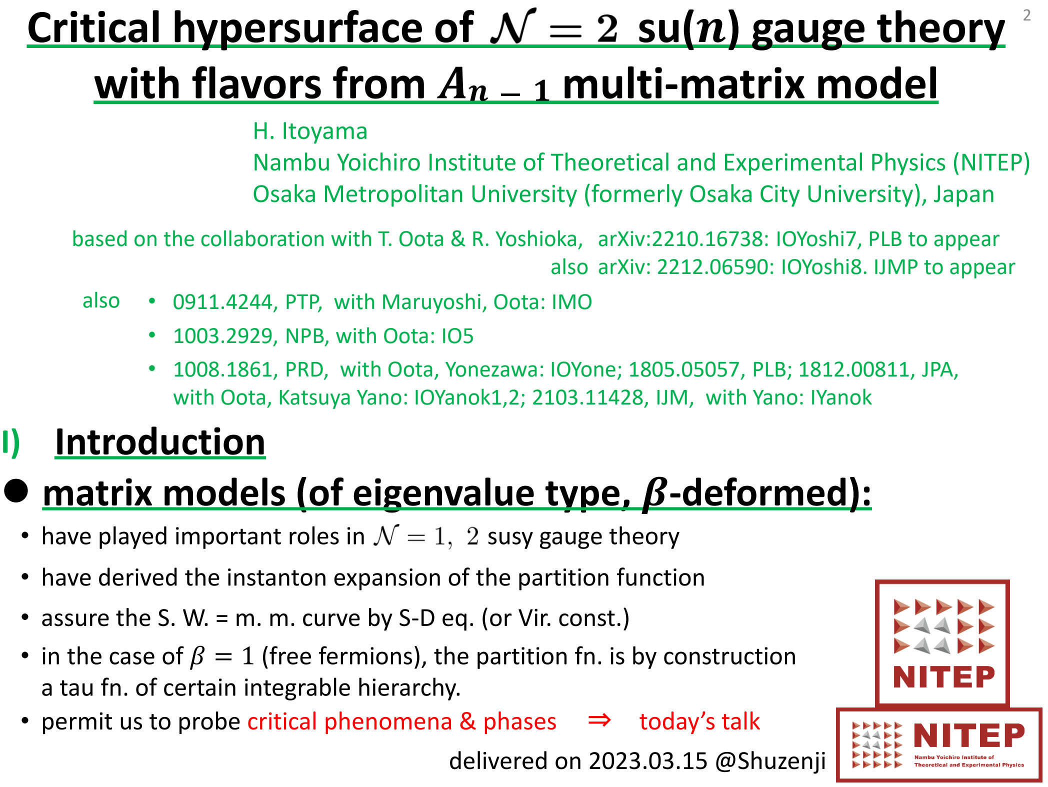
Things to remember & recall

- K. M.
- OCAMI brother of NITEP
- wksp “duality, integrability & matrix model”

  JSPS/RFBR collaboration

  2016 3/7-9@Shuzenji

speakers: Mironov, Seki, Sleptsov, Matsumoto, And. Morozov,
Okuyama, Moriyama, Bajnok, K. Yoshida, Zenkevich,
K. Ito, A. Morozov



Critical hypersurface of $\mathcal{N}=2$ su(n) gauge theory with flavors from A_{n-1} multi-matrix model

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based on the collaboration with T. Oota & R. Yoshioka, arXiv:2210.16738: IOYoshi7, PLB to appear
also arXiv: 2212.06590: IOYoshi8. IJMP to appear

also
- 0911.4244, PTP, with Maruyoshi, Oota: IMO
- 1003.2929, NPB, with Oota: IO5
- 1008.1861, PRD, with Oota, Yonezawa: IOYone; 1805.05057, PLB; 1812.00811, JPA, with Oota, Katsuya Yano: IOYanok1,2; 2103.11428, IJM, with Yano: IYanok

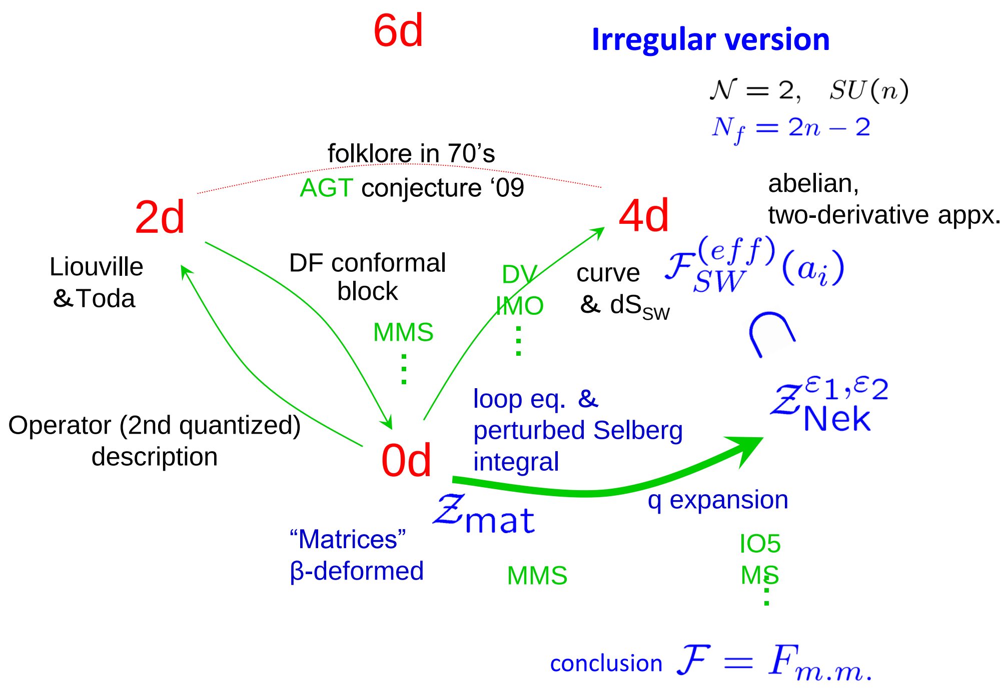
I) Introduction

● matrix models (of eigenvalue type, β-deformed):

- have played important roles in $\mathcal{N}=1,\ 2$ susy gauge theory
- have derived the instanton expansion of the partition function
- assure the S. W. = m. m. curve by S-D eq. (or Vir. const.)
- in the case of $\beta=1$ (free fermions), the partition fn. is by construction a tau fn. of certain integrable hierarchy.
- permit us to probe critical phenomena & phases ⇒ today’s talk

delivered on 2023.03.15 @Shuzenji

2d-4d connection & AGT relation



6d

Irregular version

$\mathcal{N}=2, \quad SU(n)$

$N_f = 2n-2$

folklore in 70's

AGT conjecture '09

2d

4d

abelian,
two-derivative appx.

Liouville
& Toda

DF conformal
block

MMS

DV
IMO

curve
& $\mathrm{dS}_{\mathrm{SW}}$

$\mathcal{F}^{(eff)}_{SW}(a_i)$

$\cap$

$\mathcal{Z}^{\varepsilon_1,\varepsilon_2}_{\text{Nek}}$

loop eq. &
perturbed Selberg
integral

Operator (2nd quantized)
description

0d

$\mathcal{Z}_{\text{mat}}$

q expansion

"Matrices"
β-deformed

MMS

IO5
MS

conclusion $\mathcal{F} = F_{m.m.}$

some remarks in advance:

- integral representation 2d Virasoro block is provided by one-matrix model of multi-log potential: $\log(z_i - z_j)$ both for 2d CFT and $\log(\text{v.d.m. det})$
- in the cases of W_n block, the series of multi- matrix models of quiver type which obey the W_n constraints by construction is available.
- irregular limit of this series of models of the multi-log type:
  a place to look at to explore new critical phenomena
- an interplay between symmetry and phase structure ala Landau
- investigate broken symmetry structure of the model enforced by the automorphism of the Dynkin diagram and its possible connection to the critical hypersurface of the system, namely the nature of singularities of Argyres-Douglas type

cf. Argyres-Douglas; Argyres-Plesser-Seiberg-Witten;
Eguchi-Hori-Ito-Yang; Kubota-Yokoi; …

● maximal symmetry ⇒ a set of flavor mass relations:

our conclusion: $m_i = m_{2n+1-i}, \qquad i = 2, \cdots, n$

which is our proposed critical hypersurface

● In fact

- in IOYosh8, we checked that the S-W curve of Hanany-Oz maximally degenerates under this rel. no need to set all masses being equal.

● contents:

II) one matrix model irregular C.B./su(2) $N_f = 2, 3$: our past work as review
III) A_{n-1} extension
IV) constraints from maximal sym.
V)

II) ● Generating q-expansion coeff. from Selberg-type m.m.

IO5

- the 3-log type m. m. in general
- q here is “the cross ratio”, namely, the location of the third V. O.
- $(c_0 \log z + c_2 \log(z-1) + c_3 \log(z-q))' = 0$ has two roots.
- ⇒ ∴ two cut distributions of e. v., but at $q = 0$ collapse to one-cut
- still need to keep N_L, N_R finite for q-expansion
- factorizes into left & right, each factor being the Selberg integral, the generalization of Euler’s beta fn. dx_i, dy_j integrals over [0,1]
- formula for q-expansion ⇒ $$\mathcal{A}(q) = \left\langle\!\!\left\langle \exp\left[-\sum_{k=1}^{\infty}\frac{q^k}{k}\left(\alpha_2 + b_E\sum_{I=1}^{N_L}x_I^k\right)\left(b_E\sum_{J=1}^{N_R}y_J^k\right)\right.\right.\right.$$ $$\left.\left.\left. -\sum_{k=1}^{\infty}\frac{q^k}{k}\left(b_E\sum_{I=1}^{N_L}x_I^k\right)\left(\alpha_3 + b_E\sum_{J=1}^{N_R}y_J^k\right)\right]\right\rangle\!\!\right\rangle_{N_L,N_R}.$$ $$\mathcal{A}(q) = \sum_{k=0}^{\infty} q^k \sum_{|Y_1|+|Y_2|=k} \mathcal{A}_{Y_1,Y_2}.$$
- $^{\exists}$a pair of partitions (Y_1, Y_2)
- to evaluate, finite $N_{L,R}$ S-D eq. & “Kadel formula” have been used.
- The lowest order gives the mm-4d dictionary

filling fraction necessary

$$b_E N_L = \frac{a - m_2}{g_s}, \qquad b_E N_R = -\frac{a + m_3}{g_s},$$
$$\alpha_1 = \frac{1}{g_s}(m_2 - m_1 + \epsilon), \qquad \alpha_2 = \frac{1}{g_s}(m_2 + m_1),$$
$$\alpha_3 = \frac{1}{g_s}(m_3 + m_4), \qquad \alpha_4 = \frac{1}{g_s}(m_3 - m_4 + \epsilon), \qquad \epsilon = \epsilon_1 + \epsilon_2$$

- Higher orders in q computable

● “Irregular block” from $N_f = 4 \ \to 3, 2$

IOYone, IOYanok
cf. Eguchi-Maruyoshi; Nishinaka-Rim

- possible to obtain from the $N_f = 2N_c$ case (β fn $= 0$) by sending some of the masses to ∞
- can be done 3 log → 2 log → 1 log
- the potential then contains $1/z$ and the original path of the real axis must be deformed to a complex path for analytic continuation

$$W^{(4)}(z) = \alpha_1 \log|z| + \alpha_2 \log|z - q_0| + \alpha_3 \log|z - 1|$$

$$\to \quad W^{(3)}(z) = \alpha_{1+2} \log z - \frac{q_{03}}{z} + +\alpha_3 \log(z-1),$$

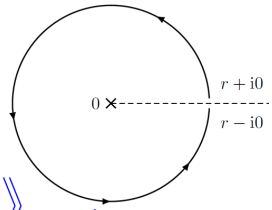
$$\to \quad W^{(2)}(z) = -q_{02}\left(z + \frac{1}{z}\right) + \alpha_{1+2} \log z$$

$m_4 \to \infty, \quad \Lambda_3 \equiv 4q_0 m_4$ fixed
$\alpha_{1+2} \equiv \alpha_1 + \alpha_2$ fixed
$q_{03} = \Lambda_3/(4g_s)$
$m_3 \to \infty, \quad \Lambda_2 = (m_3\Lambda_3)^{1/2}$ fixed
$\alpha_{3+4} \equiv \alpha_3 + \alpha_4$ fixed
$q_{02} = \Lambda_2/(2g_s)$

- best to use $\displaystyle\int_0^\infty dw w^\alpha \left(\sum_{n=0}^\infty c_n w^n\right) = \frac{1}{(\mathrm{e}^{2\pi i\alpha} - 1)} \int_{C(q_0)} dw w^\alpha \left(\sum_{n=0}^\infty c_n w^{-n}\right)$ for $\mathrm{Re}\,\alpha > 0$

  and $\displaystyle\int_1^\infty dw w^\alpha \left(\sum_{n=0}^\infty c_n w^{-n}\right) = \frac{1}{(1 - \mathrm{e}^{2\pi i\alpha})} \int_{C(1)} dw w^\alpha \left(\sum_{n=0}^\infty c_n w^n\right)$ for $\mathrm{Re}\,\alpha < 0$

- instanton expansion in $q_{02,03}$, using the left part $Z_L^{(2)}$ and the right part $Z_R^{(2)}$ of $Z^{(2)}$



$$\frac{Z^{(2)}}{\mathcal{N}^{(2)}\, Z_L^{(2)}\, Z_R^{(2)}} = \left\langle\!\!\left\langle \prod_{I=1}^{N_L} \prod_{J=1}^{N_R} \left(1 - {q_{02}}^2 \frac{w_I}{u_J}\right)^{2\beta} \prod_{I=1}^{N_L} \exp\left(-\sqrt{\beta}{q_{02}}^2 w_I\right) \right.\right.$$

$$\left.\left. \times \prod_{J=1}^{N_R} \exp\left(-\sqrt{\beta}{q_{02}}^2 \frac{1}{u_J}\right) \right\rangle\!\!\right\rangle_{N_L, N_R},$$

- can be evaluated from finite N loop eq.

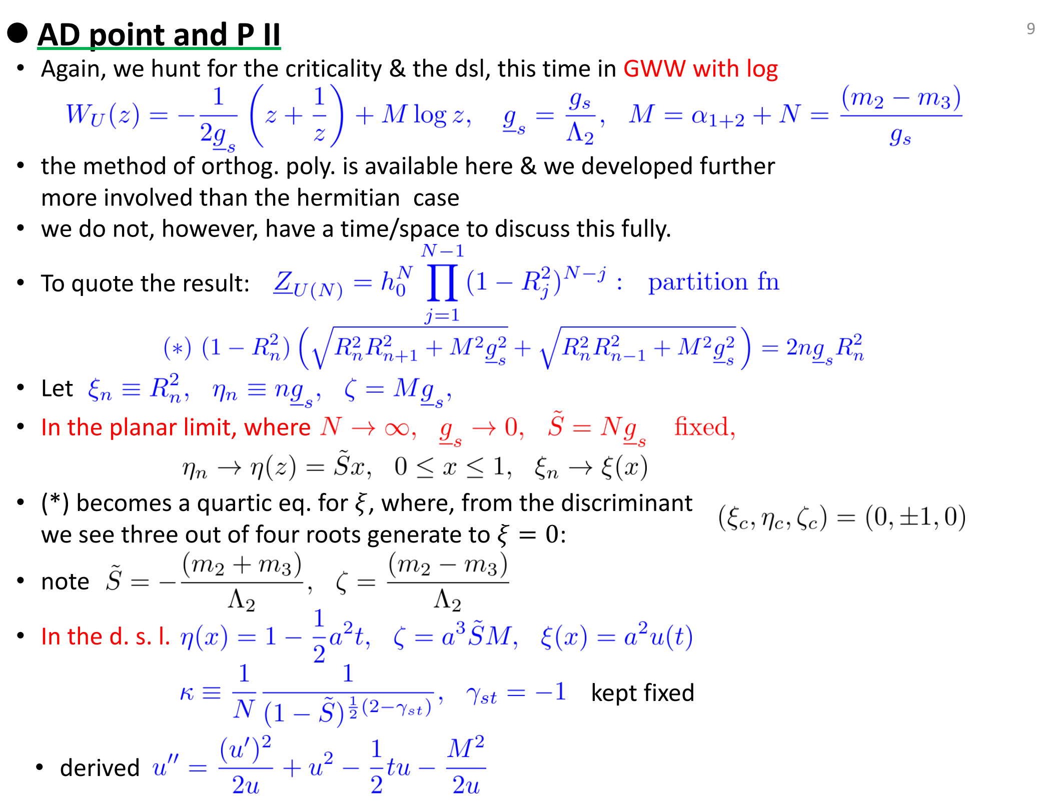
● Unitary m. m. of GWW type at $\beta \in \mathbb{Z}_+$

IOYanok

- the cut disappears if $\sqrt{\beta}\alpha_{1+2} \in \mathbb{Z}$ and the both paths can be deformed to the unit circle.
- In particular, at $\beta = 1$, is a unitary m. m. with a log potential

$$Z^{(2)} = \left(\prod_{I=1}^N \int_{C_I^{(2)}} \frac{dw_I}{w_I}\right) \Delta(w)^\beta \Delta(w^{-1})^\beta \exp\left[\sqrt{\beta} \sum_{I=1}^N \left\{-q_{02}\left(w_I + \frac{1}{w_I}\right) + M \log w_I\right\}\right].$$

- vev a dependence disappears



AD point and P II

9

- Again, we hunt for the criticality & the dsl, this time in GWW with log

$$W_U(z) = -\frac{1}{2\underline{g}_s}\left(z+\frac{1}{z}\right) + M\log z,\quad \underline{g}_s = \frac{g_s}{\Lambda_2},\quad M = \alpha_{1+2}+N = \frac{(m_2-m_3)}{g_s}$$

- the method of orthog. poly. is available here & we developed further more involved than the hermitian case
- we do not, however, have a time/space to discuss this fully.
- To quote the result: $\underline{Z}_{U(N)} = h_0^N \prod_{j=1}^{N-1}(1-R_j^2)^{N-j}$: partition fn

$$(*)\ (1-R_n^2)\left(\sqrt{R_n^2R_{n+1}^2+M^2\underline{g}_s^2}+\sqrt{R_n^2R_{n-1}^2+M^2\underline{g}_s^2}\right) = 2n\underline{g}_s R_n^2$$

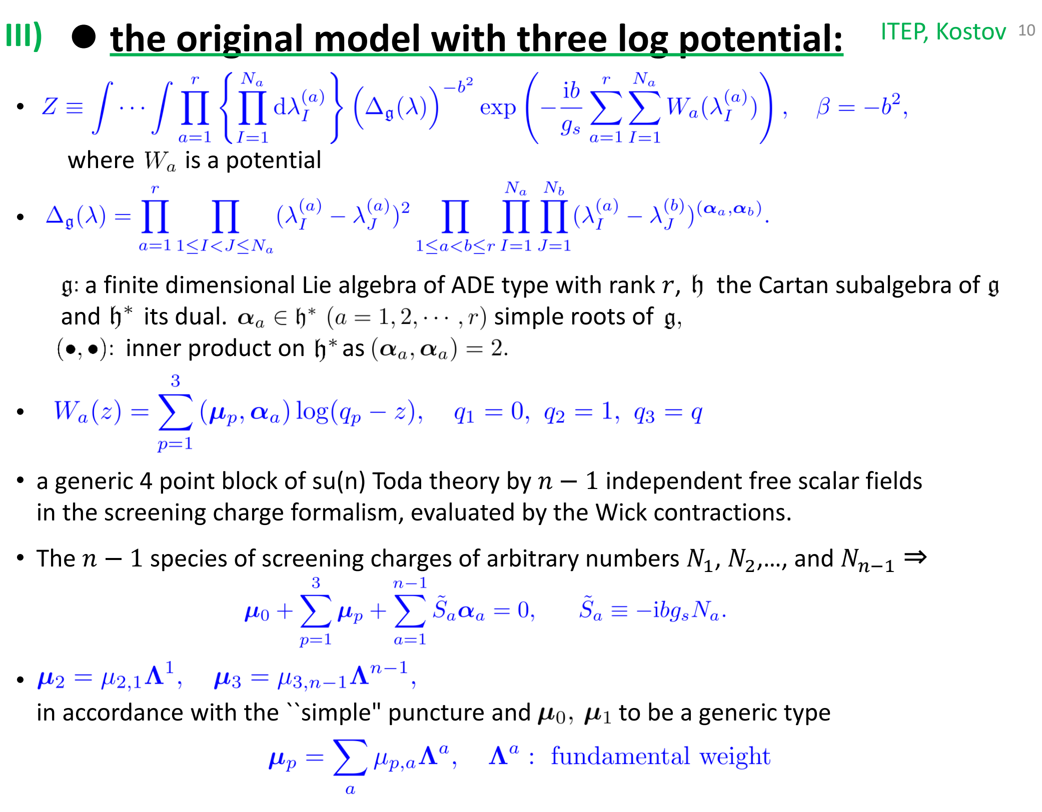
- Let $\xi_n \equiv R_n^2,\quad \eta_n \equiv n\underline{g}_s,\quad \zeta = M\underline{g}_s,$
- In the planar limit, where $N\to\infty,\quad \underline{g}_s\to 0,\quad \tilde{S} = N\underline{g}_s$ fixed,

$$\eta_n \to \eta(z) = \tilde{S}x,\quad 0\le x\le 1,\quad \xi_n \to \xi(x)$$

- (*) becomes a quartic eq. for ξ, where, from the discriminant we see three out of four roots generate to $\xi = 0$: $(\xi_c,\eta_c,\zeta_c) = (0,\pm1,0)$
- note $\tilde{S} = -\frac{(m_2+m_3)}{\Lambda_2},\quad \zeta = \frac{(m_2-m_3)}{\Lambda_2}$
- In the d. s. l. $\eta(x) = 1-\frac{1}{2}a^2t,\quad \zeta = a^3\tilde{S}M,\quad \xi(x) = a^2u(t)$

$$\kappa \equiv \frac{1}{N}\frac{1}{(1-\tilde{S})^{\frac{1}{2}(2-\gamma_{st})}},\quad \gamma_{st} = -1$$ kept fixed

  - derived $u'' = \frac{(u')^2}{2u}+u^2-\frac{1}{2}tu-\frac{M^2}{2u}$



III) ● the original model with three log potential:

ITEP, Kostov 10

- $Z \equiv \int\cdots\int\prod_{a=1}^{r}\left\{\prod_{I=1}^{N_a}d\lambda_I^{(a)}\right\}\left(\Delta_{\mathfrak{g}}(\lambda)\right)^{-b^2}\exp\left(-\frac{ib}{g_s}\sum_{a=1}^{r}\sum_{I=1}^{N_a}W_a(\lambda_I^{(a)})\right),\quad \beta = -b^2,$
  where W_a is a potential
- $\Delta_{\mathfrak{g}}(\lambda) = \prod_{a=1}^{r}\prod_{1\le I<J\le N_a}(\lambda_I^{(a)}-\lambda_J^{(a)})^2\prod_{1\le a<b\le r}\prod_{I=1}^{N_a}\prod_{J=1}^{N_b}(\lambda_I^{(a)}-\lambda_J^{(b)})^{(\boldsymbol{\alpha}_a,\boldsymbol{\alpha}_b)}.$

  $\mathfrak{g}$: a finite dimensional Lie algebra of ADE type with rank r, $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{h}^*$ its dual. $\boldsymbol{\alpha}_a \in \mathfrak{h}^*$ $(a = 1,2,\cdots,r)$ simple roots of $\mathfrak{g}$,
  $(\bullet,\bullet)$: inner product on $\mathfrak{h}^*$ as $(\boldsymbol{\alpha}_a,\boldsymbol{\alpha}_a) = 2.$
- $W_a(z) = \sum_{p=1}^{3}(\boldsymbol{\mu}_p,\boldsymbol{\alpha}_a)\log(q_p-z),\quad q_1 = 0,\ q_2 = 1,\ q_3 = q$
- a generic 4 point block of su(n) Toda theory by $n-1$ independent free scalar fields in the screening charge formalism, evaluated by the Wick contractions.
- The $n-1$ species of screening charges of arbitrary numbers N_1, N_2,…, and N_{n-1} ⇒

$$\boldsymbol{\mu}_0+\sum_{p=1}^{3}\boldsymbol{\mu}_p+\sum_{a=1}^{n-1}\tilde{S}_a\boldsymbol{\alpha}_a = 0,\qquad \tilde{S}_a \equiv -ibg_sN_a.$$

- $\boldsymbol{\mu}_2 = \mu_{2,1}\boldsymbol{\Lambda}^1,\quad \boldsymbol{\mu}_3 = \mu_{3,n-1}\boldsymbol{\Lambda}^{n-1},$
  in accordance with the ``simple" puncture and $\boldsymbol{\mu}_0,\ \boldsymbol{\mu}_1$ to be a generic type

$$\boldsymbol{\mu}_p = \sum_a \mu_{p,a}\boldsymbol{\Lambda}^a,\quad \boldsymbol{\Lambda}^a:\ \text{fundamental weight}$$

irregular limit to $N_f = 5, 4$, su(3):

- the 0d-4d dictionary at $n = 3$, $N_f = 6$ from Gaiotto vs. m.m. curve

$$(\boldsymbol{\mu}_1, \boldsymbol{\alpha}_1) = m_4 - m_5, \quad (\boldsymbol{\mu}_2, \boldsymbol{\alpha}_1) = m_1 + m_2 + m_3, \quad (\boldsymbol{\mu}_3, \boldsymbol{\alpha}_1) = 0,$$
$$(\boldsymbol{\mu}_1, \boldsymbol{\alpha}_2) = m_5 - m_6, \quad (\boldsymbol{\mu}_2, \boldsymbol{\alpha}_2) = 0, \quad (\boldsymbol{\mu}_3, \boldsymbol{\alpha}_2) = m_4 + m_5 + m_6.$$

IMO

- the $N_f = 6 \to 5$ limit: $m_6 \to \infty$, $q_0 \to 0$ with $\Lambda_5 = \lim q_0 m_6$ (up to constant) kept finite.

$$q_{05} \equiv \lim q_0 (\boldsymbol{\mu}_3, \boldsymbol{\alpha}_2), \qquad \text{so that } q_{05} = \Lambda_5 / g_s \text{ (up to constant).}$$

- the potential $W_1^{(N_f=5)}(z) = (\boldsymbol{\mu}_1, \boldsymbol{\alpha}_1) \log z + (\boldsymbol{\mu}_2, \boldsymbol{\alpha}_2) \log(1 - z),$

$$W_2^{(N_f=5)}(z) = (\boldsymbol{\mu}_1 + \boldsymbol{\mu}_3, \boldsymbol{\alpha}_2) \log z - \frac{q_{05}}{z}.$$

- the subsequent limit $N_f = 5 \to 4$: $m_1 \to \infty$, $q_{05} \to 0$
with $\Lambda_4 \equiv \lim (m_1 \Lambda)^{1/2}$ (up to constant) kept finite.

$$q_{04}^2 \equiv \lim q_{05} (\boldsymbol{\mu}_2, \boldsymbol{\alpha}_1), \qquad q_{04} = \Lambda_4 / g_s \text{ up to some constant.}$$

- the potential $W_1^{(N_f=4)}(z) = (\boldsymbol{\mu}_1, \boldsymbol{\alpha}_1) \log z - q_{04} z,$

$$W_2^{(N_f=4)}(z) = (\boldsymbol{\mu}_1 + \boldsymbol{\mu}_3, \boldsymbol{\alpha}_2) \log z - \frac{q_{04}}{z}.$$

generalization to A_{n-1}: straightforward

- the limit $N_f = 2n \to 2n - 1$: $W_1(z), W_a(z), a = 2 \cdots, n-2$ the same as $N_f = 2n$

$$W_{n-1}(z) = (\boldsymbol{\mu}_1 + \boldsymbol{\mu}_3, \boldsymbol{\alpha}_{n-1}) \log z - \frac{q_{05}}{z}, \qquad q_{05} \equiv \lim q_0 (\boldsymbol{\mu}_3, \boldsymbol{\alpha}_{n-1}).$$

- Further limit $N_f = 2n - 1 \to 2n - 2$: $q_{04}^2 = \lim q_{05} (\boldsymbol{\mu}_2, \boldsymbol{\alpha}_1)$.
- The potentials $W_1(w^{(1)}) = (\boldsymbol{\mu}_1, \boldsymbol{\alpha}_1) \log w^{(1)} - q_{04} w^{(1)}, \quad W_a(w^{(a)}) = (\boldsymbol{\mu}_1, \boldsymbol{\alpha}_a) \log w^{(a)}, \quad a = 2, \cdots, n-2,$

$$W_{n-1}(w^{(n-1)}) = (\boldsymbol{\mu}_1 + \boldsymbol{\mu}_3, \boldsymbol{\alpha}_{n-1}) \log w^{(n-1)} - \frac{q_{04}}{w^{(n-1)}}.$$

- the (broken) symmetry based on the "evenness" of the potential and
the automorphism of the Dynkin diagram still not manifest.
- The 0d-4d dictionary $\boldsymbol{\mu}_0 = \sum_{a=1}^{n-1} (-m_a + m_{a+1}) \boldsymbol{\Lambda}^a, \quad \boldsymbol{\mu}_1 = \sum_{a=1}^{n-1} (m_{n+a} - m_{n+a+1}) \boldsymbol{\Lambda}^a,$ IMO

$$\boldsymbol{\mu}_2 = \left(\sum_{i=1}^{n} m_i \right) \boldsymbol{\Lambda}^1, \quad \boldsymbol{\mu}_3 = \left(\sum_{i=1}^{n} m_{i+n} \right) \boldsymbol{\Lambda}^{n-1}.$$

recasting into the unitary form ("unitarization"):

- $Z_{A_{n-1},U}^{N_f=2n-2} \equiv \int \cdots \int \prod_{a=1}^{n-1} \left\{ \prod_{I_a=1}^{N_a} \frac{\mathrm{d}w_{I_a}^{(a)}}{w_{I_a}} \right\}_{C_{I,a}^{N_f=2n-2}} (\Delta_{A_{n-1}}(w))^{\frac{\beta}{2}} \left(\Delta_{A_{n-1}} \left(\frac{1}{w} \right) \right)^{\frac{\beta}{2}} \times \cdots$
- The potentials $W_{U,1}(w^{(1)}) = c_1 \log w^{(1)} - q_{04} w^{(1)}, \quad W_{U,a}(w^{(a)}) = c_a \log w^{(a)}, \quad a = 2, \cdots, n-2,$

$$W_{U,n-1}(w^{(n-1)}) = c_{n-1} \log w^{(n-1)} - \frac{q_{04}}{w^{(1)}}.$$ $^{\exists}$extra contributions

- the evaluation of c_a; again the case of su(3) $\tilde{S}_1 = -m_2 - m_3 - m_4, \quad \tilde{S}_2 = -m_3 - m_4 - m_5.$
- Hence, $c_1 = -\left(m_2 - m_4 + \frac{1}{2}(m_3 + m_4 + m_5) \right), \qquad c_2 = m_5 - m_3 + \frac{1}{2}(m_2 + m_3 + m_4).$

● dual coefficients:

- observe $\check{c}_1 \equiv (\boldsymbol{\mu}_0 + \boldsymbol{\mu}_2, \boldsymbol{\alpha}_1) + \tilde{S}_1 + \frac{(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)}{2}\tilde{S}_2.$
- actually valid for arbitrary n: $\check{c}_1 + c_1 = 0.$
- by the ``unitarization" procedure, we have restored the left-right symmetry of the original irregular 4 point block which became invisible by the SL(2,**C**) fixing.
- can proceed further to work out the coefficients $c_1 = -\check{c}_1, c_2, \cdots, c_{n-1}$

$$\begin{pmatrix} \tilde{S}_1 \\ \tilde{S}_2 \\ \vdots \\ \tilde{S}_{n-1} \end{pmatrix} = -\mathcal{C}^{-1} \begin{pmatrix} (\boldsymbol{\alpha}_1, \sum_i \hat{\boldsymbol{\mu}}_i) \\ (\boldsymbol{\alpha}_2, \sum_i \hat{\boldsymbol{\mu}}_i) \\ \vdots \\ (\boldsymbol{\alpha}_{n-1}, \sum_i \hat{\boldsymbol{\mu}}_i) \end{pmatrix}, \qquad \mathcal{C}^{-1}_{a,a'} = (\boldsymbol{\Lambda}_a, \boldsymbol{\Lambda}_{a'}) = \frac{1}{n}\min(a,a')\{n - \max(a,a')\},$$

the inverse of the Cartan matrix

● dual coeff for general n:

- $c_a = (\boldsymbol{\mu}_1 + \boldsymbol{\mu}_3, \boldsymbol{\alpha}_a) + \frac{(\boldsymbol{\alpha}_a, \boldsymbol{\alpha}_{a-1})}{2}\tilde{S}_{a-1} + \tilde{S}_a + \frac{(\boldsymbol{\alpha}_a, \boldsymbol{\alpha}_{a+1})}{2}\tilde{S}_{a+1},$

  $\text{with } (\boldsymbol{\mu}_3, \boldsymbol{\alpha}_a) = 0, \quad a = 2, \cdots, n-2, \quad \cdots$
- $\check{c}_a \equiv (\boldsymbol{\mu}_0 + \boldsymbol{\mu}_2, \boldsymbol{\alpha}_a) + \frac{(\boldsymbol{\alpha}_a, \boldsymbol{\alpha}_{a-1})}{2}\tilde{S}_{a-1} + \tilde{S}_a + \frac{(\boldsymbol{\alpha}_a, \boldsymbol{\alpha}_{a+1})}{2}\tilde{S}_{a+1},, \quad a = 2, \cdots, n-2, \quad \cdots$
- checked $c_a = -\check{c}_a, \quad \text{for } a = 1, \cdots, n-1,$ a dual expression
- $\tilde{S}_1 = -(m_2 + \cdots + m_{n+1}), \quad \tilde{S}_2 = -(m_3 + \cdots + m_{n+2}), \quad \cdots,$

$$\tilde{S}_{n-2} = -(m_{n-1} + \cdots + m_{2n-2}), \quad \tilde{S}_{n-1} = -(m_n + \cdots + m_{2n-1}),$$

$$\check{c}_1 = m_2 - m_{n+1} + \tfrac{1}{2}\textstyle\sum_{i=1}^n m_{2+i}, \quad \cdots$$

$$\begin{aligned} \check{c}_a &= (-m_a + m_{a+1}) + \frac{1}{2}(m_a - m_{a+1}) - \frac{1}{2}(m_{a+n} - m_{a+n+1}) \\ &= -\frac{1}{2}(m_a - m_{a+1}) - \frac{1}{2}(m_{a+n} - m_{a+n+1}), \quad a = 2, \cdots, n-2, \end{aligned}$$

$$c_{n-1} = m_{2n-1} - m_n + \tfrac{1}{2}\textstyle\sum_{j=1}^n m_{2n-1-j}.$$

IV) ● constraints from maximal symmetry:

- impose the maximal symmetry in the parameter space and derive a set of relations among $m_2, m_3, \cdots \text{ and } m_{2n-1}.$
- in $n = 2$, demand evenness of the potential ⇒ $c_1 = -\check{c}_1 = 0$ ⇒ $m_2 = m_3$ ⇒ the A-D point
- $n = 3$, demand the combination of evenness and folding of the Dynkin diagram:

$$w^{(1)} = \frac{1}{w^{(2)\prime}}, \quad w^{(2)} = \frac{1}{w^{(1)\prime}} \quad \Rightarrow \quad \tilde{S}_1 = \tilde{S}_2 \quad \text{and} \quad \check{c}_1 = c_2,$$

  which imply $m_2 = m_5, \quad m_3 = m_4.$
- $n = 4$, the case of su(4), demand $w^{(1)} = \frac{1}{w^{(3)\prime}}, \quad w^{(3)} = \frac{1}{w^{(1)\prime}}, \quad w^{(2)} = \frac{1}{w^{(2)\prime}}.$

$$\Rightarrow \quad \tilde{S}_1 = \tilde{S}_3, \quad \check{c}_1 = c_3, \quad \check{c}_2 = c_2 = 0.$$

$$\Rightarrow \quad m_2 - m_7 = -(m_3 - m_6), \quad m_2 - m_7 = -(m_4 - m_5), \quad m_2 - m_7 = m_3 - m_6.$$

$$\therefore m_2 - m_7 = m_3 - m_6 = m_4 - m_5 = 0.$$

- $n = 5, \quad \tilde{S}_1 = \tilde{S}_4, \quad \tilde{S}_2 = \tilde{S}_3, \quad \check{c}_1 = c_4, \quad \check{c}_2 = c_3.$

$$\begin{aligned} \Rightarrow \quad & (m_2 - m_4) + (m_3 - m_8) + (m_4 - m_7) = 0, \\ & m_3 - m_8 = 0, \quad m_2 - m_9 = -(m_5 - m_6), \quad m_2 - m_9 = m_3 - m_8 \\ & \therefore m_2 - m_9 = m_3 - m_8 = m_4 - m_7 = m_5 - m_6 = 0. \end{aligned}$$

- At general n, the number of net free parameters is $n - 1$. Do the same analysis to conclude

$$m_i = m_{2n+1-i}, \qquad i = 2, \cdots, n.$$

- after imposing the maximal symmetry of the system, we still do not reach the point of all masses being equal, which is sometimes assumed in the analysis of the S-W curve in general. the same conclusion has been drawn at the level of the S-W curve perse. ⇒ IOYosh8

● Hanany-Oz curve at the proposed critical hypersurface:

- $\tilde{m}_a = \tilde{m}_{2n-1-a}, \quad a = 1, 2, \cdots, n-1$
- $y^2 = C(\tilde{x})^2 - G(\tilde{x})^2,$

  where $C(\tilde{x}) = \tilde{x}^n + \sum_{i=2}^{n} \tilde{s}_i \tilde{x}^{n-i}, \qquad G(\tilde{x}) = \Lambda \prod_{a=1}^{n-1} (\tilde{x} + \tilde{m}_a).$

  $\tilde{s}_i$ the moduli parameter and Λ the scale parameter
- $n = 3$

  Put $x = \tilde{x} + \tilde{m}_2. \qquad y^2 = (x^3 + s_1 x^2 + s_2 x + s_3)^2 - \Lambda^2 x^2 (x+m)^2,$

  where $m = \tilde{m}_1 - \tilde{m}_2$

  Set $S_{31\pm} := s_1 \pm \Lambda = 0,$
  $S_{32\pm} := s_2 \pm \Lambda m = 0,$
  $S_{33} := s_3 = 0$

  we obtain $(*) \quad y^2 = x^4 (x^2 \mp 2\Lambda x \mp 2\Lambda m).$

  Hence the curve is degenerate with multiplicity four.

  $$\left. \frac{\Delta_3}{S_{31\pm}^2 S_{32\pm}^2 S_{33}^2} \right|_{S_{31\pm}=S_{32\pm}=S_{33}=0} = (2\Lambda m)^8 \tilde{\Delta}_{3\pm},$$

  where $\tilde{\Delta}_{3\pm} = 4\Lambda(\Lambda \pm 2m)$ is the discriminant of $x^2 \mp 2\Lambda x \mp 2\Lambda m.$

  The condition $\tilde{\Delta}_{3\pm} = 0$ reduces $(*)$ to $y^2 = x^4 (x \mp \Lambda)^2$
- can be generalized to arbitrary n.

V) ● HW set (partial) to myself:

- $0 \leq N_f \leq 2n-3$ case: not possible in the current procedure
  while keeping the size of the matrices finite
- operator analysis: A_1 type even potential $\leftrightarrow$ (A_1, A_{4k-1}) series
  multicritical pts. IOYanok2,3

  generalization

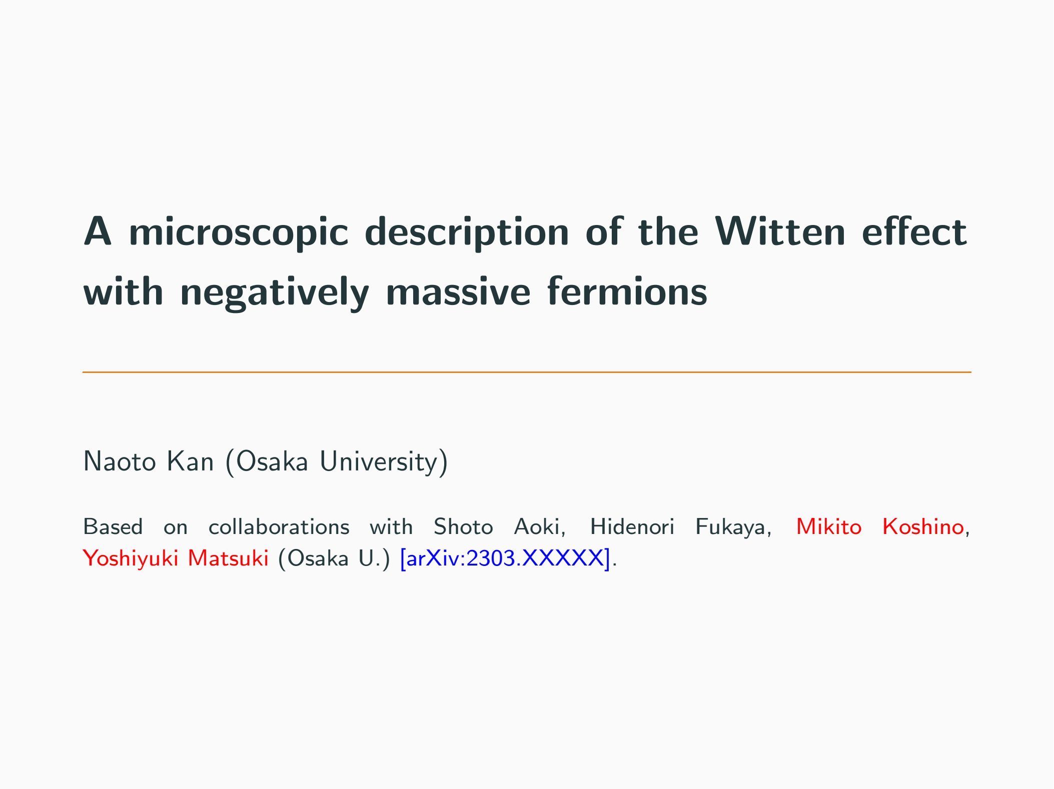
  $\vdots$

A microscopic description of the Witten effect with negatively massive fermions

Naoto Kan

Abstract. Inside topological insulators or in the theta=pi vacuum, magnetic monopoles gain fractional electric charges, which is known as the Witten effect. In this work, we try to give a microscopic description for this phenomenon, solving a "negatively" massive Dirac equation. The "Wilson term" plays a key role in 1) identifying the sign of the fermion mass, 2) confirming evidence for dynamical domain-wall creations, and 3) understanding why the electric charge is fractional.

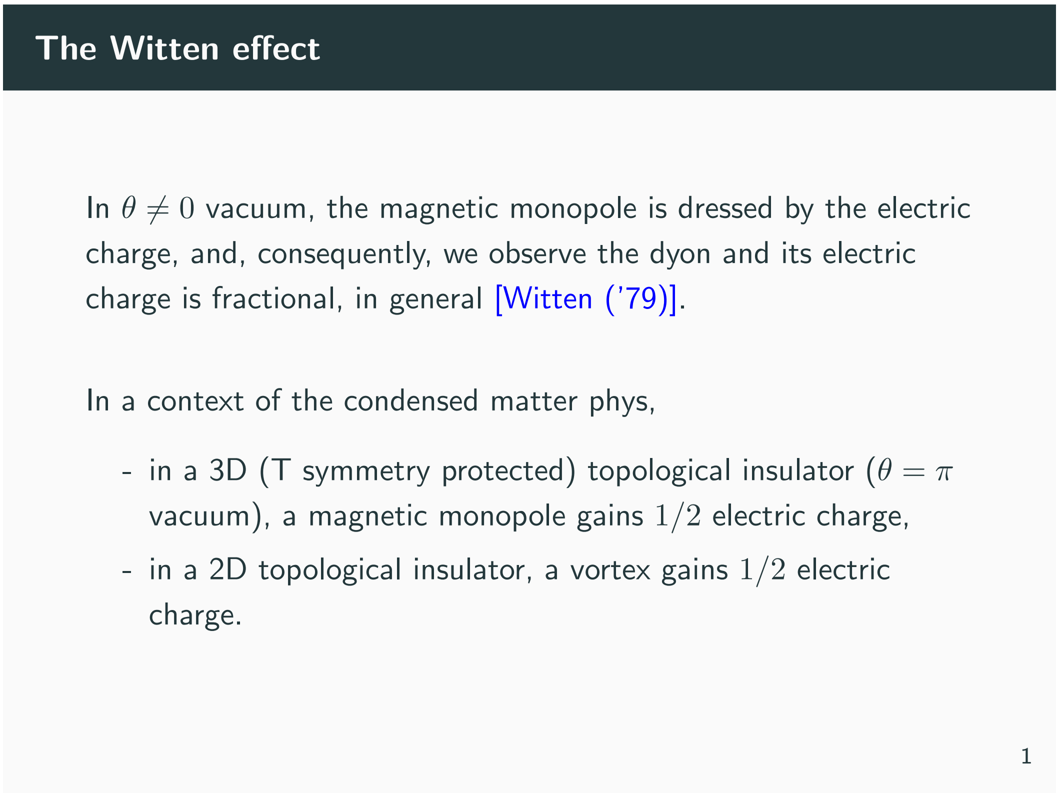
(N. Kan) Osaka University



A microscopic description of the Witten effect with negatively massive fermions

Naoto Kan (Osaka University)

Based on collaborations with Shoto Aoki, Hidenori Fukaya, Mikito Koshino, Yoshiyuki Matsuki (Osaka U.) [arXiv:2303.XXXXX].

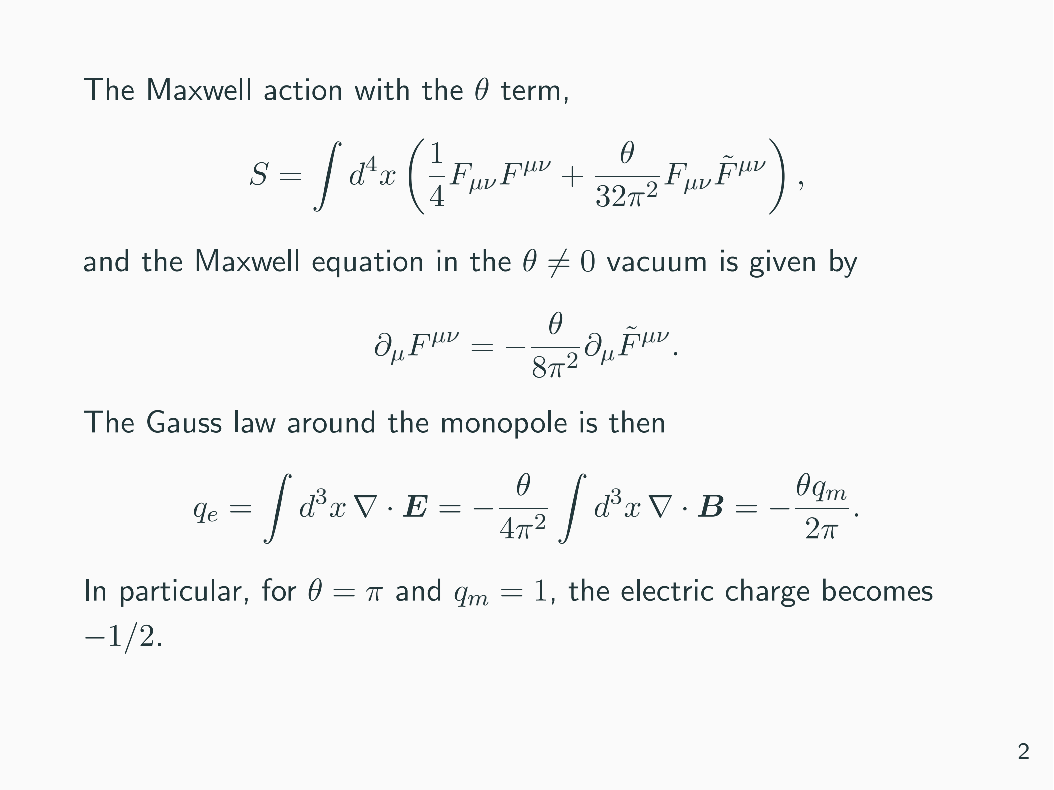


The Witten effect

In $\theta \neq 0$ vacuum, the magnetic monopole is dressed by the electric charge, and, consequently, we observe the dyon and its electric charge is fractional, in general [Witten ('79)].

In a context of the condensed matter phys,

- in a 3D (T symmetry protected) topological insulator ($\theta = \pi$ vacuum), a magnetic monopole gains $1/2$ electric charge,
- in a 2D topological insulator, a vortex gains $1/2$ electric charge.



The Maxwell action with the θ term,

$$S = \int d^4x \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\theta}{32\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} \right),$$

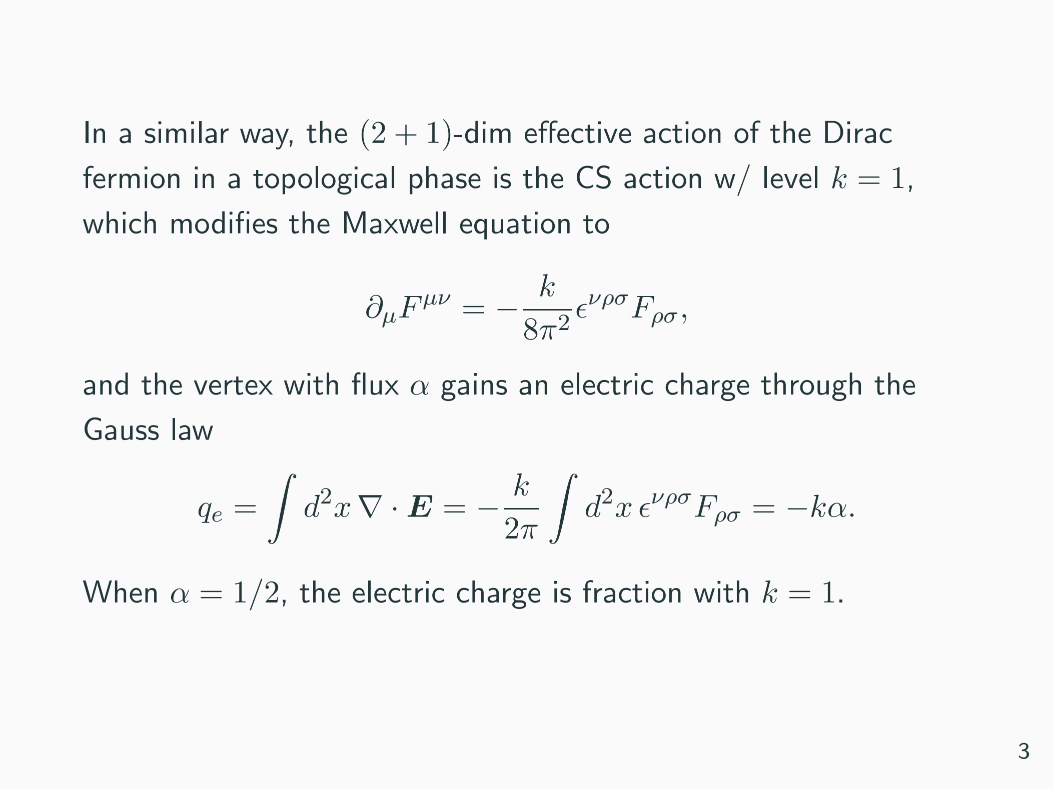
and the Maxwell equation in the $\theta \neq 0$ vacuum is given by

$$\partial_\mu F^{\mu\nu} = -\frac{\theta}{8\pi^2} \partial_\mu \tilde{F}^{\mu\nu}.$$

The Gauss law around the monopole is then

$$q_e = \int d^3x \, \nabla \cdot \boldsymbol{E} = -\frac{\theta}{4\pi^2} \int d^3x \, \nabla \cdot \boldsymbol{B} = -\frac{\theta q_m}{2\pi}.$$

In particular, for $\theta = \pi$ and $q_m = 1$, the electric charge becomes $-1/2$.



In a similar way, the $(2+1)$-dim effective action of the Dirac fermion in a topological phase is the CS action w/ level $k = 1$, which modifies the Maxwell equation to

$$\partial_\mu F^{\mu\nu} = -\frac{k}{8\pi^2} \epsilon^{\nu\rho\sigma} F_{\rho\sigma},$$

and the vertex with flux α gains an electric charge through the Gauss law

$$q_e = \int d^2x \, \nabla \cdot \boldsymbol{E} = -\frac{k}{2\pi} \int d^2x \, \epsilon^{\nu\rho\sigma} F_{\rho\sigma} = -k\alpha.$$

When $\alpha = 1/2$, the electric charge is fraction with $k = 1$.

The effective theory description above is quite simple, but can't answer to the following questions:

(1) what is the origin of the electric charge? (must be electrons)

(2) if the origin is the electrons, why is it bound to monopole/vortex?

(3) why is the electric charge fractional?

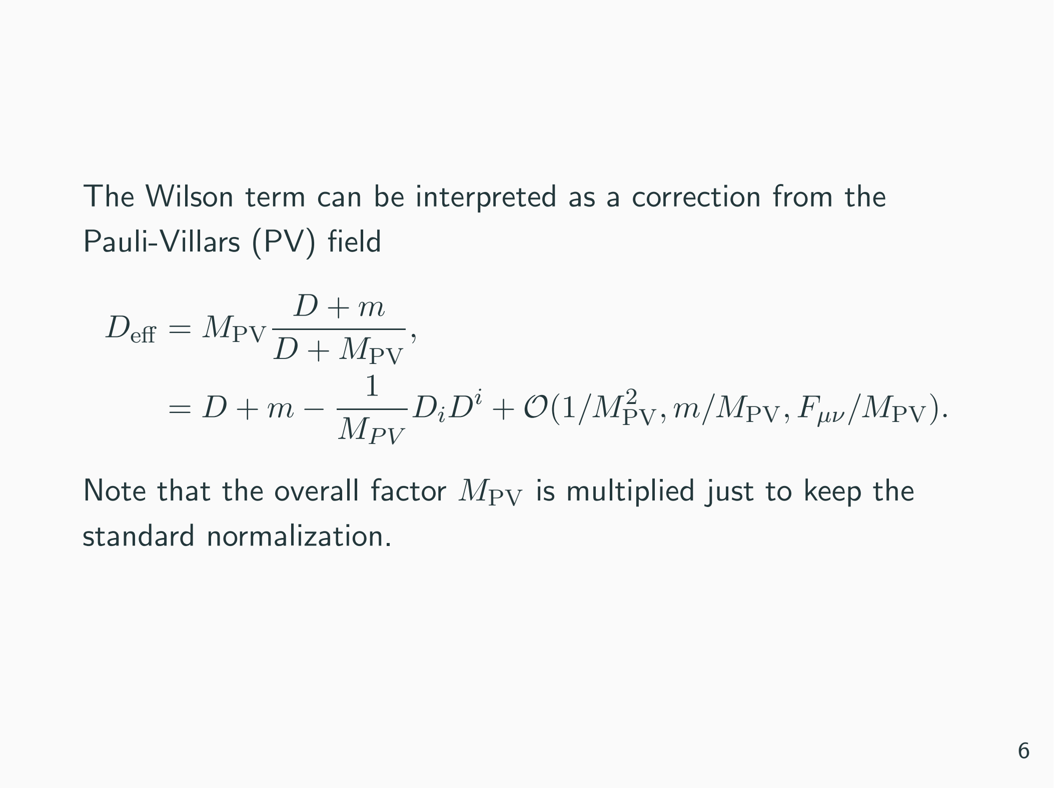
In this our work, we try to give answers to the questions from in terms of a microscopic description.

In order to give a microscopic description of the Witten effect, we solve the Dirac equation,

$$\gamma_5 \left(D + m - \frac{1}{M_{\mathrm{PV}}} D_i D^i \right) \psi = E\psi,$$

which contains the Wilson term .

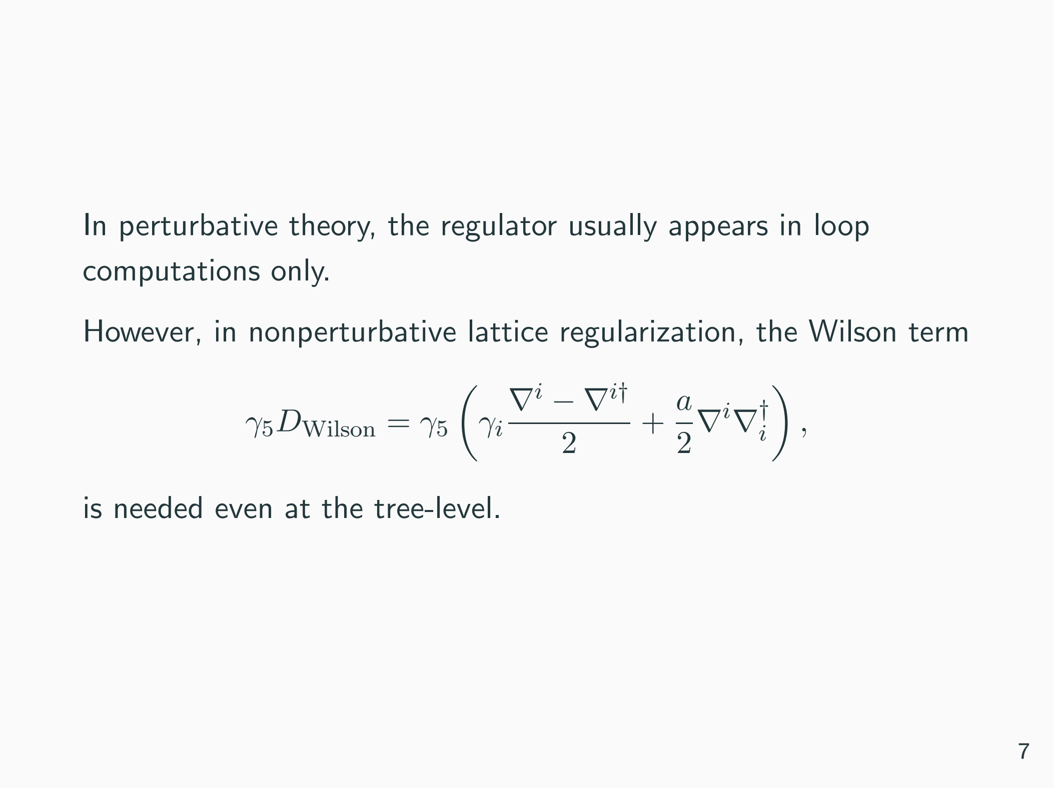
Here $D = \gamma^i D_i$.



The Wilson term can be interpreted as a correction from the Pauli-Villars (PV) field

$$\begin{aligned} D_{\text{eff}} &= M_{\text{PV}} \frac{D+m}{D+M_{\text{PV}}}, \\ &= D + m - \frac{1}{M_{PV}} D_i D^i + \mathcal{O}(1/M_{\text{PV}}^2, m/M_{\text{PV}}, F_{\mu\nu}/M_{\text{PV}}). \end{aligned}$$

Note that the overall factor M_{PV} is multiplied just to keep the standard normalization.



In perturbative theory, the regulator usually appears in loop computations only.

However, in nonperturbative lattice regularization, the Wilson term

$$\gamma_5 D_{\text{Wilson}} = \gamma_5 \left(\gamma_i \frac{\nabla^i - \nabla^{i\dagger}}{2} + \frac{a}{2} \nabla^i \nabla_i^\dagger \right),$$

is needed even at the tree-level.

Contents

2. 2D vortex

Solving a naive Dirac equation

We first consider the Dirac Hamiltonian with a mass m. We put a $U(1)$ gauge flux located at the origin describing the vortex:

$$A_x(x,y) = -\alpha\frac{y}{r^2}, \quad A_y(x,y) = \alpha\frac{x}{r^2}, \quad F_{xy}(x,y) = 2\pi\alpha\delta(x)\delta(y).$$

The naive Dirac Hamiltonian is

$$\begin{aligned} H &= \sigma_z \left(\sum_{i=x,y} \sigma_i \left(\partial_i - iA_i\right) + m \right), \\ &= \sigma_z \begin{pmatrix} m & e^{-i\theta}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\theta} - \frac{\alpha}{r}\right) \\ e^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\theta} + \frac{\alpha}{r}\right) & m \end{pmatrix}. \end{aligned}$$

The general solution for $H\psi^{E,j} = E\psi^{E,j}$ is

$$\psi^{E,j}(r,\theta) = C \begin{pmatrix} (m+E)K_{j-1/2-\alpha}(\sqrt{m^2-E^2}r)e^{i(j-1/2)\theta} \\ \sqrt{m^2-E^2}K_{j+1/2-\alpha}(\sqrt{m^2-E^2}r)e^{i(j+1/2)\theta} \end{pmatrix},$$

where $j \in \mathbb{Z} + 1/2$ is an eigenvalue of the total angular momentum J. The solution is finite at $r = \infty$.

We also impose the normalizable condition at $r = 0$. Then, we find $\alpha = j$, and the solution with $E = 0$ is

$$\psi^{E=0,n+1/2}(r,\theta) = C \begin{pmatrix} m \\ |m|e^{i\theta} \end{pmatrix} K_{1/2}(|m|r)e^{in\theta},$$

where $n \in \mathbb{Z}$.

It is important to note that the zero mode $\psi^{E=0,n+1/2}$ is a "chiral" eigenstate:

$$\sigma_r \psi^{E=0,n+1/2}(r,\theta) = \mathrm{sign}(m)\psi^{E=0,n+1/2}(r,\theta),$$

where

$$\sigma_r = \frac{x}{r}\sigma_x + \frac{y}{r}\sigma_y = \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}.$$

The solution explains that the vortex captures an electron w/o energy loss: "charged" vortex.

However, the solution doesn't describe

(i) why this happens in topological insulators ($m < 0$) but doesn't in normal insulator ($m > 0$) (w/o imposing a boundary condition by hand [Yamagishi ('83)]),

(ii) why the chiral zero mode appears (why the chiral boundary condition is imposed),

(iii) why the charge becomes fractional.

The regularized Dirac equation

Let's modify the vector potential to regularize the size of $U(1)$ flux:

$$A_\theta = \begin{cases} \frac{\alpha r}{r_1^2} & (\text{for } r \leq r_1), \\ \frac{\alpha}{r} & (\text{for } r > r_1). \end{cases}$$

The Hamiltonian with the Wilson term is

$$H = \begin{pmatrix} m - \frac{1}{M_{\rm PV}}\Delta_- & \frac{\partial}{\partial r} + \left(\frac{j+1/2}{r} - A_\theta\right) \\ -\frac{\partial}{\partial r} + \left(\frac{j-1/2}{r} - A_\theta\right) & -m + \frac{1}{M_{\rm PV}}\Delta_+ \end{pmatrix},$$

with

$$\Delta_\pm = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \left(\frac{j \pm 1/2}{r} - A_\theta\right)^2.$$

The solution outside r_1:

$$\psi^{E,j}_{\rm out}(r,\theta) = C \begin{pmatrix} \left(m + E - \frac{\kappa_-^2}{M_{\rm PV}}\right) K_{j-1/2-\alpha}(\kappa_- r) e^{i(j-1/2)\theta} \\ \kappa_- K_{j+1/2-\alpha}(\kappa_- r) e^{i(j+1/2)\theta} \end{pmatrix} + D \begin{pmatrix} \left(m + E - \frac{\kappa_+^2}{M_{\rm PV}}\right) K_{j-1/2-\alpha}(\kappa_+ r) e^{i(j-1/2)\theta} \\ \kappa_+ K_{j+1/2-\alpha}(\kappa_+ r) e^{i(j+1/2)\theta} \end{pmatrix},$$

with

$$\kappa_\pm = M_{\rm PV}\sqrt{\frac{(1+2m/M_{\rm PV}) \pm \sqrt{(1+2m/M_{\rm PV})^2 - 4(m^2-E^2)/M_{\rm PV}^2}}{2}}.$$

Note in the large $M_{\rm PV}$ and $1/r_1$, the solution converses to the naive one.

The solution inside r_1:

$$\psi^{E,j}_{\text{in}}(r,\theta) = A\begin{pmatrix}\left(m+E+\frac{L_-}{M_{\text{PV}}}-\frac{2\alpha}{M_{\text{PV}}r_1^2}\right)f^j_-(r)e^{i(j-1/2)\theta}\\ g^j_-(r)e^{i(j+1/2)\theta}\end{pmatrix}$$
$$+B\begin{pmatrix}\left(m+E+\frac{L_+}{M_{\text{PV}}}-\frac{2\alpha}{M_{\text{PV}}r_1^2}\right)f^j_+(r)e^{i(j-1/2)\theta}\\ g^j_+(r)e^{i(j+1/2)\theta}\end{pmatrix},$$

with

$$f_\pm(r)=\begin{cases} r^{j-1/2}e^{-\frac{\alpha r^2}{2r_1^2}}\,{}_1F_1(-r_1^2L_\pm/4\alpha, j+1/2; \alpha r^2/r_1^2) & (\text{for } j>0),\\ r^{-j+1/2}e^{-\frac{\alpha r^2}{2r_1^2}}\,{}_1F_1(-r_1^2L_\pm/4\alpha-j+1/2, -j+3/2; \alpha r^2/r_1^2) & (\text{for } j<0),\end{cases}$$

$$g_\pm(r)=\begin{cases} \frac{L_\pm}{2j+1}r^{j+1/2}e^{-\frac{\alpha r^2}{2r_1^2}}\,{}_1F_1(-r_1^2L_\pm/4\alpha+1, j+3/2; \alpha r^2/r_1^2) & (\text{for } j>0),\\ (2j-1)r^{-j-1/2}e^{-\frac{\alpha r^2}{2r_1^2}}\,{}_1F_1(-r_1^2L_\pm/4\alpha-j+1/2, -j+1/2; \alpha r^2/r_1^2) & (\text{for } j<0),\end{cases}$$

$$L_\pm = M_{\text{PV}}^2\left[-\left(1+\frac{2m}{M_{\text{PV}}}\right)\pm\sqrt{\left(1+\frac{2m}{M_{\text{PV}}}\right)^2-4\left\{m^2-\left(E-\frac{2\alpha}{M_{\text{PV}}r_1^2}\right)^2\right\}/M_{\text{PV}}^2}\right]/2.$$

Let's connect the exact solutions inside and outside at $r=r_1$. One requires the wavefunction and its first derivative to be continuous:

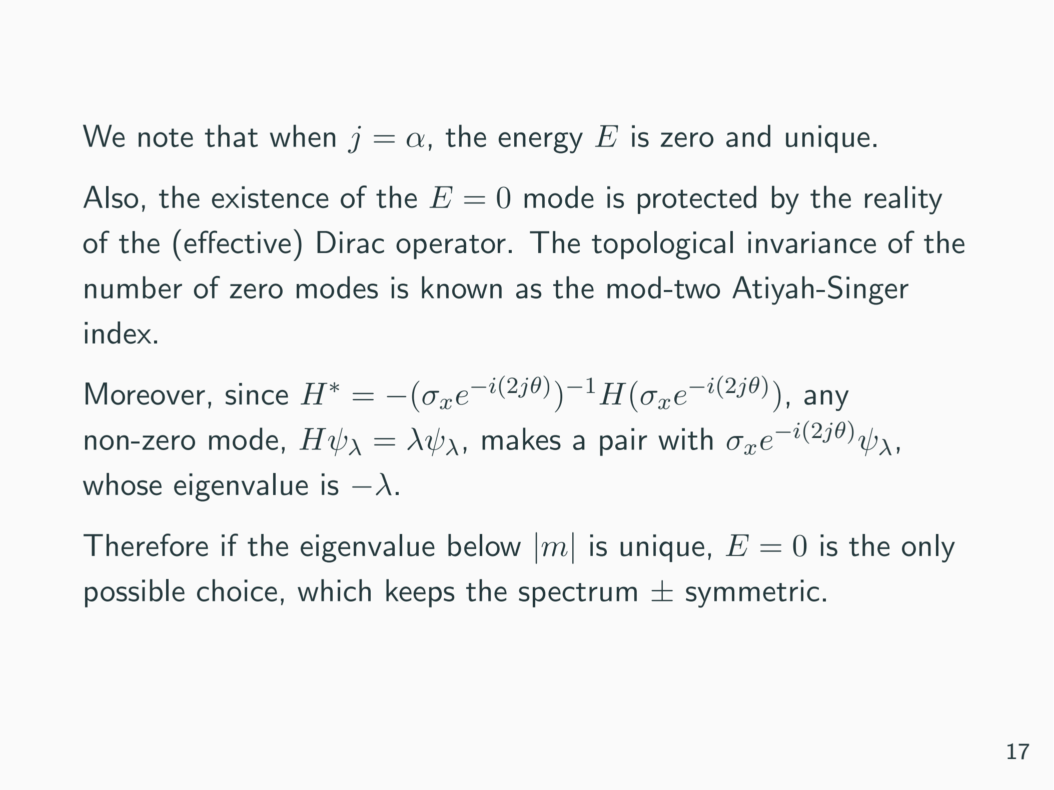
$$\psi^{E,j}_{\text{in}}(r_1,\theta)=\psi^{E,j}_{\text{out}}(r_1,\theta),\qquad \partial_r\psi^{E,j}_{\text{in}}(r_1,\theta)=\partial_r\psi^{E,j}_{\text{out}}(r_1,\theta),$$

determines the coefficients and energy.

In the limit $|m|\ll|M_{\text{PV}}|\ll 1/r_1$, we can obtain E approximately but analytically:

$$E\simeq\begin{cases} -|m| & (\text{for } j-\alpha\sim 1/2),\\ 2|m|\log\left(\frac{|m|}{M_{\text{PV}}}\right)(j-\alpha) & (\text{for } j-\alpha\sim 0),\\ |m| & (\text{for } j-\alpha\sim -1/2).\end{cases}$$

The solution exists only when $mM_{\text{PV}}<0$.

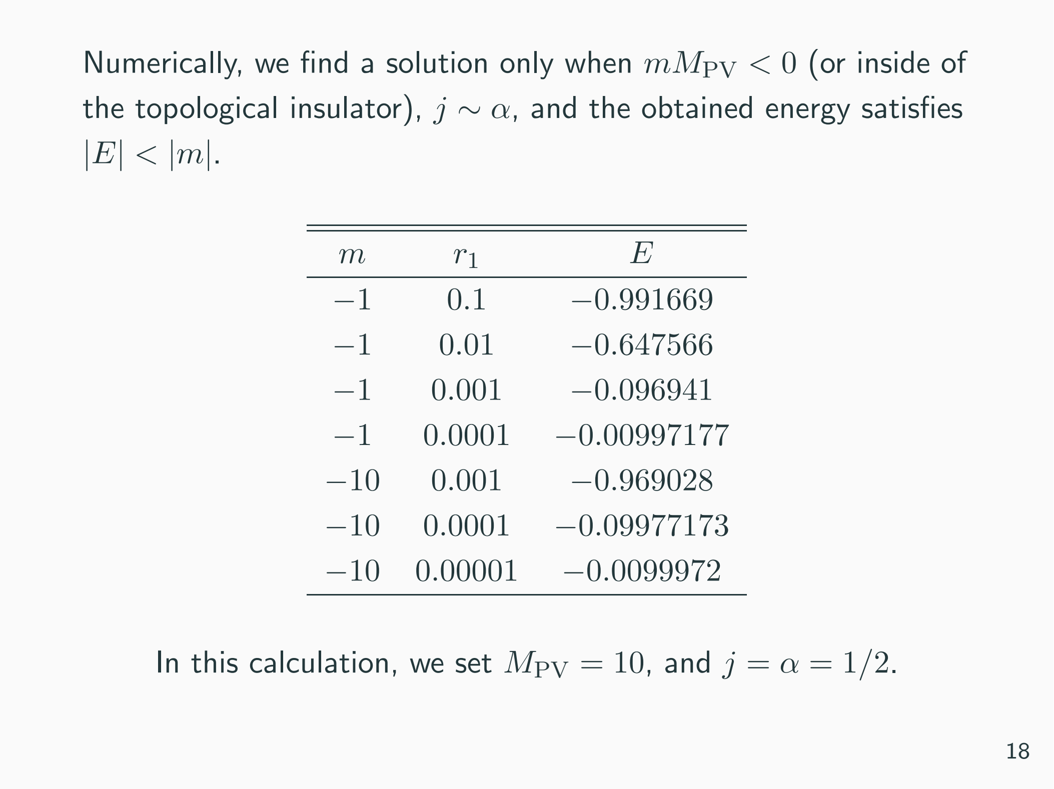


We note that when $j = \alpha$, the energy E is zero and unique.

Also, the existence of the $E = 0$ mode is protected by the reality of the (effective) Dirac operator. The topological invariance of the number of zero modes is known as the mod-two Atiyah-Singer index.

Moreover, since $H^* = -(\sigma_x e^{-i(2j\theta)})^{-1} H(\sigma_x e^{-i(2j\theta)})$, any non-zero mode, $H\psi_\lambda = \lambda\psi_\lambda$, makes a pair with $\sigma_x e^{-i(2j\theta)}\psi_\lambda$, whose eigenvalue is $-\lambda$.

Therefore if the eigenvalue below $|m|$ is unique, $E = 0$ is the only possible choice, which keeps the spectrum ± symmetric.



Numerically, we find a solution only when $mM_{\mathrm{PV}} < 0$ (or inside of the topological insulator), $j \sim \alpha$, and the obtained energy satisfies $|E| < |m|$.

m	r_1	E
-1	0.1	-0.991669
-1	0.01	-0.647566
-1	0.001	-0.096941
-1	0.0001	-0.00997177
-10	0.001	-0.969028
-10	0.0001	-0.09977173
-10	0.00001	-0.0099972

In this calculation, we set $M_{\mathrm{PV}} = 10$, and $j = \alpha = 1/2$.

Microscopic interpretation

Since the Laplacian $-D_iD^i$ is always positive, the mass shift due to the Wilson term is always positive when we take $M_{\rm PV}$ positive.

For $m < 0$ (or inside topological insulators), it is possible to locally flip the sign of the "effective" mass

$$m < 0 \quad \rightarrow \quad m_{\rm eff} = m + \frac{-D_iD^i}{M_{\rm PV}} \sim m + \frac{1}{M_{\rm PV}r_1^2} > 0,$$

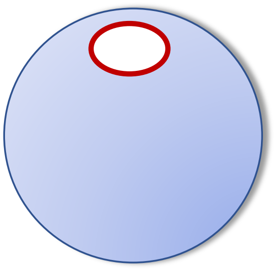
in the region $r < r_1$.

It's implies that the inside region $r < r_1$ becomes a normal insulator, and the domain-wall is dynamically created and the chiral edge-mode appears on it!

So far, we considered a $\mathbb{R}^2$ space, but in order to discuss topological feature of the fermion zero mode, we also need an IR regularization, such as the one-point compactification.

Then the topological insulator region with $(m_{\rm eff} < 0)$ have topology of a disk with a small S^1 boundary at $r = r_1$.

However, due to the cobordism invariance of the mod-two AS index, the disk is not possible.

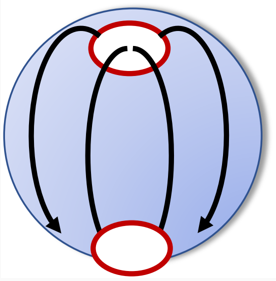


A resolution is: to create another domain-wall at, say, $r = r_0$, outside of the topological insulator.

Another zero mode is localized at the outside domain-wall, and the index is kept trivial (We will see explicitly in the case of 3D topological insulator).

Then the two zero modes is mixed by the tunneling effect, (and the eignevalues may split to $\pm\epsilon$), and the 50% of the state is located at the vortex, while the other 50% is sit at the domain-wall.

Thus the dressed electric charge of the vortex becomes $1/2$!



3. 3D monopole

The vector potential of the Dirac monopole with the magnetic charge q_m is given by

$$A_x = \frac{-q_m y}{r(r+z)}, \quad A_y = \frac{q_m x}{r(r+z)}, \quad A_z = 0,$$

of which field strength is

$$F_{ij} = q_m \epsilon_{ijk} \frac{x_k}{r^3} - 4\pi q_m \delta(x)\delta(y)\theta(-z)\epsilon_{ij3},$$

where the second term represents the Dirac string. Due to the Dirac quantization, we assume $q_m = n/2$ with $n \in \mathbb{Z}$.

Solving a naive Dirac equation

The solution of the Dirac equation $H\psi = E\psi$ for $j \neq |n/2| - 1/2$ is

$$\psi_{j,j_3,\pm} = \frac{C_{j,j_3,\pm}}{\sqrt{r}} \begin{pmatrix} (m+E)K_{\nu\mp1/2}(\sqrt{m^2-E^2}r)\ \chi_{j,j_3,\pm}(\theta,\phi) \\ \sqrt{m^2-E^2}K_{\nu\pm1/2}(\sqrt{m^2-E^2}r)\ \sigma_r\chi_{j,j_3,\pm}(\theta,\phi) \end{pmatrix},$$

where $\sigma_r = x^i\sigma_i/r$, $\nu = \sqrt{(j+1/2)^2 - n^2/4}$, and the $\chi_{j,j_3,\pm}$ is the eigenstates,

$$\sigma^i \left(L_i + \frac{n}{2}\frac{x_i}{r} \right) \chi_{j,j_3,\pm} = (-1 \pm \nu)\, \chi_{j,j_3,\pm}.$$

However, the solution is not normalizable at $r = 0$.

The only normalizable solution is the state with $j = |n|/2 - 1/2$ and $E = 0$:

$$\psi_{j,j_3,0} = \frac{C_{j,j_3,0}}{r} \exp(-|m|r) \begin{pmatrix} 1 \\ \mathrm{sign}(m)\mathrm{sign}(n) \end{pmatrix} \otimes \chi_{j,j_3,0}(\theta, \phi),$$

where $\chi_{j,j_3,0}(\theta, \phi)$ satisfies

$$\sigma_r \chi_{j,j_3,0} = \mathrm{sign}(n) \chi_{j,j_3,0}.$$

The state is a chiral eigenstate of $\sigma_x \otimes \sigma_r$ with the eigenvalue $\mathrm{sign}(m)$.

For example, for a unit magnetic charge $n = 1$, the possible bound state is unique (the degeneracy is $|n|$), but we can't distinguish the normal/topological insulator unless we specify the boundary condition at $r = 0$.

The Dirac equation with the Wilson term

We introduce the Wilson term,

$$H = \gamma_0 \left(\gamma^i D_i + m - \frac{D_i D^i}{M_{\mathrm{PV}}} \right),$$

The Hamiltonian still anticommutes with $\bar{\gamma} = \sigma_x \otimes 1$, which indicates that if the bound state around the monopole is unique, its energy eigenvalue E must be zero and it must be a chiral eigenstate of $\bar{\gamma}$.

Assuming $j = |n|/2 - 1/2$ and $r_1 \to 0$, let's solve the Dirac equation w/ the Wilson term. The solution of the zero modes is given by

$$\psi^{\mathrm{mono}}_{j,j_3} = \frac{Be^{-M_{\mathrm{PV}} r/2}}{\sqrt{r}} I_\nu(\kappa r) \begin{pmatrix} 1 \\ -s \end{pmatrix} \otimes \chi_{j,j_3,0}(\theta, \phi),$$

where $\nu = \sqrt{2|n|+1}/2$, $\kappa = M_{\mathrm{PV}}\sqrt{1 + 4m/M_{\mathrm{PV}}}/2$, and $s = \mathrm{sing}(n)$. The zero modes should have $2j + 1 = |n|$ degeneracy, having different values of j_3.

The state becomes previous one $\psi_{j,j_3,0}$ in the limit $M_{\mathrm{PV}} \to \infty$.

The solution is also the zero modes of the operator $\sigma^i \left(L_i + \frac{n}{2}\frac{x_i}{r} \right) + 1$. In fact, this operator can be identified as the effective Dirac operator on the two-dim sphere around the monopole with an infinitesimal radius r_1.

With a local Lorentz (or $Spin^c$ to be precise) transformation $R(\theta, \phi) = \exp(i\theta\sigma_y/2)\exp(i\phi(\sigma_z + 1)/2)$, we obtain

$$\begin{aligned} D^{S^2} &= R(\theta, \phi) \left[\sigma^i \left(L_i + \frac{n}{2}\frac{x_i}{r} \right) + 1 \right] R(\theta, \phi)^{-1}, \\ &= -\sigma_z \left[\sigma_x \frac{\partial}{\partial\theta} + \sigma_y \left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} + i\hat{A}_\phi + i\hat{A}^s_\phi \right) \right], \end{aligned}$$

where $\hat{A}_\phi = \frac{n}{2}\frac{\sin\theta}{1+\cos\theta}$ is the vector potential (in units of r_1) generated by the monopole.

The second connection,

$$\hat{A}^s_\phi = \frac{1}{2\sin\theta} - \frac{\cos\theta}{2\sin\theta}\sigma_z,$$

is the induced $Spin^c$ connection on the sphere which is strongly curved with the small radius r_1.

Namely, the zero modes are the chiral zero modes of not only 3D but also D^{S^2}.

Stability of the chiral zero modes is topologically protected by the AS index theorem. On the two dim sphere, (noting that the Spinc connection As does not contributes,) the total flux reproduces the index:

$$\frac{1}{4\pi}\int_{S^2} d^2x\epsilon^{\mu\nu}F_{\mu\nu} = n.$$

Since the AS index is a cobordism invariant, the long-range discussion in the case of the vortex in two dim works here:

We need a normal insulator region with $m > 0$ outside the topological insulator with $m < 0$ and $|n|$ zero modes with the opposite chirality must appear on the outer domain-wall.

To see this, let's solve the regularized Dirac equation around the outside domain-wall at $r = r_0$.

The edge-localized state is obtained as

$$\psi^{\mathrm{DW}}_{j,j_3} = \begin{cases} \frac{\exp\left(\frac{M_{\mathrm{PV}} r}{2}\right)}{\sqrt{r}} \left(B' K_\nu(\kappa_- r) + C' I_\nu(\kappa_- r)\right) \begin{pmatrix} 1 \\ s \end{pmatrix} \otimes \chi_{j,j_3,0}(\theta,\phi) & (r < r_0), \\ \frac{D' \exp\left(\frac{M_{\mathrm{PV}} r}{2}\right)}{\sqrt{r}} K_\nu(\kappa_+ r) \begin{pmatrix} 1 \\ s \end{pmatrix} \otimes \chi_{j,j_3,0}(\theta,\phi) & (r > r_0), \end{cases},$$

where $\kappa_\pm = \frac{M_{\mathrm{PV}}}{2}\sqrt{1 \pm 4|m|/M_{\mathrm{PV}}}$.

This edge-localized modes have the same $|n| = 2j + 1$ degeneracy and the opposite chirality with the zero modes captured by the monopole.

So far we considered only $r_1 \to 0$ limit: how about finite r_1?

We calculate numerically!

4. Numerical analysis of monopole

Lattice setup

On a three-dimensional hyper-cubic lattice with size $L = 31$ with open boundary conditions, we put a monopole at $\boldsymbol{x}_m = (L/2, L/2, L/2)$ with a magnetic charge $n/2$. We also put an anti-monopole at $\boldsymbol{x}_a = (L/2, L/2, 1/2)$ with the opposite charge $n/2$.

The continuum vector potential at $\boldsymbol{x} = (x, y, z)$ is then given by

$$A_x(\boldsymbol{x}) = q_m \left[\frac{-(y - y_m)}{|\boldsymbol{x} - \boldsymbol{x}_m|(|\boldsymbol{x} - \boldsymbol{x}_m| + (z - z_m))} - \frac{-(y - y_a)}{|\boldsymbol{x} - \boldsymbol{x}_a|(|\boldsymbol{x} - \boldsymbol{x}_a| + (z - z_a))} \right],$$

$$A_y(\boldsymbol{x}) = q_m \left[\frac{x - x_m}{|\boldsymbol{x} - \boldsymbol{x}_m|(|\boldsymbol{x} - \boldsymbol{x}_m| + (z - z_m))} - \frac{x - x_a}{|\boldsymbol{x} - \boldsymbol{x}_a|(|\boldsymbol{x} - \boldsymbol{x}_a| + (z - z_a))} \right],$$

$$A_z(\boldsymbol{x}) = 0,$$

with $q_m = n/2$. Note that the Dirac string extends from $\boldsymbol{x}_a$ to $\boldsymbol{x}_m$.

For the fermion field, we assign a position-dependent mass term to be $m(\boldsymbol{x}) = -m_0$ with $m_0 = 14/L$ for $\sqrt{|\boldsymbol{x} - \boldsymbol{x}_m|} < r_0 = 3L/8$, and $m(\boldsymbol{x}) = +m_0$ otherwise.

Namely, the monopole is located at the center of a spherical topological insulator with radius r_0 surrounded by a normal insulator with the gap m_0, while the anti-monopole sits in the normal insulator region.

We assume that outside of the lattice with open boundary condition corresponds to a "laboratory" with $m(\boldsymbol{x}) = +\infty$.

The Wilson Dirac Hamiltonian is given by

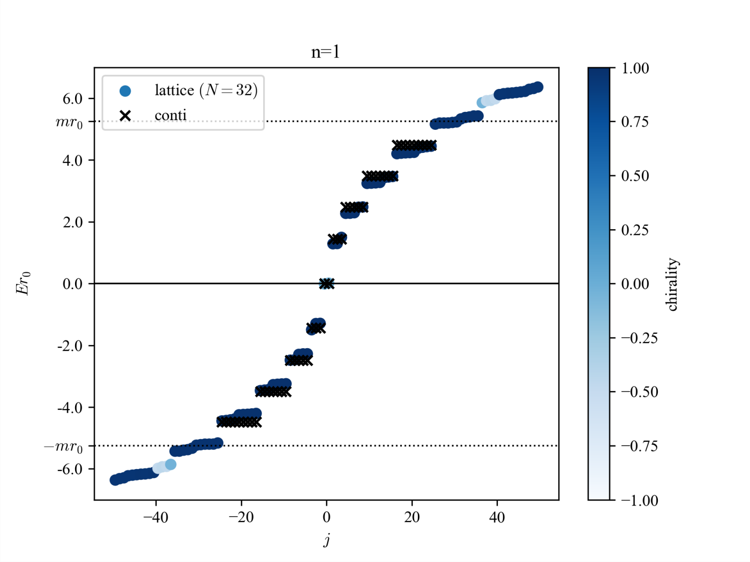
$$H_W = \gamma^0 \left(\sum_{i=1}^{3} \left[\gamma_i \frac{\nabla_i^f + \nabla_i^b}{2} - \frac{1}{2} \nabla_i^f \nabla_i^b \right] + m(\boldsymbol{x}) \right),$$

where $\nabla_i^f \psi(\boldsymbol{x}) = U_i(\boldsymbol{x})\psi(\boldsymbol{x} + \boldsymbol{e}_i) - \psi(\boldsymbol{x})$ denotes the forward covariant difference and $\nabla_i^b \psi(\boldsymbol{x}) = \psi(\boldsymbol{x}) - U_i^\dagger(\boldsymbol{x} - \boldsymbol{e}_i)\psi(\boldsymbol{x} - \boldsymbol{e}_i)$ is the backward difference. Also, $U_i(\boldsymbol{x}) = \exp\left(i \int_0^1 A_i(\boldsymbol{x} + \boldsymbol{e}_i l) dl \right)$ is the link variables.

Note that H_W anti-commutes with $\bar{\gamma} = \gamma_x \otimes 1$ even on a lattice.

Numerical results

We plot the eigenvalue spectrum of H_W w/ $n = 1$ on the $L = 31$ lattice:



We see that:

- the circle symbols are the numerical results,
- the cross symbols are the continuum results,

Let's focus on the near-zero modes which are apparently not chiral. The number of these modes is doubled compared to the continuum prediction of the edge-localized modes around the domain-wall at $r = r_0$.

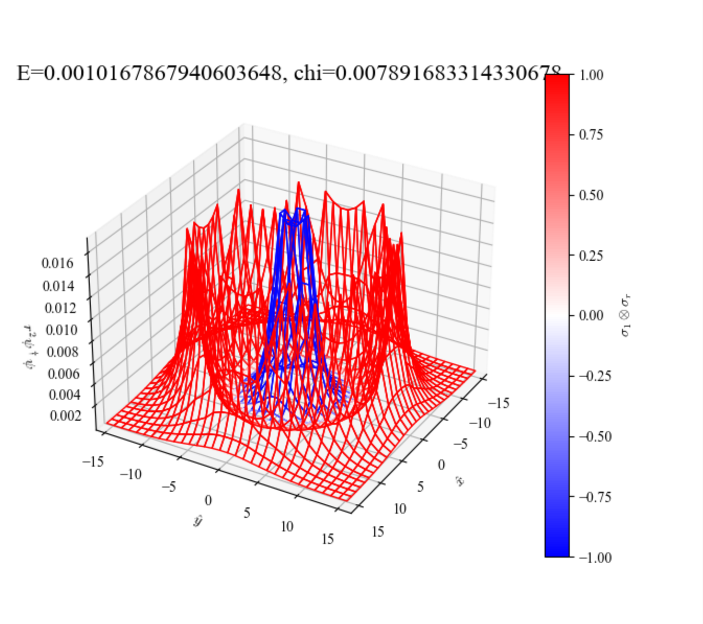
We expect that any chiral zero mode localized at the domain-wall must appear in pair with the mode with the opposite chirality localized at the domain-wall dynamically created by the monopole.

To see this, we plot the amplitude,

$$A_0(\boldsymbol{x}) = \phi_0(\boldsymbol{x})^\dagger \phi_0(\boldsymbol{x}) r^2,$$

for the positive nearest-zero mode.

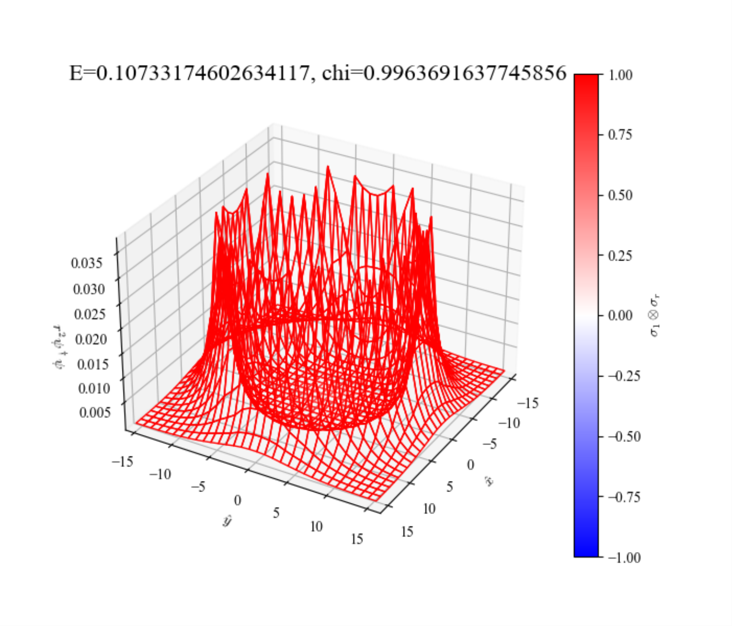
The amplitude of ϕ_0 for $n = 1$ in $z = 0$ slice:



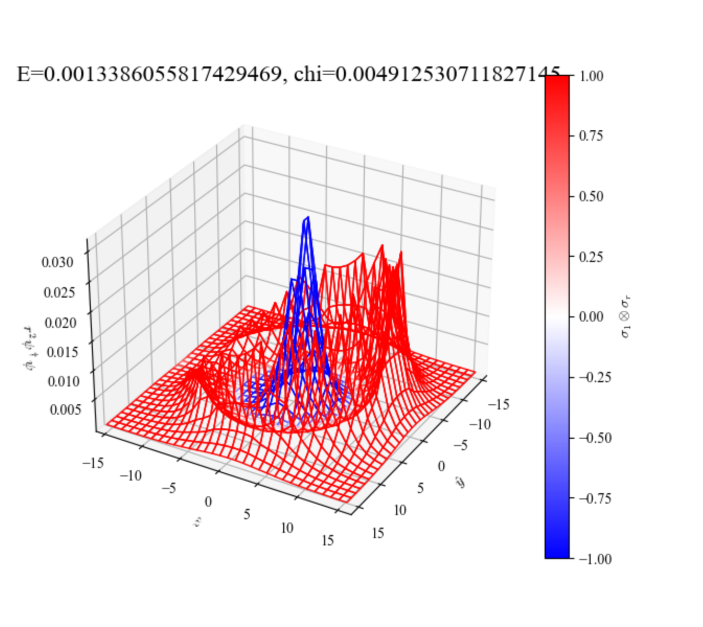
We see that:

- for the nearest zeromode, the amplitude has two peaks around $r = |\boldsymbol{x} - \boldsymbol{x}_m| = 0$ and $r = r_0$,
- the local chirality near each peak is ~ -1 and $+1$, respectively, although the total chiraity is near zero,
- the 50% of the state is located around the monopole, while the other 50% is located at $r = r_0$: the half electric charge,
- this is only for the nearest zeromode, e.g., for the second nearest zero mode, we have only the edge-localized modes:

The amplitude of the second nearest zero mode:



The same amplitude $A_0(\boldsymbol{x})$ but with $n = -2$ on $x = 0$ slice:

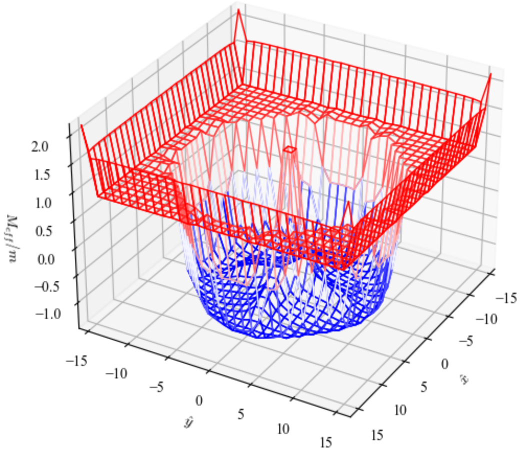


To directly confirm creation of the domain-wall near the monopole, we plot distribution of the "effective mass" (normalized by m_0),

$$m_{\text{eff}}(\boldsymbol{x}) = \phi_k(\boldsymbol{x})^\dagger \left[- \sum_{i=1,2,3} \frac{1}{2} \nabla_i^f \nabla_i^b + m(\boldsymbol{x}) \right] \Big/ \phi_k(\boldsymbol{x}) \phi_k(\boldsymbol{x})^\dagger \phi_i(\boldsymbol{x}),$$

on the $z = 0$ slice.

The effective mass for $k = 0$ with $n = 1$ on $z = 0$ slice:



We see that:

- the small island of the normal insulator (or a positive mass region) appears around the monopole: the domain-wall is dynamically created,
- even for non-zero edge localized modes, (which has almost no amplitude at the monopole) the measured mass $m_{\mathrm{eff}}(\boldsymbol{x})$ is clearly positive at the monopole.

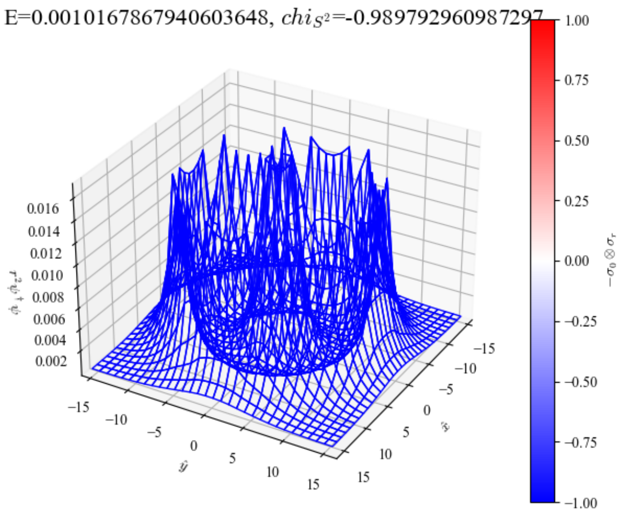
As discussed above, the distribution of the chirality $\sigma_x \otimes \sigma_r$ has no significant difference between $n = 1$ and $n = -2$, i.e., the wavefunction near the monopole is $\sigma_x \otimes \sigma_r \sim -1$, while that near the domain-wall at $r = r_0$ has $\sim +1$ chirality.

However, the two-dimensional chirality $1 \otimes \sigma_r$ is sensitive to $\mathrm{sign}(n)$.

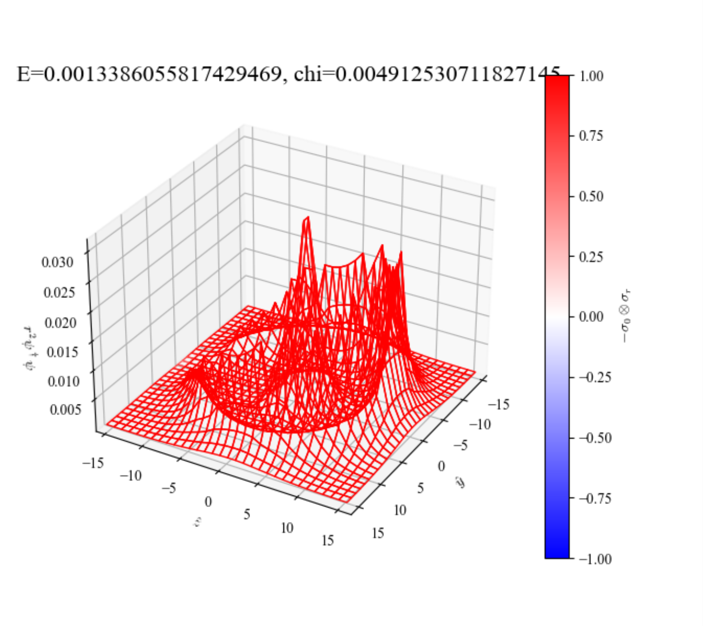
We plot the distribution of

$$\phi_0(\boldsymbol{x})^\dagger \left[1 \otimes \sigma_r\right] \phi_0(\boldsymbol{x}) / \phi_0(\boldsymbol{x})^\dagger \phi_0(\boldsymbol{x}).$$

The plot of $A_0(\boldsymbol{x})$ and $1 \otimes \sigma_r$ chirality w/ $n = 1$ on $z = 0$ slice:



The plot of $A_0(\boldsymbol{x})$ and $1 \otimes \sigma_r$ chirality w/ $n = -2$ on $x = 0$ slice:



We see that:

- the local chirality is $\sim +1$ for $n = 1$, and ~ -1 for $n = -2$,
- this is consistent with the cobordism property of the AS index that the spherical domain-wall at $r = r_0$ and that near the monopole must share the same value.

Summary

We discussed a microscopic description of the Witten effect with the Wilson term.

How do we distinguish between the normal insulator ($m > 0$) and topological insulator ($m < 0$)?

- It is the topological insulator if the mass is relatively negative compared to the PV mass.

Why are electrons bound to monopole/vertex?

- Because of the positive mass correction from the magnetic field of the monopole/vortex, the domain-wall is dynamically created (only for the negative mass).

Why does the chiral zero mode appear?

- Because the zero modes localized at the domain-wall are protected by the AS index.

Why is the electric charge fractional?

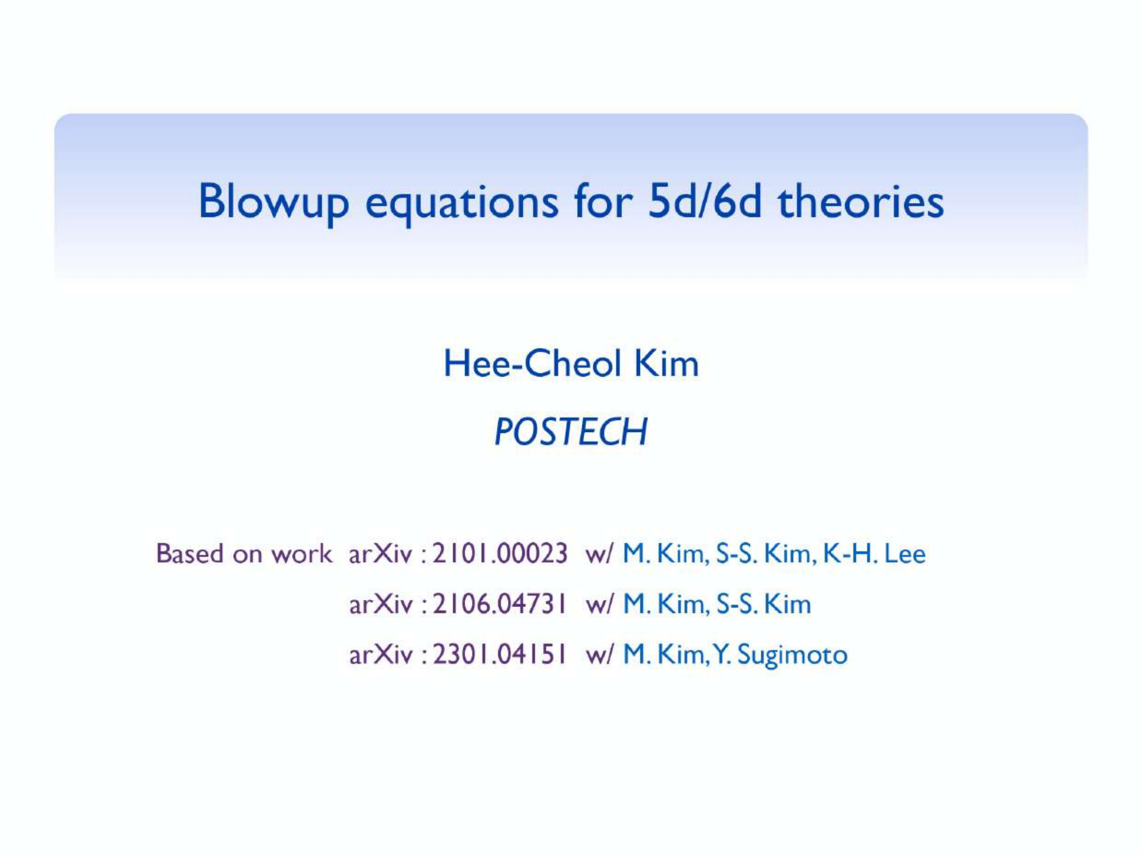
- Because the 50% of the wavefunction is located around the monopole/vortex (the other 50% is located at the surface of the topological insulator).

Blowup Equations for 5d/6d theories

Hee-Cheol Kim

Abstract. I will talk about the blowup equations for 5d/6d supersymmetric QFTs and little string theories which generalize Nakajima-Yoshioka's blowup equations for the instanton partition functions of the 4d/5d gauge theories on Omega background.

(H. Kim) POSTECH

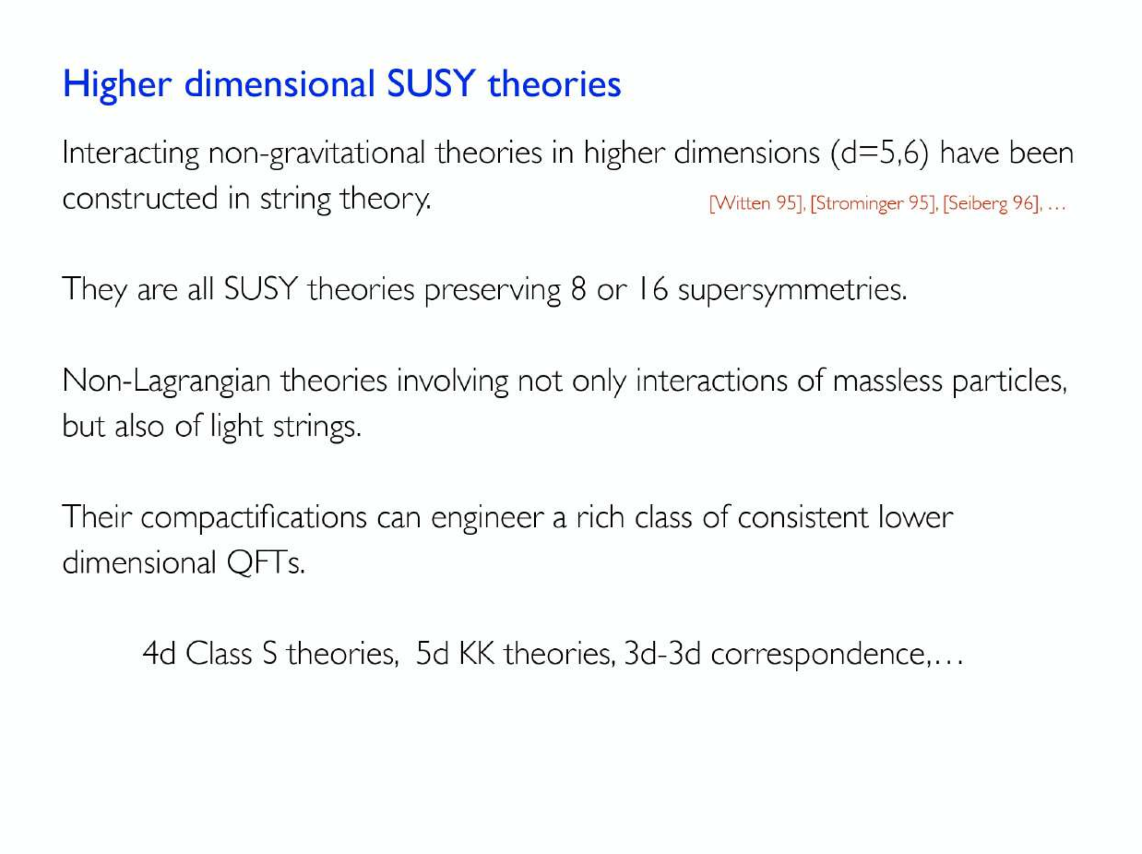


Blowup equations for 5d/6d theories

Hee-Cheol Kim

POSTECH

Based on work arXiv : 2101.00023 w/ M. Kim, S-S. Kim, K-H. Lee
arXiv : 2106.04731 w/ M. Kim, S-S. Kim
arXiv : 2301.04151 w/ M. Kim, Y. Sugimoto



Higher dimensional SUSY theories

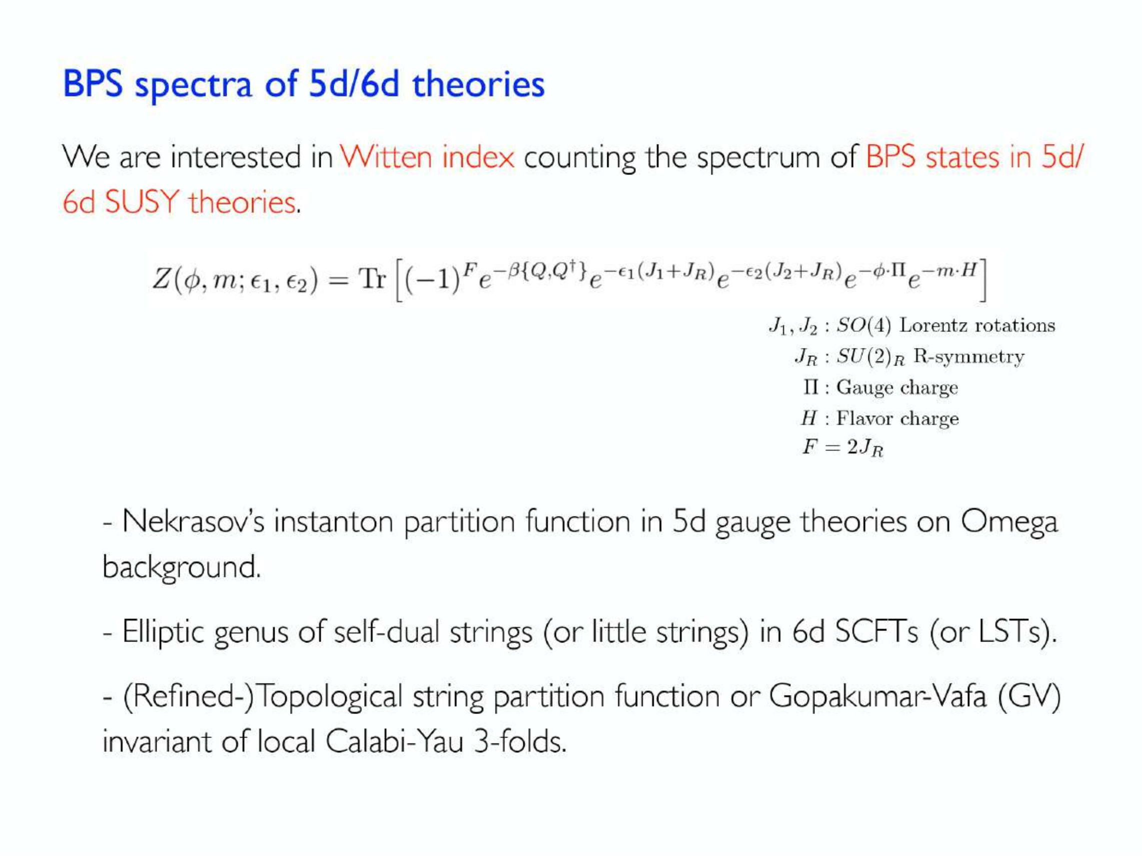
Interacting non-gravitational theories in higher dimensions (d=5,6) have been constructed in string theory. [Witten 95], [Strominger 95], [Seiberg 96], …

They are all SUSY theories preserving 8 or 16 supersymmetries.

Non-Lagrangian theories involving not only interactions of massless particles, but also of light strings.

Their compactifications can engineer a rich class of consistent lower dimensional QFTs.

4d Class S theories, 5d KK theories, 3d-3d correspondence,…



BPS spectra of 5d/6d theories

We are interested in Witten index counting the spectrum of BPS states in 5d/6d SUSY theories.

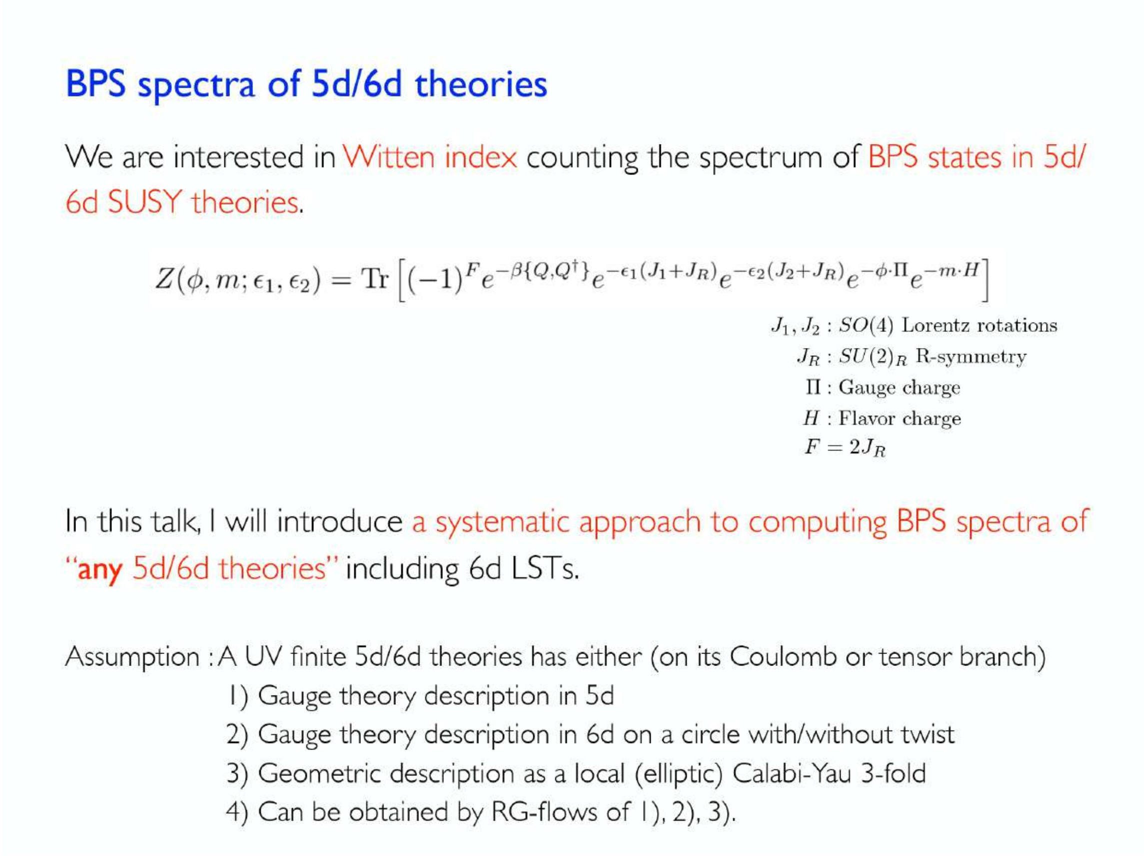
$$Z(\phi, m; \epsilon_1, \epsilon_2) = \mathrm{Tr}\left[(-1)^F e^{-\beta\{Q,Q^\dagger\}} e^{-\epsilon_1(J_1+J_R)} e^{-\epsilon_2(J_2+J_R)} e^{-\phi\cdot\Pi} e^{-m\cdot H}\right]$$

$J_1, J_2 : SO(4)$ Lorentz rotations
$J_R : SU(2)_R$ R-symmetry
$\Pi :$ Gauge charge
$H :$ Flavor charge
$F = 2J_R$

- Nekrasov's instanton partition function in 5d gauge theories on Omega background.

- Elliptic genus of self-dual strings (or little strings) in 6d SCFTs (or LSTs).

- (Refined-)Topological string partition function or Gopakumar-Vafa (GV) invariant of local Calabi-Yau 3-folds.



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$J_1, J_2 : SO(4)$ Lorentz rotations
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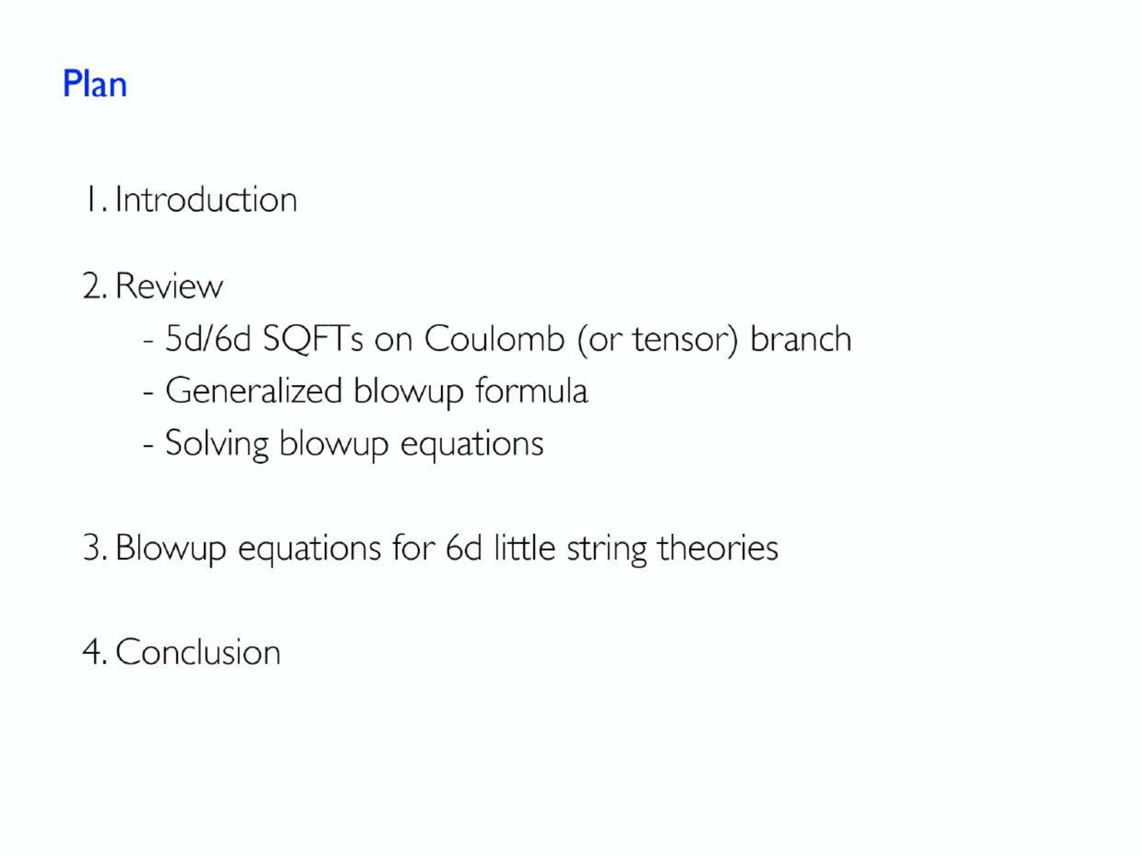
$F = 2J_R$

In this talk, I will introduce a systematic approach to computing BPS spectra of "**any** 5d/6d theories" including 6d LSTs.

Assumption : A UV finite 5d/6d theories has either (on its Coulomb or tensor branch)

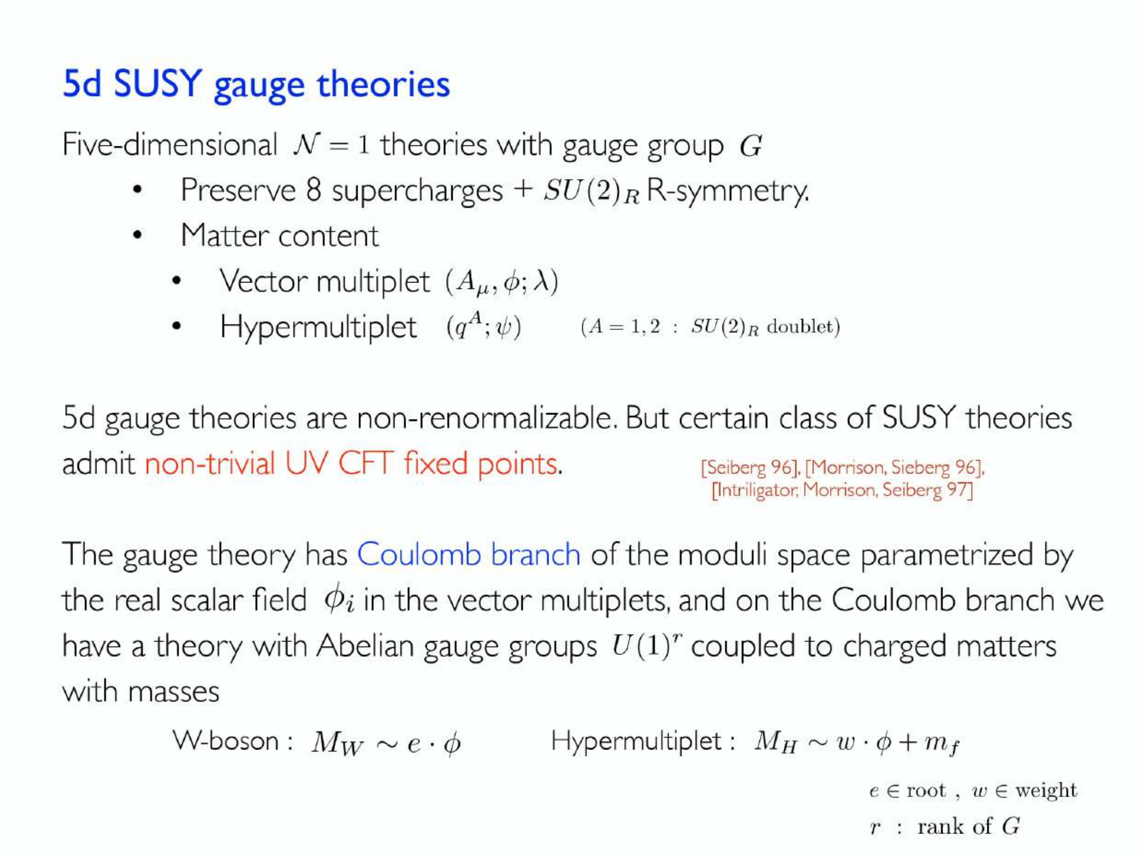
1) Gauge theory description in 5d
2) Gauge theory description in 6d on a circle with/without twist
3) Geometric description as a local (elliptic) Calabi-Yau 3-fold
4) Can be obtained by RG-flows of 1), 2), 3).



Plan

1. Introduction
2. Review
    - 5d/6d SQFTs on Coulomb (or tensor) branch
    - Generalized blowup formula
    - Solving blowup equations
3. Blowup equations for 6d little string theories
4. Conclusion

5d/6d SQFTs on Coulomb branch



5d SUSY gauge theories

Five-dimensional $\mathcal{N}=1$ theories with gauge group G

- Preserve 8 supercharges + $SU(2)_R$ R-symmetry.
- Matter content
  - Vector multiplet $(A_\mu, \phi; \lambda)$
  - Hypermultiplet $(q^A; \psi)$ $(A=1,2\ :\ SU(2)_R \text{ doublet})$

5d gauge theories are non-renormalizable. But certain class of SUSY theories admit non-trivial UV CFT fixed points.

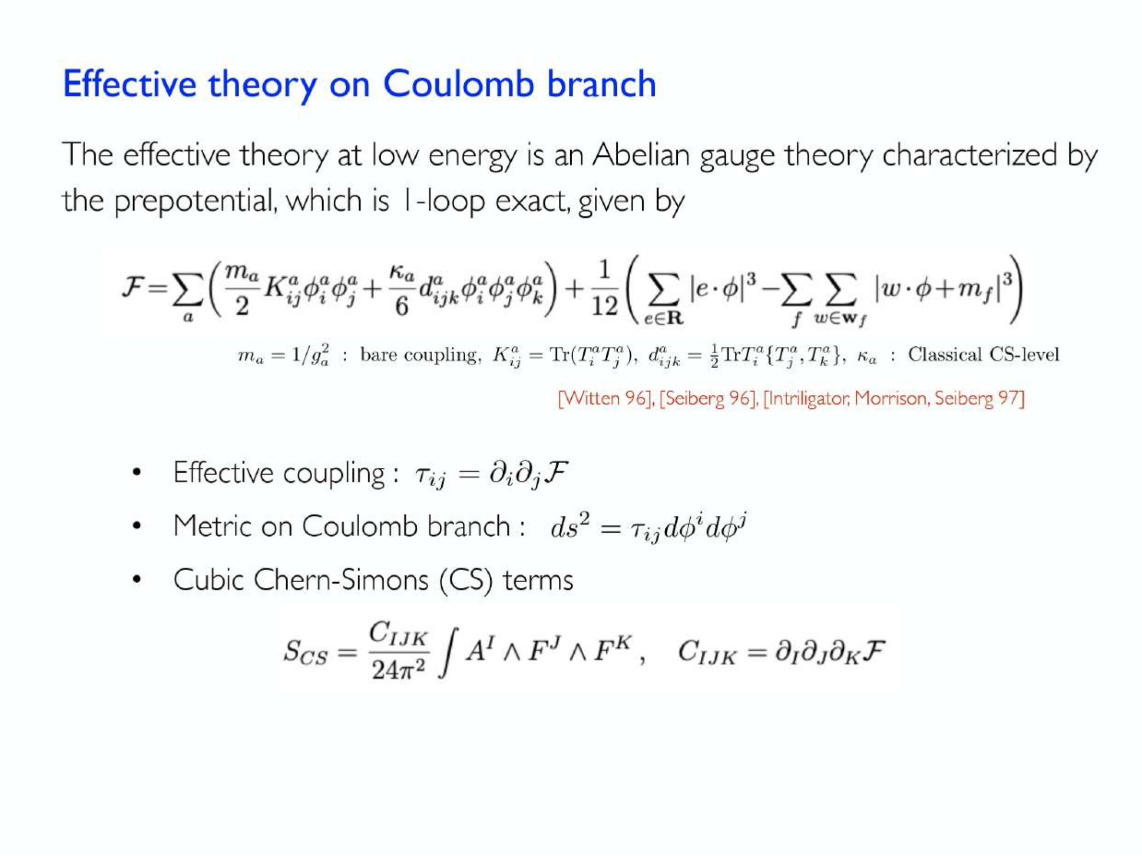
[Seiberg 96], [Morrison, Sieberg 96], [Intriligator, Morrison, Seiberg 97]

The gauge theory has Coulomb branch of the moduli space parametrized by the real scalar field ϕ_i in the vector multiplets, and on the Coulomb branch we have a theory with Abelian gauge groups $U(1)^r$ coupled to charged matters with masses

W-boson : $M_W \sim e \cdot \phi$ Hypermultiplet : $M_H \sim w \cdot \phi + m_f$

$e \in \text{root}\ ,\ w \in \text{weight}$

$r\ :\ \text{rank of } G$



Effective theory on Coulomb branch

The effective theory at low energy is an Abelian gauge theory characterized by the prepotential, which is 1-loop exact, given by

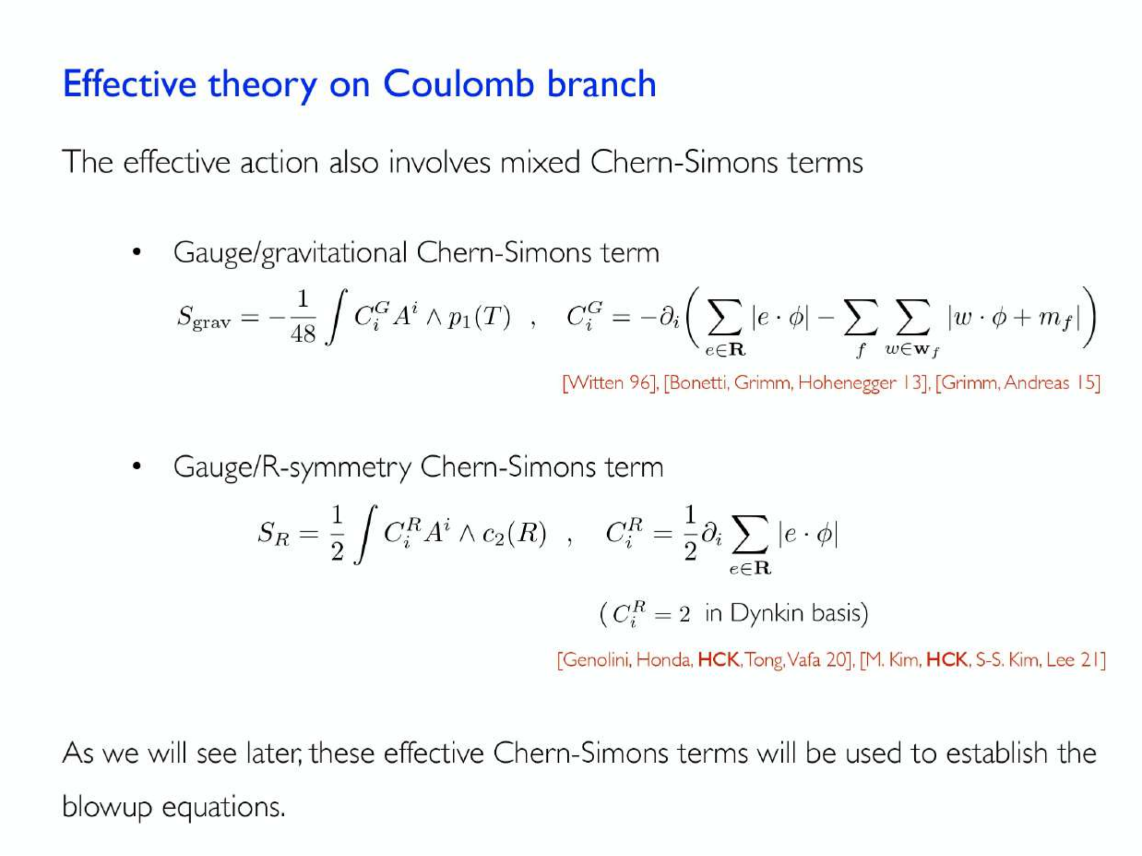
$$\mathcal{F}=\sum_a\left(\frac{m_a}{2}K^a_{ij}\phi^a_i\phi^a_j+\frac{\kappa_a}{6}d^a_{ijk}\phi^a_i\phi^a_j\phi^a_k\right)+\frac{1}{12}\left(\sum_{e\in\mathbf{R}}|e\cdot\phi|^3-\sum_f\sum_{w\in\mathbf{w}_f}|w\cdot\phi+m_f|^3\right)$$

$m_a=1/g_a^2$: bare coupling, $K^a_{ij}=\mathrm{Tr}(T^a_iT^a_j)$, $d^a_{ijk}=\frac{1}{2}\mathrm{Tr}T^a_i\{T^a_j,T^a_k\}$, κ_a : Classical CS-level

[Witten 96], [Seiberg 96], [Intriligator, Morrison, Seiberg 97]

- Effective coupling : $\tau_{ij}=\partial_i\partial_j\mathcal{F}$
- Metric on Coulomb branch : $ds^2=\tau_{ij}d\phi^id\phi^j$
- Cubic Chern-Simons (CS) terms

$$S_{CS}=\frac{C_{IJK}}{24\pi^2}\int A^I\wedge F^J\wedge F^K\,,\quad C_{IJK}=\partial_I\partial_J\partial_K\mathcal{F}$$



Effective theory on Coulomb branch

The effective action also involves mixed Chern-Simons terms

- Gauge/gravitational Chern-Simons term

$$S_{\text{grav}} = -\frac{1}{48}\int C_i^G A^i \wedge p_1(T) \quad , \quad C_i^G = -\partial_i\Big(\sum_{e\in\mathbf{R}}|e\cdot\phi| - \sum_f \sum_{w\in\mathbf{w}_f}|w\cdot\phi + m_f|\Big)$$

[Witten 96], [Bonetti, Grimm, Hohenegger 13], [Grimm, Andreas 15]

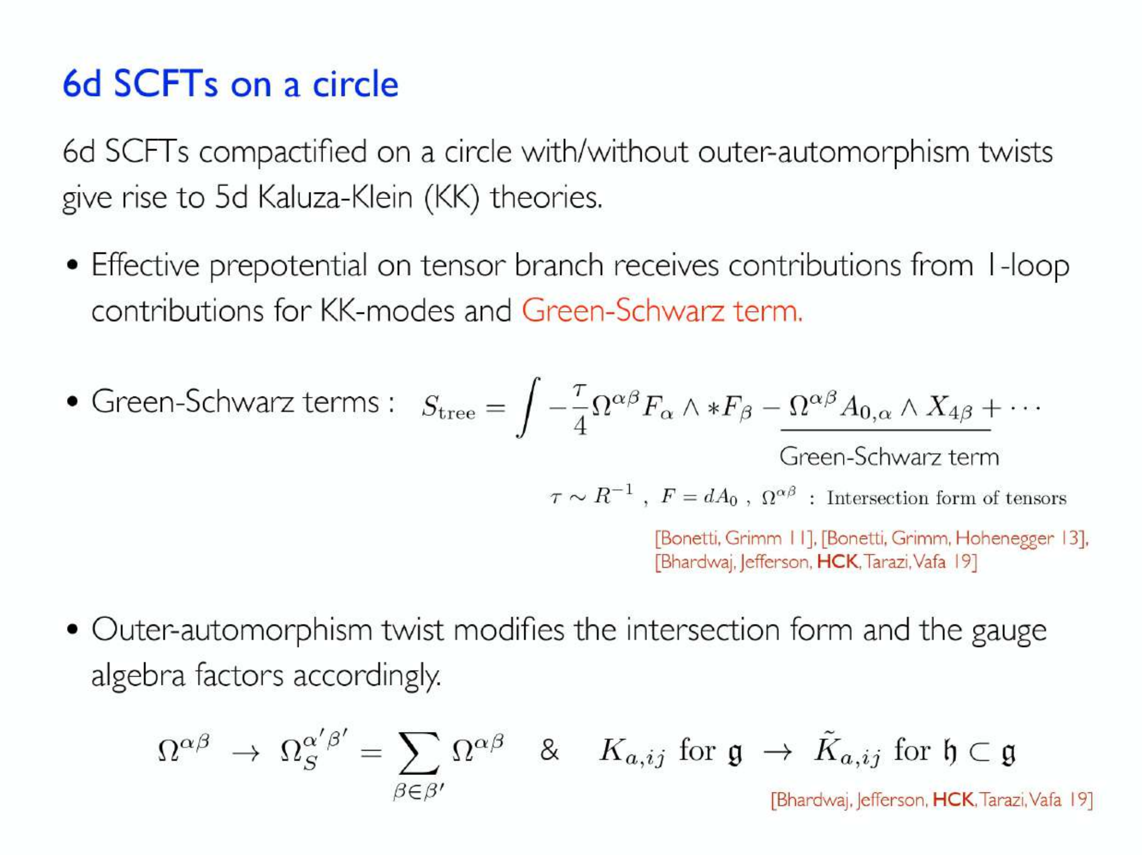
- Gauge/R-symmetry Chern-Simons term

$$S_R = \frac{1}{2}\int C_i^R A^i \wedge c_2(R) \quad , \quad C_i^R = \frac{1}{2}\partial_i \sum_{e\in\mathbf{R}}|e\cdot\phi|$$

($C_i^R = 2$ in Dynkin basis)

[Genolini, Honda, **HCK**, Tong, Vafa 20], [M. Kim, **HCK**, S-S. Kim, Lee 21]

As we will see later, these effective Chern-Simons terms will be used to establish the blowup equations.



6d SCFTs on a circle

6d SCFTs compactified on a circle with/without outer-automorphism twists give rise to 5d Kaluza-Klein (KK) theories.

- Effective prepotential on tensor branch receives contributions from 1-loop contributions for KK-modes and Green-Schwarz term.

- Green-Schwarz terms : $S_{\text{tree}} = \int -\frac{\tau}{4}\Omega^{\alpha\beta}F_\alpha \wedge *F_\beta - \underbrace{\Omega^{\alpha\beta}A_{0,\alpha} \wedge X_{4\beta}}_{\text{Green-Schwarz term}} + \cdots$

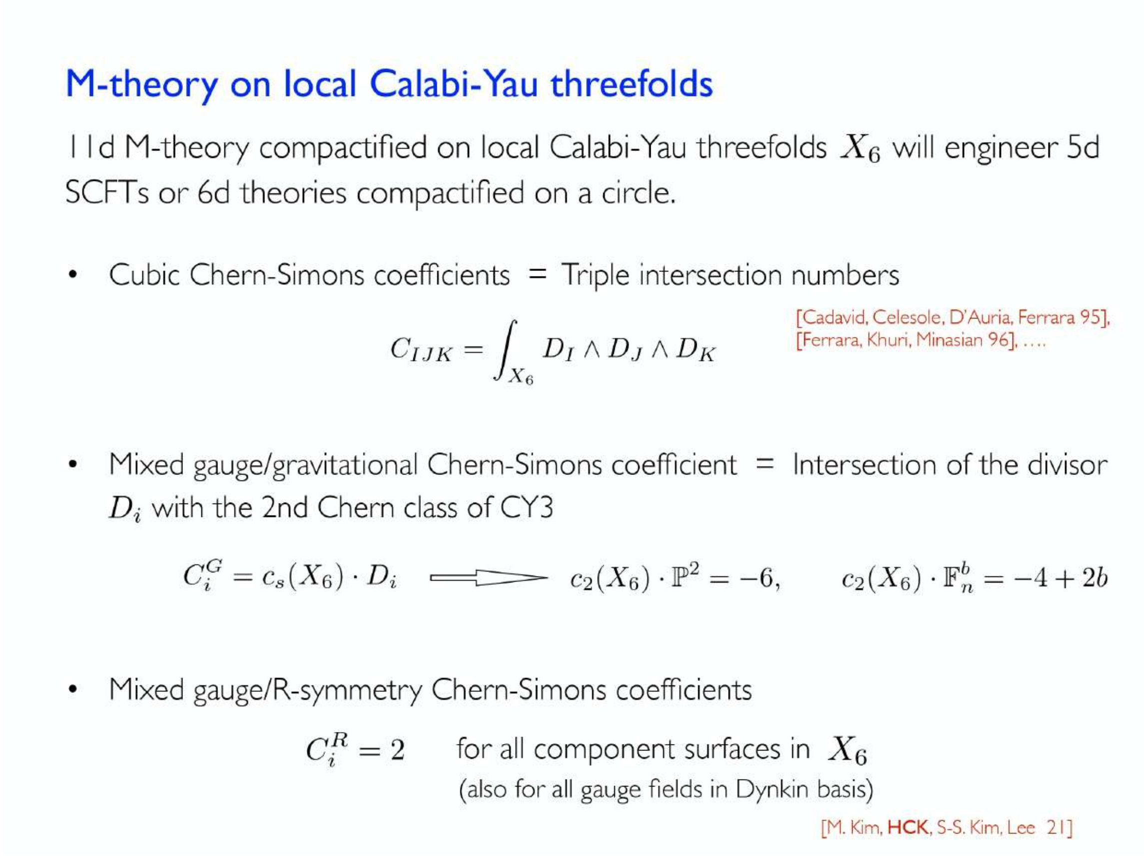
$\tau \sim R^{-1}$, $F = dA_0$, $\Omega^{\alpha\beta}$: Intersection form of tensors

[Bonetti, Grimm 11], [Bonetti, Grimm, Hohenegger 13],
[Bhardwaj, Jefferson, **HCK**, Tarazi, Vafa 19]

- Outer-automorphism twist modifies the intersection form and the gauge algebra factors accordingly.

$$\Omega^{\alpha\beta} \;\to\; \Omega_S^{\alpha'\beta'} = \sum_{\beta\in\beta'} \Omega^{\alpha\beta} \quad \& \quad K_{a,ij} \text{ for } \mathfrak{g} \;\to\; \tilde{K}_{a,ij} \text{ for } \mathfrak{h} \subset \mathfrak{g}$$

[Bhardwaj, Jefferson, **HCK**, Tarazi, Vafa 19]



M-theory on local Calabi-Yau threefolds

11d M-theory compactified on local Calabi-Yau threefolds X_6 will engineer 5d SCFTs or 6d theories compactified on a circle.

- Cubic Chern-Simons coefficients = Triple intersection numbers

$$C_{IJK} = \int_{X_6} D_I \wedge D_J \wedge D_K$$

[Cadavid, Celesole, D'Auria, Ferrara 95], [Ferrara, Khuri, Minasian 96], ….

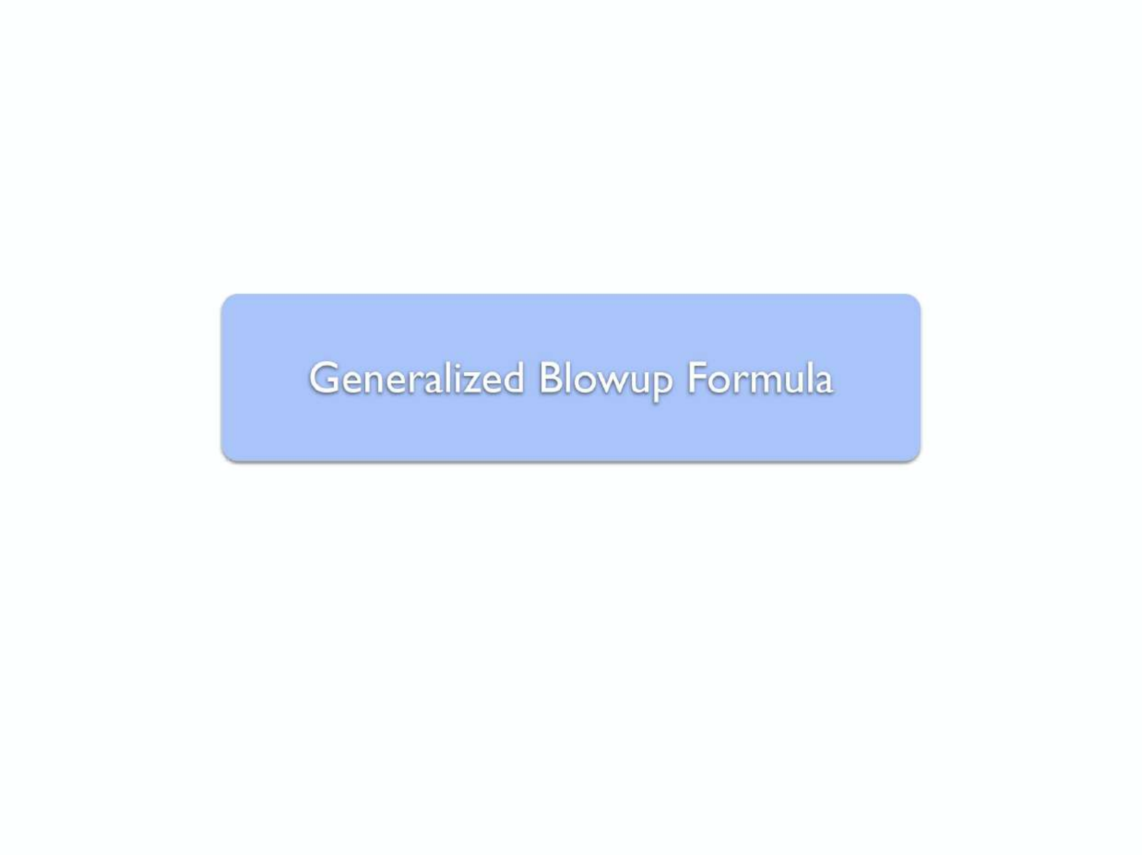
- Mixed gauge/gravitational Chern-Simons coefficient = Intersection of the divisor D_i with the 2nd Chern class of CY3

$$C_i^G = c_s(X_6) \cdot D_i \quad \Longrightarrow \quad c_2(X_6) \cdot \mathbb{P}^2 = -6, \qquad c_2(X_6) \cdot \mathbb{F}_n^b = -4 + 2b$$

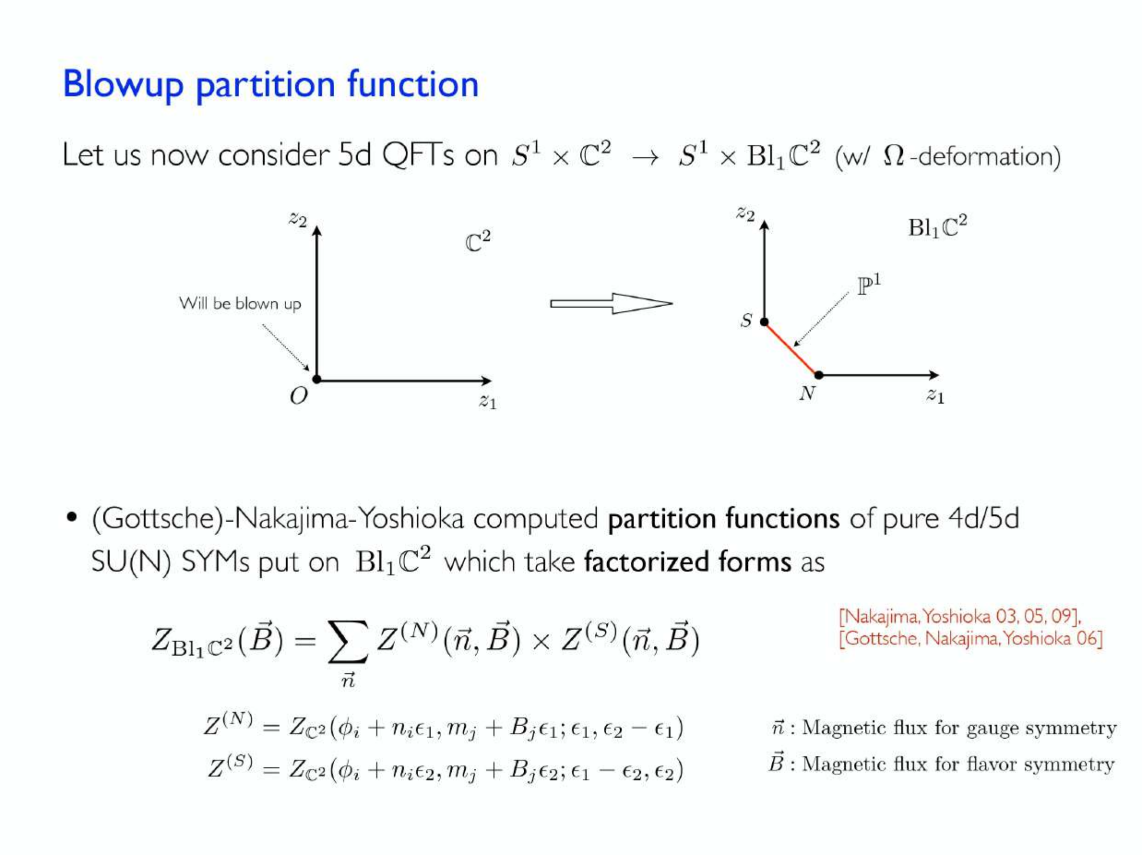
- Mixed gauge/R-symmetry Chern-Simons coefficients

$C_i^R = 2$ for all component surfaces in X_6
(also for all gauge fields in Dynkin basis)

[M. Kim, **HCK**, S-S. Kim, Lee 21]



Generalized Blowup Formula



Blowup partition function

Let us now consider 5d QFTs on $S^1 \times \mathbb{C}^2 \;\rightarrow\; S^1 \times \mathrm{Bl}_1\mathbb{C}^2$ (w/ Ω-deformation)

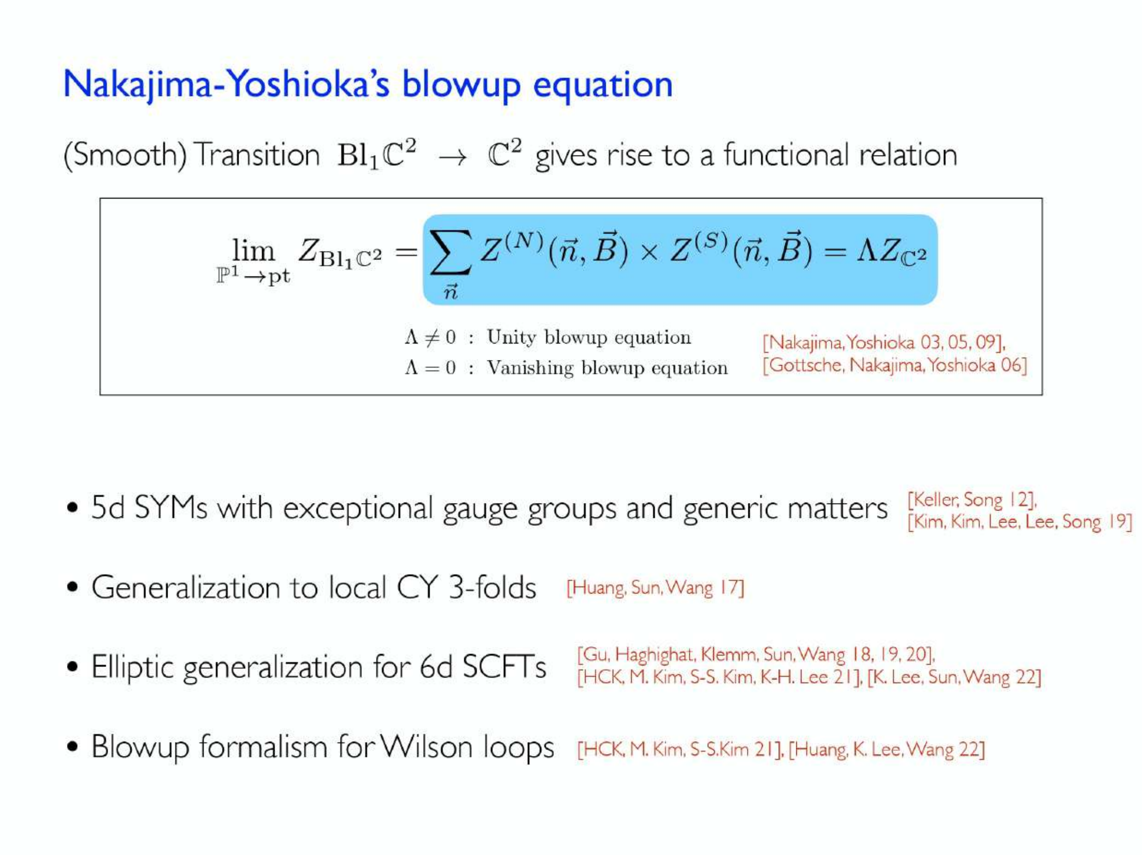
- (Gottsche)-Nakajima-Yoshioka computed **partition functions** of pure 4d/5d SU(N) SYMs put on $\mathrm{Bl}_1\mathbb{C}^2$ which take **factorized forms** as

$$Z_{\mathrm{Bl}_1\mathbb{C}^2}(\vec{B}) = \sum_{\vec{n}} Z^{(N)}(\vec{n},\vec{B}) \times Z^{(S)}(\vec{n},\vec{B})$$

[Nakajima, Yoshioka 03, 05, 09],
[Gottsche, Nakajima, Yoshioka 06]

$$Z^{(N)} = Z_{\mathbb{C}^2}(\phi_i + n_i\epsilon_1, m_j + B_j\epsilon_1; \epsilon_1, \epsilon_2 - \epsilon_1)$$
$$Z^{(S)} = Z_{\mathbb{C}^2}(\phi_i + n_i\epsilon_2, m_j + B_j\epsilon_2; \epsilon_1 - \epsilon_2, \epsilon_2)$$

$\vec{n}$: Magnetic flux for gauge symmetry
$\vec{B}$: Magnetic flux for flavor symmetry



Nakajima-Yoshioka's blowup equation

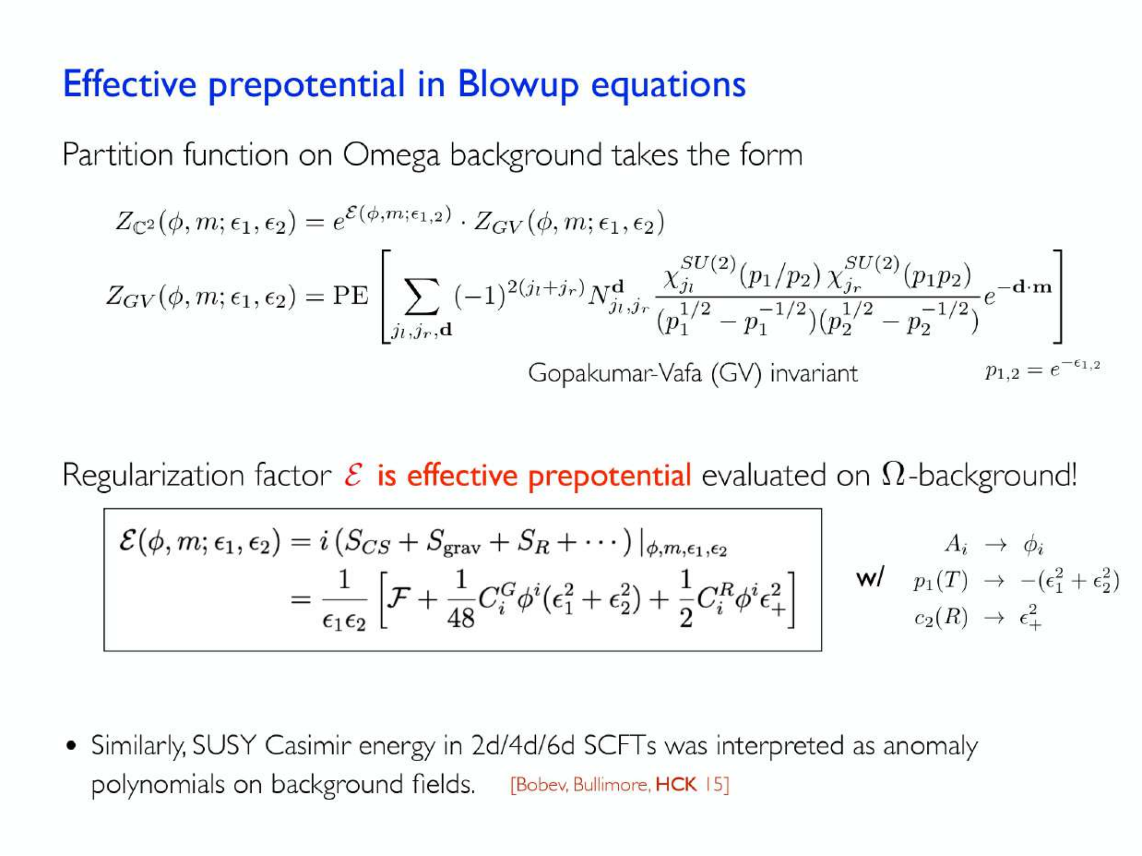
(Smooth) Transition $\mathrm{Bl}_1\mathbb{C}^2 \;\to\; \mathbb{C}^2$ gives rise to a functional relation

$$\lim_{\mathbb{P}^1\to \mathrm{pt}} Z_{\mathrm{Bl}_1\mathbb{C}^2} = \sum_{\vec{n}} Z^{(N)}(\vec{n},\vec{B}) \times Z^{(S)}(\vec{n},\vec{B}) = \Lambda Z_{\mathbb{C}^2}$$

$\Lambda \neq 0$: Unity blowup equation
$\Lambda = 0$: Vanishing blowup equation

[Nakajima, Yoshioka 03, 05, 09],
[Gottsche, Nakajima, Yoshioka 06]

- 5d SYMs with exceptional gauge groups and generic matters [Keller, Song 12], [Kim, Kim, Lee, Lee, Song 19]
- Generalization to local CY 3-folds [Huang, Sun, Wang 17]
- Elliptic generalization for 6d SCFTs [Gu, Haghighat, Klemm, Sun, Wang 18, 19, 20], [HCK, M. Kim, S-S. Kim, K-H. Lee 21], [K. Lee, Sun, Wang 22]
- Blowup formalism for Wilson loops [HCK, M. Kim, S-S.Kim 21], [Huang, K. Lee, Wang 22]



Effective prepotential in Blowup equations

Partition function on Omega background takes the form

$$Z_{\mathbb{C}^2}(\phi, m; \epsilon_1, \epsilon_2) = e^{\mathcal{E}(\phi, m; \epsilon_{1,2})} \cdot Z_{GV}(\phi, m; \epsilon_1, \epsilon_2)$$

$$Z_{GV}(\phi, m; \epsilon_1, \epsilon_2) = \mathrm{PE}\left[\sum_{j_l, j_r, \mathbf{d}} (-1)^{2(j_l+j_r)} N^{\mathbf{d}}_{j_l, j_r} \frac{\chi^{SU(2)}_{j_l}(p_1/p_2)\, \chi^{SU(2)}_{j_r}(p_1 p_2)}{(p_1^{1/2} - p_1^{-1/2})(p_2^{1/2} - p_2^{-1/2})} e^{-\mathbf{d}\cdot\mathbf{m}}\right]$$

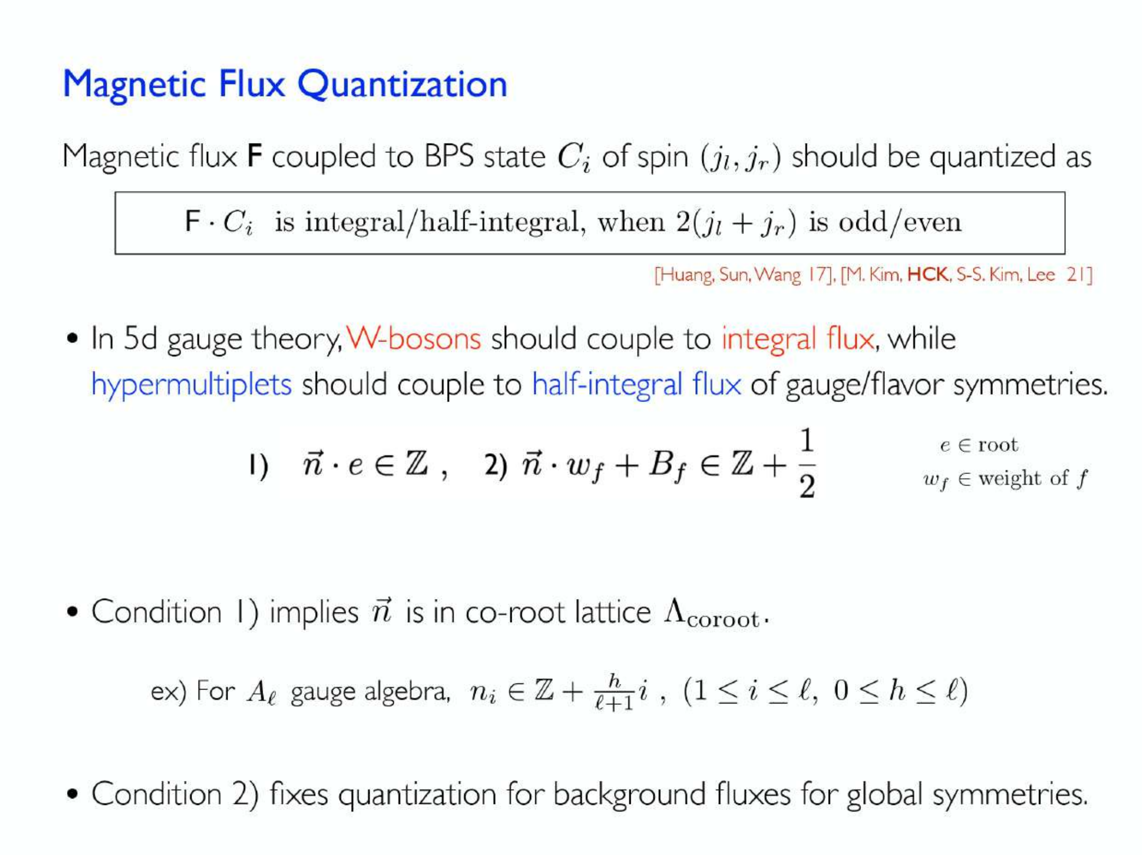
Gopakumar-Vafa (GV) invariant $p_{1,2} = e^{-\epsilon_{1,2}}$

Regularization factor $\mathcal{E}$ **is effective prepotential** evaluated on Ω-background!

$$\boxed{\begin{aligned}\mathcal{E}(\phi, m; \epsilon_1, \epsilon_2) &= i\,(S_{CS} + S_{\text{grav}} + S_R + \cdots)\,|_{\phi, m, \epsilon_1, \epsilon_2} \\ &= \frac{1}{\epsilon_1 \epsilon_2}\left[\mathcal{F} + \frac{1}{48} C^G_i \phi^i (\epsilon_1^2 + \epsilon_2^2) + \frac{1}{2} C^R_i \phi^i \epsilon_+^2\right]\end{aligned}}$$

w/ $$\begin{aligned} A_i &\to \phi_i \\ p_1(T) &\to -(\epsilon_1^2 + \epsilon_2^2) \\ c_2(R) &\to \epsilon_+^2 \end{aligned}$$

- Similarly, SUSY Casimir energy in 2d/4d/6d SCFTs was interpreted as anomaly polynomials on background fields. [Bobev, Bullimore, **HCK** 15]



Magnetic Flux Quantization

Magnetic flux **F** coupled to BPS state C_i of spin (j_l, j_r) should be quantized as

$$\boxed{\mathsf{F}\cdot C_i \ \ \text{is integral/half-integral, when } 2(j_l+j_r) \text{ is odd/even}}$$

[Huang, Sun, Wang 17], [M. Kim, **HCK**, S-S. Kim, Lee 21]

- In 5d gauge theory, W-bosons should couple to integral flux, while hypermultiplets should couple to half-integral flux of gauge/flavor symmetries.

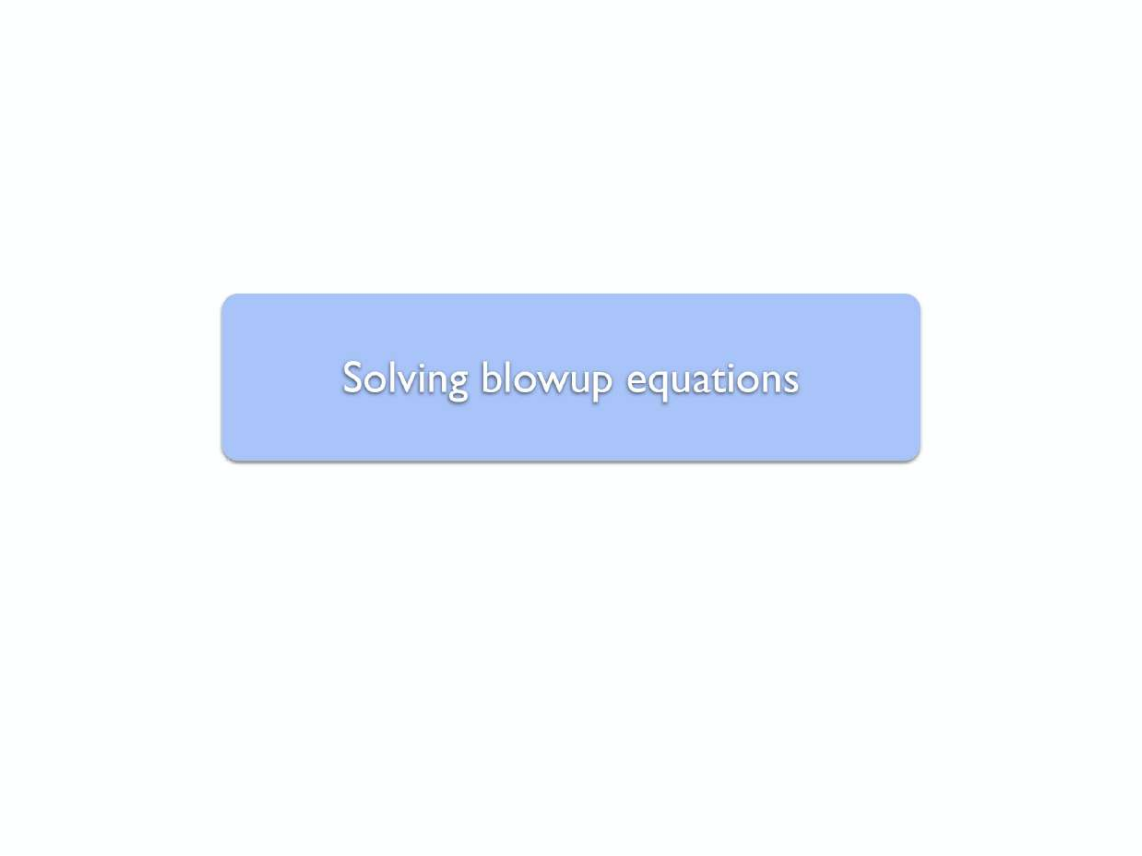
$$\text{1)} \quad \vec{n}\cdot e \in \mathbb{Z}\ , \quad \text{2)}\ \vec{n}\cdot w_f + B_f \in \mathbb{Z}+\frac{1}{2}$$

$e \in$ root
$w_f \in$ weight of f

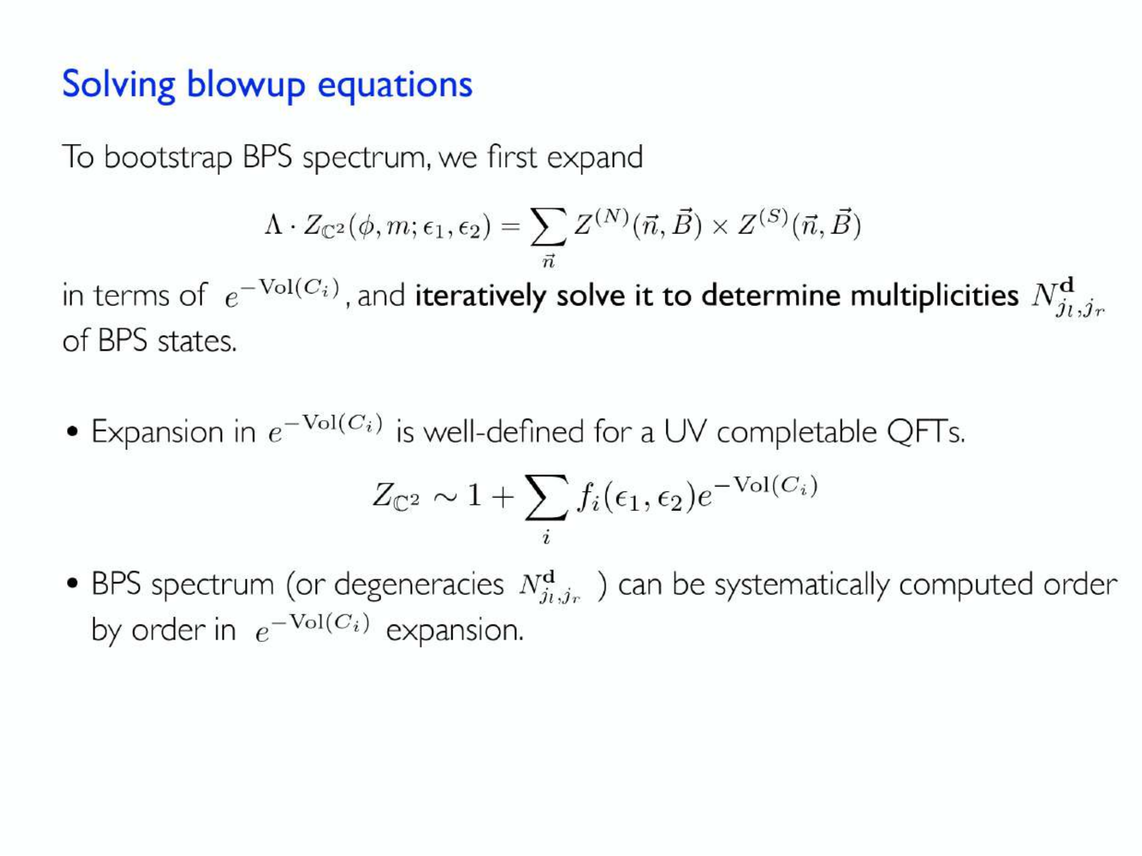
- Condition 1) implies $\vec{n}$ is in co-root lattice Λ_{coroot}.

ex) For A_ℓ gauge algebra, $n_i \in \mathbb{Z}+\frac{h}{\ell+1}i$, $(1 \le i \le \ell,\ 0 \le h \le \ell)$

- Condition 2) fixes quantization for background fluxes for global symmetries.



Solving blowup equations



Solving blowup equations

To bootstrap BPS spectrum, we first expand

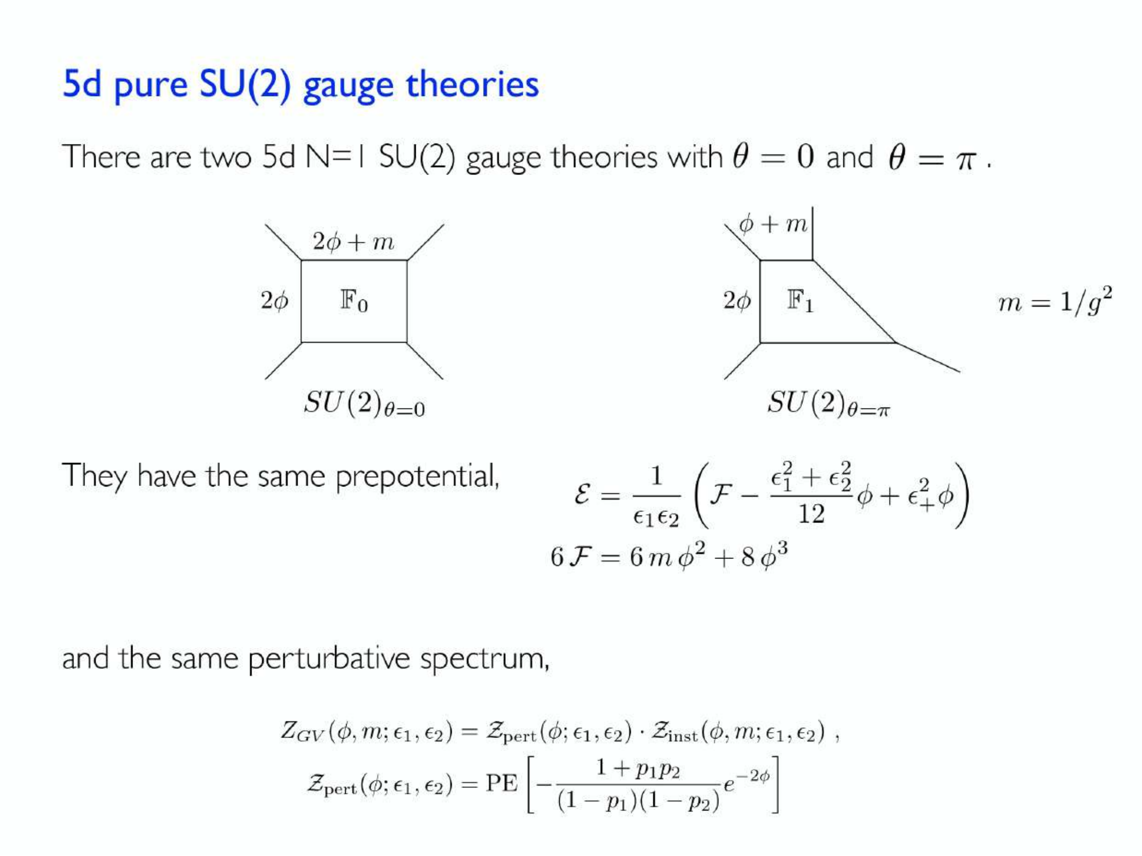
$$\Lambda \cdot Z_{\mathbb{C}^2}(\phi, m; \epsilon_1, \epsilon_2) = \sum_{\vec{n}} Z^{(N)}(\vec{n}, \vec{B}) \times Z^{(S)}(\vec{n}, \vec{B})$$

in terms of $e^{-\mathrm{Vol}(C_i)}$, and **iteratively solve it to determine multiplicities** $N^{\mathbf{d}}_{j_l, j_r}$ of BPS states.

- Expansion in $e^{-\mathrm{Vol}(C_i)}$ is well-defined for a UV completable QFTs.

$$Z_{\mathbb{C}^2} \sim 1 + \sum_i f_i(\epsilon_1, \epsilon_2) e^{-\mathrm{Vol}(C_i)}$$

- BPS spectrum (or degeneracies $N^{\mathbf{d}}_{j_l, j_r}$) can be systematically computed order by order in $e^{-\mathrm{Vol}(C_i)}$ expansion.



5d pure SU(2) gauge theories

There are two 5d N=1 SU(2) gauge theories with $\theta = 0$ and $\theta = \pi$.

$SU(2)_{\theta=0}$ $SU(2)_{\theta=\pi}$

$m = 1/g^2$

They have the same prepotential,

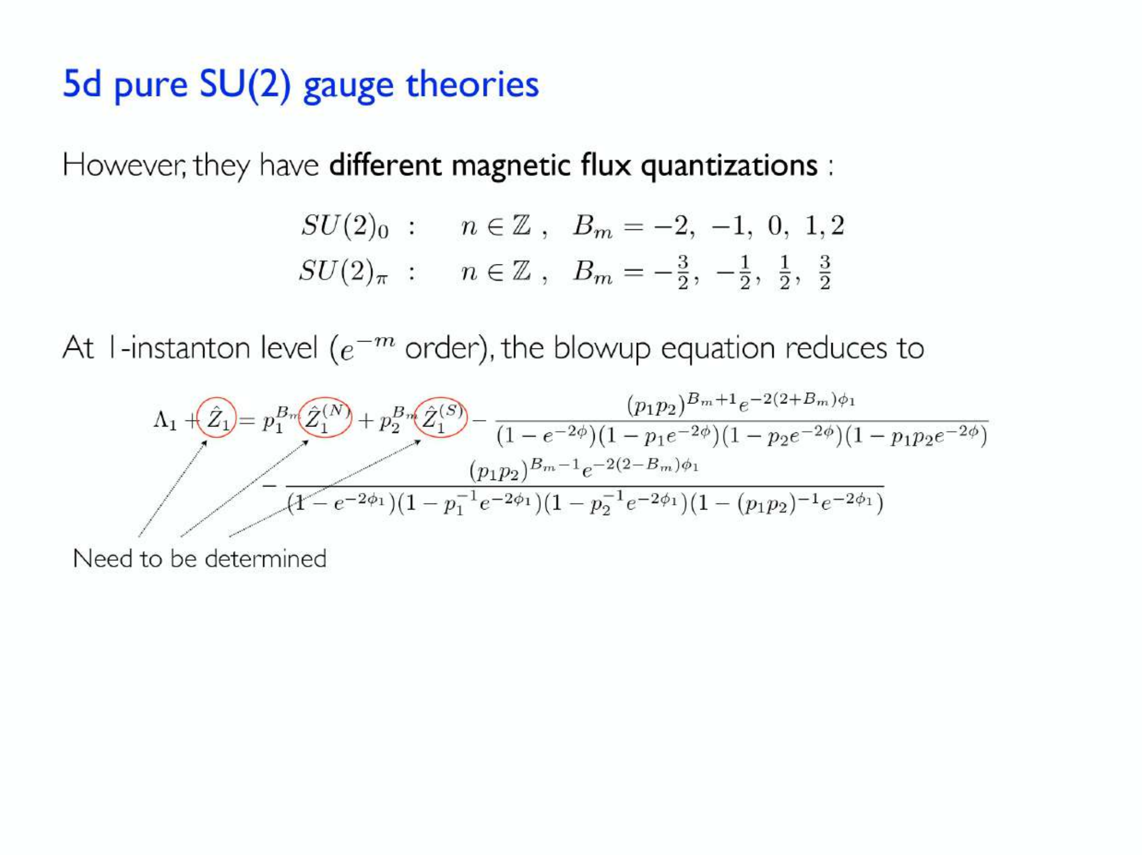
$$\mathcal{E} = \frac{1}{\epsilon_1\epsilon_2}\left(\mathcal{F} - \frac{\epsilon_1^2+\epsilon_2^2}{12}\phi + \epsilon_+^2\phi\right)$$

$$6\,\mathcal{F} = 6\,m\,\phi^2 + 8\,\phi^3$$

and the same perturbative spectrum,

$$Z_{GV}(\phi, m; \epsilon_1, \epsilon_2) = \mathcal{Z}_{\text{pert}}(\phi; \epsilon_1, \epsilon_2) \cdot \mathcal{Z}_{\text{inst}}(\phi, m; \epsilon_1, \epsilon_2) \ ,$$

$$\mathcal{Z}_{\text{pert}}(\phi; \epsilon_1, \epsilon_2) = \text{PE}\left[-\frac{1+p_1p_2}{(1-p_1)(1-p_2)}e^{-2\phi}\right]$$



5d pure SU(2) gauge theories

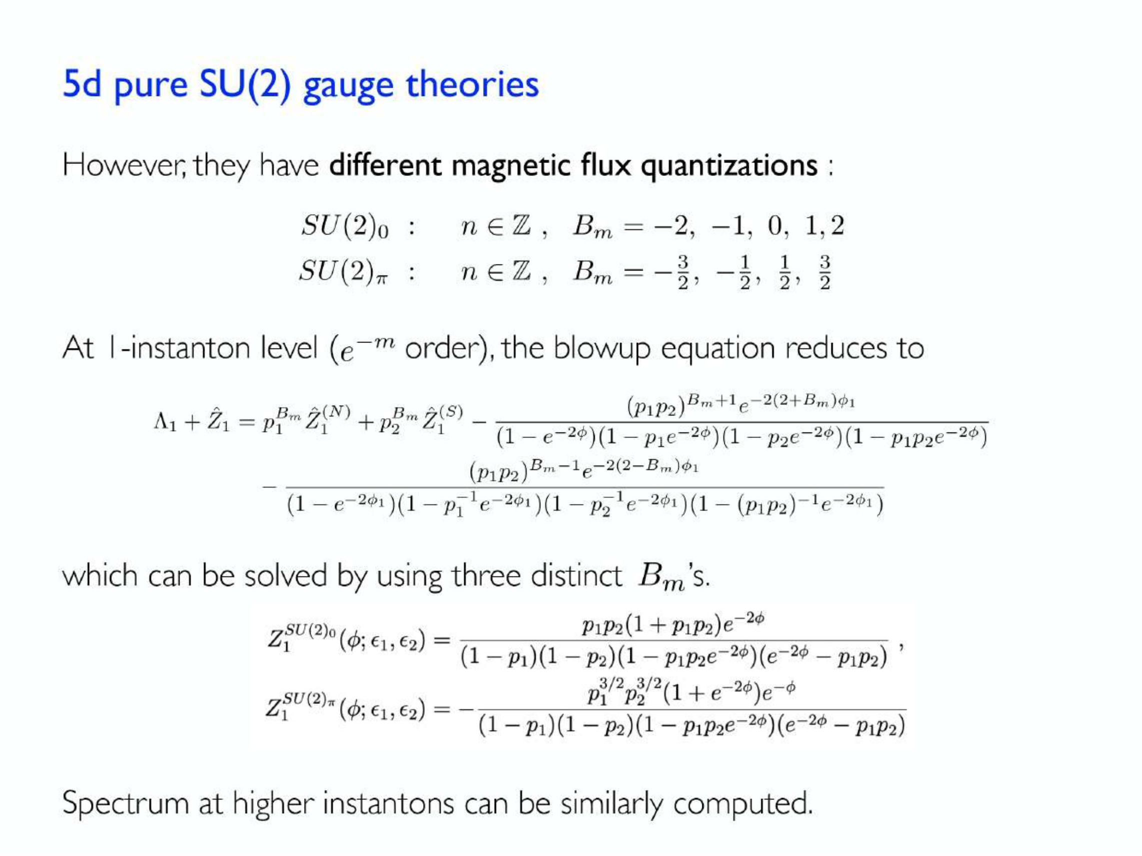
However, they have **different magnetic flux quantizations** :

$$SU(2)_0 \;:\quad n\in\mathbb{Z}\;,\quad B_m=-2,\;-1,\;0,\;1,2$$
$$SU(2)_\pi \;:\quad n\in\mathbb{Z}\;,\quad B_m=-\tfrac{3}{2},\;-\tfrac{1}{2},\;\tfrac{1}{2},\;\tfrac{3}{2}$$

At 1-instanton level (e^{-m} order), the blowup equation reduces to

$$\Lambda_1+\hat{Z}_1=p_1^{B_m}\hat{Z}_1^{(N)}+p_2^{B_m}\hat{Z}_1^{(S)}-\frac{(p_1p_2)^{B_m+1}e^{-2(2+B_m)\phi_1}}{(1-e^{-2\phi})(1-p_1e^{-2\phi})(1-p_2e^{-2\phi})(1-p_1p_2e^{-2\phi})}$$
$$-\frac{(p_1p_2)^{B_m-1}e^{-2(2-B_m)\phi_1}}{(1-e^{-2\phi_1})(1-p_1^{-1}e^{-2\phi_1})(1-p_2^{-1}e^{-2\phi_1})(1-(p_1p_2)^{-1}e^{-2\phi_1})}$$

Need to be determined



5d pure SU(2) gauge theories

However, they have **different magnetic flux quantizations** :

$$SU(2)_0 \;:\quad n \in \mathbb{Z}\;,\quad B_m = -2,\; -1,\; 0,\; 1, 2$$
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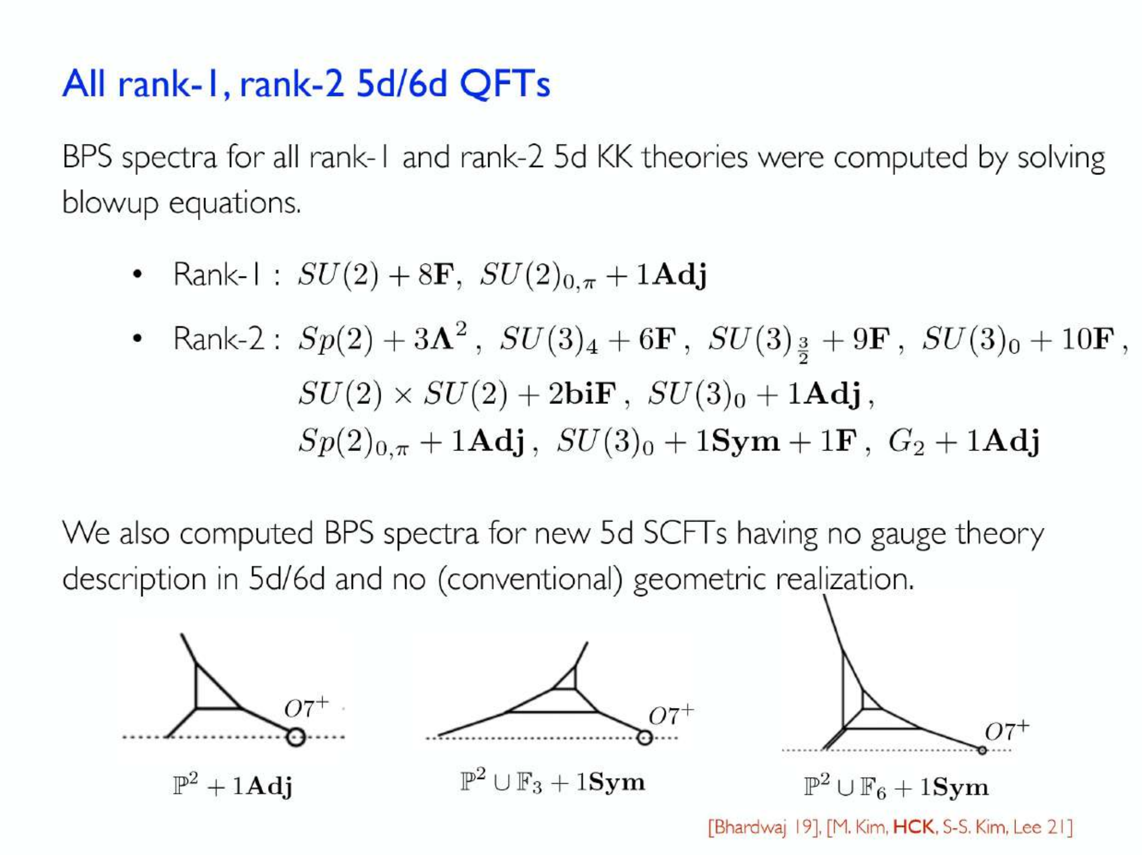
At 1-instanton level (e^{-m} order), the blowup equation reduces to

$$\Lambda_1 + \hat{Z}_1 = p_1^{B_m}\hat{Z}_1^{(N)} + p_2^{B_m}\hat{Z}_1^{(S)} - \frac{(p_1p_2)^{B_m+1}e^{-2(2+B_m)\phi_1}}{(1-e^{-2\phi})(1-p_1e^{-2\phi})(1-p_2e^{-2\phi})(1-p_1p_2e^{-2\phi})}$$
$$- \frac{(p_1p_2)^{B_m-1}e^{-2(2-B_m)\phi_1}}{(1-e^{-2\phi_1})(1-p_1^{-1}e^{-2\phi_1})(1-p_2^{-1}e^{-2\phi_1})(1-(p_1p_2)^{-1}e^{-2\phi_1})}$$

which can be solved by using three distinct B_m's.

$$Z_1^{SU(2)_0}(\phi;\epsilon_1,\epsilon_2) = \frac{p_1p_2(1+p_1p_2)e^{-2\phi}}{(1-p_1)(1-p_2)(1-p_1p_2e^{-2\phi})(e^{-2\phi}-p_1p_2)}\;,$$
$$Z_1^{SU(2)_\pi}(\phi;\epsilon_1,\epsilon_2) = -\frac{p_1^{3/2}p_2^{3/2}(1+e^{-2\phi})e^{-\phi}}{(1-p_1)(1-p_2)(1-p_1p_2e^{-2\phi})(e^{-2\phi}-p_1p_2)}$$

Spectrum at higher instantons can be similarly computed.



All rank-1, rank-2 5d/6d QFTs

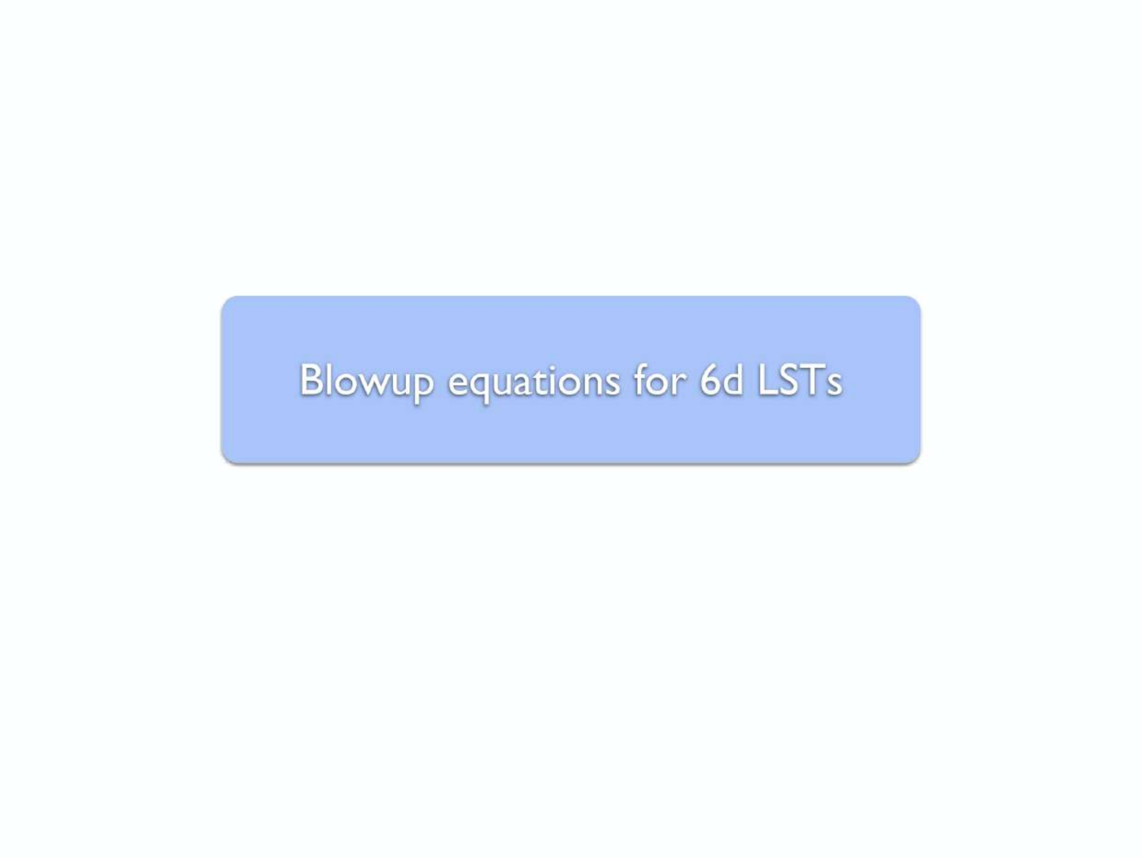
BPS spectra for all rank-1 and rank-2 5d KK theories were computed by solving blowup equations.

- Rank-1 : $SU(2) + 8\mathbf{F},\ SU(2)_{0,\pi} + 1\mathbf{Adj}$
- Rank-2 : $Sp(2) + 3\mathbf{\Lambda}^2\,,\ SU(3)_4 + 6\mathbf{F}\,,\ SU(3)_{\frac{3}{2}} + 9\mathbf{F}\,,\ SU(3)_0 + 10\mathbf{F}\,,$
  $SU(2) \times SU(2) + 2\mathbf{biF}\,,\ SU(3)_0 + 1\mathbf{Adj}\,,$
  $Sp(2)_{0,\pi} + 1\mathbf{Adj}\,,\ SU(3)_0 + 1\mathbf{Sym} + 1\mathbf{F}\,,\ G_2 + 1\mathbf{Adj}$

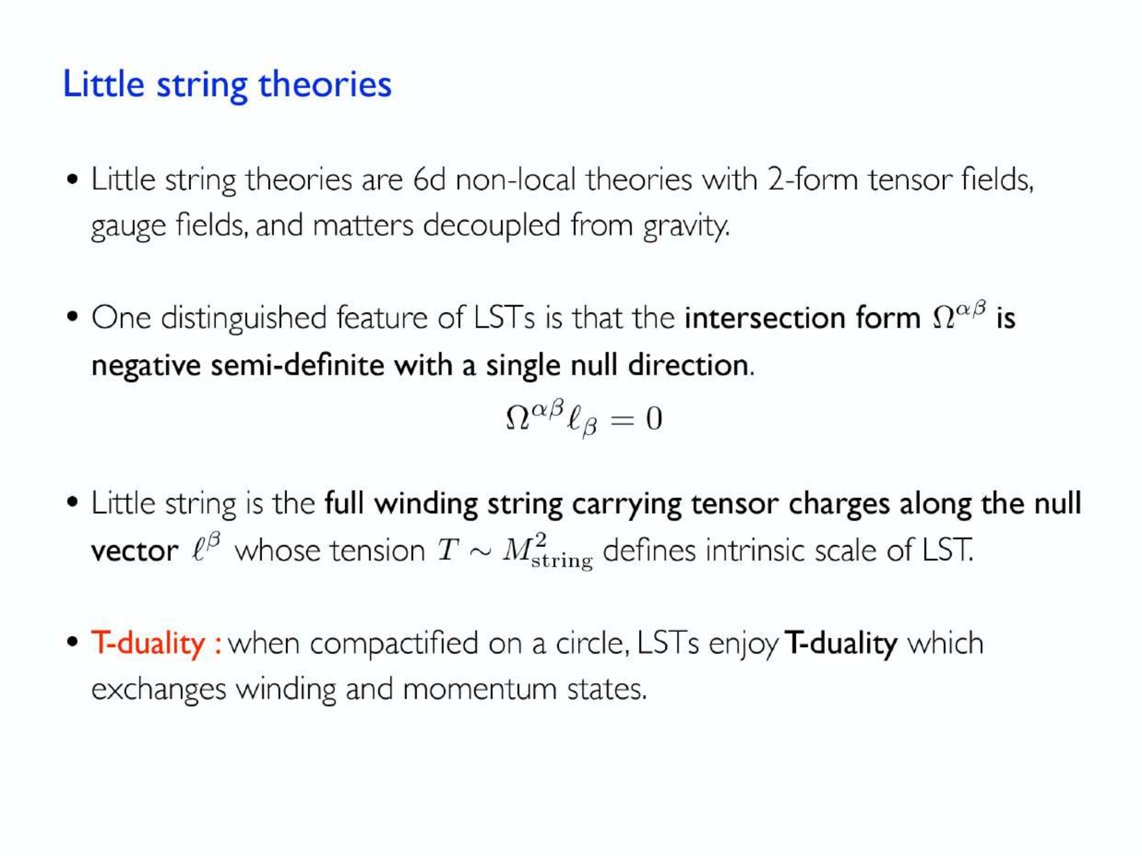
We also computed BPS spectra for new 5d SCFTs having no gauge theory description in 5d/6d and no (conventional) geometric realization.

$\mathbb{P}^2 + 1\mathbf{Adj}$ $\mathbb{P}^2 \cup \mathbb{F}_3 + 1\mathbf{Sym}$ $\mathbb{P}^2 \cup \mathbb{F}_6 + 1\mathbf{Sym}$

[Bhardwaj 19], [M. Kim, **HCK**, S-S. Kim, Lee 21]



Blowup equations for 6d LSTs



Little string theories

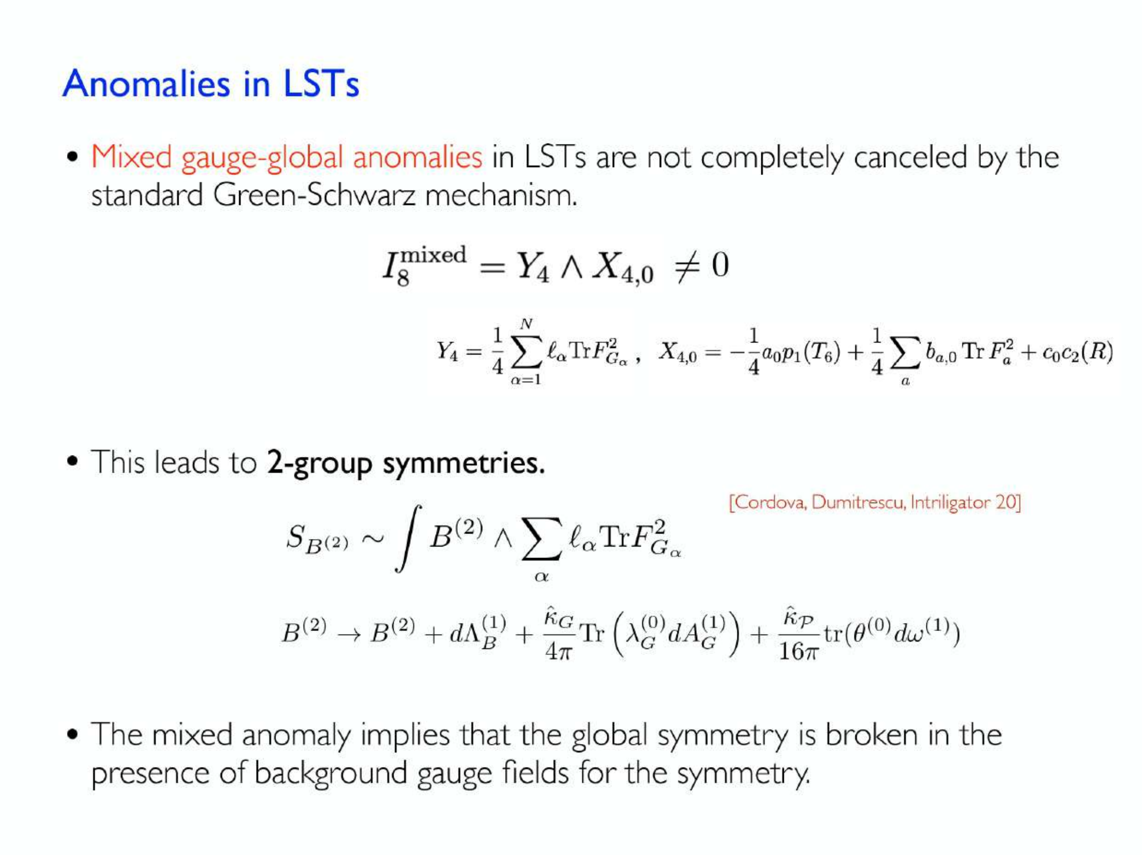
- Little string theories are 6d non-local theories with 2-form tensor fields, gauge fields, and matters decoupled from gravity.

- One distinguished feature of LSTs is that the **intersection form $\Omega^{\alpha\beta}$ is negative semi-definite with a single null direction**.

$$\Omega^{\alpha\beta}\ell_\beta = 0$$

- Little string is the **full winding string carrying tensor charges along the null vector ℓ^β** whose tension $T \sim M_{\text{string}}^2$ defines intrinsic scale of LST.

- **T-duality :** when compactified on a circle, LSTs enjoy **T-duality** which exchanges winding and momentum states.



Anomalies in LSTs

- Mixed gauge-global anomalies in LSTs are not completely canceled by the standard Green-Schwarz mechanism.

$$I_8^{\text{mixed}} = Y_4 \wedge X_{4,0} \neq 0$$

$$Y_4 = \frac{1}{4}\sum_{\alpha=1}^{N} \ell_\alpha \mathrm{Tr} F_{G_\alpha}^2\,, \quad X_{4,0} = -\frac{1}{4}a_0 p_1(T_6) + \frac{1}{4}\sum_a b_{a,0}\,\mathrm{Tr}\,F_a^2 + c_0 c_2(R)$$

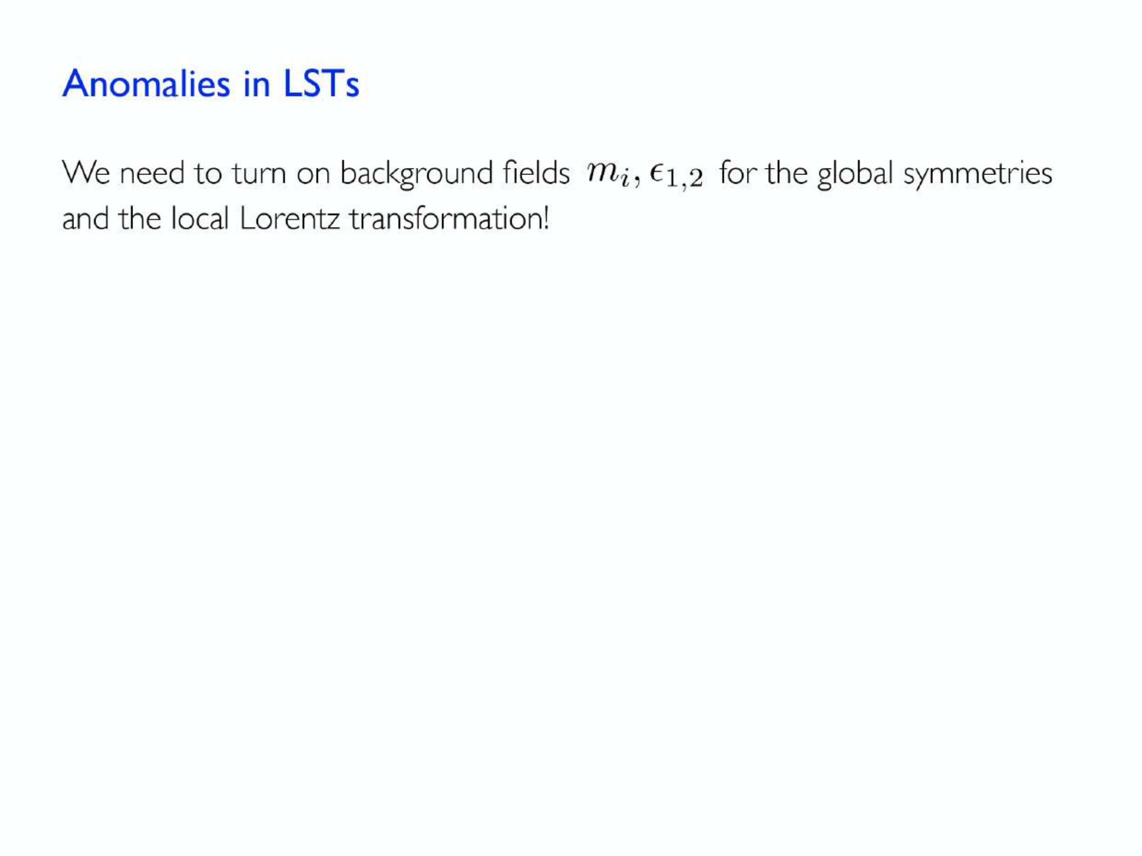
- This leads to **2-group symmetries.**

[Cordova, Dumitrescu, Intriligator 20]

$$S_{B^{(2)}} \sim \int B^{(2)} \wedge \sum_\alpha \ell_\alpha \mathrm{Tr} F_{G_\alpha}^2$$

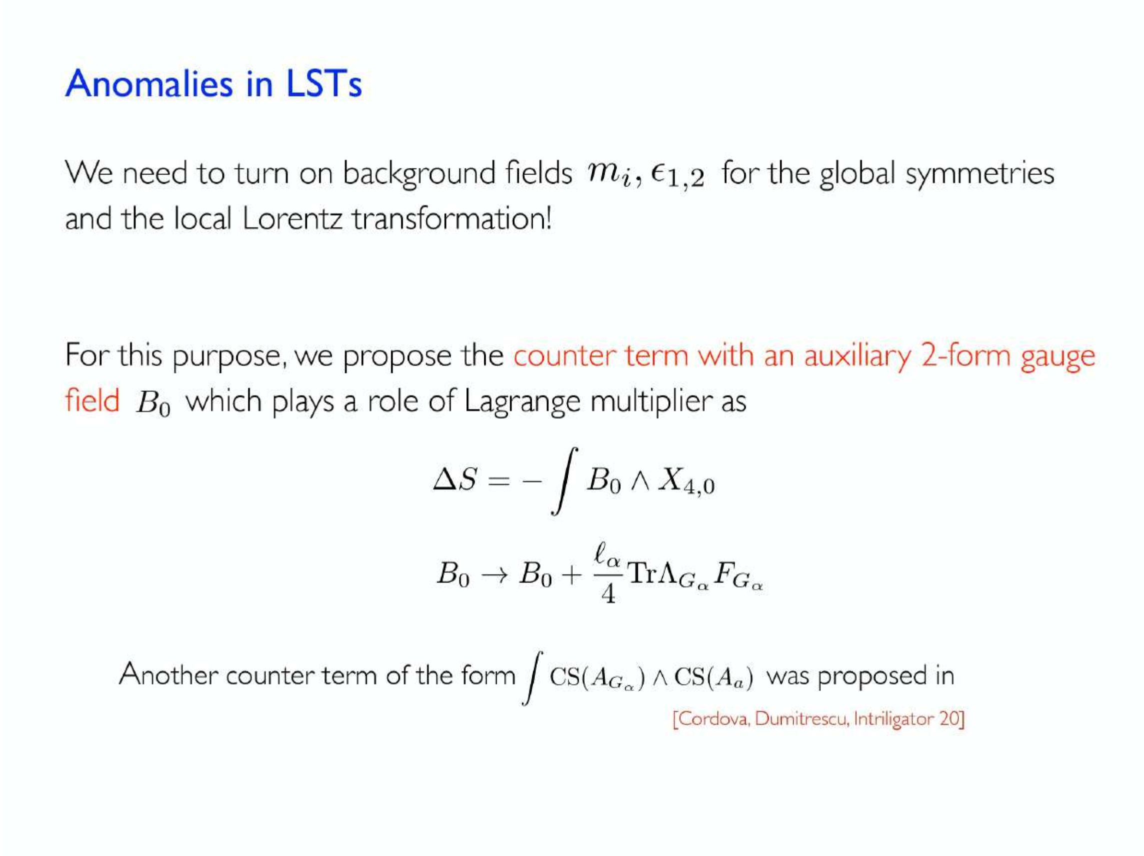
$$B^{(2)} \to B^{(2)} + d\Lambda_B^{(1)} + \frac{\hat{\kappa}_G}{4\pi}\mathrm{Tr}\left(\lambda_G^{(0)} dA_G^{(1)}\right) + \frac{\hat{\kappa}_{\mathcal{P}}}{16\pi}\mathrm{tr}(\theta^{(0)} d\omega^{(1)})$$

- The mixed anomaly implies that the global symmetry is broken in the presence of background gauge fields for the symmetry.



Anomalies in LSTs

We need to turn on background fields $m_i, \epsilon_{1,2}$ for the global symmetries and the local Lorentz transformation!



Anomalies in LSTs

We need to turn on background fields $m_i, \epsilon_{1,2}$ for the global symmetries and the local Lorentz transformation!

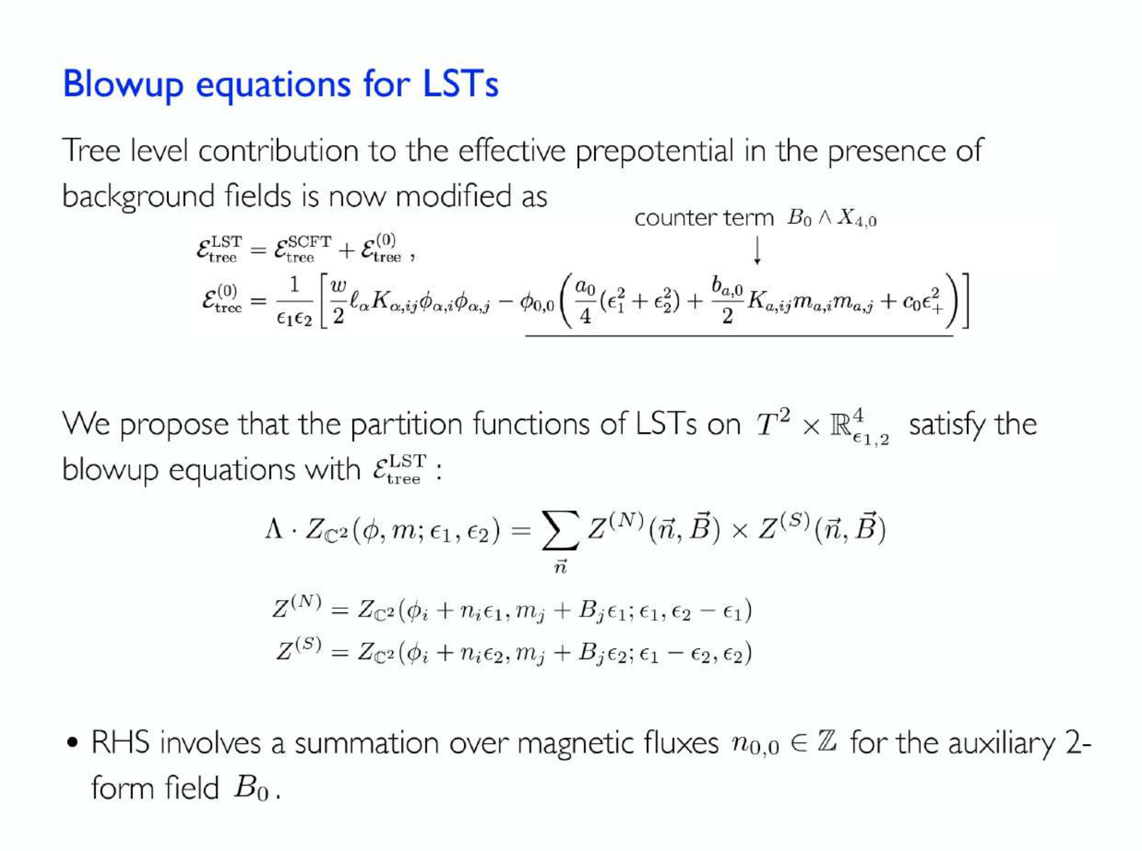
For this purpose, we propose the counter term with an auxiliary 2-form gauge field B_0 which plays a role of Lagrange multiplier as

$$\Delta S = -\int B_0 \wedge X_{4,0}$$

$$B_0 \to B_0 + \frac{\ell_\alpha}{4} \mathrm{Tr}\Lambda_{G_\alpha} F_{G_\alpha}$$

Another counter term of the form $\int \mathrm{CS}(A_{G_\alpha}) \wedge \mathrm{CS}(A_a)$ was proposed in

[Cordova, Dumitrescu, Intriligator 20]



Blowup equations for LSTs

Tree level contribution to the effective prepotential in the presence of background fields is now modified as

counter term $B_0 \wedge X_{4,0}$

$\downarrow$

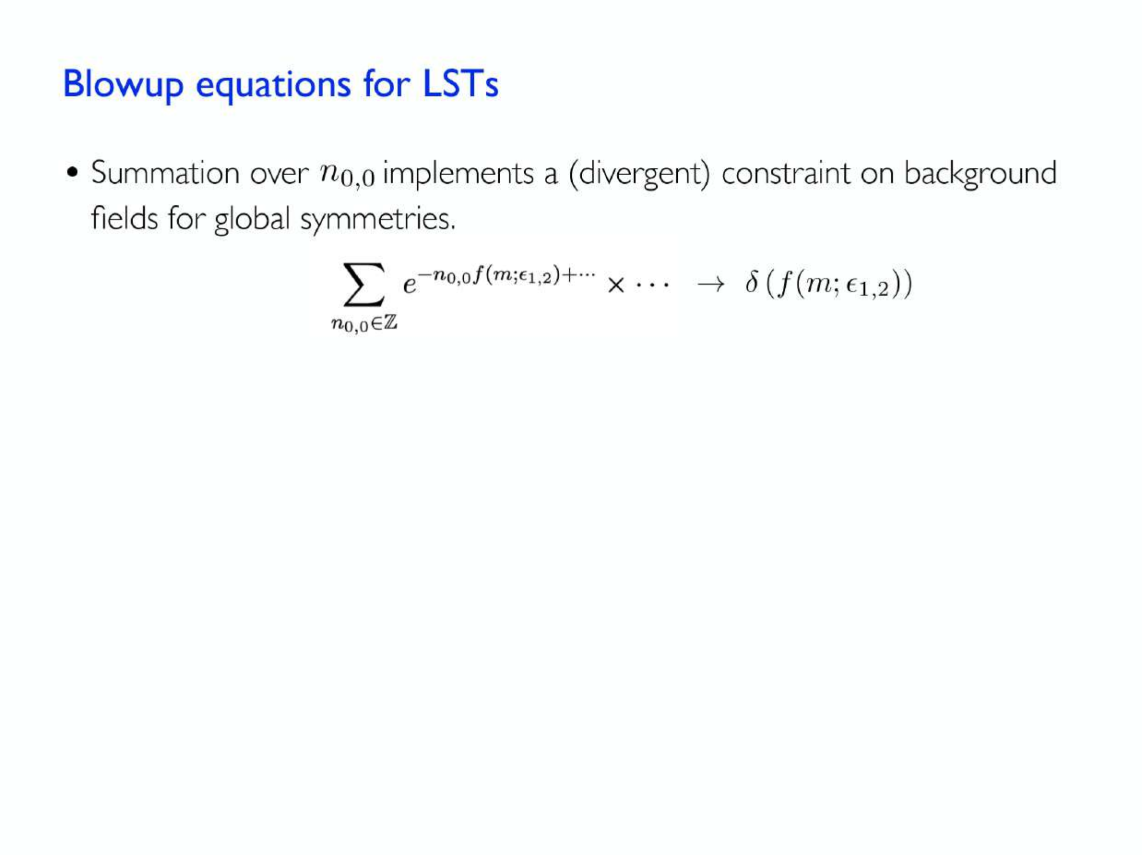
$$\begin{aligned}
\mathcal{E}_{\text{tree}}^{\text{LST}} &= \mathcal{E}_{\text{tree}}^{\text{SCFT}} + \mathcal{E}_{\text{tree}}^{(0)} \, , \\
\mathcal{E}_{\text{tree}}^{(0)} &= \frac{1}{\epsilon_1\epsilon_2}\left[\frac{w}{2}\ell_\alpha K_{\alpha,ij}\phi_{\alpha,i}\phi_{\alpha,j} - \underline{\phi_{0,0}\left(\frac{a_0}{4}(\epsilon_1^2+\epsilon_2^2) + \frac{b_{a,0}}{2}K_{a,ij}m_{a,i}m_{a,j} + c_0\epsilon_+^2\right)}\right]
\end{aligned}$$

We propose that the partition functions of LSTs on $T^2 \times \mathbb{R}^4_{\epsilon_{1,2}}$ satisfy the blowup equations with $\mathcal{E}_{\text{tree}}^{\text{LST}}$:

$$\Lambda \cdot Z_{\mathbb{C}^2}(\phi, m; \epsilon_1, \epsilon_2) = \sum_{\vec{n}} Z^{(N)}(\vec{n}, \vec{B}) \times Z^{(S)}(\vec{n}, \vec{B})$$

$$\begin{aligned}
Z^{(N)} &= Z_{\mathbb{C}^2}(\phi_i + n_i\epsilon_1, m_j + B_j\epsilon_1; \epsilon_1, \epsilon_2 - \epsilon_1) \\
Z^{(S)} &= Z_{\mathbb{C}^2}(\phi_i + n_i\epsilon_2, m_j + B_j\epsilon_2; \epsilon_1 - \epsilon_2, \epsilon_2)
\end{aligned}$$

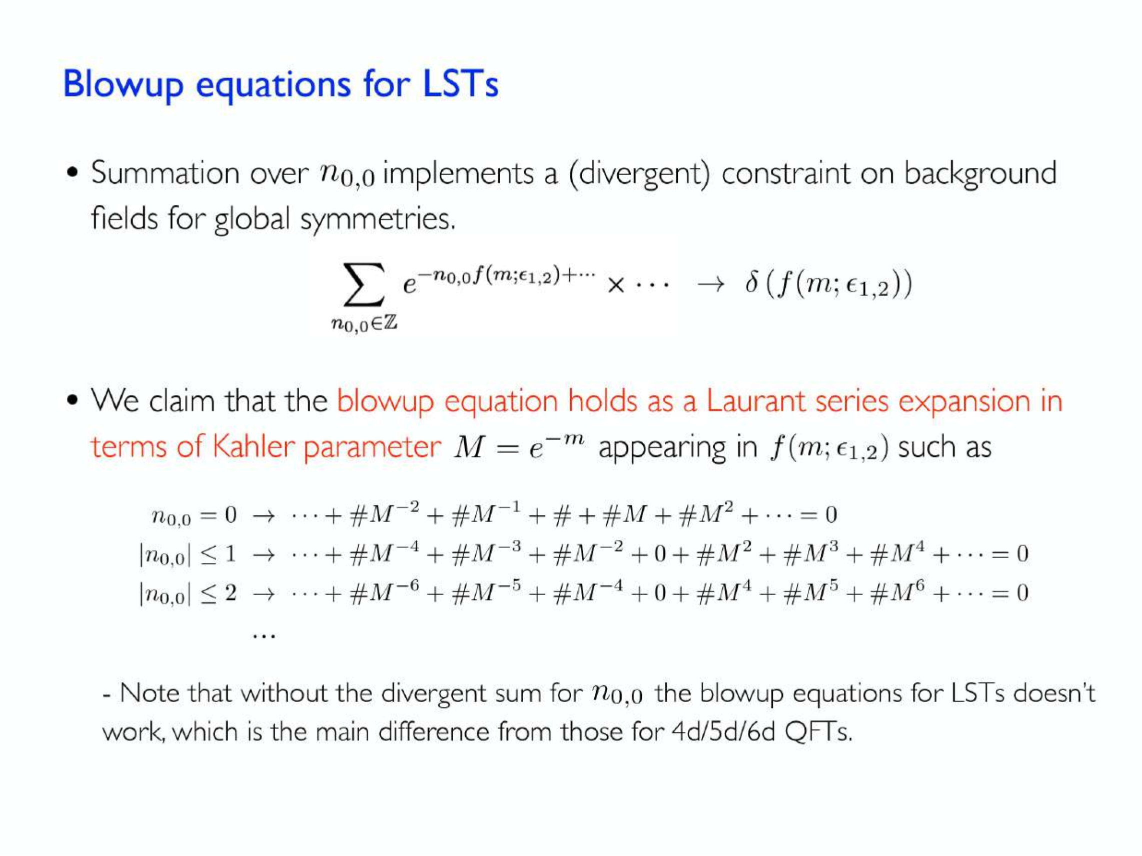
- RHS involves a summation over magnetic fluxes $n_{0,0} \in \mathbb{Z}$ for the auxiliary 2-form field B_0.



Blowup equations for LSTs

- Summation over $n_{0,0}$ implements a (divergent) constraint on background fields for global symmetries.

$$\sum_{n_{0,0}\in\mathbb{Z}} e^{-n_{0,0}f(m;\epsilon_{1,2})+\cdots} \times \cdots \;\to\; \delta\left(f(m;\epsilon_{1,2})\right)$$



Blowup equations for LSTs

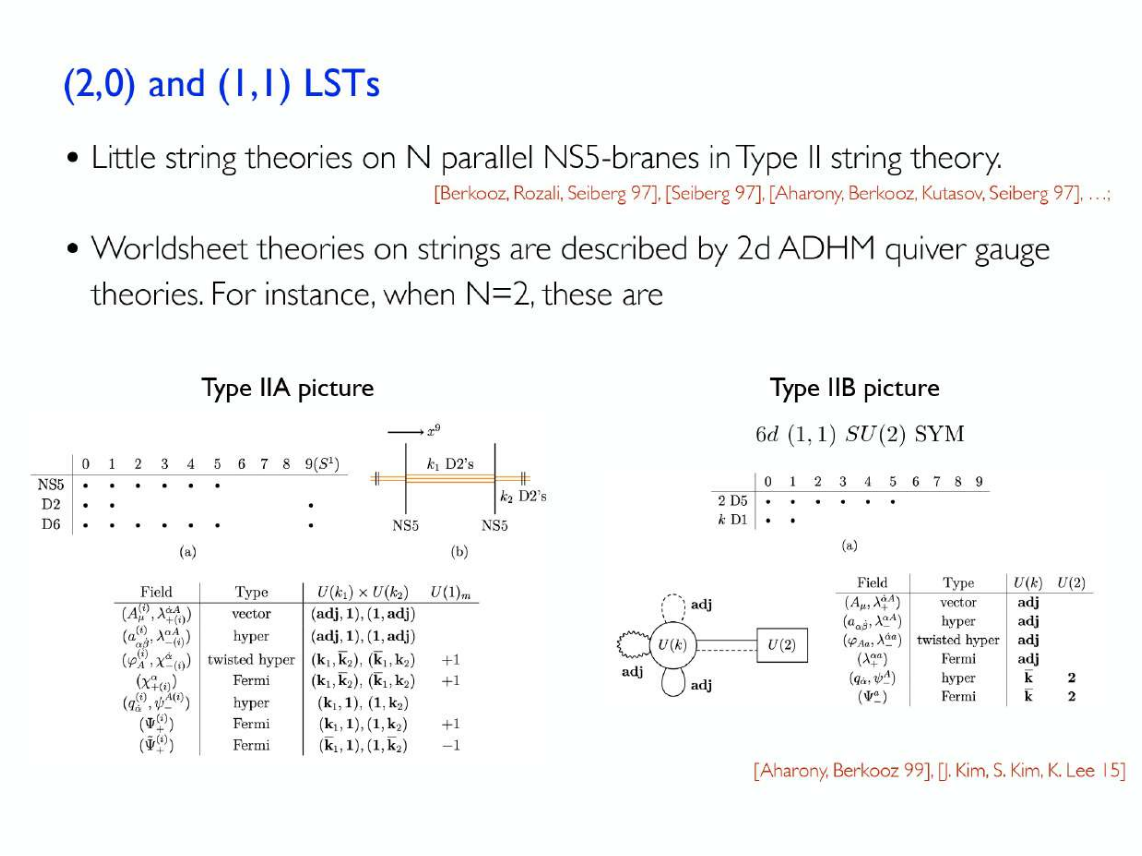
- Summation over $n_{0,0}$ implements a (divergent) constraint on background fields for global symmetries.

$$\sum_{n_{0,0}\in\mathbb{Z}} e^{-n_{0,0}f(m;\epsilon_{1,2})+\cdots} \times \cdots \quad \to \quad \delta\left(f(m;\epsilon_{1,2})\right)$$

- We claim that the blowup equation holds as a Laurant series expansion in terms of Kahler parameter $M = e^{-m}$ appearing in $f(m;\epsilon_{1,2})$ such as

$$\begin{aligned} n_{0,0} = 0 \;&\to\; \cdots + \#M^{-2} + \#M^{-1} + \# + \#M + \#M^2 + \cdots = 0 \\ |n_{0,0}| \leq 1 \;&\to\; \cdots + \#M^{-4} + \#M^{-3} + \#M^{-2} + 0 + \#M^2 + \#M^3 + \#M^4 + \cdots = 0 \\ |n_{0,0}| \leq 2 \;&\to\; \cdots + \#M^{-6} + \#M^{-5} + \#M^{-4} + 0 + \#M^4 + \#M^5 + \#M^6 + \cdots = 0 \\ &\cdots \end{aligned}$$

- Note that without the divergent sum for $n_{0,0}$ the blowup equations for LSTs doesn't work, which is the main difference from those for 4d/5d/6d QFTs.



(2,0) and (1,1) LSTs

- Little string theories on N parallel NS5-branes in Type II string theory.
  [Berkooz, Rozali, Seiberg 97], [Seiberg 97], [Aharony, Berkooz, Kutasov, Seiberg 97], …;
- Worldsheet theories on strings are described by 2d ADHM quiver gauge theories. For instance, when N=2, these are

Type IIA picture

	0	1	2	3	4	5	6	7	8	$9(S^1)$
NS5	•	•	•	•	•	•				
D2	•	•								•
D6	•	•	•	•	•	•				•

(a)

(b)

Field	Type	$U(k_1)\times U(k_2)$	$U(1)_m$
$(A_\mu^{(i)},\lambda_{+(i)}^{\dot\alpha A})$	vector	$(\mathbf{adj},\mathbf{1}),(\mathbf{1},\mathbf{adj})$	
$(a_{\alpha\dot\beta}^{(i)},\lambda_{-(i)}^{\alpha A})$	hyper	$(\mathbf{adj},\mathbf{1}),(\mathbf{1},\mathbf{adj})$	
$(\varphi_A^{(i)},\chi_{-(i)}^{\dot\alpha})$	twisted hyper	$(\mathbf{k}_1,\bar{\mathbf{k}}_2),(\bar{\mathbf{k}}_1,\mathbf{k}_2)$	$+1$
$(\chi_{+(i)}^{\alpha})$	Fermi	$(\mathbf{k}_1,\bar{\mathbf{k}}_2),(\bar{\mathbf{k}}_1,\mathbf{k}_2)$	$+1$
$(q_{\dot\alpha}^{(i)},\psi_-^{A(i)})$	hyper	$(\mathbf{k}_1,\mathbf{1}),(\mathbf{1},\mathbf{k}_2)$	
$(\Psi_+^{(i)})$	Fermi	$(\mathbf{k}_1,\mathbf{1}),(\mathbf{1},\mathbf{k}_2)$	$+1$
$(\tilde\Psi_+^{(i)})$	Fermi	$(\bar{\mathbf{k}}_1,\mathbf{1}),(\mathbf{1},\bar{\mathbf{k}}_2)$	-1

Type IIB picture

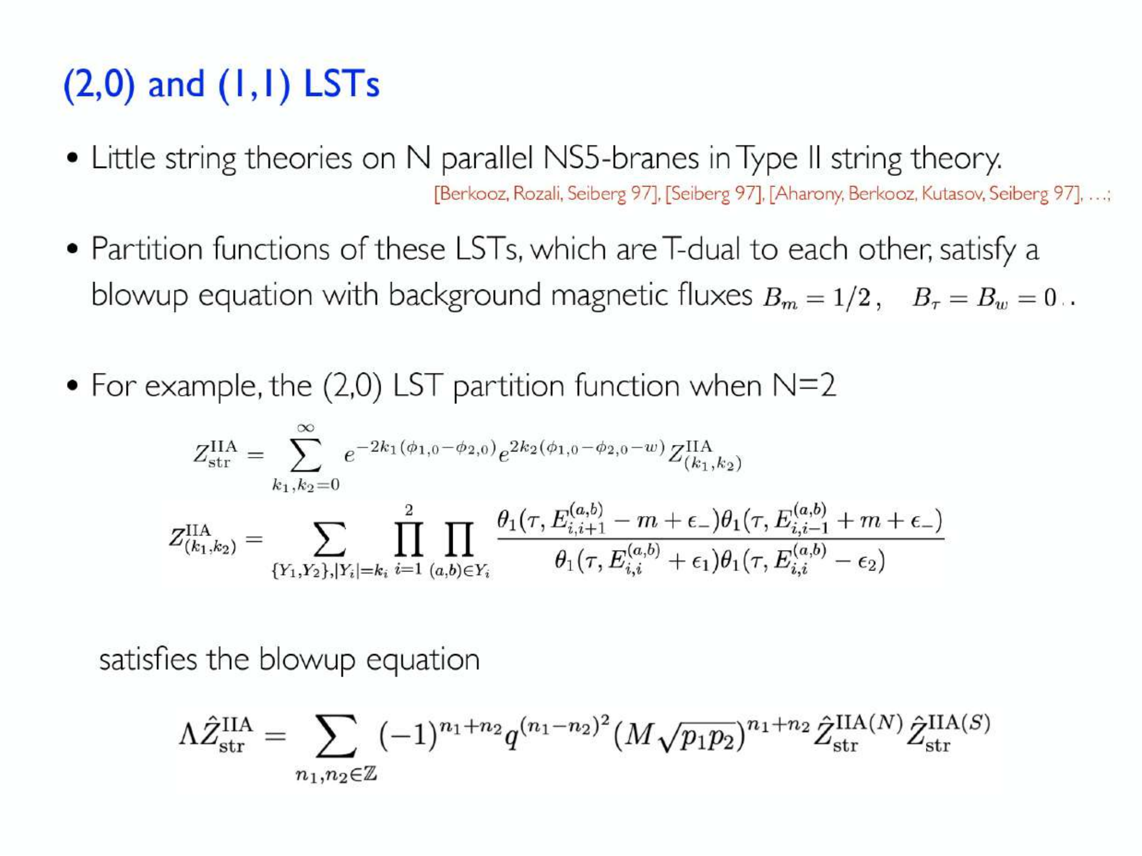
$6d\ (1,1)\ SU(2)$ SYM

	0	1	2	3	4	5	6	7	8	9
2 D5	•	•	•	•	•	•				
k D1	•	•								

(a)

Field	Type	$U(k)$	$U(2)$
$(A_\mu,\lambda_+^{\dot\alpha A})$	vector	**adj**	
$(a_{\alpha\dot\beta},\lambda_-^{\alpha A})$	hyper	**adj**	
$(\varphi_{Aa},\lambda_-^{\dot\alpha a})$	twisted hyper	**adj**	
$(\lambda_+^{\alpha a})$	Fermi	**adj**	
$(q_{\dot\alpha},\psi_-^A)$	hyper	$\bar{\mathbf{k}}$	$\mathbf{2}$
(Ψ_-^a)	Fermi	$\bar{\mathbf{k}}$	$\mathbf{2}$

[Aharony, Berkooz 99], [J. Kim, S. Kim, K. Lee 15]



(2,0) and (1,1) LSTs

- Little string theories on N parallel NS5-branes in Type II string theory.

  [Berkooz, Rozali, Seiberg 97], [Seiberg 97], [Aharony, Berkooz, Kutasov, Seiberg 97], …;

- Partition functions of these LSTs, which are T-dual to each other, satisfy a blowup equation with background magnetic fluxes $B_m = 1/2\,, \quad B_\tau = B_w = 0\,.$

- For example, the (2,0) LST partition function when N=2

$$Z^{\rm IIA}_{\rm str} = \sum_{k_1,k_2=0}^{\infty} e^{-2k_1(\phi_{1,0}-\phi_{2,0})} e^{2k_2(\phi_{1,0}-\phi_{2,0}-w)} Z^{\rm IIA}_{(k_1,k_2)}$$

$$Z^{\rm IIA}_{(k_1,k_2)} = \sum_{\{Y_1,Y_2\},|Y_i|=k_i} \prod_{i=1}^{2} \prod_{(a,b)\in Y_i} \frac{\theta_1(\tau, E^{(a,b)}_{i,i+1} - m + \epsilon_-)\theta_1(\tau, E^{(a,b)}_{i,i-1} + m + \epsilon_-)}{\theta_1(\tau, E^{(a,b)}_{i,i} + \epsilon_1)\theta_1(\tau, E^{(a,b)}_{i,i} - \epsilon_2)}$$

  satisfies the blowup equation

$$\Lambda \hat{Z}^{\rm IIA}_{\rm str} = \sum_{n_1,n_2\in\mathbb{Z}} (-1)^{n_1+n_2} q^{(n_1-n_2)^2} (M\sqrt{p_1 p_2})^{n_1+n_2} \hat{Z}^{{\rm IIA}(N)}_{\rm str} \hat{Z}^{{\rm IIA}(S)}_{\rm str}$$

Solving blowup equations for LSTs

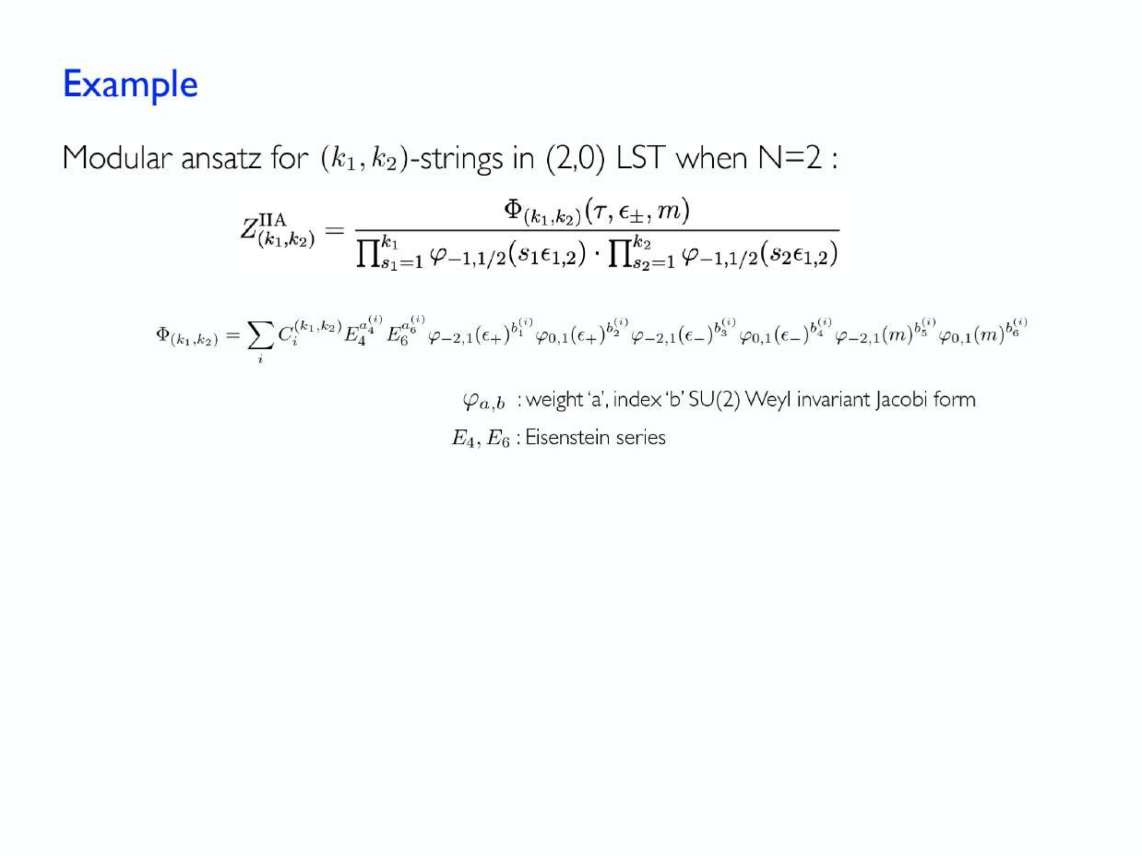
A single blowup equation for LSTs does not seem to be enough to be solved unlike the blowup equations for 5d/6d SQFTs.

Modular ansatz for elliptic genera for strings plugged into the blowup equation allows us to **compute the full partition function of LSTs.**

Proper modular ansatz is determined by the anomaly polynomial I_4 of strings which can be computed by anomaly inflow mechanism!

$$Z_{\vec{k}}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = \epsilon(a,b,c,d)^{c_R-c_L}\exp\left(\frac{2\pi i c}{c\tau+d}f(z)\right)Z_{\vec{k}}(\tau,z)$$

where $f(z) = \int_{\text{eq}} I_4$



Example

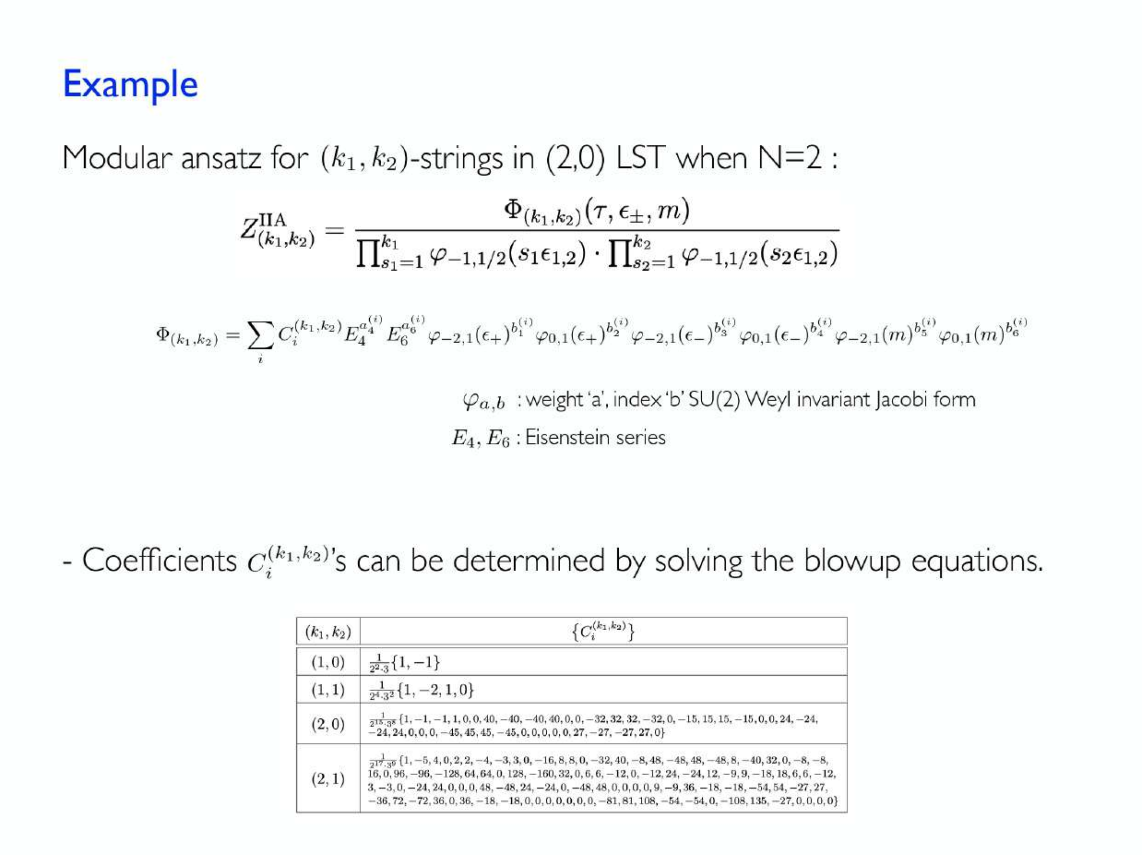
Modular ansatz for (k_1, k_2)-strings in (2,0) LST when N=2 :

$$Z^{\mathrm{IIA}}_{(k_1,k_2)} = \frac{\Phi_{(k_1,k_2)}(\tau, \epsilon_\pm, m)}{\prod_{s_1=1}^{k_1} \varphi_{-1,1/2}(s_1\epsilon_{1,2}) \cdot \prod_{s_2=1}^{k_2} \varphi_{-1,1/2}(s_2\epsilon_{1,2})}$$

$$\Phi_{(k_1,k_2)} = \sum_i C_i^{(k_1,k_2)} E_4^{a_4^{(i)}} E_6^{a_6^{(i)}} \varphi_{-2,1}(\epsilon_+)^{b_1^{(i)}} \varphi_{0,1}(\epsilon_+)^{b_2^{(i)}} \varphi_{-2,1}(\epsilon_-)^{b_3^{(i)}} \varphi_{0,1}(\epsilon_-)^{b_4^{(i)}} \varphi_{-2,1}(m)^{b_5^{(i)}} \varphi_{0,1}(m)^{b_6^{(i)}}$$

$\varphi_{a,b}$: weight 'a', index 'b' SU(2) Weyl invariant Jacobi form

E_4, E_6 : Eisenstein series



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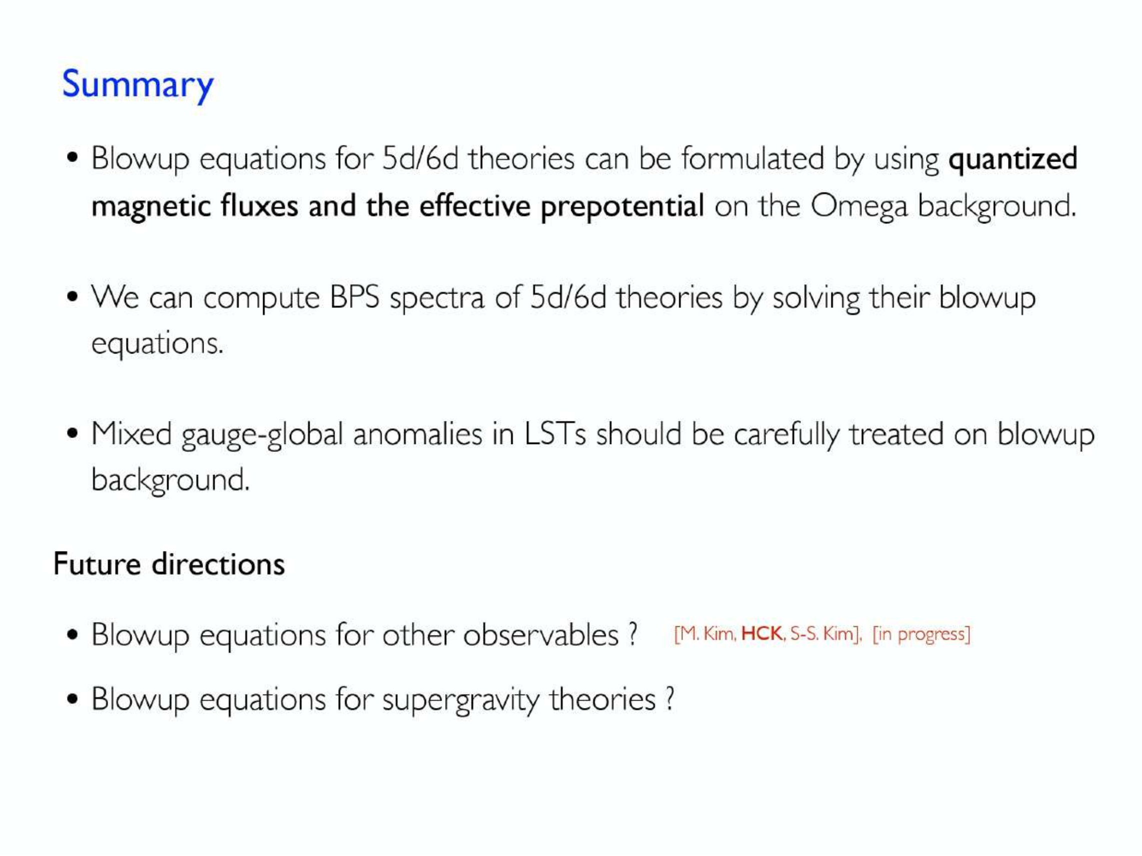
$\varphi_{a,b}$: weight 'a', index 'b' SU(2) Weyl invariant Jacobi form

E_4, E_6 : Eisenstein series

- Coefficients $C_i^{(k_1,k_2)}$'s can be determined by solving the blowup equations.

(k_1, k_2)	$\{C_i^{(k_1,k_2)}\}$
$(1,0)$	$\frac{1}{2^2 \cdot 3}\{1, -1\}$
$(1,1)$	$\frac{1}{2^4 \cdot 3^2}\{1, -2, 1, 0\}$
$(2,0)$	$\frac{1}{2^{15} \cdot 3^8}\{1, -1, -1, 1, 0, 0, 40, -40, -40, 40, 0, 0, -32, 32, 32, -32, 0, -15, 15, 15, -15, 0, 0, 24, -24, -24, 24, 0, 0, 0, -45, 45, 45, -45, 0, 0, 0, 0, 0, 27, -27, -27, 27, 0\}$
$(2,1)$	$\frac{1}{2^{17} \cdot 3^9}\{1, -5, 4, 0, 2, 2, -4, -3, 3, 0, -16, 8, 8, 0, -32, 40, -8, 48, -48, 48, -48, 8, -40, 32, 0, -8, -8, 16, 0, 96, -96, -128, 64, 64, 0, 128, -160, 32, 0, 6, 6, -12, 0, -12, 24, -24, 12, -9, 9, -18, 18, 6, 6, -12, 3, -3, 0, -24, 24, 0, 0, 0, 48, -48, 24, -24, 0, -48, 48, 0, 0, 0, 0, 9, -9, 36, -18, -18, -54, 54, -27, 27, -36, 72, -72, 36, 0, 36, -18, -18, 0, 0, 0, 0, 0, 0, 0, -81, 81, 108, -54, -54, 0, -108, 135, -27, 0, 0, 0, 0\}$

Summary



Summary

- Blowup equations for 5d/6d theories can be formulated by using **quantized magnetic fluxes and the effective prepotential** on the Omega background.
- We can compute BPS spectra of 5d/6d theories by solving their blowup equations.
- Mixed gauge-global anomalies in LSTs should be carefully treated on blowup background.

Future directions

- Blowup equations for other observables ? [M. Kim, **HCK**, S-S. Kim], [in progress]
- Blowup equations for supergravity theories ?

Thank you !

S^1 Reduction of 4D $\mathcal{N} = 3$ SCFTs and Squashing Independence of ABJM theories

Tomoki Nakanishi

Abstract. We study the compactification of 4D $\mathcal{N} = 3$ superconformal field theories (SCFTs) on S^1, focusing on the relation between the 4D superconformal index and 3D partition function on the squashed sphere S_b^3. Since the center $u(1)$ of the $u(3)$ R-symmetry of the 4D theory can mix with an $\mathcal{N} = 6$ abelian flavor symmetry in three dimensions, the precise 4D/3D relation for the global symmetry is not obvious. Focusing on the case in which the 3D theory is the ABJM theory, we demonstrate that the above R-symmetry mixing can be precisely identified by considering the Schur limit (and/or its $\mathcal{N} = 3$ cousin) of the 4D index. As a result, we generalize to the ABJM theories recent discussions on the connection between supersymmetry enhancement of the 4D index and squashing independence of the S_b^3 partition function.

(T. Nakanishi) Osaka Metropolitan University

S^1 Reduction of 4D $\mathcal{N} = 3$ SCFTs and Squashing Independence of ABJM theories

Tomoki Nakanishi

Osaka Metropolitan University

Based on arXiv:2211.07421, w/ T. Nishinaka,
and on a work in progress.

QFT and Related Mathematical Aspects
@ Shuzenji Sogo Kaikan, Shizuoka

Introduction

4D $\mathcal{N}=3$ SCFTs

- We have no known Lagrangian description of 4D $\mathcal{N}=3$.

⇒ 4D genuine $\mathcal{N}=3$ SCFTs are **strongly interacting**.
(Known 4D $\mathcal{N}=3$: S-fold construction).
[Garcia-Etxebarria and Regalado, 1512.06434, 1611.05769]

- 4D genuine $\mathcal{N}=3$ SCFTs have no marginal deformations.
[Aharony-Evtikhiev, 1512.03524]

⇒ 4D $\mathcal{N}=3$ is **isolated** at strong coupling fixed point.

Q: How can we analyze that?

Introduction

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⇒ 4D $\mathcal{N}=3$ is **isolated** at strong coupling fixed point.

Q: How can we analyze that?

Introduction

Many works on 4D $\mathcal{N}=3$ SCFTs :

- Analyzing from field theoretic perspective.
  [Aharony-Evtikhiev, 1512.03524],
  [Nishinaka-Tachikawa, 1602.01503],
  [Aharony-Tachikawa, 1602.08638],
  [Argyres-Bourget-Martone, 1904.10969, 1912.04926], etc.

- Calculating the superconformal index (SCI).
  [Imamura-Yokoyama, 1603.00851],
  [Arai-Fujiwara-Imamura, 1901.00023], [Arai-Imamura, 1904.09776],
  [Agarwal *et al.*, 2103.00985], etc.

I will analyze 4D $\mathcal{N}=3$ SCFTs a bit differently.

Introduction

The key is **dimensional reduction**.

4D $\mathcal{N}=3$ $\xrightarrow{\text{dim. reduction}}$ **3D $\mathcal{N}=6$ (ABJM)**

- $U(3)_R$
- 4D $\mathcal{N}=3$ SCI
- on $S^1_\beta \times S^3_b$

- $SO(6)_R \times U(1)_B$
- Matrix model
- on S^3_b

By using "**squashing independence**",

we found a **non-trivial** relation

between $U(3)_R$ in 4D and $SO(6)_R \times U(1)_B$ in 3D.

Outline

1. 4D $\mathcal{N}=3$ SCI and Squashing indep.
2. Relation to ABJM
3. Application

4D $\mathcal{N} = 3$ Superconformal Index

$R = \frac{1}{2}(\mathcal{R}^1{}_1 - \mathcal{R}^2{}_2)$,

$r = \mathcal{R}^1{}_1 + \mathcal{R}^2{}_2$,

$f = \mathcal{R}^1{}_1 + \mathcal{R}^2{}_2 + 2\mathcal{R}^3{}_3$.

4D $\mathcal{N}=3$ SCI

$$\mathcal{I} := \mathrm{Tr}(-1)^F p^{j_2-j_1-r} q^{j_2+j_1-r} t^{r+R} a^f \ .$$

(j_1, j_2): $\mathfrak{so}(4)$ spins. (R, r, f): R-charges.

- $\mathcal{I}$ preserves one pair $(\mathcal{Q}, \mathcal{Q}^\dagger)(= (\widetilde{\mathcal{Q}}_{2\dot{-}}, \widetilde{\mathcal{Q}}^\dagger_{2\dot{-}}))$.
- We can consider this as 4D $\mathcal{N}=2$ SCI.
- For 4D $\mathcal{N}=2$, f is a flavor charge.

4D $\mathcal{N} = 3$ Superconformal Index

$R = \frac{1}{2}(\mathcal{R}^1{}_1 - \mathcal{R}^2{}_2)$,

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~~$f = \mathcal{R}^1{}_1 + \mathcal{R}^2{}_2 + 2\mathcal{R}^3{}_3$~~ .

4D $\mathcal{N}=2$ SCI

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(j_1, j_2): $\mathfrak{so}(4)$ spins. (R, r): R-charges.

f: $\mathfrak{u}(1)$ flavor charge.

- $\mathcal{I}$ preserves one pair $(\mathcal{Q}, \mathcal{Q}^\dagger)(= (\widetilde{\mathcal{Q}}_{2\dot{-}}, \widetilde{\mathcal{Q}}^\dagger_{2\dot{-}}))$.
- We can consider this as 4D $\mathcal{N}=2$ SCI.
- For 4D $\mathcal{N}=2$, f is a flavor charge.

Schur limit: $t \to q$ ($\mathcal{N}=2$)

The Schur limit is special limits on 4D $\mathcal{N}=2$ index: $t \to q$,

$$\begin{aligned}\mathcal{I} &= \mathrm{Tr}(-1)^F p^{j_2-j_1-r} q^{j_2+j_1-r} t^{r+R} a^f \ , \\ &\longrightarrow \mathrm{Tr}(-1)^F p^{j_2-j_1-r} q^{j_2+j_1+R} a^f = \mathrm{Tr}(-1)^F q^{j_2+j_1+R} a^f .\end{aligned}$$

$$(\ j_2 - j_1 - r = \{\mathcal{Q}', \mathcal{Q}'^\dagger\} - \{\mathcal{Q}, \mathcal{Q}^\dagger\} \ , \ [\mathcal{Q}', q^{j_2+j_1+R}] = [\mathcal{Q}', a^f] = 0 \ .)$$

SUSY enhancement does occur: the index becomes

- preserving one pair $(\mathcal{Q}', \mathcal{Q}'^\dagger)$ additionally and
- **p-independent**.

Analogous independence occurs when we take $t \to pq$.

$t \to pq$ ($\mathcal{N}=2$)

Taking $t \to pq$,

$$\begin{aligned}\mathcal{I} &= \mathrm{Tr}(-1)^F p^{j_2-j_1-r} q^{j_2+j_1-r} t^{r+R} a^f \ , \\ &\longrightarrow \mathrm{Tr}(-1)^F (pq)^{j_2+R} \left(\frac{q}{p}\right)^{j_1} a^f = \mathrm{Tr}(-1)^F \left(\frac{q}{p}\right)^{j_1} a^f .\end{aligned}$$

$$(\ j_2 + R = \frac{1}{2}\{\mathcal{Q}'', \mathcal{Q}''^\dagger\} - \frac{1}{2}\{\mathcal{Q}, \mathcal{Q}^\dagger\} \ , \ [\mathcal{Q}'', j_1] = [\mathcal{Q}'', f] = 0 \ .)$$

SUSY enhancement does occur: the index becomes

- preserving one pair $(\mathcal{Q}'', \mathcal{Q}''^\dagger)$ additionally and
- **pq-independent**.

Squashing independence

These limits can cause Squashing independence.
To explain this, we use the following parametrization:

$$p = e^{-\beta b}\ , \quad q = e^{-\beta b^{-1}}\ , \quad \frac{t}{\sqrt{pq}} = e^{-i\beta m}\ , \quad a = e^{-i\beta M}\ ,$$

β : radius of S^1 , b : squashing parameter of S^3_b ,
(m, M) : mass paramters .

$$\left(S^3_b : \omega_1^2|z_1|^2 + \omega_2^2|z_2|^2 = 1, b^2 := \tfrac{\omega_1}{\omega_2}\right)$$

Squashing independence

1. Taking the special limits causing an independence,
2. and redefining parameters appropriately,

$\Rightarrow$ **b is removed from SCI.**

Squashing independence (Check)

$p = e^{-\beta b}$, $q = e^{-\beta b^{-1}}$,
$\frac{t}{\sqrt{pq}} = e^{-i\beta m}$, $a = e^{-i\beta M}$.

1. Schur limit $t \to q \quad \Rightarrow \quad m \to i\frac{b-b^{-1}}{2}$.

$$\mathrm{Tr}(-1)^F q^{j_2+j_1+R} a^f\ .$$

Def: $(\beta', M') \equiv (\beta b^{-1}, bM)$
$\Rightarrow (q, a) = (e^{-\beta b^{-1}}, e^{-i\beta M}) \to (e^{-\beta'}, e^{-i\beta' M'})$. $\quad (b^{-1} \equiv 1)$

2. $t \to pq \quad \Rightarrow \quad m \to -i\frac{b+b^{-1}}{2}$.

$$\mathrm{Tr}(-1)^F \left(\frac{q}{p}\right)^{j_1} a^f\ , \quad \left(\frac{q}{p} = e^{-\beta(b^{-1}-b)}\right)\ .$$

Def: $(\beta', M') \equiv (\beta(b^{-1} - b), \frac{1}{b^{-1}-b} \cdot M)$
$\Rightarrow (q/p, a) = (e^{-\beta(b^{-1}-b)}, e^{-i\beta M}) \to (e^{-\beta'}, e^{-i\beta' M'})$. $(b^{-1} - b \equiv 1)$

One finds these turn out to be on $S^1_{\beta'} \times S^3_{b=1}$ and $S^1_{\beta'} \times S^3_{b=\frac{-1\pm\sqrt{5}}{2}}$.

$a \to \frac{\sqrt{t}}{p}$ ($\mathcal{N}=3$)

$R = \frac{1}{2}(\mathcal{R}^1{}_1 - \mathcal{R}^2{}_2)$,
$r = \mathcal{R}^1{}_1 + \mathcal{R}^2{}_2$,
$f = \mathcal{R}^1{}_1 + \mathcal{R}^2{}_2 + 2\mathcal{R}^3{}_3$.

In 4D $\mathcal{N}=3$, we get a new limit $a \to \frac{\sqrt{t}}{p}$:

$$\mathcal{I} = \mathrm{Tr}(-1)^F p^{j_2-j_1-r} q^{j_2+j_1-r} t^{r+R} a^f \ ,$$

$$\longrightarrow \mathrm{Tr}(-1)^F (pt)^{\delta_1} q^{j_2+j_1-r} \left(\frac{p}{t}\right)^{\delta_2} = \mathrm{Tr}(-1)^F q^{j_2+j_1-r} \left(\frac{p}{t}\right)^{\delta_2} .$$

$$\delta_1 = \frac{1}{2}\{\mathcal{Q}, \mathcal{Q}^\dagger\} - \frac{1}{2}\{\mathcal{Q}''', \mathcal{Q}'''^\dagger\} \ , \ \delta_2 = \frac{1}{2}(j_2 - j_1 - R - 2r - \frac{3}{2}f) \ ,$$

$$[\mathcal{Q}''', j_2 + j_1 - r] = [\mathcal{Q}''', \delta_2] = 0 \ .$$

SUSY enhancement does occur: the index becomes

- preserving one pair ($\mathcal{Q}'''$, $\mathcal{Q}'''^\dagger$) additionally and
- **pt-independent**.

Squashing independence (Check)

$p = e^{-\beta b}$, $q = e^{-\beta b^{-1}}$,
$\frac{t}{\sqrt{pq}} = e^{-i\beta m}$, $a = e^{-i\beta M}$.

$$a \to \frac{\sqrt{t}}{p} \quad \Rightarrow \quad M \to \frac{m}{2} + i\frac{3b - b^{-1}}{4} \ .$$

$$\mathrm{Tr}(-1)^F q^{j_2+j_1-r} \left(\frac{p}{t}\right)^{\delta_2} \ , \quad \left(\frac{p}{t} = e^{i\beta\left(m + i\frac{b-b^{-1}}{2}\right)}\right) \ .$$

Redefining $(\beta', m') \equiv \left(\beta b^{-1}, b(m + i\frac{b-b^{-1}}{2})\right)$

$\Rightarrow \left(q, \frac{p}{t}\right) = (e^{-\beta b^{-1}}, e^{i\beta(m+i\frac{b-b^{-1}}{2})}) \to (e^{-\beta'}, e^{i\beta' m'})$.

($b^{-1} \equiv 1$ and $b - b^{-1} \equiv 0 \to b \equiv 1$)

We have reached that the 4D $\mathcal{N}=3$ SCI must have
three types of Squashing independece associated with

$t \to q$, **$t \to pq$** and **$a \to \frac{\sqrt{t}}{p}$**.

Outline

1. 4D $\mathcal{N} = 3$ SCI and Squashing indep.
2. Relation to ABJM
3. Application

Relation to ABJM

- To observe the squashing independencies from another view, let's consider S^1-reduction of the 4D $\mathcal{N}=3$ SCI.
- As the reduced objects we will consider the ABJM matrix model.
- Recently, the squashing independence of mass-deformed ABJM matrix model was analyzed by [Chester-Kalloor-Sharon, 2102.05643],
- and interpreted as the SUSY enhancement on 4D index by [Minahan-Naseer-Thull, 2107.07151].
- However, naive dim. reduction of the 4D $\mathcal{N}=3$ contradicts these results.
- We will resolve this by carefully observing the relation between 4D and 3D.

Relation to ABJM

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- We will resolve this by carefully observing the relation between 4D and 3D.

Mass-deformed ABJM matrix model on S_b^3

ABJM: 3D $\mathcal{N}=6$ $U(N)_k \times U(N)_{-k}$ CS w/ four bi-fund matters.

$$
\begin{aligned}
&Z^{\mathrm{ABJM}}_{S_b^3,N,k}(m_1,m_2,m_3)\\
&=\int \frac{d^N\mu\ d^N\nu}{(N!)^2} e^{-i\pi k(\sum_i \mu_i^2-\sum_j \nu_j^2)}\\
&\times\prod_{i<j} 2\sinh(\pi b(\mu_i-\mu_j))2\sinh(\pi b^{-1}(\mu_i-\mu_j))2\sinh(\pi b(\nu_i-\nu_j))2\sinh(\pi b^{-1}(\nu_i-\nu_j))\\
&\times\prod_{i,j}\Bigg[s_b\left(-\mu_i+\nu_j+i\frac{Q}{4}-\frac{m_1+m_2+m_3}{2}\right)\ s_b\left(-\mu_i+\nu_j+i\frac{Q}{4}-\frac{m_1-m_2-m_3}{2}\right)\\
&\qquad s_b\left(\mu_i-\nu_j+i\frac{Q}{4}-\frac{-m_1-m_2+m_3}{2}\right)\ s_b\left(\mu_i-\nu_j+i\frac{Q}{4}-\frac{-m_1+m_2-m_3}{2}\right)\Bigg]\ .
\end{aligned}
$$

$s_b(x)$: double sine function, $\quad Q := b + b^{-1}$.

(m_2, m_3): masses associated with $\mathfrak{so}(2)^2 \subset \mathfrak{so}(4) \subset \mathfrak{so}(6)_R$,

m_1: mass associated with $\mathfrak{u}(1)_B$.

$\mathfrak{u}(1)_B$ is an accidental Baryon number symmetry.

(topological/monopole $U(1)$ in recent terms.)

Contradiction & Resolution

4D $\mathcal{N}=3$

- $U(3)_R$
- 4D $\mathcal{N}=3$ SCI
- on $S^1_\beta \times S^3_b$
- (m, M)

$\xrightarrow{\text{dim. reduction}}$

3D $\mathcal{N}=6$ (ABJM)

- $SO(6)_R \times U(1)_B$
- Matrix model
- on S^3_b
- (m_1, m_2, m_3)

In 4D $\mathcal{N}=3$ there are **two** masses while **three** in ABJM.

This difference contradicts naive dimensional reduction.

Note that the center $\mathfrak{u}(1) \subset \mathfrak{u}(3)_R$ can be **mixed** with $\mathfrak{u}(1)_\mathrm{B}$.

To resolve the contradiction, we can identify the mixing.

R-symmetry mixing

We can identify 4D $(\mathcal{Q}_\alpha, \widetilde{\mathcal{Q}}_{I\dot\alpha})$ and 3D (Q_α) supercharges:

$$\mathcal{Q}^I_{4D} \propto Q^{2I-1}_{3D} + Q^{2I}_{3D}\,, \quad \widetilde{\mathcal{Q}}_{4D\,I} \propto Q^{2I-1}_{3D} - Q^{2I}_{3D}\,, \quad (I = 1, 2, 3)\,.$$

These allow us to equate three Cartans in $U(3)_R$ with the three in $SO(6)_R$ directly, up to R-symmetry mixing with $u(1)_B$:

$$\boxed{R^{\text{4D}}_I = R^{\text{3D}}_I + \xi J^{\text{3D}}_{U(1)_B}\,, \quad \xi\text{: mixing parameter}}$$

$R^{\text{4D/3D}}_I$: three Cartans in 4D/3D , $J^{\text{3D}}_{U(1)_B}$: charge of $U(1)_B$,

where $R^{\text{4D/3D}}_I$ are defined by

$$[R^{\text{4D/3D}}_I, \mathcal{Q}^I_\alpha] = +\mathcal{Q}^I_\alpha\,, \quad [R^{\text{4D/3D}}_I, \widetilde{\mathcal{Q}}_{I\dot\alpha}] = -\widetilde{\mathcal{Q}}_{I\dot\alpha}\,, \quad [R^{\text{4D/3D}}_{J(\neq I)}, \mathcal{Q}^I_\alpha\,(\widetilde{\mathcal{Q}}_{I\dot\alpha})] = 0\,.$$

Mass identification

$R = \frac{1}{2}(\mathcal{R}^1{}_1 - \mathcal{R}^2{}_2)$,
$r = \mathcal{R}^1{}_1 + \mathcal{R}^2{}_2$,
$f = \mathcal{R}^1{}_1 + \mathcal{R}^2{}_2 + 2\mathcal{R}^3{}_3$.

One can find

$$R_1^{\text{4D}} = R + r + \frac{f}{2} \ , \quad R_2^{\text{4D}} = -R + r + \frac{f}{2} \ , \quad R_3^{\text{4D}} = f \ .$$

Then the 4D $\mathcal{N}=3$ SCI is written by

$$\mathcal{I} = \mathrm{Tr}(-1)^F p^{j_2-j_1} q^{j_2+j_1} \left(\frac{t}{\sqrt{pq}}\right)^{R_1^{\text{4D}}} \left(\frac{1}{\sqrt{pq}}\right)^{R_2^{\text{4D}}} \left(a\sqrt{\frac{pq}{t}}\right)^{R_3^{\text{4D}}} \ .$$

$$\Downarrow \ R_I^{\text{4D}} = R_I^{\text{3D}} + \xi J^{\text{3D}}_{U(1)_B}$$

$$\mathrm{Tr}(-1)^F p^{j_2-j_1} q^{j_2+j_1} \left(\frac{t}{\sqrt{pq}}\right)^{R_1^{\text{3D}}} \left(\frac{1}{\sqrt{pq}}\right)^{R_2^{\text{3D}}} \left(a\sqrt{\frac{pq}{t}}\right)^{R_3^{\text{3D}}} \left(a\sqrt{\frac{t}{pq}}\right)^{\xi J^{\text{3D}}_{U(1)_B}} \ .$$

We can read fugacities associated with the 3D masses.

Mass identification

$p = e^{-\beta b}$, $q = e^{-\beta b^{-1}}$,
$\frac{t}{\sqrt{pq}} = e^{-i\beta m}$, $a = e^{-i\beta M}$.

$$\mathrm{Tr}(-1)^F p^{j_2-j_1} q^{j_2+j_1} \left(\frac{t}{\sqrt{pq}}\right)^{R_1^{\text{3D}}} \left(\frac{1}{\sqrt{pq}}\right)^{R_2^{\text{3D}}} \left(a\sqrt{\frac{pq}{t}}\right)^{R_3^{\text{3D}}} \left(a\sqrt{\frac{t}{pq}}\right)^{\xi J^{\text{3D}}_{U(1)_B}} \ .$$

- Since this index preserves $\mathcal{Q} = \widetilde{\mathcal{Q}}_{2\dot{-}}$, $R_2^{\text{4D/3D}}$ can not be associated with mass parameters.
- Remains are associated with $\mathfrak{so}(4) \subset \mathfrak{so}(6)_R$ and $\mathfrak{u}(1)_B$:

$$\frac{t}{\sqrt{pq}} = e^{-i\beta m_2} \ , \quad a\sqrt{\frac{pq}{t}} = e^{-i\beta m_3} \ , \quad \left(a\sqrt{\frac{t}{pq}}\right)^{\xi} = e^{-i\beta m_1} \ .$$

We find an identification between (m_1, m_2, m_3) and (m, M),

$$\boxed{m_1 = \xi\left(M + \frac{m}{2} + i\frac{b+b^{-1}}{4}\right) \ , \quad m_2 = m \ , \quad m_3 = M - \frac{m}{2} - i\frac{b+b^{-1}}{4} \ .}$$

Mass identification

$$m_1 = \xi\left(M + \tfrac{m}{2} + i\tfrac{b+b^{-1}}{4}\right)\,,\ m_2 = m\,,$$
$$m_3 = M - \tfrac{m}{2} - i\tfrac{b+b^{-1}}{4}\,.$$

Next, let's fix ξ by using squashing independence.
Taking the Schur limit $m \to i\frac{b-b^{-1}}{2}$, (m_1, m_2, m_3) become

$$m_1 = \xi\left(M + i\frac{b}{2}\right)\,,\quad m_2 = i\frac{b-b^{-1}}{2}\,,\quad m_3 = M - i\frac{b}{2}\,.$$

Redefining $(\beta', M') \equiv (\beta b^{-1}, bM)$ removes b.

In [Chester *et al.*, '21], they showed the corresponding ABJM matrix model depends only on $b^{\mp}(m_3 \pm m_1)$, which can be now written as

$$b\left((\xi-1)M + (\xi+1)i\frac{b}{2}\right)\,,\quad b^{-1}\left((\xi+1)M + (\xi-1)i\frac{b}{2}\right)\,.$$

To reproduce b indep. by the redefinition, we get $\xi \equiv -1$.

Short Summary

We considered the S^1 reduction of 4D $\mathcal{N}=3$ SCI to 3D $\mathcal{N}=6$, ABJM matrix model.
We found R-symmetry mixing $\xi \equiv -1$:

$$R_I^{\rm 4D} = R_I^{\rm 3D} - J^{\rm 3D}_{U(1)_B} \quad (I = 1, 2, 3)\,,$$

and mass identification between 4D and 3D:

$$\begin{aligned} m_1 &= -M - \frac{m}{2} - i\frac{b+b^{-1}}{4}\,,\\ m_2 &= m\,,\\ m_3 &= M - \frac{m}{2} - i\frac{b+b^{-1}}{4}\,. \end{aligned}$$

Outline

1. 4D $\mathcal{N}=3$ SCI and Squashing indep.
2. Relation to ABJM
3. Application

S^1-reduction

- I'd like to demonstrate the S^1-reduction from 4D $\mathcal{N}=3$ SCI to ABJM matrix model.
- Unfortunately, no one knows the full expression of 4D $\mathcal{N}=3$ SCI.
- But, ABJM with $k=1,2$ have 3D $\mathcal{N}=8$, and its 4D uplifting should have $\mathcal{N}=4$, which is correct from S-folds.
- We can test the S^1-reduction of 4D $\mathcal{N}=4$ SCI.

S^1-reduction

Let's consider the Schur index, which is automatically on $S^3_{b=1}$.

The 4D $\mathcal{N}=4$ ($\mathcal{N}=2$*) Schur index

$$I^G_{\text{Schur}} := \mathrm{Tr}(-1)^F q^{j_2+j_1+R} a^f$$

can be given in a following form.

$$I^G_{\text{Schur}} = \frac{u^{\frac{d_G}{2}}}{|W(G)|}\left(\frac{(q;q)^3_\infty}{\theta(u;q)}\right)^{2r_G-d_G} \oint_{|z_i|=1} \prod_{i=1}^{r_G}\left(\frac{dz_i}{2\pi i z_i}\right) \prod_{\alpha\in\Delta_G} F(u, z^\alpha u^{-1}; q) \ ,$$

$$u := \frac{a}{\sqrt{q}} \ , \quad r_G, d_G : \text{rank and dim. of } G \ , \quad \Delta_G : \text{set of roots} \ ,$$

$$\theta(x;q) := -x^{-\frac{1}{2}}(q;q)_\infty(x;q)_\infty(x^{-1}q;q)_\infty \ , F(x,y;q) := \frac{\theta(xy;q)(q;q)^3_\infty}{\theta(x;q)\theta(y;q)} \ .$$

Next, we take S^1 reduction of this index with $G = U(1), U(2)$.

$G = U(1)$

$$q = e^{-\beta b^{-1}} \ , \ a = e^{-i\beta M} \ , \quad u = \frac{a}{\sqrt{q}} \ .$$

When $G = U(1)$, we have

$$\begin{aligned} I^{U(1)}_{\text{Schur}} &= u^{\frac{1}{2}}\frac{(q;q)^3_\infty}{\theta(u;q)}\oint\frac{dz}{2\pi i z}\cdot 1 = u^{\frac{1}{2}}\frac{(q;q)^3_\infty}{\theta(u;q)} \\ &= \frac{u^{\frac{1}{2}}}{u^{\frac{1}{2}}-u^{-\frac{1}{2}}}\prod_{n=1}^\infty\frac{(1-q^n)^2}{(1-uq^n)(1-u^{-1}q^n)} \ . \end{aligned}$$

Taking $\beta \to 0$ and picking up a leading contribution, we obtain

$$I^{U(1)}_{\text{Schur}}\Big|_{\beta\to 0} = \frac{\pi}{\beta\cosh\pi M} + \mathcal{O}(1) \ .$$

$G = U(2)$

When $G = U(2)$, we have

$$I^{U(2)}_{\mathrm{Schur}} = \frac{u^2}{2} \oint \frac{dz_1 dz_2}{(2\pi i)^2 z_1 z_2} F(u, (z_1/z_2)u^{-1}; q)\, F(u, (z_1/z_2)^{-1}u^{-1}; q) \ .$$

To integrate, we can use an expansion form of $F(x, y; q)$:

$$F(x, y; q) = \sum_{m\in\mathbb{Z}} \frac{y^{-m}}{1 - x^{-1}q^m} \ , \quad \text{for } |q| < |y| < 1 < |x| \ .$$

Taking $\beta \to 0$ and picking up a leading contribution, we obtain

$$I^{U(2)}_{\mathrm{Schur}}\Big|_{\beta\to 0} = \frac{1}{2!}\left(\frac{\pi}{\beta \cosh \pi M}\right)^2 + \mathcal{O}(1) \ .$$

Generalization

Generally, we can calculate the S^1 reduction for $G = U(N)$:

$$I^{U(N)}_{\mathrm{Schur}}\Big|_{\beta\to 0} = \frac{1}{N!}\left(\frac{\pi}{\beta \cosh \pi M}\right)^N + \mathcal{O}(1) \ .$$

According to our analysis, this must be reproduced by substituting

$$m_1 = -M - i\frac{b}{2} \ , \quad m_2 = i\frac{b - b^{-1}}{2} \ , \quad m_3 = M - i\frac{b}{2} \ ,$$

in $Z^{\mathrm{ABJM}}_{S^3,N,k}(m_1, m_2, m_3)$. This is indeed correct for $k = 1$ and 2.

$$Z^{\mathrm{ABJM}}_{S^3}(M) = \frac{1}{k^N N!}\left(\frac{1}{2\sinh \pi 0}\right)^N \left(\frac{1}{2\cosh \pi M}\right)^N \ .$$

One can expect this is also correct for 4D genuine $\mathcal{N}\!=3$ cases corresponding to $k = 3, 4, 6$. (This is constructed by S-folds.)

More generalization

We found the exact method for S^1 reduction of 4D $\mathcal{N}=4$ SCI.

The S^1 reduction of I^G_{Schur} for $\mathfrak{g} = ABCDEFG$ become

$$I^G_{\text{Schur}}\Big|_{\beta\to 0} = \frac{1}{|W(G)|}\left(\frac{\pi}{\beta\cosh\pi M}\right)^{r_G} + \mathcal{O}(1)\ .$$

In general, the S^1 reductions of I^G_{Schur} have this type of divergent structure.

Summary

Between the 4D $\mathcal{N}=3$ and the 3D $\mathcal{N}=6$ (ABJM),

- We identified the R-symmetry mixing with $U(1)_B$

$$R^{4\text{D}}_I = R^{3\text{D}}_I - J^{3\text{D}}_{U(1)_B} \quad (I=1,2,3)\ ,$$

- and mass parameters

$$m_1 = -M - \frac{m}{2} - i\frac{b+b^{-1}}{4}\ , \quad m_2 = m\ , \quad m_3 = M - \frac{m}{2} - i\frac{b+b^{-1}}{4}\ .$$

Observing the S^1 reduction of 4D $\mathcal{N}=4$ SCI,

- We checked the above result for $k=1,2$ ($\mathcal{N}=4$) and
- expected the form of divergent structure for $k=3,4,6$ ($\mathcal{N}=3$) and for $\mathfrak{g} = ABCDEFG$ ($\mathcal{N}=4$).

$$I^G_{\text{Schur}}\Big|_{\beta\to 0} = \frac{1}{|W(G)|}\left(\frac{\pi}{\beta\cosh\pi M}\right)^{r_G} + \mathcal{O}(1)\ .$$

Probing Anomalies of Non-Invertible Symmetries with Symmetry TFTs

Emily Nardoni

Abstract. 't Hooft anomalies provide crucial insight into the properties of quantum field theories, imposing powerful constraints on their low energy dynamics. For invertible global symmetries, it is known that the 't Hooft anomalies can be characterized by an invertible TQFT in one higher dimension. However, the analogous statement remains to be understood for non-invertible symmetries. In this talk we will discuss how the linking invariants in a non-invertible TQFT known as the Symmetry TFT can be used as a diagnostic for the anomalies of non-invertible symmetries. We will illustrate this proposal through examples in two and four dimensions, including 4d adjoint QCD, and comment on how knowledge of these anomalies can impose constraints on the dynamics.

(E. Nardoni) IPMU, University of Tokyo

Probing Anomalies of Non-Invertible Symmetries with Symmetry TFTs

Emily Nardoni

based on [2301.07112]
with Justin Kaidi, Gabi Zafrir, Yunqin Zheng

KAVLI IPMU

QFT and Related Mathematical Aspects | Shuzenji | 3/13/2023

(Generalized) Global Symmetries = Topological Defect Operators

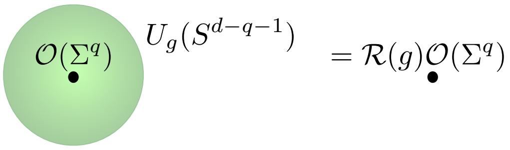
- *Example:* continuous q-form $G^{(q)}$, with conserved $d\star j=0$.

$$U_g(\Sigma^{d-q-1}) = \exp\left[i\theta \oint_{\Sigma^{d-q-1}} \star j\right] \qquad g(\theta)\in G \qquad \Sigma^{d-q-1}\subset M_d$$

= operator that implements the symmetry,
depends topologically on Σ^{d-q-1}

- Also exists in general! For finite symmetries,…

- The symmetry defects $U_g(\Sigma^{d-q-1})$ act on local operators $\mathcal{O}(\Sigma^q)$ charged under $G^{(q)}$:



[Gaiotto, Kapustin, Seiberg, Willett '14]

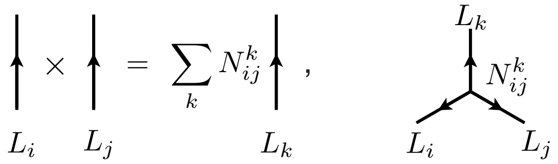
Non-invertible (categorical) symmetries

- For group-like symmetries the symmetry operators satisfy the group multiplication law ↔ fusion of defects.

$$U_g U_{g'} = U_{g\cdot g'} \qquad U_g\, U_{g^{-1}} = \mathbb{1}$$

(*e.g.* 2d 0-form)

- *Generalization*: relax invertibility, and consider defects governed by a (higher) fusion category, rather than group.



e.g. in 2*d*, [Frölich, Fuchs, Runkel, Schweigert '09][Bhardwaj, Tachikawa '17][Chang, Lin, Shan, Wang, Yin '18][Komargodski, Ohmori, Roumpedakis, Seifnashri '20]…; in *d*>2, [Nguyen, Tanizaki, Unsal '21][Koide, Nagoya, Yamaguchi '21][Choi, Cordova, Hsin, Lam, Shao '21][Kaidi, Ohmori, Zheng '21][Bhardwaj, Bottini, Schafer-Nameki, Tiwari '22]…

Goal today: Discuss a set of easy-to-compute observables that constitute a sufficient condition for a 't Hooft anomaly of a non-invertible symmetry in *d*-dimensions.

I) Anomalies for invertible symmetries

II) Anomalies for non-invertible symmetries, and the Symmetry TFT

III) Examples in 4d gauge theories

I. ’t Hooft anomalies for invertible symmetries

an inflow perspective

’t Hooft anomalies are obstructions to gauging

- Let $\mathcal{T}[M_d]$ a QFT with an (invertible) global symmetry *G*.
- There is a ’t Hooft anomaly if $\mathcal{T}$ cannot be coupled to a background gauge field A ($\mathcal{L} \sim A \wedge j$ for continuous *G)* in a *G*-invariant way:

$$Z_{\mathcal{T}}[A] \quad \overset{g\in G}{\longrightarrow} \quad Z_{\mathcal{T}}[A] e^{2\pi i \int_{M_d} \alpha(A,g)}$$

- The anomaly is an obstruction to gauging *G* (promoting *A* → dynamical).

$$\underset{\substack{\uparrow \\ \text{quantum } \widehat{G}^{(d-2-p)}}}{Z_{\mathcal{T}/G}[B]} = \sum_a Z_{\mathcal{T}}[a]\, e^{2\pi i \int_{M_d} a \cup B}$$

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$$Z_{\mathcal{T}/G}[B] = \sum_a Z_{\mathcal{T}}[a]\, e^{2\pi i \int_{M_d} a \cup B}$$

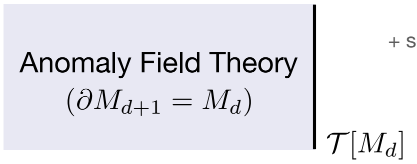
$\uparrow$

quantum $\widehat{G}^{(d-2-p)}$

- Anomaly matching provides powerful constraints on RG flows, since anomalies are preserved under G-preserving continuous deformations.
  - *e.g.* a QFT with a nontrivial anomaly cannot flow to a unique, trivially gapped vacuum.

Anomaly inflow: a modern perspective

- The anomaly is naturally described by inflow from $(d+1)$-d, whereby the anomalous phase is canceled by an invertible TQFT.

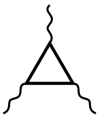

Anomaly Field Theory
$(\partial M_{d+1} = M_d)$
$\mathcal{T}[M_d]$

[Callan, Harvey ’85]
+ see [Freed ’14, Monnier ‘19]

$$Z_{\text{Anom}}[A] = e^{2\pi i \int_{M_{d+1}} \omega(A)} \quad \text{such that} \quad Z_{\mathcal{T}}[A] \cdot Z_{\text{Anom}}[A] \text{ is invariant}$$

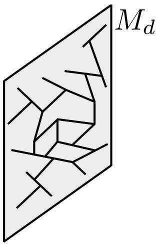
- *Special case:* chiral fermion anomalies in even-*d* are encoded in the anomaly polynomial $I_{d+2}(A) = d\omega_{d+1}(A)$; but inflow holds in general!

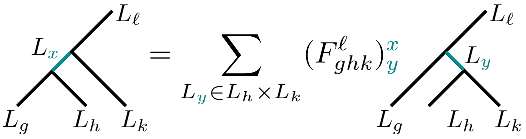
e.g. for *U*(1) in 4*d*, $\omega_5(A) \sim A \wedge dA \wedge dA$



’t Hooft anomalies for finite *G*

- Turning on *A* ↔ inserting a network of *G*-defects
- Gauging ↔ fixing a triangulation and summing over configurations
- Anomaly ↔ phase from change of topology of the junctions





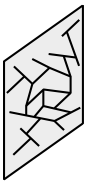
capture gauge transformations
(change of basis of network)

see [Kapustin, Thorngren ’14,’14][Gaiotto, Kapustin, Seiberg, Willett ’14][Tachikawa ’17]

Extension to non-invertible symmetries

Suppose $\mathcal{T}[M_d]$ has a (finite) non-invertible symmetry associated to a fusion category $\mathcal{C}$.

It can still have a ’t Hooft anomaly (obstruction to gauging), where gauging = fixing a triangulation of M_d and summing over defect configurations. But questions remain!



- What is the background field that couples to $\mathcal{C}$?
- Is there a picture of anomaly inflow by coupling to a bulk TQFT?
- How do I characterize the anomaly?
  In 2d, equivalent to lack of a fiber functor [Thorngren, Wang ’19,’21] ;
  also see [Décoppet, Yu ’22] for the higher-categorical perspective.

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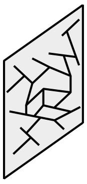

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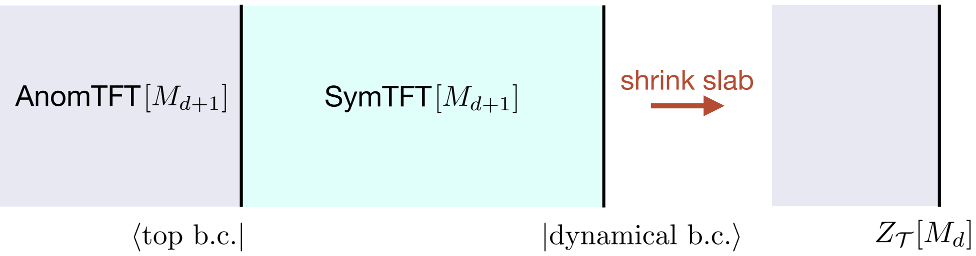
Our main result: a sufficient diagnostic for the existence of a ’t Hooft anomaly, from computing correlation functions in the Symmetry TFT.

II. ’t Hooft anomalies for non-invertible symmetries

from the Symmetry TFT

Symmetry TFT

The Symmetry TFT is a TQFT which yields $\mathcal{T}[M_d]$ when compactified on an interval with appropriate boundary conditions.



AnomTFT$[M_{d+1}]$

SymTFT$[M_{d+1}]$

shrink slab

$\langle\text{top b.c.}|$

$|\text{dynamical b.c.}\rangle$

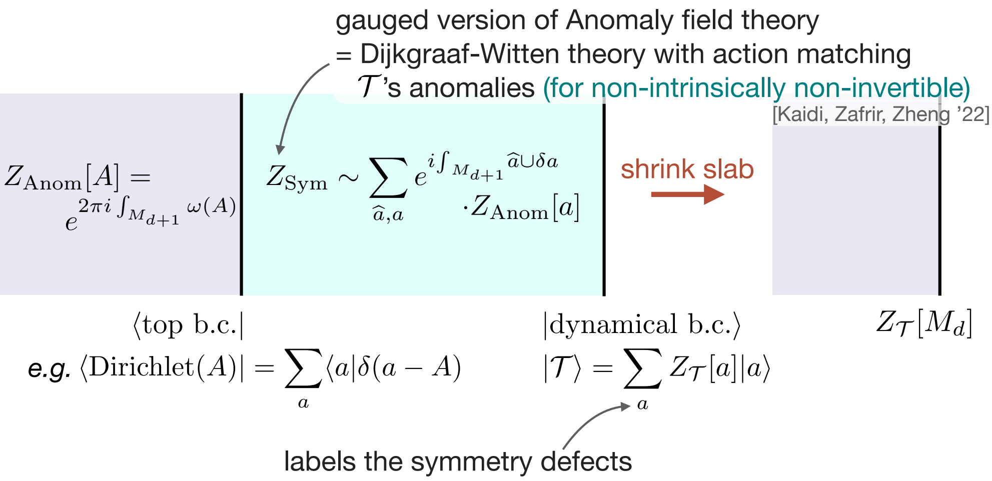
$Z_{\mathcal{T}}[M_d]$

- SymTFT + top b.c. capture the full symmetry structure $\mathcal{C}$ of $\mathcal{T}[M_d]$.
- SymTFT is invariant under topological manipulations of $\mathcal{T}[M_d]$.

[Kong, Wen, Zheng '15][Freed, Teleman '18][Gaiotto, Kulp '20][Burbano, Kulp, Neuser '21]
[Apruzzi, Bonetti, Garcia-Etxebarria, Hosseini, Schafer-Nameki '21][Freed, Moore, Teleman '22]...

*Picture drawn with AnomTFT for invertible *G*, but the SymTFT can be defined for general categorical $\mathcal{C}$. [Kaidi, Ohmori, Zheng '22]

Symmetry TFT, in detail



gauged version of Anomaly field theory
= Dijkgraaf-Witten theory with action matching
$\mathcal{T}$'s anomalies (for non-intrinsically non-invertible)

[Kaidi, Zafrir, Zheng '22]

$Z_{\text{Anom}}[A] = e^{2\pi i \int_{M_{d+1}} \omega(A)}$

$Z_{\text{Sym}} \sim \sum_{\widehat{a},a} e^{i\int_{M_{d+1}} \widehat{a}\cup\delta a} \cdot Z_{\text{Anom}}[a]$

shrink slab

$\langle\text{top b.c.}|$

e.g. $\langle\text{Dirichlet}(A)| = \sum_a \langle a|\delta(a-A)$

$|\text{dynamical b.c.}\rangle$

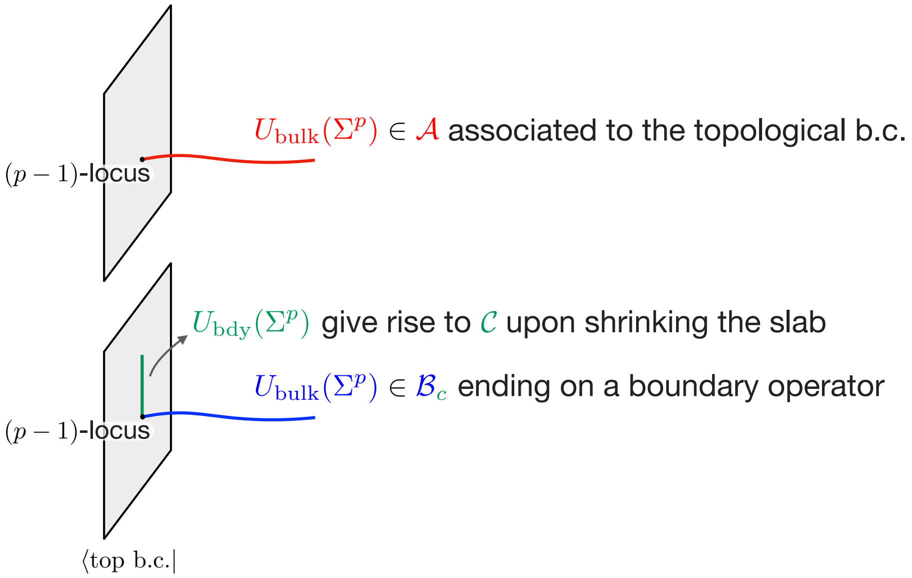
$|\mathcal{T}\rangle = \sum_a Z_{\mathcal{T}}[a]|a\rangle$

$Z_{\mathcal{T}}[M_d]$

labels the symmetry defects

*Picture drawn with AnomTFT for invertible *G*, but the SymTFT can be defined for general categorical $\mathcal{C}$. [Kaidi, Ohmori, Zheng '22]

Topological defects of the SymTFT → symmetries of $\mathcal{T}$



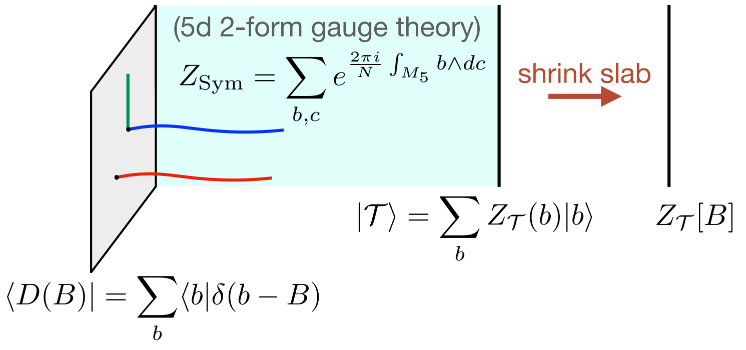
$U_{\text{bulk}}(\Sigma^p) \in \mathcal{A}$ associated to the topological b.c.

$U_{\text{bdy}}(\Sigma^p)$ give rise to $\mathcal{C}$ upon shrinking the slab

$U_{\text{bulk}}(\Sigma^p) \in \mathcal{B}_c$ ending on a boundary operator

An invertible 4d example

- $\mathcal{T} = 4d\ SU(N)$ Yang-Mills, $\mathbb{Z}_N^{(1)}$ generated by $U(\Sigma^2) = e^{\frac{2\pi i}{N}\oint_{\Sigma^2} B}$



$$Z_{\text{Sym}} = \sum_{b,c} e^{\frac{2\pi i}{N}\int_{M_5} b\wedge dc}$$

$$|\mathcal{T}\rangle = \sum_b Z_{\mathcal{T}}(b)|b\rangle \qquad Z_{\mathcal{T}}[B]$$

$$\langle D(B)| = \sum_b \langle b|\delta(b-B)$$

e.g. see discussion in [Gaiotto, Kapustin, Seiberg, Willett '14]

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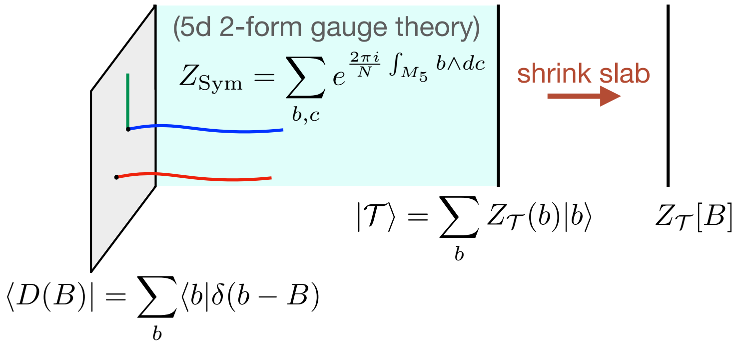


(5d 2-form gauge theory)

$$Z_{\text{Sym}} = \sum_{b,c} e^{\frac{2\pi i}{N}\int_{M_5} b\wedge dc}$$

shrink slab

$$|\mathcal{T}\rangle = \sum_b Z_{\mathcal{T}}(b)|b\rangle \qquad Z_{\mathcal{T}}[B]$$

$$\langle D(B)| = \sum_b \langle b|\delta(b-B)$$

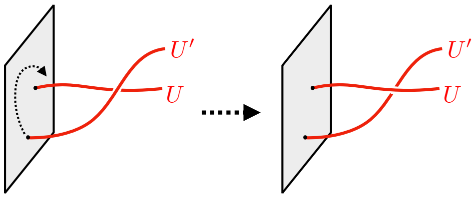
- Bulk surface defects: $\{\mathbb{1},\ \ U_b(\Sigma^2) = e^{\frac{2\pi i}{N}\oint_{\Sigma^2} b},\ \ U_c(\Sigma^2) = e^{\frac{2\pi i}{N}\oint_{\Sigma^2} c},\ \ldots\}$

$\Rightarrow U_c(\Sigma^2) \longrightarrow$ boundary $\mathbb{Z}_N^{(1)}$ symmetry defect $U(\Sigma^2)$

e.g. see discussion in [Gaiotto, Kapustin, Seiberg, Willett '14]

Probing 't Hooft anomalies with link invariants

- Suppose we can gauge $\mathcal{C}$ (it is anomaly free). Gauging $\mathcal{C}$ changes the topological b.c., but the SymTFT is unchanged.
- The left boundary is now a condensate of operators in $\mathcal{C}$. For each $U_{\text{bdy}} \in \mathcal{C}$, a corresponding $U_{\text{bulk}} \in \mathcal{B}_c$ should move to $\mathcal{A}$.
- Topological operators in $\mathcal{A}$ have trivial link invariant in the bulk:



$$\langle U(\Sigma)\, U'(\Sigma')\rangle_{\text{bulk}} \sim 1$$

Main observation: If $\mathcal{C}$ is anomaly-free, then for each $U_{\text{bdy}} \in \mathcal{C}$ there must be bulk topological operators $U_{\text{bulk}} \in \mathcal{B}_c$ with trivial link invariants.

$\Leftrightarrow$ If one cannot find a representative $U_{\text{bulk}} \in \mathcal{B}_c$ with trivial link invariants, then $\mathcal{C}$ is anomalous.

*up to equivalence relations

Comments

- This is a sufficient (not necessary) condition for a nontrivial anomaly.
- It is *strictly weaker* than the fiber functor condition.
  [Thorngren, Wang '19,'21][Décoppet, Yu '22]
- This is especially useful when the non-invertible symmetry arises by topological manipulations from a QFT with only invertible symmetry. Then the SymTFT is a DW theory, and the link invariants are easy to compute.
- Can go beyond links involving 2 lines.

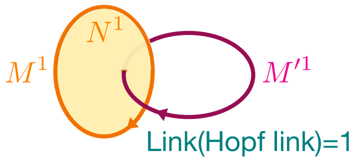
Computing link invariants

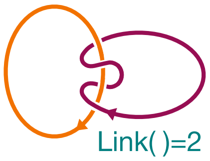
- *2-components:* closed M^{p_1}, M'^{p_2} can link in q-dim if $p_1 + p_2 + 1 = q$

$$\text{Link}(M^{p_1}, M'^{p_2}) = \int_{S^q} \text{PD}(N^{p_1+1})\, d\text{PD}(N'^{p_2+1}) = \text{Int}(N^{p_1+1}, M'^{p_2})$$

Seifert surfaces

e.g. lines in 3d:





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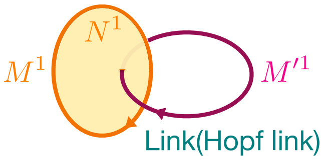
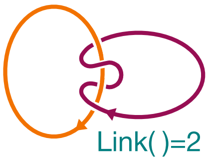
Seifert surfaces

e.g. lines in 3d:



Link(Hopf link)=1



Link( )=2

- *3-components:* closed $M^{p_1}, M'^{p_2}, M''^{p_3}$ can link in q-dim if $p_1 + p_2 + p_3 + (3 - n) = 2q \quad (n = 0, 1, 2)$

$$\mathrm{Link}(M^{p_1}, M'^{p_2}, M''^{p_3})_{n=0} = \mathrm{Int}(N^{p_1+1}, N'^{p_2+1}, N''^{p_3+1})$$

e.g. lines in 3d, *n*=0:

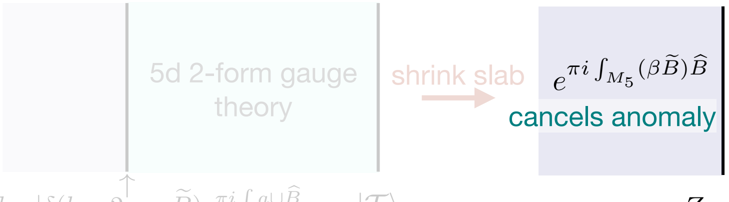
Link(Borromean ring)=1

III. Examples in 4d gauge theory

An invertible 4d example with anomaly

- $\mathcal{T} = 4d\ SU(4)$ Yang-Mills + gauge a normal subgroup $\mathbb{Z}_2^{(1)} \subset \underset{\substack{\uparrow \\ B}}{\mathbb{Z}_4^{(1)}}$

$$G = \underset{\substack{\uparrow \\ \text{quantum } \widehat{B}}}{\widehat{\mathbb{Z}}_2^{(1)}} \times \Big(\underset{\substack{\uparrow \\ \widetilde{B}}}{\widetilde{\mathbb{Z}}_2^{(1)}} \simeq \mathbb{Z}_4^{(1)}/\mathbb{Z}_2^{(1)} \Big) \text{ with mixed anomaly}$$



$\sum_{a \in \mathbb{Z}_2^{(1)},\, b \in \mathbb{Z}_4^{(1)}} \langle b, a | \delta(b - 2a - \widetilde{B}) e^{\pi i \int a \cup \widehat{B}}$ $|\mathcal{T}\rangle$ $Z_{\mathcal{T}/\mathbb{Z}_2^{(1)}}[\widehat{B}, \widetilde{B}]$

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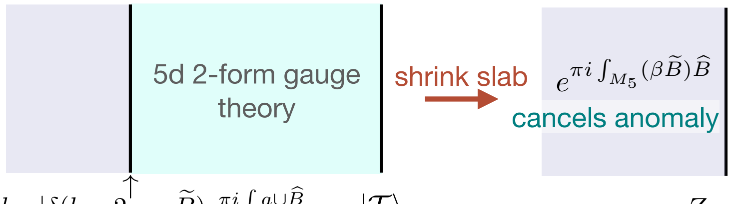
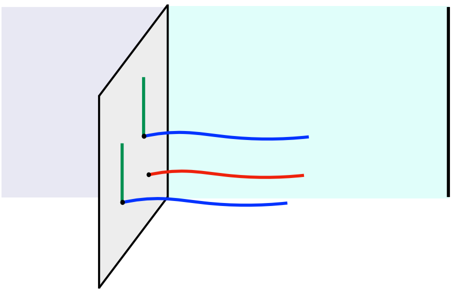


$\sum_{a \in \mathbb{Z}_2^{(1)},\, b \in \mathbb{Z}_4^{(1)}} \langle b, a | \delta(b - 2a - \widetilde{B}) e^{\pi i \int a \cup \widehat{B}}$ $|\mathcal{T}\rangle$ $Z_{\mathcal{T}/\mathbb{Z}_2^{(1)}}[\widehat{B}, \widetilde{B}]$

- Gauging changes the Dirichlet b.c. to mixed b.c.

The mixed anomaly is probed by a 2-link invariant



$U_b(\Sigma^2) = e^{\frac{\pi i}{2}\oint b}$ → $\widetilde{U}(\Sigma^2)$ generating $\widetilde{\mathbb{Z}}_2^{(1)}$

$U_c(\Sigma^2) = e^{\frac{\pi i}{2}\oint c}$ → $\widehat{U}(\Sigma^2)$ generating $\widehat{\mathbb{Z}}_2^{(1)}$ (quantum)

(Meanwhile $(U_c)^2$ moves to $\mathcal{A}$, as it reduces to the gauged $\mathbb{Z}_2^{(1)}$ defect.)

$\langle U_b(\Sigma)\, U_c(\Sigma')\rangle = (-i)^{\mathrm{Link}(\Sigma,\Sigma')}$ due to the mixed anomaly

A class of anomalous non-invertible examples

Consider $\mathcal{T}[M_d]$ with $G = \cdots \times H \times \mathbb{Z}_m^A$, where H and $\mathbb{Z}_m^A$ have a mixed anomaly, and $\mathbb{Z}_m^A$ has a self anomaly.

- *e.g.* applies for 4d $SU(N)$ adjoint QCD, 4d $\mathcal{N} = 4$ SYM, …

(1)

(2)

A class of anomalous non-invertible examples

Consider $\mathcal{T}[M_d]$ with $G = \cdots \times H \times \mathbb{Z}_m^A$, where H and $\mathbb{Z}_m^A$ have a mixed anomaly, and $\mathbb{Z}_m^A$ has a self anomaly.

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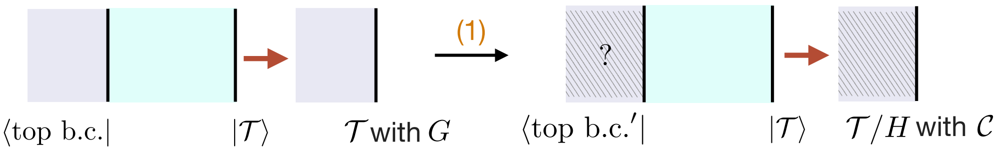
(1) Gauge H to promote $\mathbb{Z}_m^A$ to a non-invertible symmetry, implemented by non-invertible defect $\mathcal{N}_A$



(2)

A class of anomalous non-invertible examples

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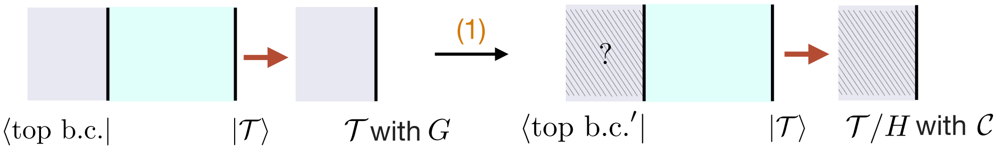
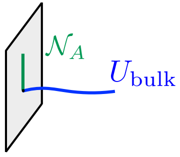


(2) The non-invertible symmetry is anomalous (as expected!), as can be probed by link invariants in the SymTFT.



$$\langle U_{\text{bulk}}(\Sigma) \ldots U_{\text{bulk}}(\Sigma') \rangle \sim e^{\frac{2\pi i}{m}\mathrm{Link}(\Sigma,\cdots,\Sigma')}$$

Example: 4d adjoint QCD

- Consider $4d$ $SU(N_c)$ gauge theory with N_f adjoint Weyl fermions

$$G = \cdots \times \mathbb{Z}_{N_c}^{(1)} \times \mathbb{Z}_{2N_f N_c}^{A}$$

$$\omega(A,B) = \frac{1}{2N_c} A\,\mathcal{P}(B) + \frac{N_c^2-1}{6N_c} A\,\beta A\,\beta A \quad \text{(even } N_c\text{)}$$

[Cordova, Dumitrescu '18], also see [Delmastro, Gomis, Hsin, Komargodski '22]

- Bulk defect operators in the SymTFT include:
  - ⁎ electric line/surface operators $\in \mathcal{A}$

$$U_a(M^1) = e^{\frac{2\pi i}{2N_f N_c} \oint_{M^1} a} \qquad U_b(M^2) = e^{\frac{2\pi i}{N_c} \oint_{M^2} b}$$

  - ⁎ magnetic non-invertible surface/volume operators $\in \mathcal{B}$

$$U_{\widehat{a}}(M^3) = e^{\frac{2\pi i}{2N_f N_c} \oint_{M^3} \widehat{a}} \cdot \mathrm{TQFT}[b] \longrightarrow \text{bdy } \mathbb{Z}_{2N_f N_c}^{A} \text{ defect}$$

$$U_{\widehat{b}}(M^2) = e^{\frac{2\pi i}{N_c} \oint_{M^2} \widehat{b}} \cdot \mathrm{TQFT}[a,b] \longrightarrow \text{bdy } \mathbb{Z}_{N_c}^{(1)} \text{ defect}$$

Example: 4d adjoint QCD

(1) Gauging $\mathbb{Z}_{N_c}^{(1)} \to PSU(N_c)$ adjoint QCD with $\widehat{\mathbb{Z}}_{N_c}^{(1)}$ symmetry, and noninvertible defect $\mathcal{N}_A$, see [Choi, Cordova, Hsin, Lam, Shao '21][Kaidi, Ohmori, Zheng '21,'22][Bhardwaj, Bottini, Schafer-Nameki, Tiwari '22]

$$\mathcal{N}_A \times \overline{\mathcal{N}}_A \sim \sum_{M^2 \in H_2(M^3, \mathbb{Z}_{N_c})} (-1)^{Q(M^2)} \widehat{U}_{\text{bdy}}(M^2), \qquad \mathcal{N}_A \times \widehat{U}_{\text{bdy}} \sim \mathcal{N}_A$$

The magnetic $U_{\widehat{b}}(M^2)$ moves to $\mathcal{A}$, and electric $U_b(M^2)$ moves to $\mathcal{B}$, with:

$$U_b(M^2) \longrightarrow \widehat{U}_{\text{bdy}}(M^2) \qquad U_{\widehat{a}}(M^3) \longrightarrow \mathcal{N}_A(M^3)$$

(2)

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$$U_b(M^2) \rightarrow \widehat{U}_{\text{bdy}}(M^2) \qquad U_{\widehat{a}}(M^3) \rightarrow \mathcal{N}_A(M^3)$$

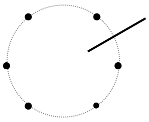
(2) $$\boxed{\langle U_{\widehat{a}}(S^3)\, U_{\widehat{a}}(S'^3)\, U_{\widehat{a}}(S''^3) \rangle \sim e^{-\frac{i\pi(N_c^2-1)}{4N_c^3 N_f^2}\text{Link}(S^3,S'^3,S''^3)_{n=2}}}$$

$\Rightarrow$4*d* $PSU(N_c)$ adjoint QCD has an anomalous non-invertible symmetry for **all** $N_c > 1, N_f \geq 1$.

(rules out special N_c, N_f with non-anomalous invertible symmetries)

Matching the anomaly in the IR

- *e.g.* consider $\mathcal{N} = 1$ SYM ($N_f = 1$ adjoint QCD).
  $\mathbb{Z}_{2N_c}^A \overset{\langle\lambda\lambda\rangle}{\to} \mathbb{Z}_2 \Rightarrow N_c$ gapped vacua.



domain walls supporting 3d $CS[B]$ match the $\mathbb{Z}_{N_c}^{(1)} \times \mathbb{Z}_{2N_c}$anomaly $\sim \int_{M^5} A \cup \mathcal{P}(B)$

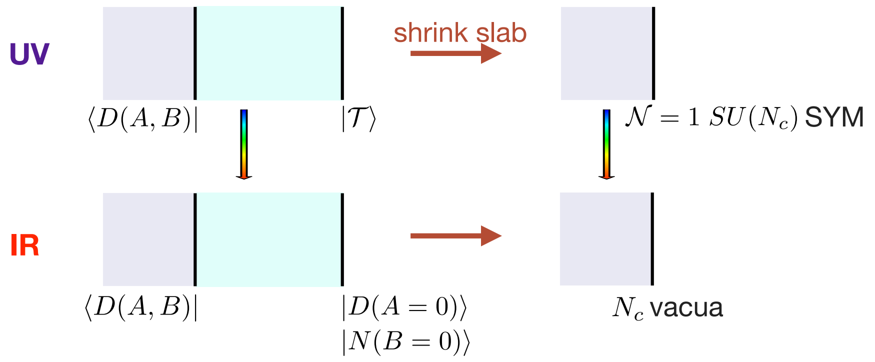
[Acharya, Vafa '01][Gaiotto, Kapustin, Komargodski, Seiberg '17][Hsin, Lam, Seiberg '18][Apruzzi, van Beest, Gould, Schafer-Nameki '21][Apruzzi, Bah, Bonetti, Schafer-Nameki '22]

How is the self-anomaly matched?

[Delmastro, Gomis, Hsin, Komargodski '22]

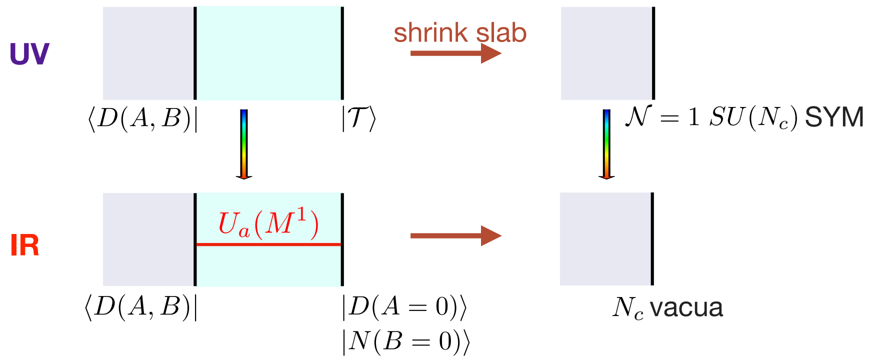
Anomaly matching via the SymTFT (schematically)

see [Apruzzi, Bah, Bonetti, Schafer-Nameki ’22] for the holographic perspective!



Anomaly matching via the SymTFT (schematically)

see [Apruzzi, Bah, Bonetti, Schafer-Nameki ’22] for the holographic perspective!



- $U_a(M^1)$ becomes top. order parameter of $\mathbb{Z}_{2N_c}^A$, labeling the N_c vacua
- $U_{\hat{a}}(M^3)$ → bdy $\mathbb{Z}_{2N_c}^A$ defect → domain wall with $CS[B]$
- The fact that the domain walls + junctions saturate the $\mathbb{Z}_{2N_c}^A$ self-anomaly is related to the nontrivial $U_{\hat{a}}(M^3)$ triple link invariants.

It would be interesting to see this in detail!

Summary

- Much like for ordinary symmetries, ’t Hooft anomalies of non-invertible (categorical) symmetries are a useful tool for constraining the dynamics of QFT.
- Linking invariants in the Symmetry TFT provide a diagnostic of ’t Hooft anomalies for generalized symmetries, which are simple to compute in many examples.
- Examples of theories with anomalous non-invertible symmetries include 4d adjoint QCD, and $\mathcal{N}=4$ super Yang-Mills.
- Tracking the defect operators in the SymTFT across RG flow can lead to insights into anomaly matching.

Future directions

- Notion of anomaly field theory for the non-invertible case?
- Complete characterization of ’t Hooft anomalies for non-invertible symmetries?
- General framework for *intrinsically* non-invertible symmetries?
- More dynamical consequences for non-invertible symmetries?

Thank you!

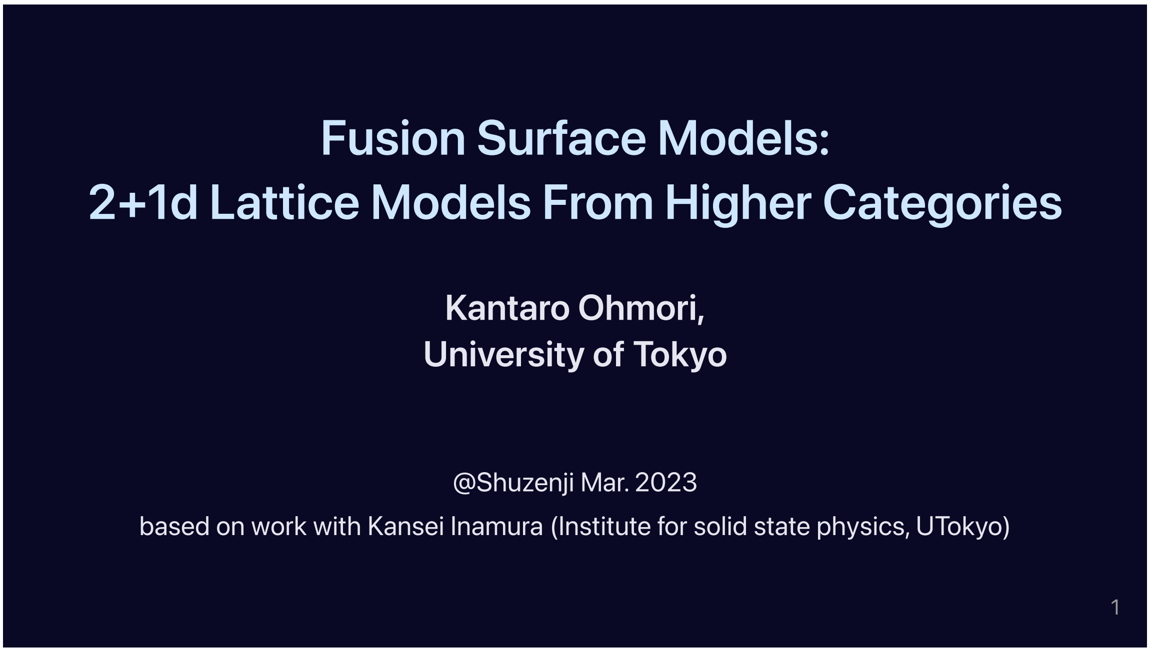
ありがとうございます

Fusion Surface Models: 2+1d Lattice Models from Higher Categories

Kantaro Ohmori

Abstract.

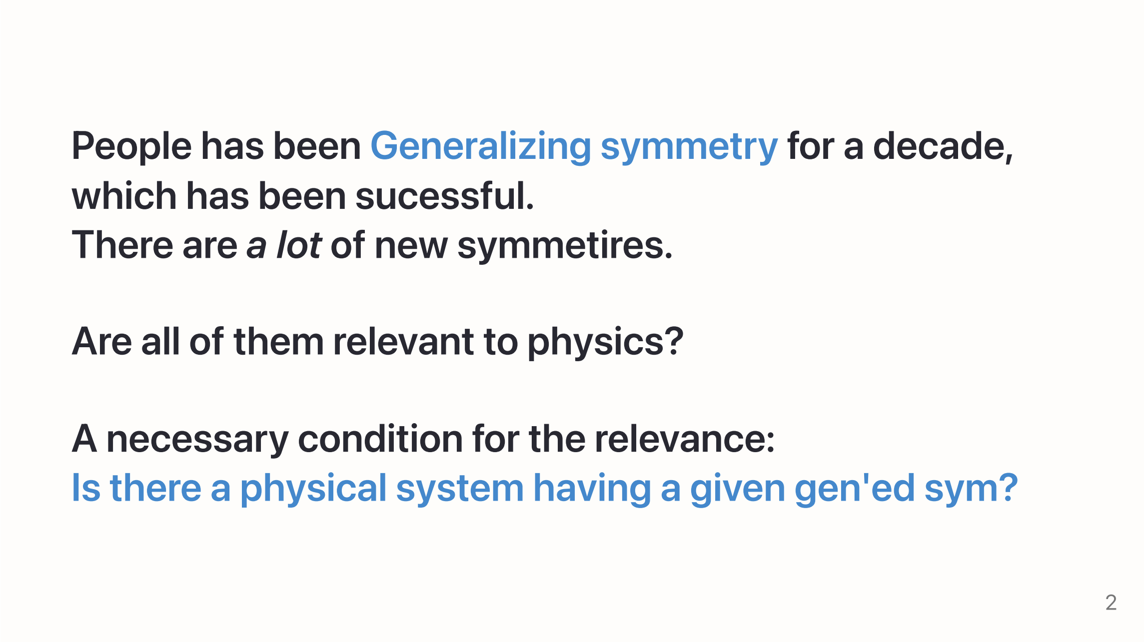
(K. Ohmori) University of Tokyo



Fusion Surface Models:
2+1d Lattice Models From Higher Categories

Kantaro Ohmori,
University of Tokyo

@Shuzenji Mar. 2023
based on work with Kansei Inamura (Institute for solid state physics, UTokyo)



**People has been Generalizing symmetry for a decade,
which has been sucessful.
There are *a lot* of new symmetires.**

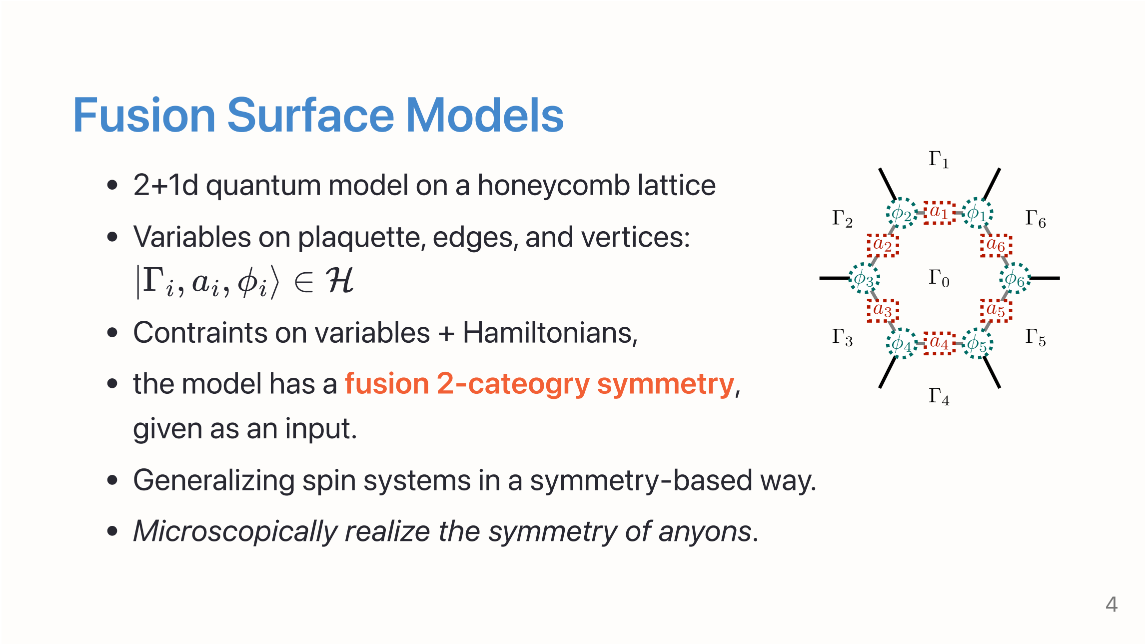
Are all of them relevant to physics?

**A necessary condition for the relevance:
Is there a physical system having a given gen'ed sym?**

Is there a physical system having a given gen'ed sym?

**Yes, at least a theoretical lattice model exists,
for a broad class of finite generalized symmetry.**

**Today we discuss systematic constructions
for gen'ed sym.s in 1+1 (review) and 2+1d.**



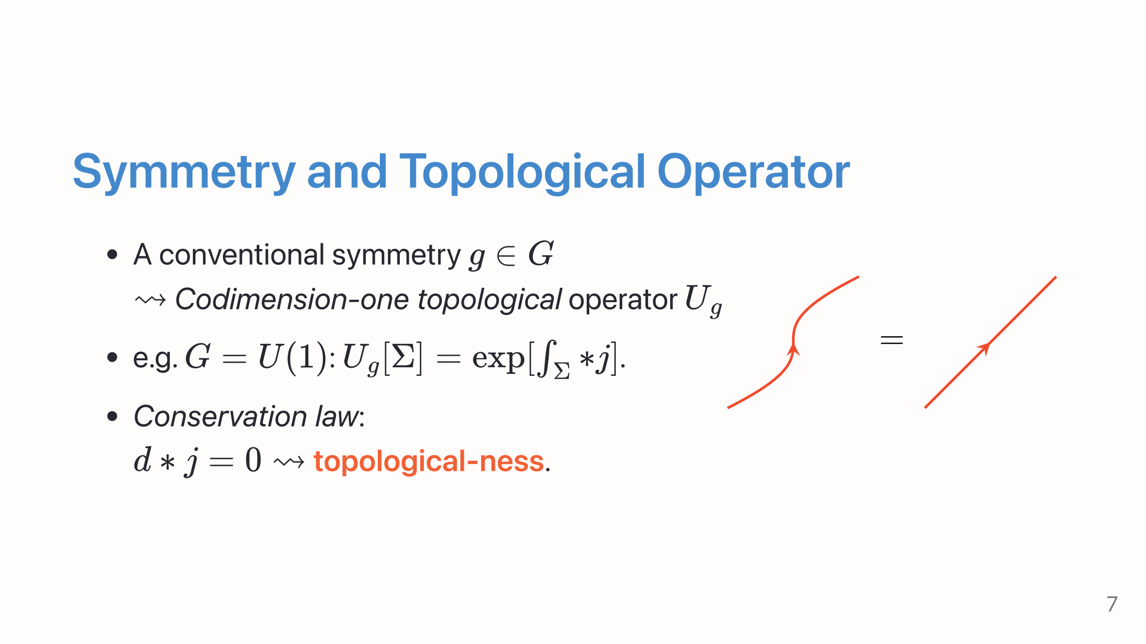
Fusion Surface Models

- 2+1d quantum model on a honeycomb lattice
- Variables on plaquette, edges, and vertices: $|\Gamma_i, a_i, \phi_i\rangle \in \mathcal{H}$
- Contraints on variables + Hamiltonians,
- the model has a **fusion 2-cateogry symmetry**, given as an input.
- Generalizing spin systems in a symmetry-based way.
- *Microscopically realize the symmetry of anyons.*

Outline

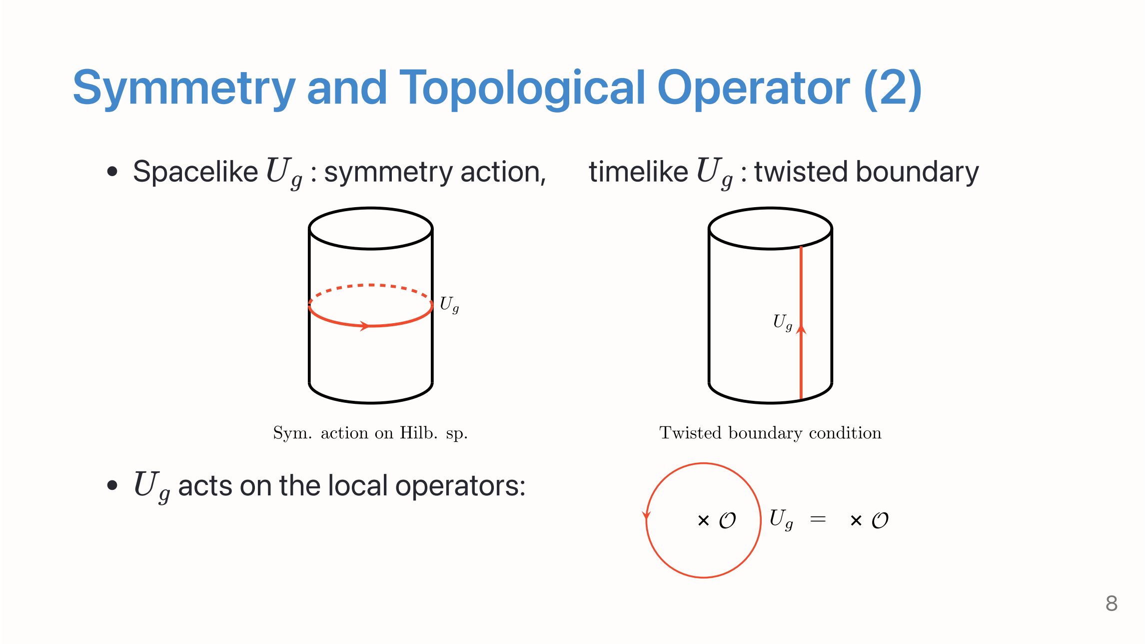
- Generalized Symmetry and Topological Order.
- 1+1d Anyon Chain [Feiguin, Trebst, Ludwig, Troyer, Kitaev, Wang, Freedman '06] [Aasen,Fendley,Mong '20]
- 2+1d Fusion Surface Model

Generalized Symmetry and Topological Order



Symmetry and Topological Operator

- A conventional symmetry $g \in G$
  $\rightsquigarrow$ *Codimension-one topological* operator U_g
- e.g. $G = U(1)$: $U_g[\Sigma] = \exp[\int_\Sigma *j]$.
- *Conservation law*:
  $d * j = 0 \rightsquigarrow$ **topological-ness**.



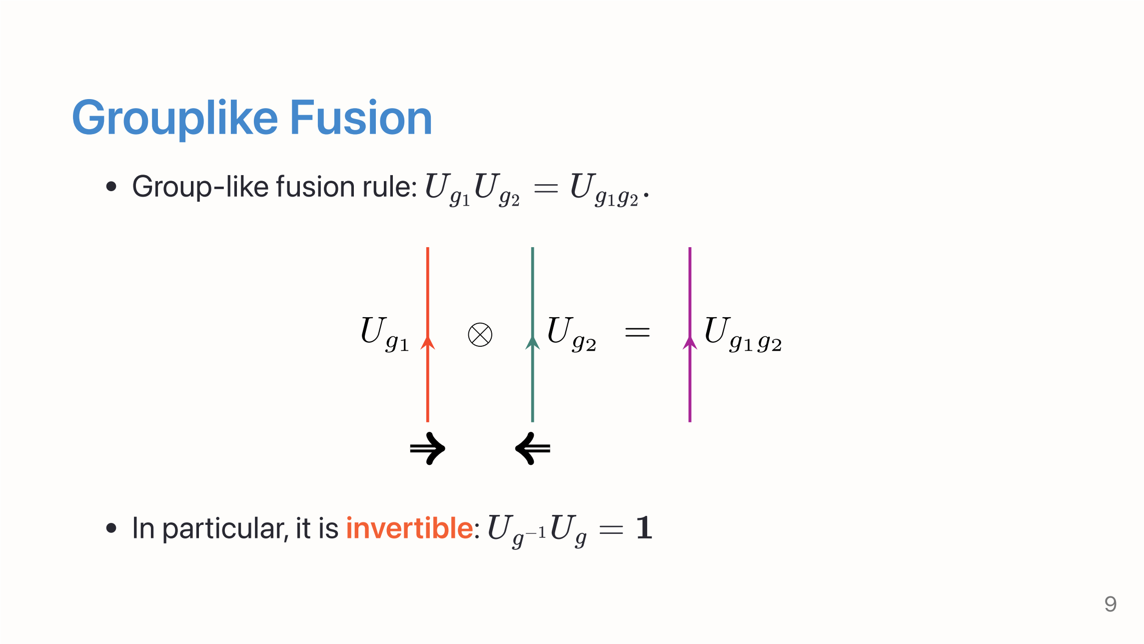
Symmetry and Topological Operator (2)

- Spacelike U_g : symmetry action, timelike U_g : twisted boundary

Sym. action on Hilb. sp.

Twisted boundary condition

- U_g acts on the local operators:

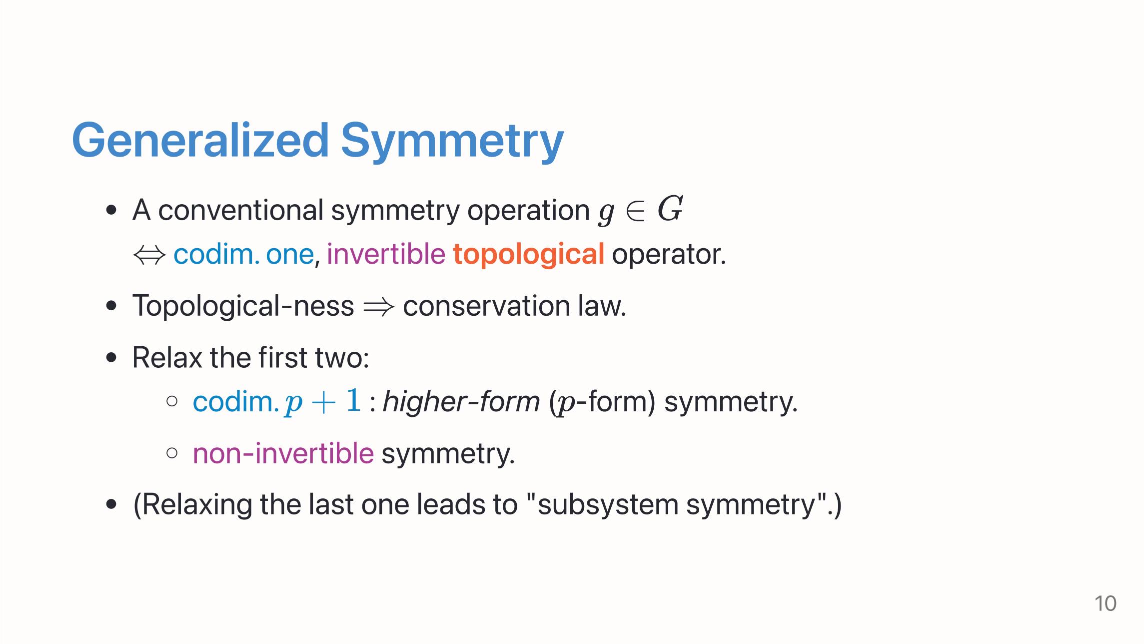


Grouplike Fusion

- Group-like fusion rule: $U_{g_1} U_{g_2} = U_{g_1 g_2}$.

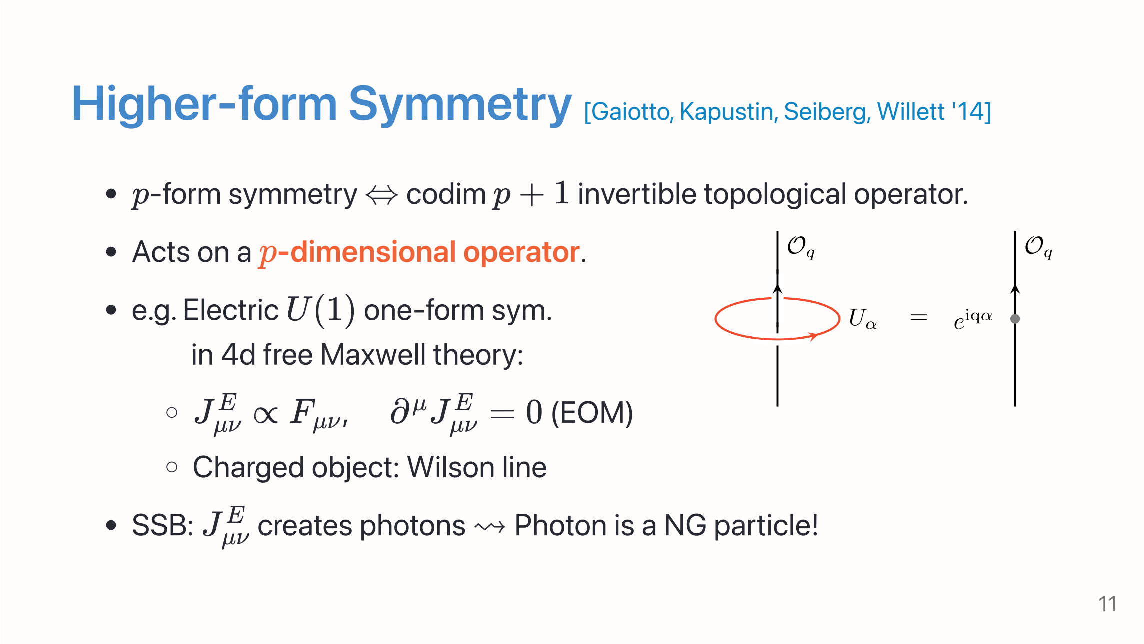
$$U_{g_1} \otimes U_{g_2} = U_{g_1 g_2}$$

- In particular, it is **invertible**: $U_{g^{-1}} U_g = \mathbf{1}$



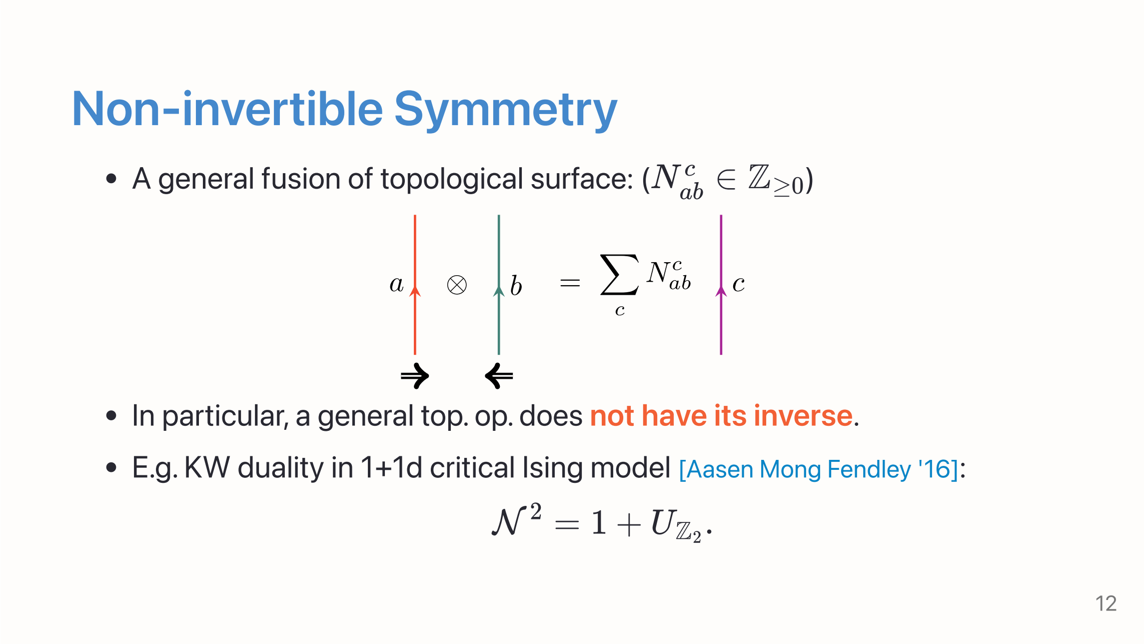
Generalized Symmetry

- A conventional symmetry operation $g \in G$
  $\Leftrightarrow$ codim. one, invertible **topological** operator.
- Topological-ness $\Rightarrow$ conservation law.
- Relax the first two:
  - codim. $p+1$: *higher-form* (p-form) symmetry.
  - non-invertible symmetry.
- (Relaxing the last one leads to "subsystem symmetry".)



Higher-form Symmetry [Gaiotto, Kapustin, Seiberg, Willett '14]

- p-form symmetry ⇔ codim $p+1$ invertible topological operator.
- Acts on a **p-dimensional operator**.
- e.g. Electric $U(1)$ one-form sym.
  in 4d free Maxwell theory:
  - $J^E_{\mu\nu} \propto F_{\mu\nu}$, $\quad \partial^\mu J^E_{\mu\nu} = 0$ (EOM)
  - Charged object: Wilson line
- SSB: $J^E_{\mu\nu}$ creates photons ⇝ Photon is a NG particle!



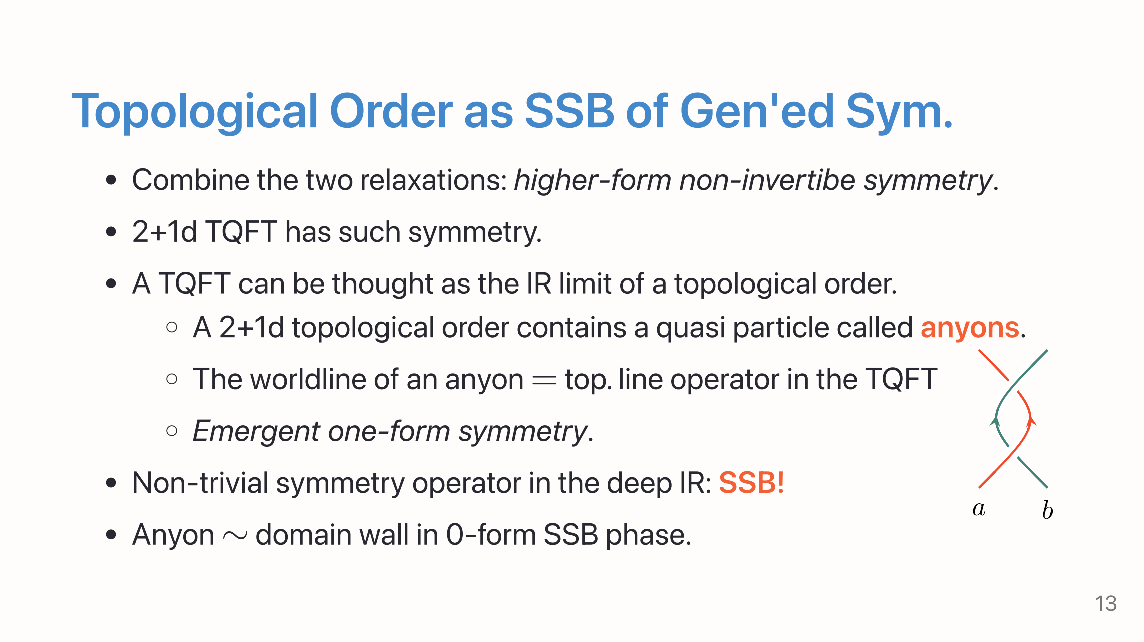
Non-invertible Symmetry

- A general fusion of topological surface: ($N^c_{ab} \in \mathbb{Z}_{\geq 0}$)

$$a \otimes b = \sum_c N^c_{ab} \; c$$

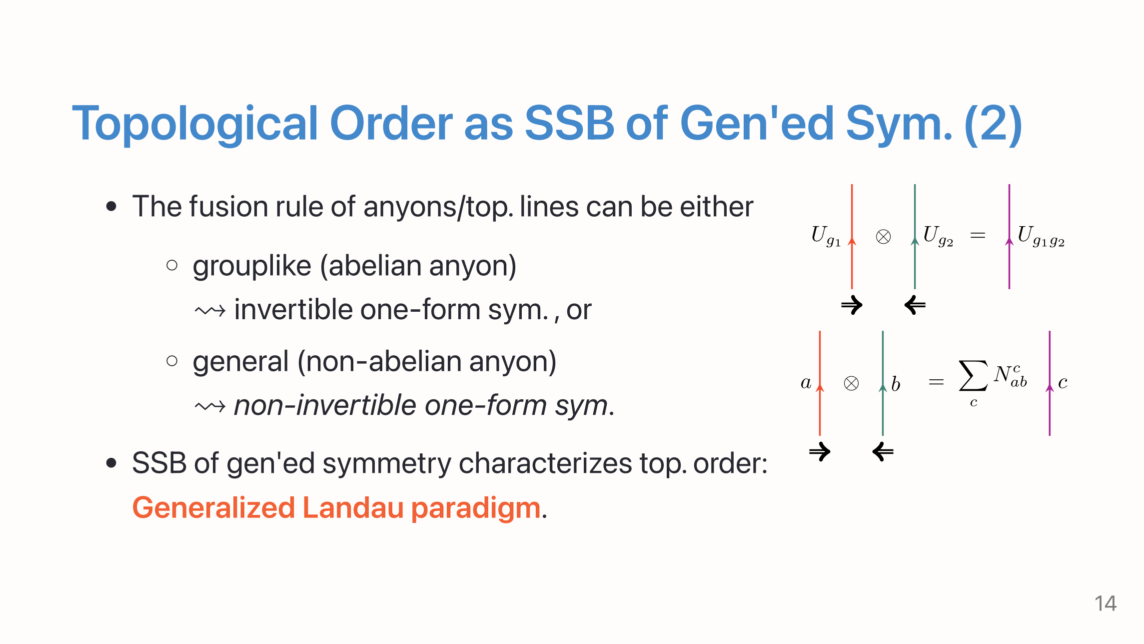
- In particular, a general top. op. does **not have its inverse**.
- E.g. KW duality in 1+1d critical Ising model [Aasen Mong Fendley '16]:

$$\mathcal{N}^2 = 1 + U_{\mathbb{Z}_2}.$$



Topological Order as SSB of Gen'ed Sym.

- Combine the two relaxations: *higher-form non-invertibe symmetry*.
- 2+1d TQFT has such symmetry.
- A TQFT can be thought as the IR limit of a topological order.
  - A 2+1d topological order contains a quasi particle called **anyons**.
  - The worldline of an anyon = top. line operator in the TQFT
  - *Emergent one-form symmetry*.
- Non-trivial symmetry operator in the deep IR: **SSB!**
- Anyon ∼ domain wall in 0-form SSB phase.



Topological Order as SSB of Gen'ed Sym. (2)

- The fusion rule of anyons/top. lines can be either
  - grouplike (abelian anyon)
    ⇝ invertible one-form sym. , or
  - general (non-abelian anyon)
    ⇝ *non-invertible one-form sym.*
- SSB of gen'ed symmetry characterizes top. order:
  Generalized Landau paradigm.

$$U_{g_1} \otimes U_{g_2} = U_{g_1 g_2}$$

$$a \otimes b = \sum_c N_{ab}^c \, c$$

Towards UV Models of topological order

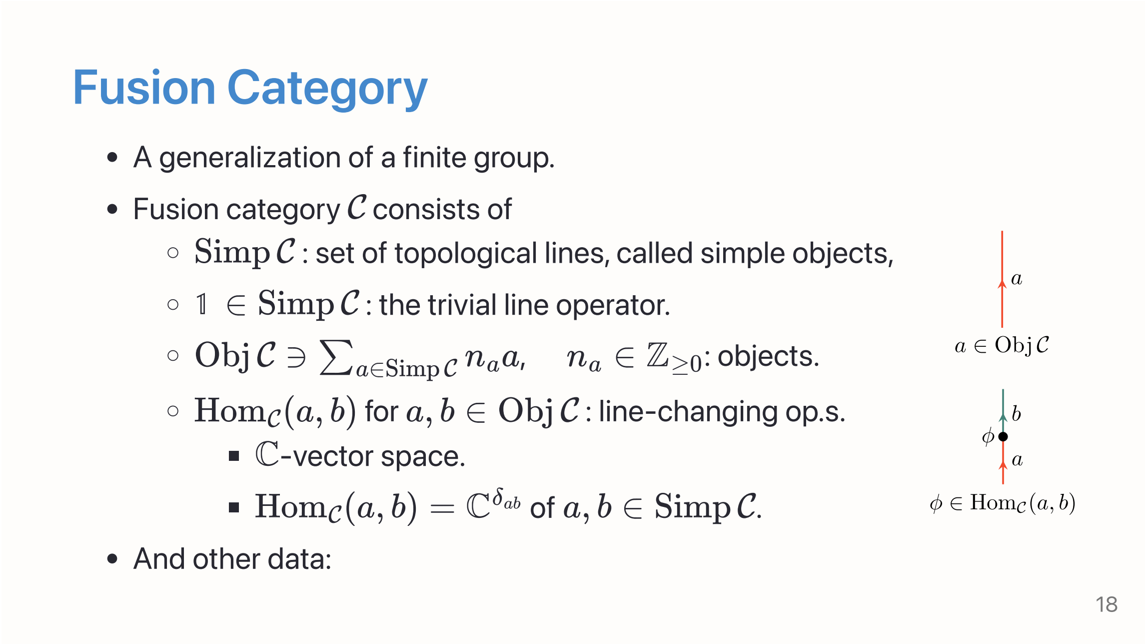
- UV lattice construction for a general top. order is unclear.
- Top. order is the **SSB** phase of a gen'ed sym.
- UV lattice model with the same gen'ed symmetry might flow to the phase! $\rightsquigarrow$ **Fusion surface model**!
- In general the models are (probably) unsolvable and strongly-interacting $\rightsquigarrow$ we need numerical study.

1+1d Anyon Chain

[Feiguin, Trebst, Ludwig, Troyer, Kitaev, Wang, Freedman '06]...[Aasen,Fendley,Mong '20]...

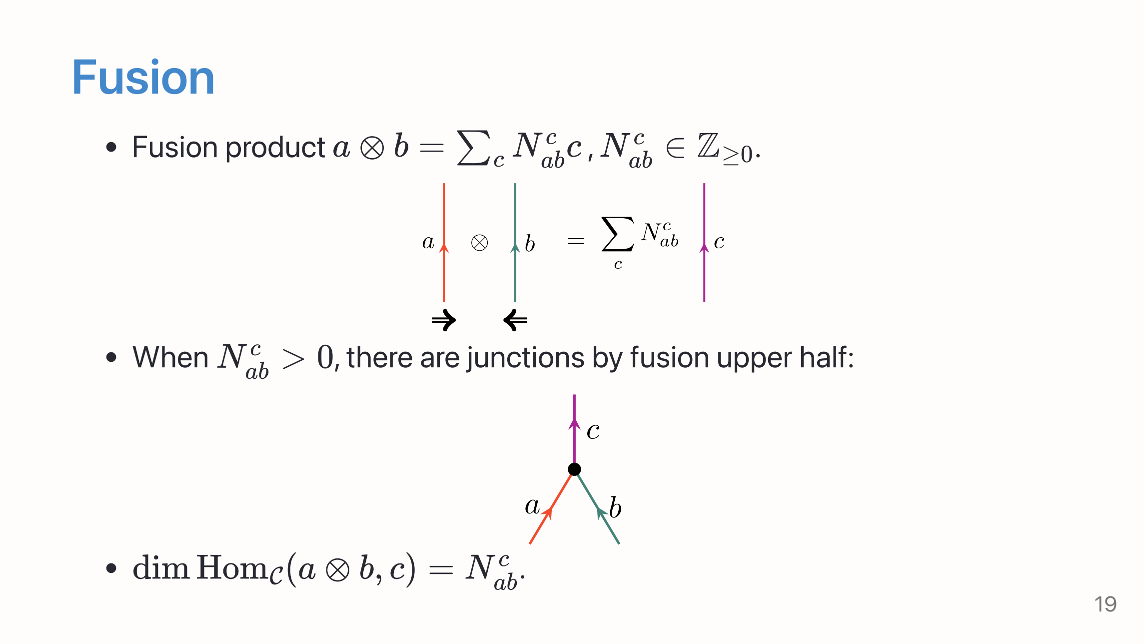
1+1d Anyon Chain

- 1+1d lattice system, generalizing spin chain.
- Acted naturally by a **fusion (1-)category** $\mathcal{C}$.
- An object $\rho \in \mathcal{C}$ instead of spin, i.e. $|\rho\rangle$ instead of $|0\rangle$,$|1\rangle$ acted by $\mathbb{Z}_2$.



Fusion Category

- A generalization of a finite group.
- Fusion category $\mathcal{C}$ consists of
  - $\mathrm{Simp}\,\mathcal{C}$: set of topological lines, called simple objects,
  - $\mathbb{1} \in \mathrm{Simp}\,\mathcal{C}$: the trivial line operator.
  - $\mathrm{Obj}\,\mathcal{C} \ni \sum_{a \in \mathrm{Simp}\,\mathcal{C}} n_a a$, $\quad n_a \in \mathbb{Z}_{\geq 0}$: objects.
  - $\mathrm{Hom}_{\mathcal{C}}(a,b)$ for $a, b \in \mathrm{Obj}\,\mathcal{C}$: line-changing op.s.
    - $\mathbb{C}$-vector space.
    - $\mathrm{Hom}_{\mathcal{C}}(a,b) = \mathbb{C}^{\delta_{ab}}$ of $a, b \in \mathrm{Simp}\,\mathcal{C}$.
- And other data:

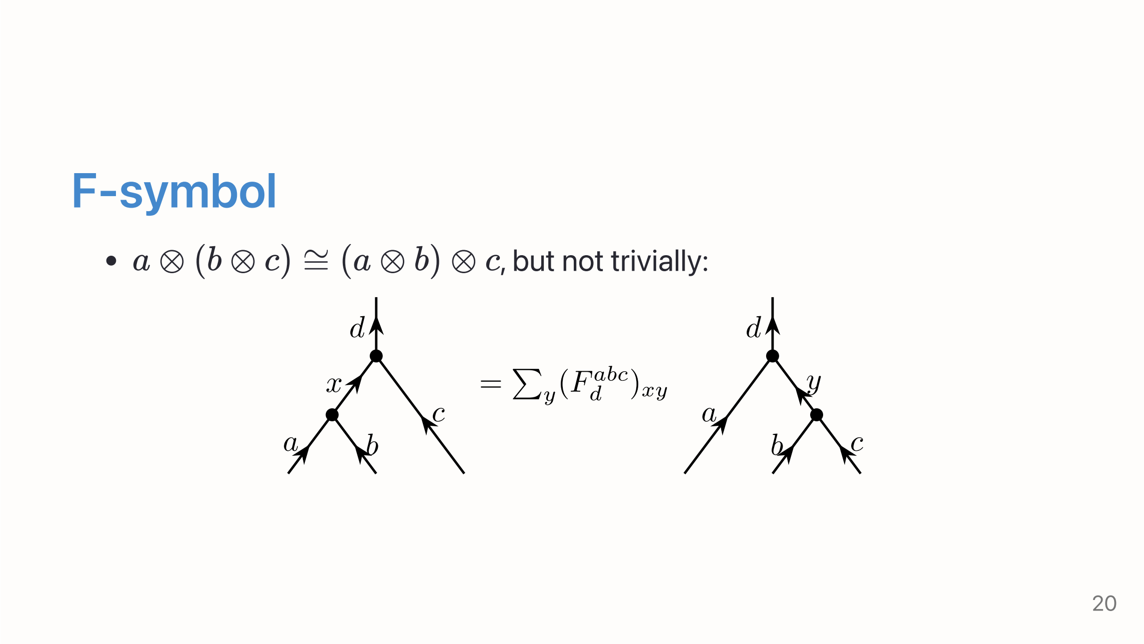


Fusion

- Fusion product $a \otimes b = \sum_c N_{ab}^c c$, $N_{ab}^c \in \mathbb{Z}_{\geq 0}$.

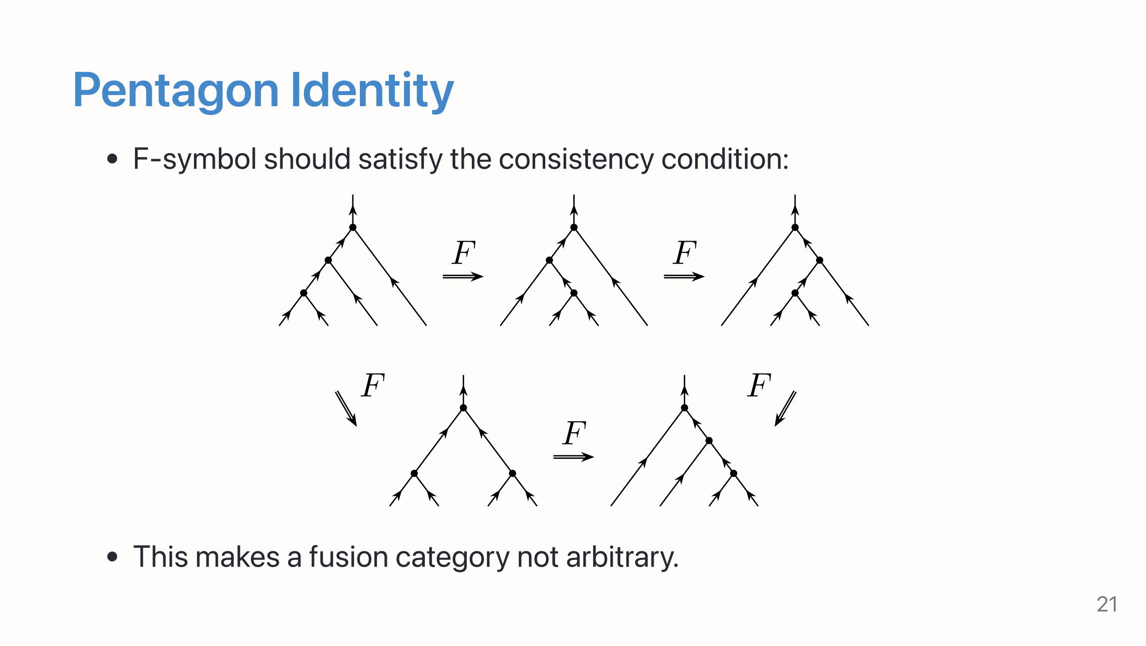
- When $N_{ab}^c > 0$, there are junctions by fusion upper half:

- $\dim \mathrm{Hom}_{\mathcal{C}}(a \otimes b, c) = N_{ab}^c$.



F-symbol

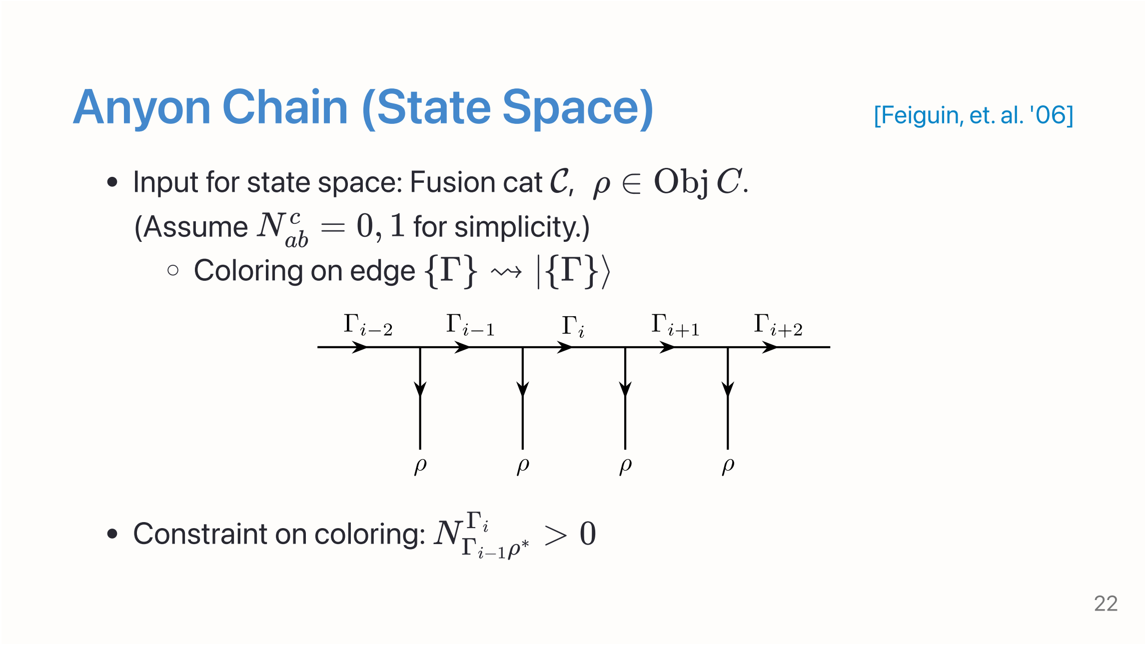
- $a \otimes (b \otimes c) \cong (a \otimes b) \otimes c$, but not trivially:



Pentagon Identity

- F-symbol should satisfy the consistency condition:

- This makes a fusion category not arbitrary.



Anyon Chain (State Space)

[Feiguin, et. al. '06]

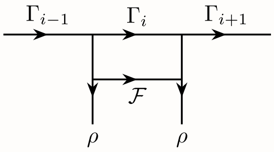
- Input for state space: Fusion cat $\mathcal{C}$, $\rho \in \mathrm{Obj}\, C$.
  (Assume $N_{ab}^{c} = 0, 1$ for simplicity.)
  - Coloring on edge $\{\Gamma\} \rightsquigarrow |\{\Gamma\}\rangle$

- Constraint on coloring: $N_{\Gamma_{i-1}\rho^*}^{\Gamma_i} > 0$

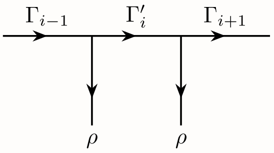
Anyon Chain (Hamiltonian)

- Hamiltonian depends on $w : \mathrm{Simp}\,\mathcal{C} \to \mathbb{C}$,
- explicit next-next-nearest neighbor interactoin:

$$H^{\langle i-1,i,i+1\rangle}|\Gamma_{i-1}\Gamma_i\Gamma_{i+1}\rangle = \sum_{\mathcal{F}} w(\mathcal{F}) \sum_{\Gamma'_i\in\mathrm{Simp}\,\mathcal{C}} \overline{(F^{\rho\mathcal{F}^*\Gamma^*_i}_{\Gamma^*_{i-1}})_{\rho\Gamma'^*_i}} (F^{\rho\mathcal{F}\Gamma_i}_{\Gamma_{i+1}})_{\rho\Gamma'_i})|\Gamma_{i-1}\Gamma'_i\Gamma_{i+1}\rangle.$$

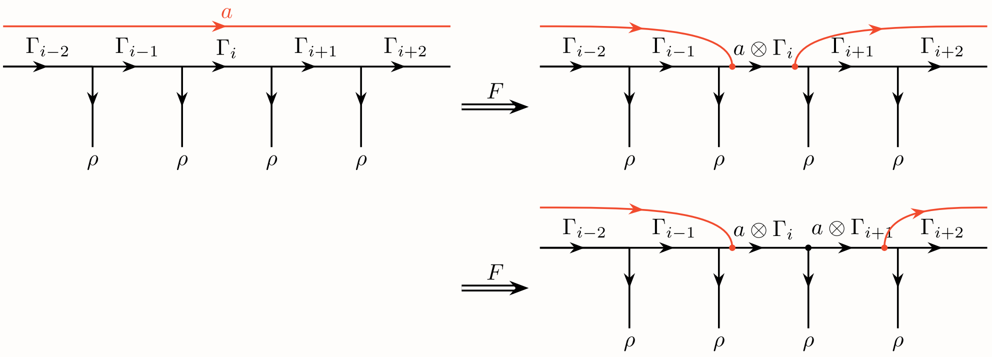


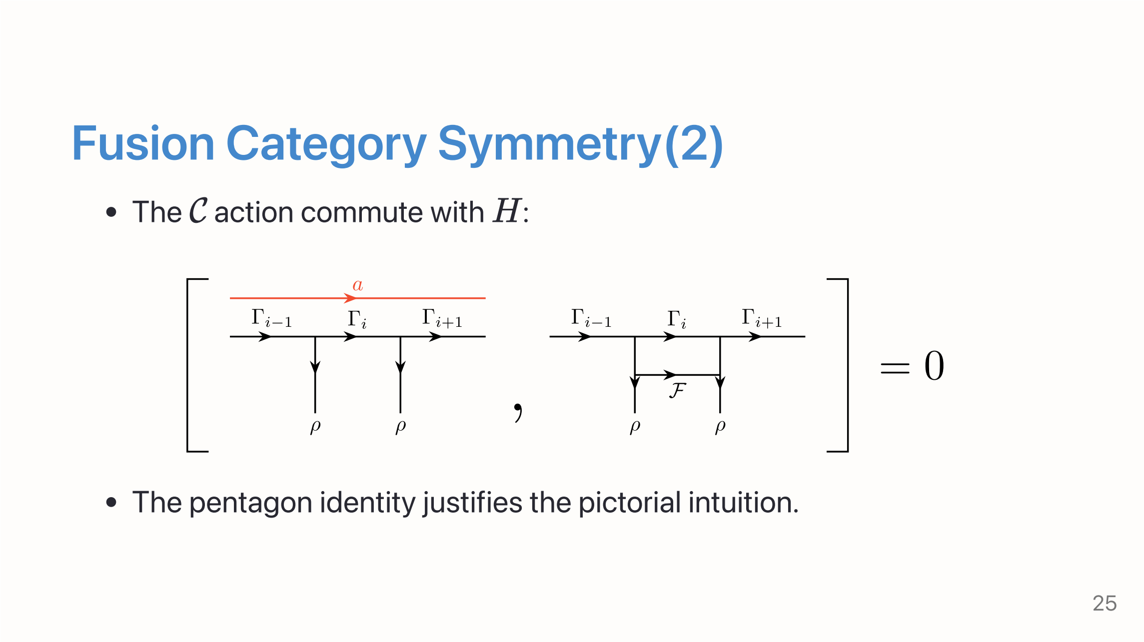
$$= \sum_{\Gamma'_i\in\Gamma_i\otimes\mathcal{F}} \overline{(F^{\rho\mathcal{F}\Gamma_i}_{\Gamma_{i-1}})_{\rho\Gamma'_i}} (F^{\Gamma_i\mathcal{F}\rho^*}_{\Gamma_{i+1}})_{\rho^*\Gamma'_i}$$



Fusion Category Symmetry

- The anyon chain based on fusion category $\mathcal{C}$ has $\mathcal{C}$-symmetry.
- The action is by "hitting from above":

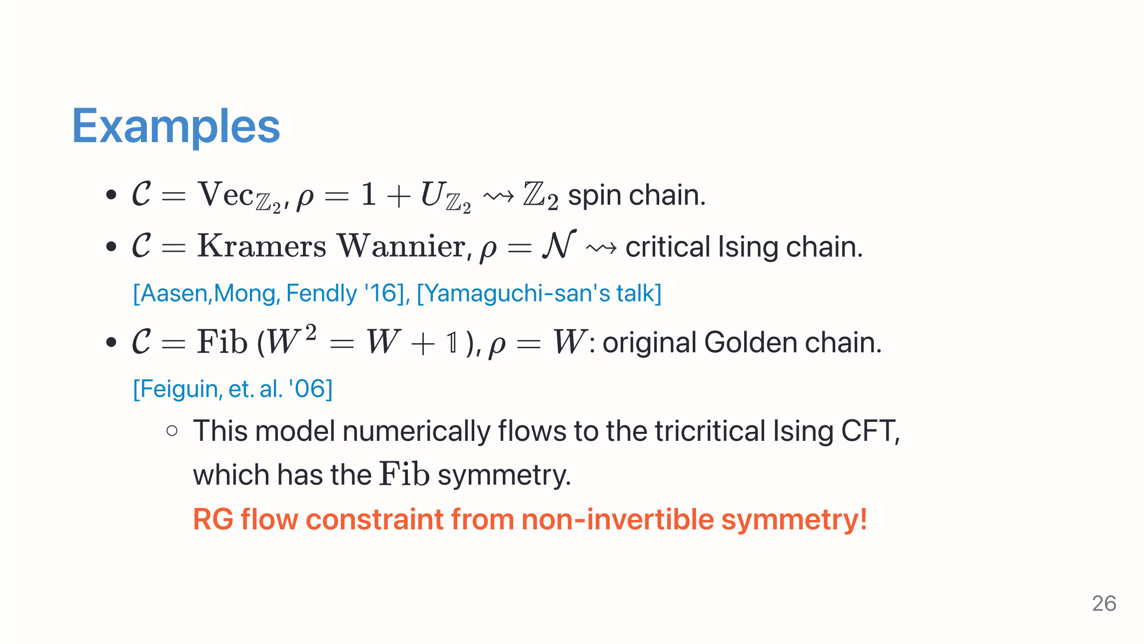




Fusion Category Symmetry(2)

- The $\mathcal{C}$ action commute with H:

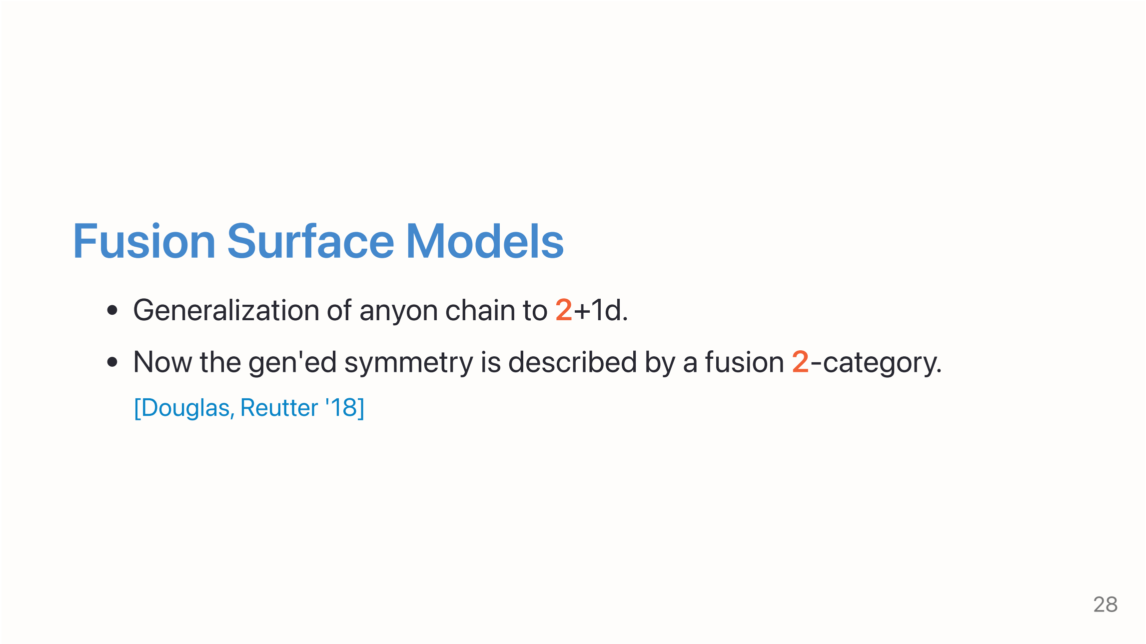
- The pentagon identity justifies the pictorial intuition.



Examples

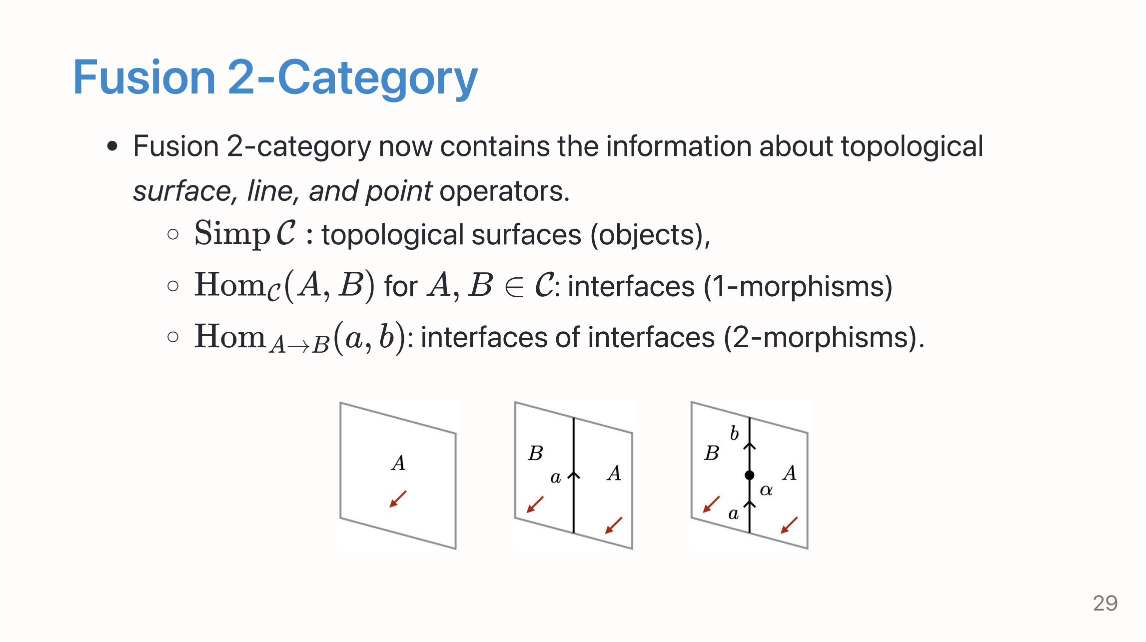
- $\mathcal{C} = \mathrm{Vec}_{\mathbb{Z}_2}$, $\rho = 1 + U_{\mathbb{Z}_2} \rightsquigarrow \mathbb{Z}_2$ spin chain.
- $\mathcal{C} = \mathrm{Kramers\ Wannier}$, $\rho = \mathcal{N} \rightsquigarrow$ critical Ising chain.
  [Aasen,Mong, Fendly '16], [Yamaguchi-san's talk]
- $\mathcal{C} = \mathrm{Fib}\,(W^2 = W + \mathbb{1})$, $\rho = W$: original Golden chain.
  [Feiguin, et. al. '06]
  - This model numerically flows to the tricritical Ising CFT, which has the Fib symmetry.
    RG flow constraint from non-invertible symmetry!

2+1d Fusion Surface Models



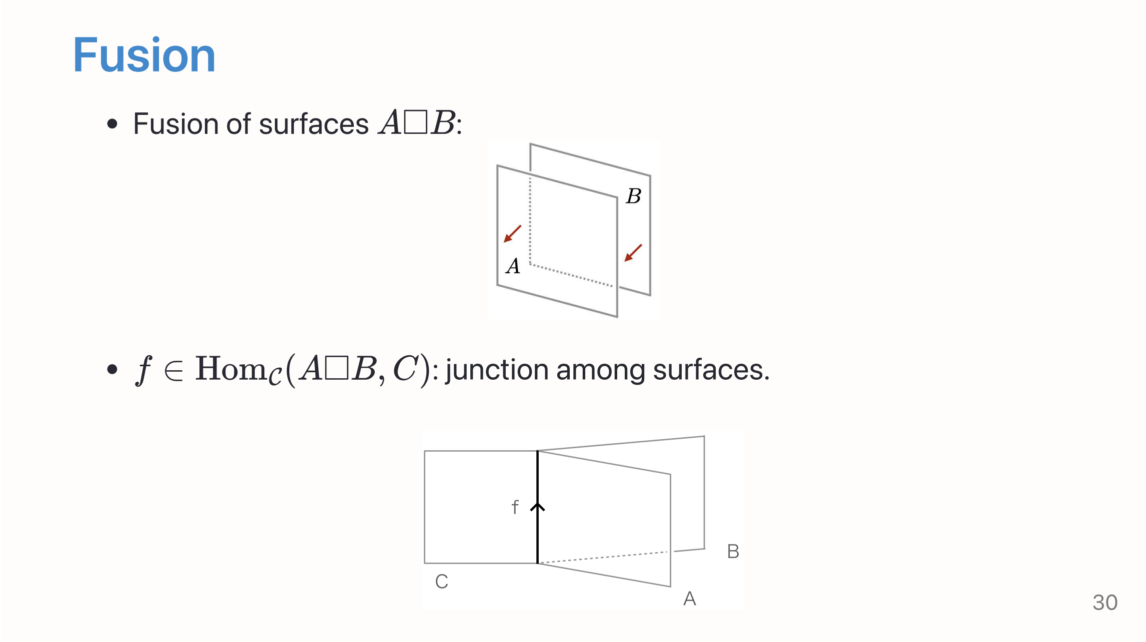
Fusion Surface Models

- Generalization of anyon chain to 2+1d.
- Now the gen'ed symmetry is described by a fusion 2-category.
  [Douglas, Reutter '18]



Fusion 2-Category

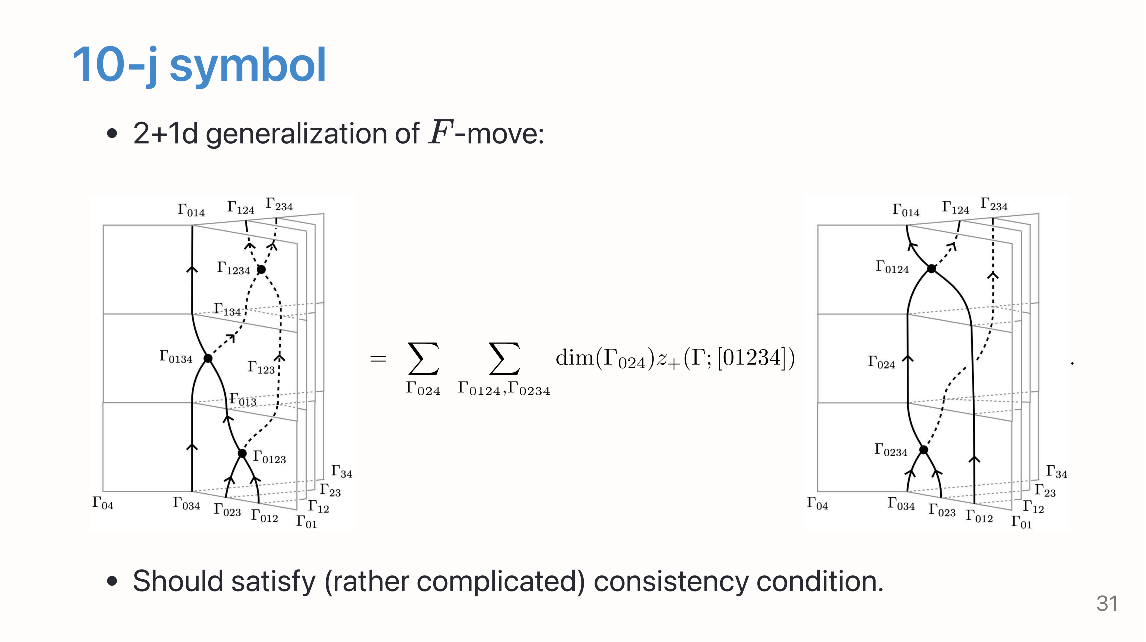
- Fusion 2-category now contains the information about topological *surface, line, and point* operators.
  - $\mathrm{Simp}\,\mathcal{C}$: topological surfaces (objects),
  - $\mathrm{Hom}_{\mathcal{C}}(A, B)$ for $A, B \in \mathcal{C}$: interfaces (1-morphisms)
  - $\mathrm{Hom}_{A\to B}(a, b)$: interfaces of interfaces (2-morphisms).



Fusion

- Fusion of surfaces $A\square B$:

- $f \in \mathrm{Hom}_{\mathcal{C}}(A\square B, C)$: junction among surfaces.

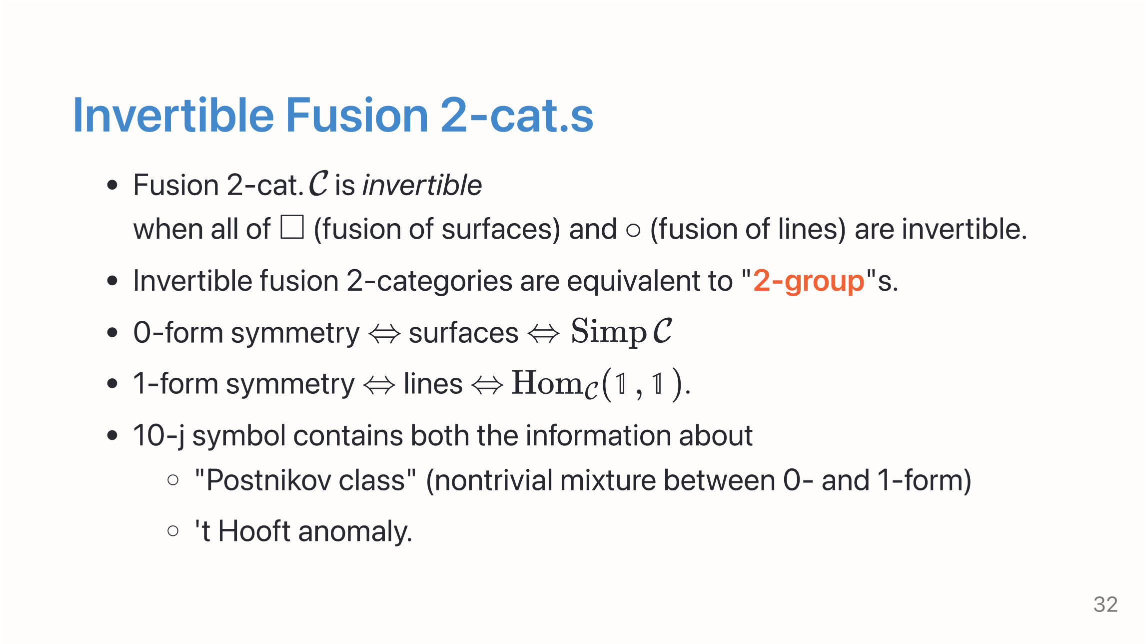


10-j symbol

- 2+1d generalization of F-move:

$$[\text{diagram}] = \sum_{\Gamma_{024}} \sum_{\Gamma_{0124},\Gamma_{0234}} \dim(\Gamma_{024}) z_+(\Gamma;[01234]) \; [\text{diagram}] .$$

- Should satisfy (rather complicated) consistency condition.

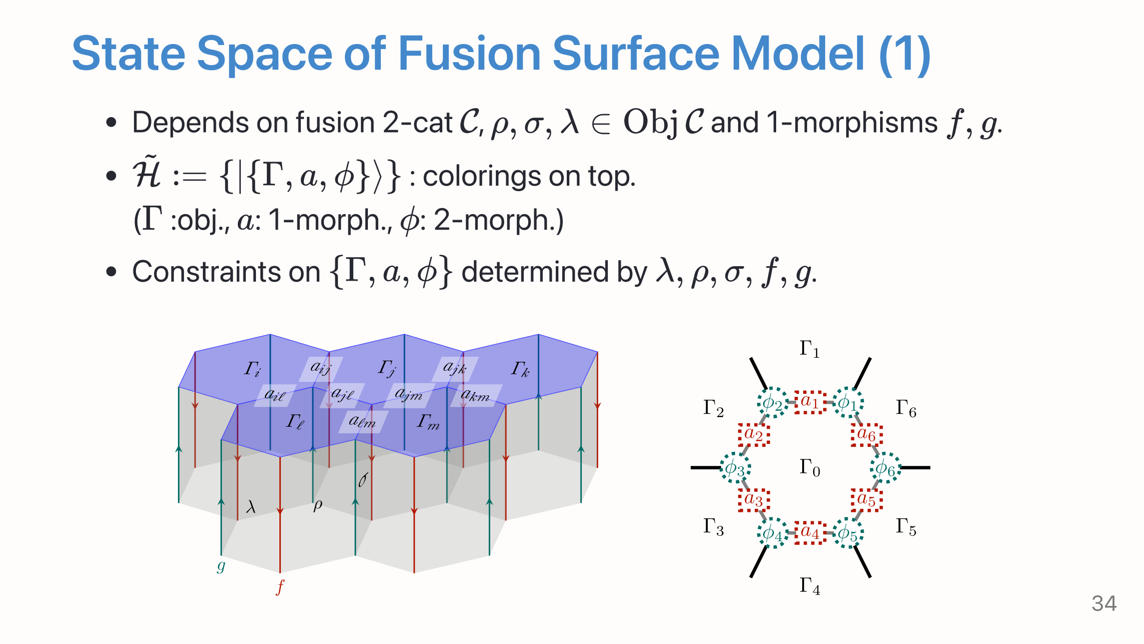


Invertible Fusion 2-cat.s

- Fusion 2-cat. $\mathcal{C}$ is *invertible* when all of $\Box$ (fusion of surfaces) and $\circ$ (fusion of lines) are invertible.
- Invertible fusion 2-categories are equivalent to "**2-group**"s.
- 0-form symmetry ⇔ surfaces ⇔ $\mathrm{Simp}\,\mathcal{C}$
- 1-form symmetry ⇔ lines ⇔ $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$.
- 10-j symbol contains both the information about
  - "Postnikov class" (nontrivial mixture between 0- and 1-form)
  - 't Hooft anomaly.

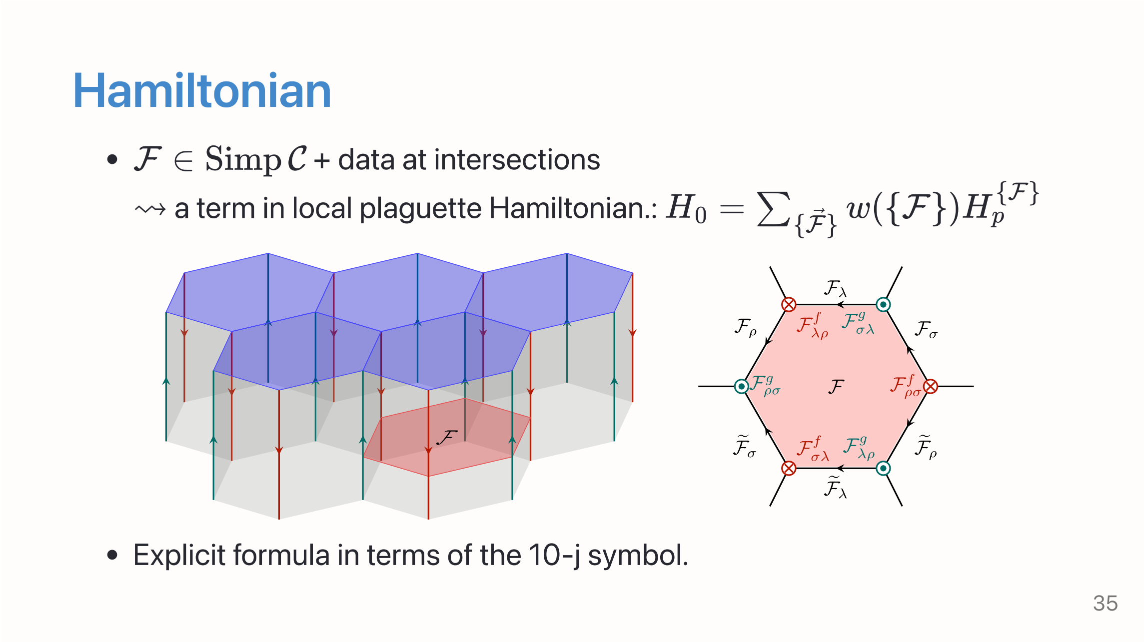
Symmetry of Anyons as Fusion 2-cat

- Usually, the data about anyons in a topological order is organized as **a modular tensor category** (MTC) $\mathcal{B}$.
- $\mathcal{B}$ (MTC) $\rightsquigarrow \mathrm{Mod}(\mathcal{B})$: fusion 2-cat
- $\mathrm{Simp}\,\mathrm{Mod}(\mathcal{B}) = \{\text{gapped self-interfaces}\}$
- $\mathrm{Hom}_{\mathrm{Mod}}(\mathcal{B})(\mathbb{1}, \mathbb{1}) = \{\text{anyons}\}$



State Space of Fusion Surface Model (1)

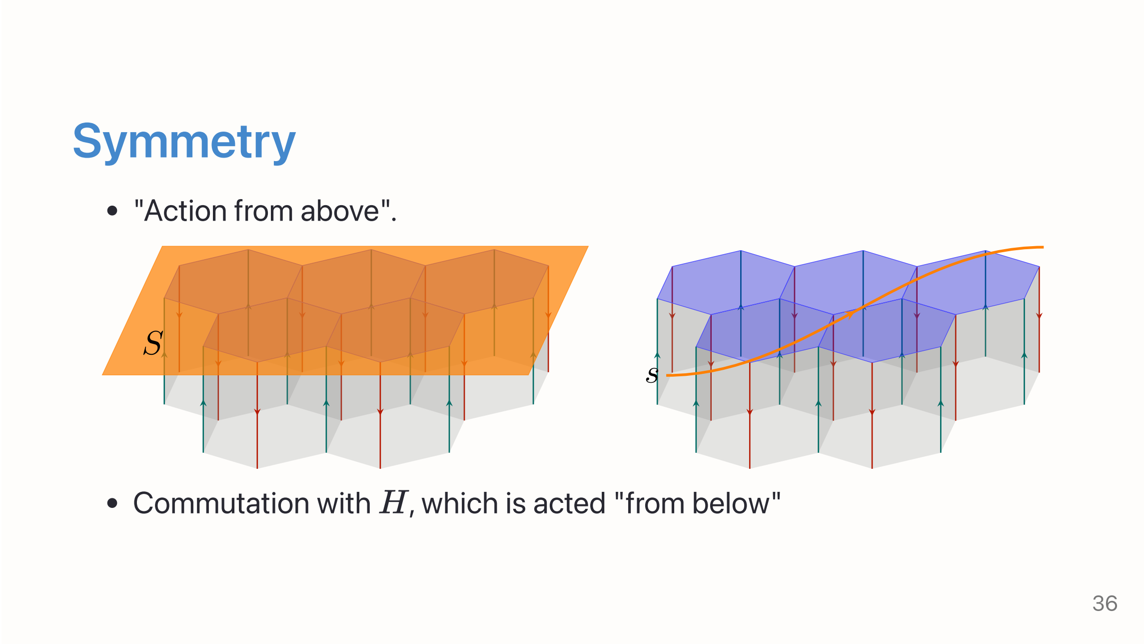
- Depends on fusion 2-cat $\mathcal{C}$, $\rho, \sigma, \lambda \in \mathrm{Obj}\,\mathcal{C}$ and 1-morphisms f, g.
- $\tilde{\mathcal{H}} := \{|\{\Gamma, a, \phi\}\rangle\}$: colorings on top.
  (Γ :obj., a: 1-morph., ϕ: 2-morph.)
- Constraints on $\{\Gamma, a, \phi\}$ determined by λ, ρ, σ, f, g.



Hamiltonian

- $\mathcal{F} \in \operatorname{Simp} \mathcal{C}$ + data at intersections
  $\rightsquigarrow$ a term in local plaguette Hamiltonian.: $H_0 = \sum_{\{\vec{\mathcal{F}}\}} w(\{\mathcal{F}\}) H_p^{\{\mathcal{F}\}}$

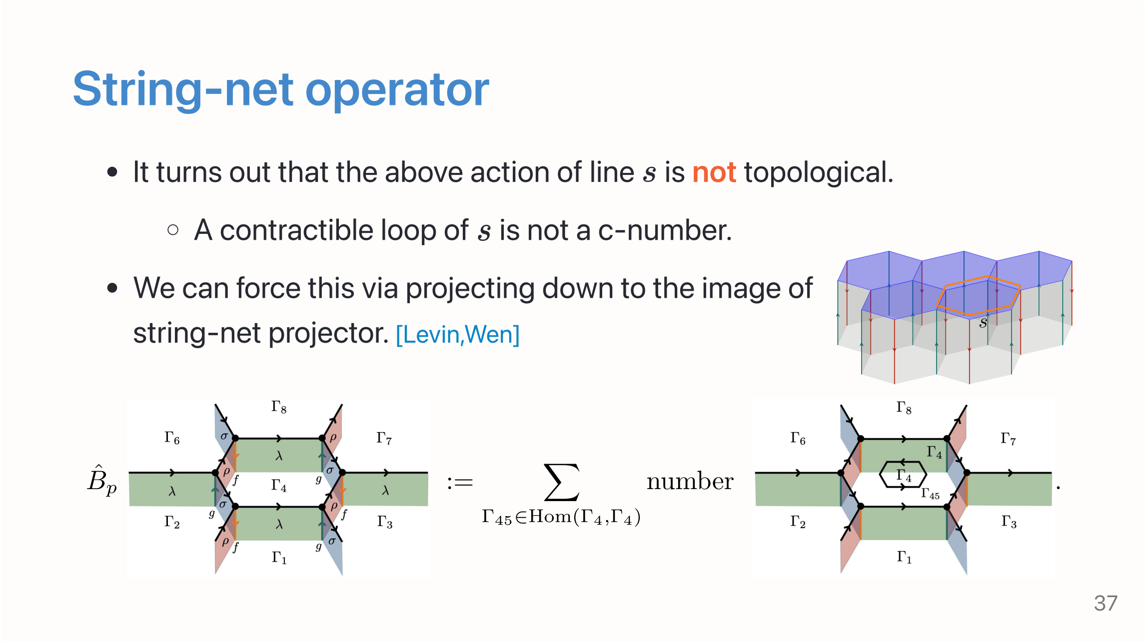
- Explicit formula in terms of the 10-j symbol.



Symmetry

- "Action from above".

- Commutation with H, which is acted "from below"



String-net operator

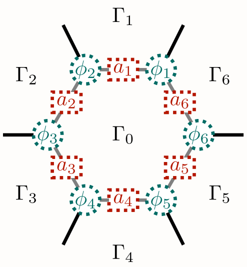
- It turns out that the above action of line s is **not** topological.
    - A contractible loop of s is not a c-number.
- We can force this via projecting down to the image of string-net projector. [Levin,Wen]

Examples

- Using invertible fusion 2-categories, we can obtain the models with
    - (anomalous) one-form symmetries
    - 2-group symmetries.
- $\mathcal{C} = \mathrm{Mod}(\mathcal{B})$ for anyon statistics $\mathcal{B}$:
  microsopic model with the symmetry of anyons!
- Reproduce Levin-Wen solvable models for non-chiral $\mathcal{B}$ as special cases.

Summary

- **Higher** and **non-invertible** sym.s are described by **fusion d-category**.
- For $d = 1$ [AFM] and $d = 2$:
  explicit lattice models with any given fusion-d category!
- This includes non-invertible one-form symmetry of **anyons**!
    - *Numerical study is desired.*



Thank you!

Backup

- Dual : $a \rightsquigarrow a^*$: orientation reversal of a.
  - Evaluation $\mathrm{ev} \in \mathrm{Hom}_{\mathcal{C}}(a \otimes a^*, \mathbb{1})$,
    coevaluation $\mathrm{coev} \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, a \otimes a^*)$.
  - **Relaxation of inverse**.

Graph Zeta Functions and Kazakov-Migdal Model

Kazutoshi Ohta

Abstract. We consider a generalized Kazakov-Migdal model defined on an arbitrary graph. The partition function of the model can be represented by the unitary matrix integral of the weighted graph zeta functions, which have series expansions by possible Wilson loops (graph cycles). The partition function of the model is expressed in two different ways according to the order of integration. A specific unitary matrix integral can be performed even at finite N, thanks to this duality. In addition, we evaluate exactly the partition function of the Kazakov-Migdal model on the graph in the large N limit and show that it is expressed by the infinite product of the graph zeta functions. We also discuss an extension including the bumps, the random matrix model approach, and the Gross-Witten-Wadia phase transition.

(K. Ohta) Meiji Gakuin University

Graph Zeta Functions
and
Kazakov-Migdal Model

Kazutoshi Ohta
Meiji Gakuin University

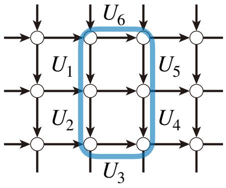
Collaboration with So Matsuura at Keio University

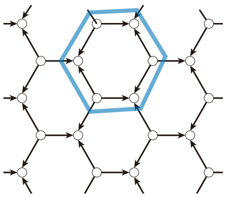
JHEP **2022**, 178 [arXiv: 2204.06424]
PTEP **2022**, 123B03 [arXiv:2208.14032]
arXiv:2303.03692

"QFT and Related Mathematical Aspects", Shuzenji Sogo Kaikan, Shizuoka (2023/3/15)

Introduction (phys)

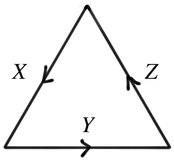
- Cycles on the *graph* play an important role in gauge and string theory
  - Wilson loops in lattice gauge theory:





$$W_C = \mathrm{Tr} U_1 U_2 U_3 U_4^\dagger U_5^\dagger U_6^\dagger \qquad W_C = \mathrm{Tr} U_1 U_2^\dagger U_3 U_4^\dagger U_5 U_6^\dagger$$

  - Gauge invariant operators in quiver gauge theory:



$$\mathcal{O} = \mathrm{Tr}\, XYZ$$

Introduction (math)

- Cycles on the *graph* can be counted by a kind of *zeta function*
    - Ihara introduced a Selberg zeta function of p-adic fields (1966)
    - Serre pointed out a relation to graph theory (1980)
    - Sunada gave a definition of Ihara zeta function for the regular graph and a graph theoretical proof for Ihara's theorem (1986)
    - Hashimoto gave a determinant expression by the edge matrix (1990)
    - Bass proved Ihara's theorem via the determinant expression for generic graphs (1992)
    - Bartholdi introduced two parameter extension of Ihara zeta function (1999)
- Question: Can we utilize the zeta function on the graph (Ihara zeta function) for problems on gauge or string theory?

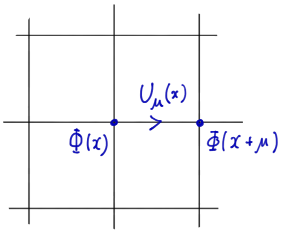
Plan of talk

1. Introduction
2. Ihara Zeta Function
3. Kazakov-Migdal Model on the graph
4. Partition Function
5. Large N Limit
6. Duality
7. Bartholdi Zeta Function and Matrix Model
8. GWW Phase Transition
9. Conclusion and Discussions

Kazakov-Migdal Model

- Kazakov-Migdal model is defined by unitary matrices $U_\mu(x)$ on links (edges) and hermite matrices $\Phi(x)$ on sites (vertices) as D-dimensional lattice gauge theory [Kazakov and Migdal (1992)]:

$$S = \sum_x N\mathrm{Tr}\left(m_0^2\Phi(x)^2 - \sum_{\mu=1,2,\cdots D} \Phi(x)U_\mu(x)\Phi(x+\mu)U_\mu^\dagger(x) \right)$$



- After eliminating $\Phi(x)$, we get

$$\int DUD\Phi\, e^{-S[U,\Phi]} \propto \int DU\, e^{-S_{\mathrm{ind}}[U]}$$

where S_{ind} is a induced action given by

$$S_{\mathrm{ind}}[U] = \frac{1}{2}\mathrm{Tr}\log\left(\delta_{x,y} - m_0^{-2}\sum_\mu U_\mu(x)\otimes U_\mu^\dagger(x)\delta_{x+\mu,y}\right)$$

Generating Function of the Wilson loops

- The induced action has the following expansion:

$$S_{\mathrm{ind}}[U] = -\frac{1}{2}\sum_C \frac{|\,\mathrm{Tr}\,W_C[U]\,|^2}{\ell(C)m_0^{2\ell(C)}}$$

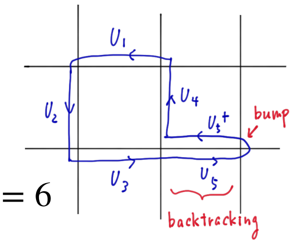
where

C : lattice loops

$\ell(C)$: length of the loops

$W_C[U]$: ordered loop product of U (Wilson loop) along C

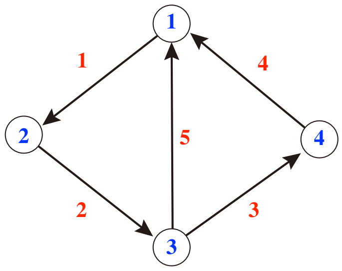
- It had been expected that the induced action reduces to the Yang-Mills action (induced QCD) in the continuum limit, but
  - C contains "bad" (collapsed) Wilson loops
  - $\ell[C]$ does not count the net length of the loop e.g. $\mathrm{Tr}U_1U_2U_5U_5^\dagger U_3U_4 = \mathrm{Tr}U_1U_2U_3U_4$ for $\ell(C)=6$



Graph Theory

- A graph G consists of the vertices V and the edges E; $G = (V, E)$
- Each edge connects two vertices between $s(e)$ and $t(e)$, where $e \in E$, $s(e), t(e) \in V$

- Graph theory gives a mapping from the graph structure to matrices; e.g. <u>double triangle</u>



adjacency matrix:

$$A = \begin{array}{cc} \begin{matrix}1&2&3&4\end{matrix} & \\ \begin{pmatrix}0&1&1&1\\1&0&1&0\\1&1&0&1\\1&0&1&0\end{pmatrix} & \begin{matrix}1\\2\\3\\4\end{matrix}\end{array}$$

$$A_{vv'} = \left\{ \begin{matrix} \sharp \text{ of edges} \\ \text{connecting } v \text{ and } v' \end{matrix} \right\}$$

incidence matrix:

$$B = \begin{array}{cc} \begin{matrix}1&2&3&4&5\end{matrix} & \\ \begin{pmatrix}1&0&0&-1&-1\\-1&1&0&0&0\\0&-1&1&0&1\\0&0&-1&1&0\end{pmatrix} & \begin{matrix}1\\2\\3\\4\end{matrix}\end{array}$$

⇒ charge matrix of
quiver gauge theory,
Dirac operator on the graph,
index theorem on the graph
[S. Matsuura and KO (2021)]

Graph Zeta Function

- The Ihara zeta function of the graph G is defined as follows [Ihara (1966)]:

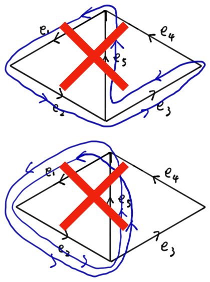
$$\boxed{\zeta_G(q) \equiv \prod_{[C]:\,\text{prime cycles}} \frac{1}{1-q^{\ell(C)}}}$$

where the product is taken over the *prime cycles*, which are

✓ Neither backtracking nor tail (reduced cycle)

✓ Not written by a power of the reduced cycle (primitive cycle) $C' \neq C^r$

✓ Defined by a equivalence classes (a fixed cyclic ordering) $C \sim C'$



$e_1e_2e_5 \sim e_2e_5e_1 \sim e_5e_1e_2$

- By definition, the Ihara zeta function counts non-collapsing cycles only
  ⇒ A coefficient of q^k are the number of the reduced cycles with the total length of k

Cf.) Riemann Zeta Function

- Recall that the Riemann zeta function (Euler product) is defined by a infinite product of all *prime* numbers

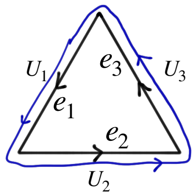
$$\zeta(s) \equiv \prod_{p:\,\text{prime numbers}} \frac{1}{1-p^{-s}} = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

The graph zeta function is an analogy to this (or Selberg zeta function)

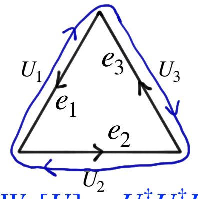
Example1: triangle graph

- A triangle graph is known as a cycle graph C_3 or $\hat{A}_2$ quiver diagram (Dynkin diagram)
- We have two prime cycles:

$$[C] = \{e_1e_2e_3, e_2e_3e_1, e_3e_1e_2\}$$
$$[\bar{C}] = \{\bar{e}_3\bar{e}_2\bar{e}_1, \bar{e}_1\bar{e}_3\bar{e}_2, \bar{e}_2\bar{e}_1\bar{e}_3\}$$



$W_C[U] = U_1U_2U_3$



$W_{\bar{C}}[U] = U_3^\dagger U_2^\dagger U_1^\dagger$

then

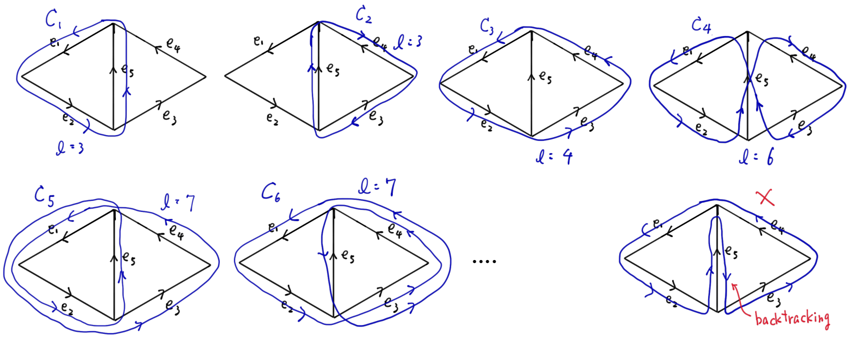
$$\zeta_{C_3}(q) = \frac{1}{(1-q^3)^2} = 1 + 2q^3 + 3q^6 + 4q^9 + 5q^{12} + \cdots$$

power of q (length)	3	6	9	12	$\cdots$
coeff	2	3	4	5	$\cdots$
cycles	C, $\bar{C}$	C^2, $C\bar{C}$, $\bar{C}^2$	C^3, $C^2\bar{C}$, $C\bar{C}^2$, $\bar{C}^3$	C^4, $C^3\bar{C}$, $C^2\bar{C}^2$ $C\bar{C}^3$, $\bar{C}^4$	$\cdots$

Easy? ⇒ For more general graph, the prime cycles are rather complicated (infinite)

Example2: double triangle graph

- For the double triangle (DT) graph, there are infinitely many prime loops



- Then, the Euler product expression of the graph zeta function becomes the infinite product in general

$$\zeta_{\mathrm{DT}}(q) = \frac{1}{(1-q^3)^4}\frac{1}{(1-q^4)^2}\frac{1}{(1-q^6)^2}\frac{1}{(1-q^7)^4}\cdots$$

- But we have another determinant (a reciprocal of a polynomial) expression of the graph zeta function

Ihara's theorem

- For a given graph G, the Ihara zeta function is given by the following determinant formula

$$\zeta_G(q) = \frac{1}{(1-q^2)^{n_E - n_V}\det\left(I - qA + q^2(D-I)\right)}$$

where

n_V : the number of the vertices
n_E : the number of the edges
I : $n_V \times n_V$ identity matrix
D : the degree matrix (♯ of the edges attached with the vertex)
A : the adjacency matrix

$$A_{vv'} = \left\{\sharp \text{ of edges connecting } v \text{ and } v'\right\}$$

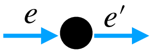
A Brief Proof

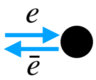
- Using an identity for the determinant between vertex and edge adjacency matrices, we can show that

$$(1-q^2)^{n_V+n_E}\det\left(I_{2n_E}-qW\right)=(1-q^2)^{2n_E}\det\left(I_{n_V}-qA+q^2(D-I_{n_V})\right)$$

where

W : the edge adjacency matrix without bumps

$W_{ee'}=1$ for 

But, $W_{e\bar{e}}=0$ for 

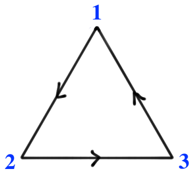
- Then, we obtain

$$\zeta_G(q)=\frac{1}{\det\left(I_{2n_E}-qW\right)}=\exp\left\{\sum_{k=1}^{\infty}\frac{q^k}{k}\mathrm{Tr}W^k\right\}=\frac{1}{(1-q^2)^{n_E-n_V}\det\left(I_{n_V}-qA+q^2(D-I_{n_V})\right)}$$

Hashimoto expression

Ihara expression

Example1: triangle graph



- For the triangle graph, $n_V=n_E=3$ and

$$D=\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix},\qquad A=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$$

then we have

$$\zeta_{C_3}(q)=\frac{1}{\det\left(I-qA-q^2\left(D-I\right)\right)}=\frac{1}{(1-q^3)^2}$$

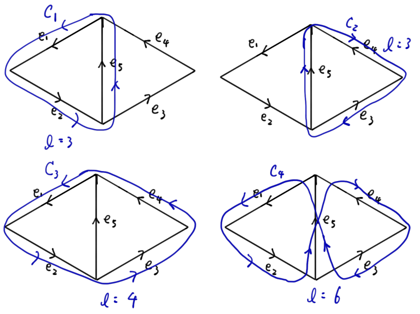
- This agrees with the previous simple observation from the graph

Example2: double triangle graph

- For the double triangle (DT) graph, $n_V = 4$, $n_E = 5$ and

$$D = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{aligned}\zeta_{\mathrm{DT}}(q) &= \frac{1}{(1-q^2)\det\left(I - qA - q^2(D-I)\right)} \\ &= \frac{1}{(1-q^4)(1+q^2-2q^3)(1-q^2-2q^3)} \\ &= 1 + 4q^3 + 2q^4 + 12q^6 + 12q^7 + 3q^8 + \cdots\end{aligned}$$



- For first few terms, the counting is as follows:

length	3	4	6	7	$\cdots$
coeff	4	2	12	12	$\cdots$
cycles	$C_1, \bar{C}_1,$ $C_2, \bar{C}_2$	$C_3, \bar{C}_3$	$C_1^2, \bar{C}_1^2, C_2^2, \bar{C}_2^2,$ $C_1\bar{C}_1, C_2\bar{C}_2,$ $C_1C_2, \bar{C}_1C_2, C_1\bar{C}_2, \bar{C}_1\bar{C}_2$ $C_4, \bar{C}_4$	$C_1C_3, \bar{C}_1C_3, C_2C_3, \bar{C}_2C_3,$ $C_1\bar{C}_3, \bar{C}_1\bar{C}_3, C_2\bar{C}_3, \bar{C}_2\bar{C}_3,$ $C_5, \bar{C}_5, C_6, \bar{C}_6$	$\cdots$

Kazakov-Migdal model on the graph

- We consider the generalized Kazakov-Migdal model defined on the graph

$$Z_{\mathrm{gKM}} = \int \prod_{v\in V} d\Phi_v \prod_{e\in E} dU_e \exp\left\{-\beta \mathrm{Tr}\left(\frac{1}{2}\sum_{v\in V} m_v^2 \Phi_v^2 - q\sum_{e\in E} \Phi_{s(e)} U_e \Phi_{t(e)} U_e^\dagger\right)\right\}$$

where

V : a set of vertices (sites) of the graph
E : a set of edges (links) of the graph
$s(e)$: a *source* of the edge e
$t(e)$: a *target* of the edge e

- We can also integrate out the scalar field Φ_v, then get

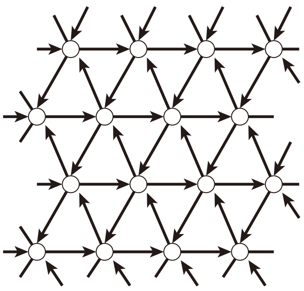
$$Z_{\mathrm{gKM}} \propto \int \prod_{e\in E} dU_e \exp\left\{-\frac{1}{2}\mathrm{Tr}\log\left(m_v^2\delta_{vv'} \otimes \mathbf{1}_{N^2} - qA[U]\right)\right\}$$

where $A[U]$ is the weighted adjacency matrix

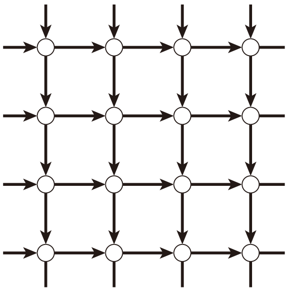
- For the general couplings, the induced action still generates the “bad” (collapsed) Wilson loops, but we will show that they can be removed by a coupling tuning

Discretization as the graph

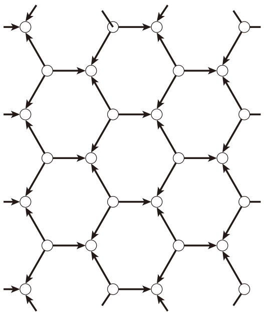
- At least, the flat plane can be discretized by using the graphs



Triangle lattice



Square lattice
(Original KM model)



Hexagon lattice

- The continuum limit and emergence of the dimensionality are difficult problem

Coupling tuning

- The partition function of the graph Kazakov-Migdal model becomes

$$\begin{aligned} Z_{\mathrm{gKM}} &= \int \prod_{v\in V} d\Phi_v \prod_{e\in E} dU_e \exp\left\{ -\beta \mathrm{Tr}\left(\frac{1}{2}\sum_{v\in V} m_v^2 \Phi_v^2 - q \sum_{e\in E} \Phi_{s(e)} U_e \Phi_{t(e)} U_e^\dagger \right) \right\} \\ &\propto \frac{1}{(1-q^2)^{\frac{N^2}{2}(n_E - n_V)}} \int \prod_{e\in E} dU_e \, \exp\left\{ -\frac{1}{2} \mathrm{Tr}\log\left(m_v^2 \delta_{vv'} \otimes \mathbf{1}_{N^2} - qA[U] \right) \right\} \\ &= \int \prod_{e\in E} dU_e \frac{1}{(1-q^2)^{\frac{N^2}{2}(n_E - n_V)} \det\left(m_v^2 I - qA[U] \right)^{\frac{1}{2}}} \end{aligned}$$

where $A[U]$ is a unitary matrix weighted adjacency matrix

- The determinant looks like the determinant formula of the Ihara zeta function. In fact, by setting $m_v^2 = 1 + (\deg v - 1)q^2$, we get

$$Z_{\mathrm{gKM}} \propto \int \prod_{e\in E} dU_e \, \zeta_G(q;U)^{\frac{1}{2}}$$

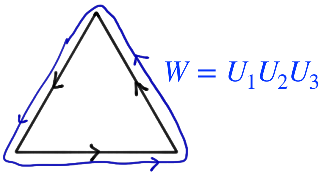
$\zeta_G(q;U)$: the unitary matrix weighted
Ihara zeta function

Example1: triangle graph

- The unitary matrix weighted adjacency matrix for $U(N)$:

$$A[U] = \begin{pmatrix} 0 & U_1 \otimes U_1^\dagger & U_3^\dagger \otimes U_3 \\ U_1^\dagger \otimes U_1 & 0 & U_2 \otimes U_2^\dagger \\ U_3 \otimes U_3^\dagger & U_2^\dagger \otimes U_2 & 0 \end{pmatrix}$$

- Recalling that the Ihara zeta is a generating function of the multi-trace Wilson loops, then we obtain



$$Z_{\mathrm{gKM}} \propto \int dU_1 dU_2 dU_3 \frac{1}{\det\left(I - qA[U] + q^2(D-I)\right)^{\frac{1}{2}}}$$

$$= \int dU_1 dU_2 dU_3 \exp\left\{ \sum_{k=1}^{\infty} \frac{q^{3k}}{k} |\mathrm{Tr}(U_1U_2U_3)^k|^2 \right\}$$

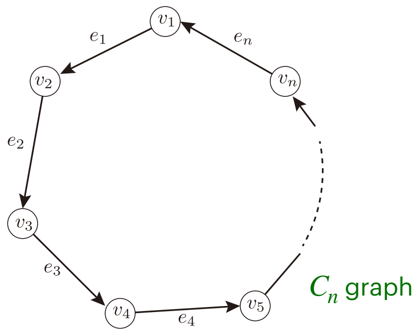
$$= \int dW \left\{ 1 + |\mathrm{Tr}W|^2 q^3 + \frac{1}{2}\left(|\mathrm{Tr}W|^4 + |\mathrm{Tr}W^2|^2 \right) q^6 + \frac{1}{6}\left(|\mathrm{Tr}W|^6 + 3|\mathrm{Tr}W|^2|\mathrm{Tr}W^2|^2 + 2|\mathrm{Tr}W^3|^2 \right) q^9 + \cdots \right\}$$

$$= \prod_{i=1}^{N} \frac{1}{1-q^{3i}}$$

where $W \equiv U_1U_2U_3$

Cycle graph C_n

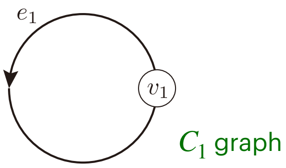
- We can generalize the previous results to cycle graphs (polygons, $\hat{A}_{n-1}$ quiver diagram)



- After integrating over the unitary groups, we obtain

$$Z_{C_n}(q) = Z_{C_1}(q^n)$$

$$= \prod_{i=1}^{N} \frac{1}{1-q^{ni}}$$



Large N limit

- It is difficult to perform the integral over the unitary matrices in general, but the situation becomes simple in the large N limit, thanks to the decomposition (clustering) of the vev of the Wilson loops

$$\begin{aligned} Z_{\mathrm{gKM}} &\propto \int \prod_{e\in E} dU_e \exp\left\{ \sum_{C\in\Pi^+} \sum_{k=1}^{\infty} \frac{q^{\ell(C)k}}{k} |\mathrm{Tr} W_C[U]^k|^2 \right\} \\ &\equiv \left\langle \prod_{C\in\Pi^+} \exp\left\{ \sum_{k=1}^{\infty} \frac{q^{\ell(C)k}}{k} |\mathrm{Tr} W_C[U]^k|^2 \right\} \right\rangle \\ &\xrightarrow[N\to\infty]{} \prod_{C\in\Pi^+} \left\langle \exp\left\{ \sum_{k=1}^{\infty} \frac{q^{\ell(C)k}}{k} |\mathrm{Tr} W_C[U]^k|^2 \right\} \right\rangle \\ &= \prod_{C\in\Pi^+} \prod_{i=1}^{\infty} \frac{1}{1-q^{i\ell(C)}} = \prod_{i=1}^{\infty} \zeta_G(q^i)^{\frac{1}{2}} \end{aligned}$$

  where Π^+ is a set of *chiral* prime loops (choose a one direction of the loops)

- The partition function of the graph Kazakov-Migdal model can be written by a infinite product of (square roots of) the Ihara zeta functions

Duality

- We can perform the U_e integral by using the Harish-Chandra-Itzykson-Zuber integral:

$$\int dU\, e^{t\mathrm{Tr}AUBU^\dagger} \propto \frac{\det_{i,j} e^{ta_i b_j}}{\Delta(a)\Delta(b)}$$

  where a_i, b_j are the eigenvalues and $\Delta(a), \Delta(b)$ are the Vandermonde determinants of A, B

- Then we obtain the multi matrix model for Φ_v as the partition function of the graph Kazakov-Migdal model

$$Z_{\mathrm{gKM}} \propto \int \prod_{v\in V} \prod_{i=1}^{N} d\phi_{v,i}\, e^{-\frac{1}{2}m_v^2\phi_{v,i}^2} \Delta(\phi_v)^{2-\deg v} \prod_{e\in E} \det_{i,j} e^{q\phi_{s(e),i}\phi_{t(e),j}}$$

- We can perform this integral exactly for simpler cases like the cycle graph and agrees with the results from the graph zeta function, but it is difficult to evaluate for the generic graphs

Extension to Bartholdi's zeta function

- It is known that there is a generalization of the Ihara zeta function, which contains one more parameter and counts the number of the bumps too
  ⇒ Bartholdi's zeta function

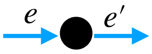
$$\zeta'_G(q,u) \equiv \prod_{C:\text{primitive (not reduced)}} \frac{1}{1-u^{cbc(C)}q^{\ell(C)}}$$
$$= \frac{1}{\left(1-(1-u)^2q^2\right)^{n_E-n_V}\det\left(I-qA+(1-u)q^2\left(D-(1-u)I\right)\right)}$$

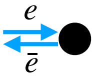
  where $cbc(C)$ is cyclic bump count (# of bumps) and $\lim_{u\to 0}\zeta'_G(q,u) = \zeta_G(q)$

- A proof is similar to the Ihara zeta function

$$\zeta'_G(q,u) = \frac{1}{\det\left(I-q(W+uJ)\right)} = \frac{1}{(1-(1-u^2)q^2)^{n_E-n_V}\det\left(I-qA+(1-u)q^2(D-(1-u)I)\right)}$$

  where

  $W_{ee'} = 1$ for 

  $J_{e\bar{e}} = 1$ for 

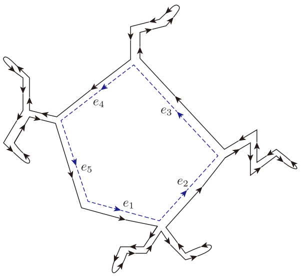
Graph Kazakov-Migdal model with bumps

- The mass and coupling of the Kazakov-Migdal model is tuned to be

$$S_{\text{gKM}} = \operatorname{Tr}\left\{\frac{1}{2}\sum_{v\in V}\left(1-q^2(1-u)^2+q^2(1-u)\deg v\right)\Phi_v^2 - q\sum_{e\in E}\Phi_{s(e)}U_e\Phi_{t(e)}U_e^\dagger\right\}$$

- Then, the partition function becomes

$$Z_{\text{gKM}} = \int\prod_{v\in V}d\Phi_v\prod_{e\in E}dU_e e^{-\beta S_{\text{gKM}}}$$
$$= \left(\frac{2\pi}{\beta}\right)^{\frac{1}{2}n_VN^2}\int_{e\in E}dU_e\frac{1}{\det\left(I-qA_U+q^2(1-u)(D-(1-u)I)\right)}$$
$$= \left(\frac{2\pi}{\beta}\right)^{\frac{1}{2}n_VN^2}\left(1-(1-u)^2q^2\right)^{\frac{1}{2}(n_E-n_V)N^2}\mathscr{V}_G(q,u)^{\frac{N^2}{2}}$$
$$\times\int_{e\in E}dU_e\prod_{C\in\Pi^+}\exp\left\{\sum_{k=1}^{\infty}\frac{1}{k}f_C(q,u)^k\left|\operatorname{Tr}W_C[U]^k\right|^2\right\}$$

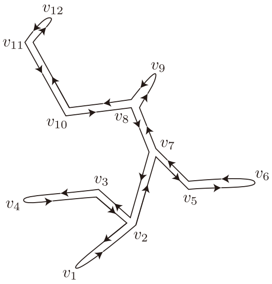


  where $f_C(q,u) = \sum_{\tilde{C}:\text{ reducible to }C} u^{cbc(\tilde{C})}q^{|\tilde{C}|}$

Contribution from the zero-area Wilson loops (collapsed cycles)

- $\mathscr{V}_G(q,u)$ is the contribution from the collapsed cycles (zero-area Wilson loops)
- For $G = C_n$,

$$\mathscr{V}_{C_n}(q,u) = \left\{ \frac{1 + (1-u^2)q^2 - \sqrt{1 - 2(1+u^2)q^2 + (1-u^2)^2 q^4}}{2q^2} \right\}^n$$

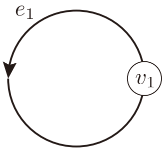


Generalized Catalan number

e.g.

$$\mathscr{V}_{C_1}(q,u) = 1 + u^2q^2 + (u^2+u^4)q^4 + (u^2 + 3u^4 + u^6)q^6 + \cdots$$

$$\leftrightarrow N + u^2\mathrm{Tr}UU^\dagger q^2 + (u^2\mathrm{Tr}UUU^\dagger U^\dagger + u^4\mathrm{Tr}UU^\dagger UU^\dagger)q^4$$
$$+(u^2\mathrm{Tr}UUUU^\dagger U^\dagger U^\dagger$$
$$+u^4(\mathrm{Tr}UU^\dagger UUU^\dagger U^\dagger + \mathrm{Tr}UUU^\dagger UU^\dagger U^\dagger + \mathrm{Tr}UUU^\dagger U^\dagger UU^\dagger)$$
$$+u^6\mathrm{Tr}UU^\dagger UU^\dagger UU^\dagger)q^6 + \cdots$$



Comparison with matrix model

- In the large N limit, $\mathscr{V}_G(q,u)$ contributes to the free energy at order N^2

$$\mathscr{F}_{\mathrm{gKM}} \sim N^2 \left(-\frac{1}{n_V} \log \mathscr{V}_G(q,u) \right) + \mathscr{O}(N)$$

- On the other hand, after eliminating U_e, we obtain the matrix model for Φ_v
- The exact semi-circle solution for this matrix model [Gross 1992] corresponds to the N^2 order contribution to the free energy
- Thus, we expect that the semi-circle solution of the matrix model at large N comes from the zero-area (collapsed) Wilson loops only $\Rightarrow$ infinite tension strings [Boulatov 1992]
- The Wilson loops with the bumps are important to understand the relation to the string theory (zigzag symmetry)

KM model with the fields in the fundamental representation

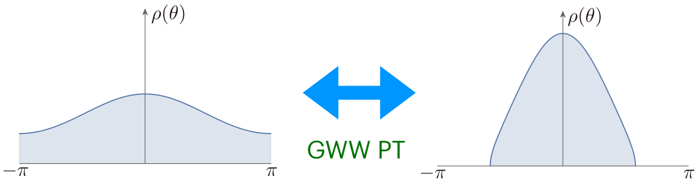
- In the original KM model, the Gross-Witten-Wadia (GWW) phase transition seems not to occur
- We replace the scalar fields in the adjoint representation with N_f fundamental fields (FKM model)

$$S = \sum_{v\in V} m_v^2 \Phi_v^{\dagger I}\Phi_{vI} - q\sum_{e\in E}\left(\Phi^{\dagger}_{s(e)}{}^{I} U_e \Phi_{t(e)I} + \Phi^{\dagger}_{t(e)}{}^{I} U_e^{\dagger}\Phi_{s(e)I}\right) \qquad (I = 1,2,\ldots,N_f)$$

- Then we get the effective action:

$$S_{\text{eff}}(U) = \sum_{C\in[\Pi_+]} N_f \sum_{n=1}^{\infty} \frac{1}{n} f_C(q,u)^n \Big(\operatorname{Tr} W_C[U]^n + \operatorname{Tr} W_C^{\dagger}[U]^n\Big)$$

- For the cycle graph C_n , we can show that there two different phases with respect to the eigenvalue distribution of U



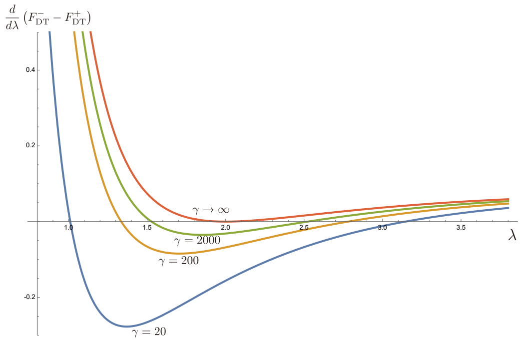
GWW Phase Transition

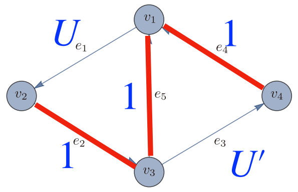
- For the general graph, it is difficult to see the GWW phase transition
- However, if we take the scaling limit:

$$q \to 0, \quad \gamma \equiv \frac{N_f}{N_c} \to \infty, \quad \lambda \equiv \frac{1}{\gamma q^l} : \text{fixed}$$

  we can see the GWW phase transition in the symmetric graph, since the action reduces to the Wilsonian one

- For example, the (derivative of) free energy on the double triangle graph behaves as:

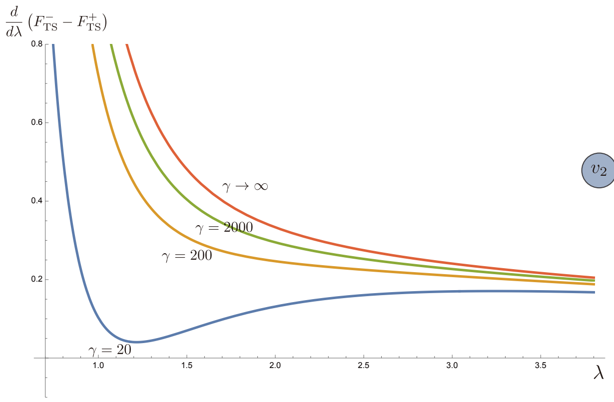


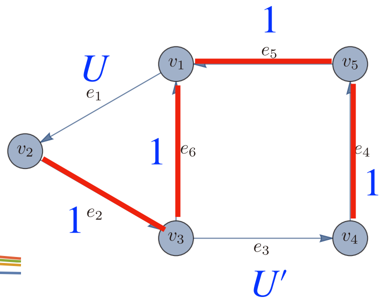


3rd order PT occurs at
$\lambda = 2$ in the $\gamma \to \infty$ limit

GWW Phase Transition

- For the triangle-square graph, the action does not reduce to the Wilsonian one and seems not to occur the 3rd order PT





Conclusion and Discussions

- We proposed a generalization of the Kazakov-Migdal model on the graph, which reproduces the weighted Ihara zeta function
- The graph Kazakov-Migdal model generates the countable Wilson loops (excluding/including the bumps)
- We can perform the unitary matrix integral exactly in the large N limit and the partition function of the graph Kazakov-Migdal model is given by the infinite product of the Ihara zeta function
- We can see the interesting "physics" like GWW PT in the graph zeta function models
- We expect much more applications of the graph zeta function to the counting (index) of the gauge invariant operators (chiral rings) in quiver gauge theory

Emergent N=4 supersymmetry from N=1

Jaewon Song

Abstract.

(J. Song) KAIST

Emergent $\mathcal{N}$=4 supersymmetry from $\mathcal{N}$=1

(or Landscape of 4d SCFTs with $a = c$)

Quantum Field Theory and Related Mathematical Aspects
@Shuzenji

Based on
collaborations with:

Monica Kang (Caltech), Craig Lawrie (Desy), Ki-Hong Lee (KAIST)

KAIST Korea Advanced Institute of
Science and Technology

Jaewon Song
Mar. 14th, 2023

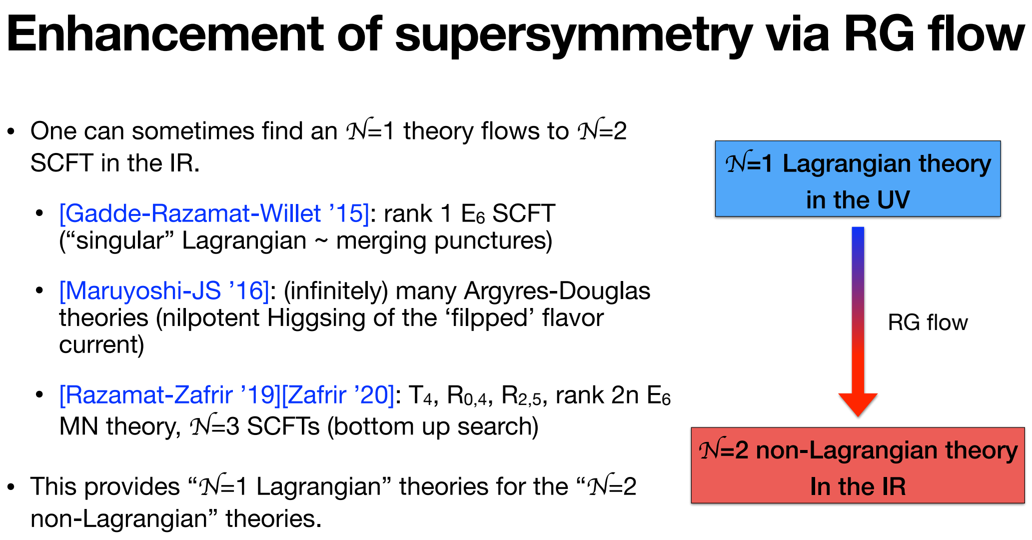
Introduction

Quantum Field Theory and Symmetry

- **Symmetry** is one of the most fundamental principle in physics.
    - Symmetry constrain the system: consequences in the observables.
    - Totalitarian Principle: “Everything not forbidden is compulsory”
    - In this sense, symmetry ***defines*** a quantum system.
- Symmetry can change along the renormalization group flow.
    - It can be spontaneously broken.
    - Symmetry can be emergent. Accidental symmetry, symmetry enhancement.

Supersymmetry is good

- Supersymmetry is a spacetime symmetry that exchanges bosons and fermions, that may explain or solve: gauge hierarchy problem, grand unification, dark matter…
- Supersymmetric field theories provide an ideal **theoretical laboratory** to study non-trivial aspects of quantum field theory.
- **Supersymmetric** quantum field theories are highly **constraining**:
    - Non-renormalization theorems
    - Certain protected quantities are exactly computable.
    - Rich mathematical structures.
- Learned lot about RG using SUSY: IR duality, conformal manifolds, symmetry enhancement, dangerously irrelevant operators, non-commuting flows, …



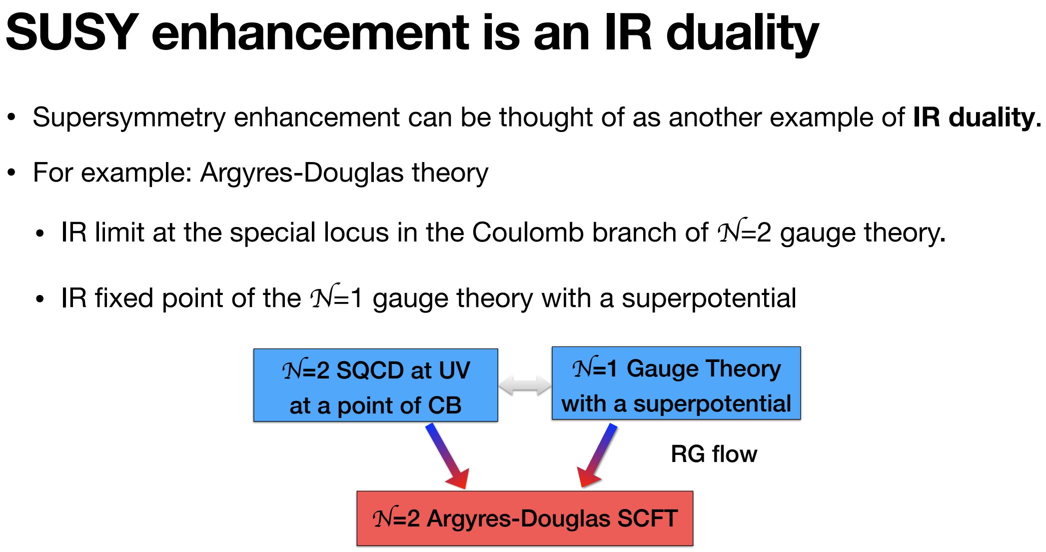
Enhancement of supersymmetry via RG flow

- One can sometimes find an $\mathcal{N}$=1 theory flows to $\mathcal{N}$=2 SCFT in the IR.
  - [Gadde-Razamat-Willet '15]: rank 1 E_6 SCFT ("singular" Lagrangian ~ merging punctures)
  - [Maruyoshi-JS '16]: (infinitely) many Argyres-Douglas theories (nilpotent Higgsing of the 'filpped' flavor current)
  - [Razamat-Zafrir '19][Zafrir '20]: T_4, $R_{0,4}$, $R_{2,5}$, rank 2n E_6 MN theory, $\mathcal{N}$=3 SCFTs (bottom up search)
- This provides "$\mathcal{N}$=1 Lagrangian" theories for the "$\mathcal{N}$=2 non-Lagrangian" theories.

$\mathcal{N}$=1 Lagrangian theory in the UV

RG flow

$\mathcal{N}$=2 non-Lagrangian theory In the IR



SUSY enhancement is an IR duality

- Supersymmetry enhancement can be thought of as another example of **IR duality**.
- For example: Argyres-Douglas theory
  - IR limit at the special locus in the Coulomb branch of $\mathcal{N}$=2 gauge theory.
  - IR fixed point of the $\mathcal{N}$=1 gauge theory with a superpotential

$\mathcal{N}$=2 SQCD at UV at a point of CB

$\mathcal{N}$=1 Gauge Theory with a superpotential

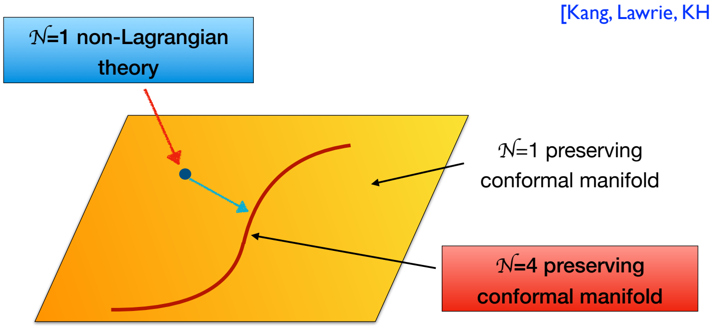
RG flow

$\mathcal{N}$=2 Argyres-Douglas SCFT

Summary:

We find a new duality between $\mathcal{N}$=1 non-Lagrangian theory and the $\mathcal{N}$=4 Super Yang-Mills theory!

[Kang, Lawrie, KH Lee, JS '23]



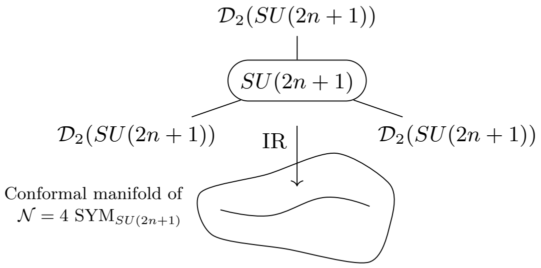
$\mathcal{N}$=1 non-Lagrangian theory

$\mathcal{N}$=1 preserving conformal manifold

$\mathcal{N}$=4 preserving conformal manifold

The Dual Theory

The $\mathcal{N}$=1 Dual of $\mathcal{N}$=4 SYM



The dual theory is built out of 3 copies of $\mathscr{D}_2(SU(2n+1))$ theory, gauging the diagonal flavor symmetry group via $\mathcal{N}$=1 gauge multiplet.

This gives an asymptotic free gauge theory that **flows to a point on the conformal manifold** of $\mathcal{N}$=4 SYM theory with G=SU(2n+1)

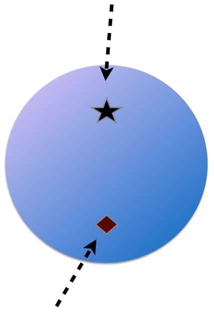
$\mathscr{D}_p[G]$ theory

[Cecotti, Del Zotto]
[Cecotti, Del Zotto, Giacomelli]
[Xie][Wang, Xie]

- It is a 4d $\mathcal{N}$=2 SCFT (Argyres-Douglas type) with **flavor symmetry** G (or larger).
- It can be realized as the 6d $\mathcal{N}$=(2, 0) theory of type G compactified on a **sphere** with one **irregular puncture** (p) and one **full regular puncture** (flavor G).
- The flavor symmetry is **exactly** G (without any U(1)'s) for *some choice* of *p*, when the irregular puncture does not possess extra flavor symmetry. Let's restrict to this case from now on.

G	$SU(N)$	$SO(2N)$	E_6	E_7	E_8
No additional symmetry	$(p,N)=1$	$p \notin 2\mathbb{Z}_{>0}$	$p \notin 3\mathbb{Z}_{>0}$	$p \notin 2\mathbb{Z}_{>0}$	$p \notin 30\mathbb{Z}_{>0}$

Irregular puncture (*p*)



Full regular puncture (*G*)

$\mathscr{D}_p[G]$ theory (cont'd)

- The flavor central charge ("amount of matter") for G: $k_G = \frac{2(p-1)}{p} h_G^\vee$
  It is defined as $-2\mathrm{Tr}(R_{\mathcal{N}=2} T^a T^b) = k_G \delta^{ab}$.
- Once coupled to ($\mathcal{N}$=1) G-gauge field, it behaves as a **fractional amount of an adjoint matter**. For the case of p=2, $k_G = h_G^\vee$ which is like a half of an adjoint matter.
- As an $\mathcal{N}$=2 SCFT, it possess $SU(2)_R \times U(1)_{R_{\mathcal{N}=2}}$ R-symmetry in addition to the flavor symmetry G.
- As an $\mathcal{N}$=1 SCFT, it contains $U(1)_R$ symmetry and a $U(1)_F$ flavor symmetry generated by

$$R_0 = \frac{1}{3} R_{\mathcal{N}=2} + \frac{4}{3} I_3 \ , \quad \mathcal{F} = -R_{\mathcal{N}=2} + 2 I_3$$

$\mathcal{N}$=1 Gauging of $\mathscr{D}_p[G]$ theories

[Kang, Lawrie, KH Lee, JS '21]

- Now, consider diagonal gauging of 3-copies of D_2[SU(2n+1)] theories.
- The 1-loop beta-function coefficient for the gauge coupling is given as
  $$\beta_g \sim -\mathrm{Tr} RGG \sim -\frac{3}{2}(2n+1) < 0 \quad : \quad \text{Asymptotically free}$$
- D_2[G] theory behaves like a **half of an adjoint chiral multiplet** in terms of the beta-function contribution. (N_f = 3/2 N_c)
- There are 3 $U(1)_F$ symmetries for each D_2[SU(2n+1)] and one of them is broken by ABJ anomaly.
- What is the IR fixed point upon RG flow? On general ground, we expect it to be an $\mathcal{N}$=1 SCFT.

Superconformal fixed point

- Necessary condition: Non-anomalous **U(1) R-symmetry**

$$\mathrm{Tr}RT^aT^b = 0 \leftrightarrow T(\text{adj}) + \sum_i T(R_i)(r_i - 1) = 0$$

- Due to the superconformal symmetry, the conformal anomalies are fixed by the trace anomalies of R-symmetry. [Anselmi, Freedman, Grisaru, Johansen]

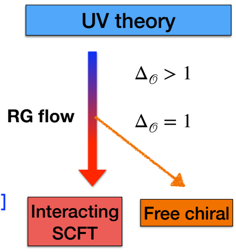
$$a = \frac{3}{32}\left(3\mathrm{Tr}R^3 - \mathrm{Tr}R\right)\ ,\quad c = \frac{1}{32}\left(9\mathrm{Tr}R^3 - 5\mathrm{Tr}R\right)$$

- The R-symmetry is not always determined via anomaly constraint. There can be a family of candidate R-symmetries.
- The superconformal R-symmetry is fixed by '**a-maximization**':

$$\frac{\partial a_{\text{trial}}}{\partial R} = 0\ ,\quad \frac{\partial^2 a_{\text{trial}}}{\partial R^2} < 0$$

[Intriligator, Wecht]

Decoupling of operators along the RG flow



- **Important caveat** in a-maximization: accidental symmetry
- Some of the gauge invariant operators may seem to violate the **unitarity bound**: $\Delta \geq 1$.
- Plausible scenario: such an operator gets **decoupled along the RG flow** and becomes free with $\Delta_{\mathcal{O}} = 1$. [Kutasov, Parnavhev, Sahakyan]
- One can remove the decoupled free field by introducing a '**flip field**' X and a superpotential coupling $W = X\mathcal{O}$.
- Redo the *a*-maximization until no operator gets decoupled.

[Barnes, Intriligator, Wecht, Wright]
[Benvenuti, Giacomelli]
[Maruyoshi, Nardoni, JS]

IR fixed point of the gauged $\mathscr{D}_2[G]^{\otimes 3}$

[Kang, Lawrie, KH Lee, JS '21]
[Kang, Lawrie, KH Lee, JS '23]

- The superconformal R-symmetry in the IR is given as

$$R = R_0 + \sum_i \epsilon_i F_i$$

- The mixing parameter is constrained by the anomaly free condition ,

$$0 = \mathrm{Tr} RGG = h_G^\vee + \sum_i \left(\left(\frac{1}{3} - \epsilon_i \right) \mathrm{Tr}_i R_{\mathcal{N}=2} GG + \left(\frac{4}{3} + 2\epsilon_i \right) \mathrm{Tr}_i I_3 GG \right)$$

$$6 - \textstyle\sum_{i=1}^{3} (1 - 3\varepsilon_i) = 0$$

- The mixing parameter is fixed by a-maximization:

$$a(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \frac{d}{32} \left(13 - 9 \sum_{i=1}^{3} \varepsilon_i^2 (\varepsilon_i + 2) \right) \longrightarrow \varepsilon := \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -\frac{1}{3}$$

Anomaly Matching

The anomaly polynomial for the IR theory is given as

$$\begin{aligned} I_6 = & \frac{1}{6} k_{RRR} c_1(R)^3 + \sum_{\alpha=1}^{2} \frac{1}{6} k_{RR\mathcal{F}_\alpha} c_1(R)^2 c_1(\mathcal{F}_\alpha) \\ & + \sum_{\alpha,\beta=1}^{2} \frac{1}{6} k_{R\mathcal{F}_\alpha \mathcal{F}_\beta} c_1(R) c_1(\mathcal{F}_\alpha) c_1(\mathcal{F}_\beta) \\ & + \sum_{\alpha,\beta,\gamma=1}^{2} \frac{1}{6} k_{\mathcal{F}_\alpha \mathcal{F}_\beta \mathcal{F}_\gamma} c_1(\mathcal{F}_\alpha) c_1(\mathcal{F}_\beta) c_1(\mathcal{F}_\gamma) \\ & - \frac{1}{24} k_R c_1(R) p_1(T) - \sum_{\alpha=1}^{2} \frac{1}{24} k_{\mathcal{F}_\alpha} c_1(\mathcal{F}_\alpha) p_1(T) \end{aligned}$$

$$k_{RRR} = \frac{8d}{9}, \quad k_{R\mathcal{F}_\alpha^2} = -\frac{2d}{3}, \quad k_{R\mathcal{F}_1\mathcal{F}_2} = \frac{d}{3},$$

$$k_{\mathcal{F}_1^2 \mathcal{F}_2} = -k_{\mathcal{F}_1 \mathcal{F}_2^2} = d, \quad a = c = \frac{1}{4} d,$$

$$d = \mathrm{dim}(SU(2n+1)) = 4n(n+1)$$

$$\mathscr{F}_\alpha \equiv F_{\alpha+1} - F_\alpha$$

It agrees with that of the $\mathcal{N}$=4 SYM theory with G=SU(2n+1)!

Matching operator spectrum

Matching of chiral operators

- Each D2[SU(2n+1)] has **Coulomb branch operators** of scaling dimensions
$$\Delta_{CB} = \frac{1}{2} R_{\mathcal{N}=2} = \left\{ \frac{3}{2}, \frac{5}{2}, \cdots, \frac{2n+1}{2} \right\}$$
and also their superpartners having $(\Delta, I_3, R_{\mathcal{N}=2}) = (\Delta_{CB} + 1,\ 1,\ r - 2)$.
- Each D2[SU(2n+1)] has the **moment map operator** μ in the adjoint of G with dimension 2 and $I_3 = 1,\ R_{\mathcal{N}=2} = 0$.
- Upon flowing to the IR, the scaling dimension of the moment map operator becomes
$$\Delta_{\text{IR}}(\mu) = \tfrac{3}{2} R = \tfrac{3}{2}\left(\tfrac{4}{3} + 2\varepsilon\right) = 1$$
- The scaling dimension of the Coulomb branch operators and their superpartners in the IR become
$$\Delta_{\text{IR}}(u) = (1 - 3\varepsilon)\, \Delta_{\text{UV}}(u)$$
$$\Delta_{\text{IR}}(Q^2 u) = 1 + 6\varepsilon + (1 - 3\varepsilon)\, \Delta_{\text{UV}}(u) \quad \Delta = \{2, 3, \cdots, 2n+1\}$$
Coulomb branch operators of $\mathcal{N}$=4 SYM!

Matching of chiral operators (cont'd)

$\mathcal{N}$=4 SYM	$\mathcal{N}$=1 dual theory
$\mathrm{Tr}\,\phi_{i_1}\phi_{i_2}\cdots\phi_{i_k}$	$\mathrm{Tr}\,\mu_{i_1}\mu_{i_2}\cdots\mu_{i_k}$
$\mathrm{Tr}(\phi_i)^k$	$\{u_i, Q^2 u_i\}$

Superfluous looking chiral operators are removed via relation:

$$\mu^2\big|_{\mathrm{adj}} = 0\,, \quad \mathrm{Tr}\mu^k = 0$$

Matching conformal manifolds

- The marginal operators of the $\mathcal{N}$=1 dual theory:
    - 3 from the Coulomb branch operators having $\Delta_{UV} = 3/2$ and 2 formed out of the moment maps $\mathrm{Tr}\mu_1\mu_2\mu_3$, $\mathrm{Tr}\mu_1\mu_3\mu_2$
    - Two of them are marginally irrelevant since it breaks $U(1)^2$ symmetry. They combine with the broken flavor symmetry currents to form a long multiplet and becomes non-BPS.

[Green, Komargodski, Seiberg, Tachikawa, Wecht]

- The marginal operators of the $\mathcal{N}$=4 SYM
    - 11 of the form $\mathrm{Tr}\phi_i\phi_j\phi_j$
    - 8 of them are marginally irrelevant as they break SU(3) flavor symmetry.
- We have 3-dimensional conformal manifold on both sides.

Matching of the superconformal index

- As a more refined check, we compute the superconformal index.
- For G=SU(3), the full index is available using the **SUSY enhancing RG flow** from an SU(2) gauge theory with and adjoint and 2 fund/anti-fundamental matters with

$$W = X\phi^2 + q_1\phi\tilde{q}_1 + Mq_2\tilde{q}_2$$

[Agarwal, Maruyoshi, JS '16]

to the $(A_1, D_4) = D_2(SU(3))$ theory.

- Using above description, we can compute the index for our $\mathcal{N}$=1 dual theory.

[Kang, Lawrie, KH Lee, JS '22]

- The index perfectly agrees with that of $\mathcal{N}$=4 SYM with G=SU(3)

$$I = \mathrm{Tr}(-1)^F t^{3(R+2j_2)} y^{2j_1} \prod_i v_i^{f_i}$$

$$\begin{aligned}\widehat{I}^{\mathfrak{su}_3} &\equiv (1-t^3y)(1-t^3/y)(I^{\mathfrak{su}_3}-1)\\ &= t^4\chi_{\mathbf{6}}^{\mathfrak{su}_3} - t^5\chi_{\mathbf{2}}^{\mathfrak{su}_2}\chi_{\mathbf{3}}^{\mathfrak{su}_3} + t^6(\chi_{\mathbf{10}}^{\mathfrak{su}_3} - \chi_{\mathbf{8}}^{\mathfrak{su}_3} + 1)\\ &\quad - t^7\chi_{\mathbf{2}}^{\mathfrak{su}_2}(\chi_{\mathbf{6}}^{\mathfrak{su}_3} - \chi_{\overline{\mathbf{3}}}^{\mathfrak{su}_3}) + t^8(\chi_{\mathbf{15'}}^{\mathfrak{su}_3} - \chi_{\mathbf{15}}^{\mathfrak{su}_3} + \chi_{\overline{\mathbf{6}}}^{\mathfrak{su}}\\ &\quad +2\chi_{\mathbf{3}}^{\mathfrak{su}_3}) - t^9\chi_{\mathbf{2}}^{\mathfrak{su}_2}(\chi_{\mathbf{10}}^{\mathfrak{su}_3} + 1) + t^{10}\chi_{\mathbf{3}}^{\mathfrak{su}_2}\chi_{\overline{\mathbf{3}}}^{\mathfrak{su}_3}\\ &\quad + t^{10}(\chi_{\overline{\mathbf{21}}}^{\mathfrak{su}_3} - \chi_{\overline{\mathbf{15}}}^{\mathfrak{su}_3} + 2\chi_{\mathbf{6}}^{\mathfrak{su}_3} - 2\chi_{\overline{\mathbf{3}}}^{\mathfrak{su}_3}) + \cdots,\end{aligned}$$

Matching of the superconformal index (cont'd)

- Schur index

- The full index is not available for n>1, but the Schur limit of the index is available.

$$I_{\mathcal{N}=2}(p,q,\mathfrak{t}) = \mathrm{Tr}(-1)^F p^{j_1+j_2+r} q^{j_2-j_1+r} \mathfrak{t}^{I_3-\frac{1}{2}r}$$

- In the $q = \mathfrak{t}$ limit, the p-dependence drops out.

[Gadde, Rastelli, Razamat, Yan '11]

- The Schur index for the D2(SU(2n+1)) theory is written in a succinct form

$$I_S^{\mathcal{D}_2(SU(2n+1))}(q;z) = \mathrm{PE}\left[\frac{q}{1-q^2}\chi_{\mathbf{adj}}(z)\right]$$

[Xie, Yan, Yau]
[JS, Xie, Yan]

which is the same as that of the free hypermultiplet upon $q \to q^2$.

- Had we known the full index for the D2(SU(2n+1) theory, we can compute the index schematically as

$$I(p,q) = \int [dz] I_{\text{vec}}(z) \prod_{i=1}^{3} I^{\mathcal{D}_2}(z)\Big|_{\mathfrak{t}\to(pq)^{\frac{2}{3}+\varepsilon_i}}$$

$p = t^3y, q = t^3/y$

Matching the index (cont'd)

We would like to compare the following two expressions:

$$I(p,q) = \int [dz] I_{\text{vec}}(z) \prod_{i=1}^{3} I^{\mathcal{D}_2}(z)\Big|_{\mathfrak{t}\to(pq)^{\frac{2}{3}+\varepsilon_i}} \qquad I^{\mathcal{N}=4}(p,q) = \int [dz] I_{\text{vec}}(z) I_{\text{chi}}(z)^3$$

$$I_{\text{chi}}(z) = \text{PE}\left[\frac{(pq)^{1/3}-(pq)^{2/3}}{(1-p)(1-q)}\chi_{\mathbf{adj}}(z)\right]$$

Now, take the limit $q = \mathfrak{t} = (pq)^{1/3}$ or equivalently $p \to q^2$. [Buican, Nishinaka '16]

Then the index for the adjoint chiral multiplet becomes identical to that of the D_2(SU(2n+1)) theory!

$$I_S^{\mathcal{D}_2(SU(2n+1))}(q;z) = \text{PE}\left[\frac{q}{1-q^2}\chi_{\mathbf{adj}}(z)\right]$$

Therefore, in this limit, the index **matches exactly**!

N=1 gauging realizes the analogy between the D_2G theory and the free theory. [Buican, Laczko '17]

Landscape of 4d SCFTs with a=c

Central charges of 4d CFT

- **Conformal anomalies** of a 4d CFT are parametrized by two parameters (central charges) *a* & *c*:

$$\langle T^{\mu}_{\mu} \rangle = \frac{c}{16\pi^2} W^2 - \frac{a}{16\pi^2} E$$

- It is now well-established that ***a*-function** is a **monotonically decreasing** function along the RG flow (*a*-theorem): [Cardy][Komargodski, Schwimmer]

$$a_{IR} < a_{UV}$$

- One can think of the *a*-function as a quantity that measures **degrees of freedom**.
- The *c*-function, on the other-hand, does ***not*** always decrease along the RG flow.

Hofman-Maldacena bound on central charges

- The ratio a/c of central charges is bounded by **unitarity**: [Hofman, Maldacena]

$$\frac{1}{3} \leq \frac{a}{c} \leq \frac{31}{18}$$ (lower/upper bound saturated by free scalar/free vector)

- For superconformal theory:
  - $\mathcal{N}$=1 SCFT: $\frac{1}{2} \leq \frac{a}{c} \leq \frac{3}{2}$ (lower/upper bound saturated by free chiral/free vector)
  - $\mathcal{N}$=2 SCFT: $\frac{1}{2} \leq \frac{a}{c} \leq \frac{5}{4}$ (lower/upper bound saturated by free hyper/free vector)
  - $\mathcal{N}$=3 or $\mathcal{N}$=4 SCFT: $a = c$ [Aharony, Evtikhiev]

The role of *a* & *c*

- Any **holographic** theories have $a = c$ (for large *N*). [Henningson, Skenderis]
- When $a \neq c$, there is a correction to the celebrated **entropy-viscosity ratio** bound of [Kovtun, Son, Starinet] to [Katz, Petrov][Buchel, Myers, Sinha]

$$\frac{\eta}{s} \geq \frac{1}{4\pi}\left(1 - \frac{c-a}{c} + \cdots\right)$$

- Appears in the Cardy-like (**high-temperature**) limit of superconformal index:

$$I(p = q = e^{-\beta}) \to \exp\left(\#\frac{3c-2a}{\beta^2}\right)$$

[J. Kim, S. Kim, JS]
[Cabo-Bizet, Cassani, Martelli, Murthy]
[Cassani, Komargodski]

This formula accounts for the **entropy of supersymmetric black holes** in AdS$_5$. [Choi, Kim, Kim, Nahmgoong] [Benini, Milan] [Cabo-Bizet, Cassani, Martelli, Murthy]

- $c - a$ appears in the universal part of **entanglement entropy**. [Perlmutter, Rangamani, Rota]

$\mathcal{N}$=2 Gauging $\mathscr{D}_p[G]$ theories

[Cecotti, Del Zotto, Giacomelli]
[Closset, Giacomelli, Schafer-Nameki, Wang]
[Kang, Lawrie, JS]

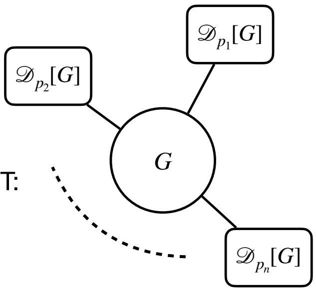
- In order to gauge the flavor and obtain SCFT, the 1-loop beta function for the gauge group should vanish:

$$\beta_G = 0 \quad \leftrightarrow \quad \sum_i k_i = 4h_G^\vee$$

flavor central charges k_i : “matter” contribution to the beta function.

- Consider gluing a number of $\mathscr{D}_p[G]$ theories to form $\mathcal{N}$=2 SCFT:

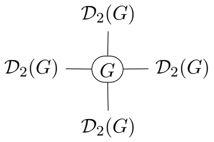
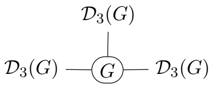
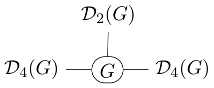
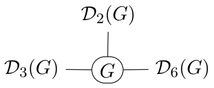
$$\sum_{i=1}^{n} \frac{2(p_i - 1)}{p_i} h_G^\vee = 4h_G^\vee \quad \to \quad \sum_{i=1}^{n} \frac{1}{p_i} = n - 2$$



- **Only 4 non-trivial solutions**: (2, 2, 2, 2), (3, 3, 3), (2, 4, 4), (2, 3, 6)

$\widehat{\Gamma}(G)$ theory with $\Gamma = D_4, E_6, E_7, E_8$

[Kang, Lawrie, JS]

(p_1, p_2, p_3, p_4)	$\widehat{\Gamma}(G)$	Quivers via gauging $\mathcal{D}_p(G)$s	$a = c$
$(2,2,2,2)$	$\widehat{D}_4(G)$		$\frac{1}{2}\dim(G)$
$(1,3,3,3)$	$\widehat{E}_6(G)$		$\frac{2}{3}\dim(G)$
$(1,2,4,4)$	$\widehat{E}_7(G)$		$\frac{3}{4}\dim(G)$
$(1,2,3,6)$	$\widehat{E}_8(G)$		$\frac{5}{6}\dim(G)$

We get $a = c$ is when the largest comark α_Γ of Γ satisfies

$$\gcd(h_G^\vee, \alpha_\Gamma) = 1 \quad \Longrightarrow \quad a = c.$$

$$\alpha_{D_4} = 2,\ \alpha_{E_6} = 3,\ \alpha_{E_7} = 4,\ \alpha_{E_8} = 6.$$

$a = c$ without any symmetry constraints! Genuinely $\mathcal{N}$=2.

In holography, it prevents $R^2_{\mu\nu\rho\sigma}$ correction in the effective supergravity action. Why???

$\mathcal{N}$=4 SYM and $\widehat{\Gamma}(G)$ theory

- The Schur index of $\widehat{\Gamma}(G)$ theory is **identical to that of the $\mathcal{N}$=4 SYM** upon rescaling!

$$I_{\widehat{\Gamma}(G)}(q) = I_G^{\mathcal{N}=4}(q^{\alpha_\Gamma}; q^{\alpha_\Gamma/2-1})$$

- For the $\hat{D}_4(SU(2\ell+1))$ theory, we find the index can be written in terms of MacMahon's generalized 'sum-of-divisor' function which is **quasi-modular**:

$$I_{\widehat{D}_4(SU(2k+1))}(q) = q^{-k(k+1)} A_k(q^2)$$
$$I^{\mathcal{N}=4}_{SU(2k+1)}(q) = q^{-\frac{k(k+1)}{2}} A_k(q)$$
$$A_k(q) = \sum_{0<m_1<m_2\cdots<m_k} \frac{q^{m_1+\cdots m_k}}{(1-q^{m_1})^2\cdots(1-q^{m_k})^2}$$

- There is an isomorphism between **associated VOA**s as a graded vector space. [Buican, Nishinaka]

- More connections to $\mathcal{N}$=4 SYM: 1 exactly marginal coupling, S-duality, 1-form center symmetry *Z(G)*, 2-group symmetry, non-invertible symmetry…

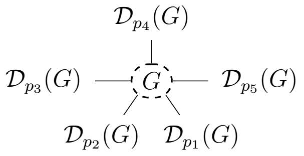
$\mathcal{N}$=1 SCFTs with $a = c$

[Kang, Lawrie, Lee, JS '21]

- One can construct even larger set of $a = c$ theories with **minimal supersymmetry**.
- Consider a number of $\mathscr{D}_p[G]$ theories gauged via **$\mathcal{N}$=1 vector** multiplet.
- It modifies the condition to be a CFT in the IR, since the theory now **RG flows**. From **asymptotic freedom** bound:

$$\sum_{i=1}^{N} \frac{2(p_i-1)}{p_i} h_G^\vee < 6h_G^\vee \qquad \sum_{i=1}^{N} \frac{1}{p_i} > N-3$$

- The IR SCFT has a number of U(1) **flavor symmetry** originates from broken R-symmetry of each block.



p_1	p_2	p_3	p_4	p_5
1	1	1	1	p_5
1	1	1	p_4	p_5
1	1	p_3	p_4	p_5
1	2	2	p_4	p_5
1	2	3	≤ 6	p_5
1	2	3	7	≤ 41
1	2	3	8	≤ 23
1	2	3	9	≤ 17

p_1	p_2	p_3	p_4	p_5
1	2	3	10	≤ 14
1	2	3	11	≤ 13
1	2	4	4	p_5
1	2	4	5	≤ 19
1	2	4	6	≤ 11
1	2	4	7	≤ 9
1	2	5	5	≤ 9
1	2	5	6	≤ 7

p_1	p_2	p_3	p_4	p_5
1	3	3	3	p_4
1	3	3	4	≤ 11
1	3	3	5	≤ 7
1	3	4	4	≤ 5
2	2	2	2	p_5
2	2	2	3	3
2	2	2	3	4
2	2	2	3	5

Tuples of (p_i)'s satisfying the asymptotic freedom bound.

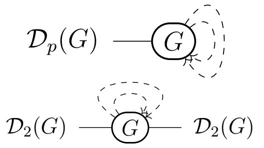
Landscape of $\mathcal{N}$=1 SCFTs with $a = c$

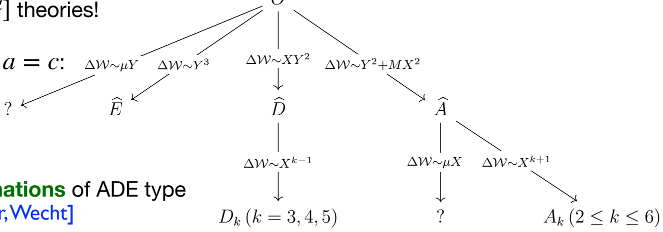
[Kang, Lawrie, Lee, JS, in progress]

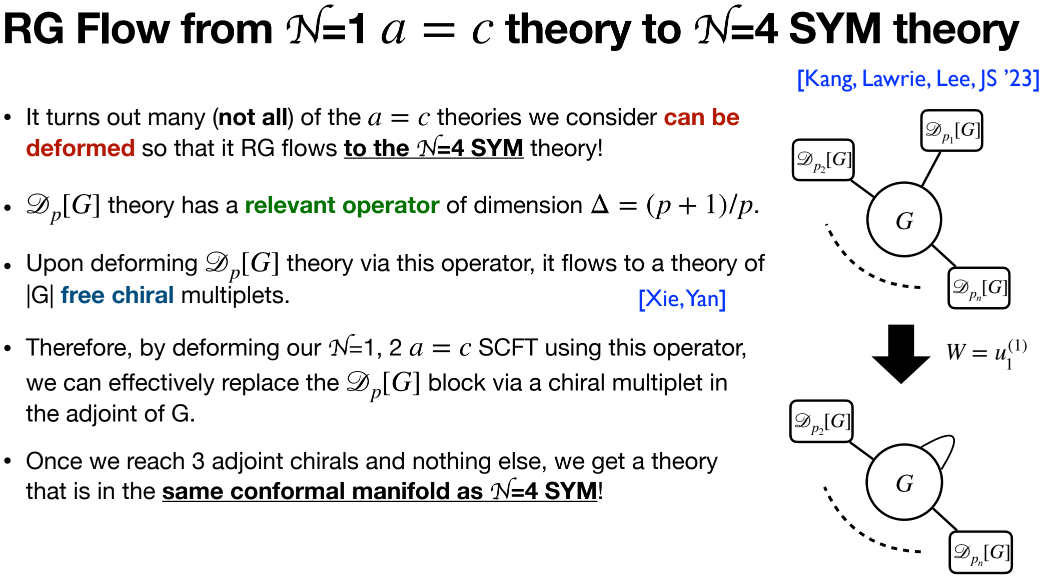
- $a = c$ theories with less SUSY not only exists, but rather common!
- One can add 1 or 2 **adjoint chiral** multiplets on top of the previous setup.
- 1 adjoint: can attach up to 4 $\mathscr{D}_p[G]$ theories.

$$p_i = (p_1, p_2),\ (2,2,p_3),\ (2,3,\leq 6),\ (2,4,4),\ (3,3,3),\ (2,2,2,2)$$

- 2 adjoints: One can even have zero $\mathscr{D}_p[G]$ theories!
  - **The simplest *Lagrangian* model** with $a = c$: $\mathcal{N}$=1 gauge theory with 2 adjoints.
  - Can attach up to 2 $\mathscr{D}_p[G]$'s
- One can consider **superpotential deformations** of ADE type as in the case of adjoint SQCD. [Intriligator, Wecht]
- How common are the $a = c$ theories in the landscape of 4d CFTs?







RG Flow from $\mathcal{N}$=1 $a = c$ theory to $\mathcal{N}$=4 SYM theory

[Kang, Lawrie, Lee, JS '23]

- It turns out many (**not all**) of the $a = c$ theories we consider **can be deformed** so that it RG flows **to the $\mathcal{N}$=4 SYM** theory!
- $\mathscr{D}_p[G]$ theory has a **relevant operator** of dimension $\Delta = (p+1)/p$.
- Upon deforming $\mathscr{D}_p[G]$ theory via this operator, it flows to a theory of |G| **free chiral** multiplets. [Xie, Yan]
- Therefore, by deforming our $\mathcal{N}$=1, 2 $a = c$ SCFT using this operator, we can effectively replace the $\mathscr{D}_p[G]$ block via a chiral multiplet in the adjoint of G.
- Once we reach 3 adjoint chirals and nothing else, we get a theory that is in the **same conformal manifold as $\mathcal{N}$=4 SYM**!

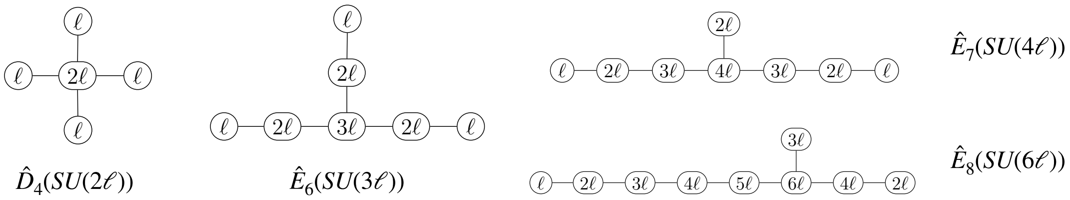
Holographic dual of a=c theories?

Lagrangian $\hat{\Gamma}(G)$ theory with $\Gamma = D_4, E_6, E_7, E_8$

- What is the **holographic dual** of such $a = c$ theories? It should forbid particular type of corrections in SUGRA action without any symmetry constraints. How?
- When $G = SU(\alpha_\Gamma \ell)$, we recover **Lagrangian affine quiver** gauge theory obtained via ℓ D3-branes probing ALE singularity $\mathbb{C}^2/\Gamma$.

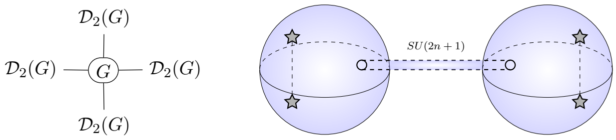
$$\mathcal{D}_p(SU(p\ell)) = \boxed{SU(p\ell)} — (SU((p-1)\ell)) — \cdots — (SU(\ell))$$

- The **holographic dual** for $\hat{\Gamma}(G)$ theories have been known for ages: It is dual to the type IIB theory on $\mathrm{AdS}_5 \times S^5/\Gamma$ with ℓ unit of 5-form flux through S^5/Γ.



Holographic dual of $\hat{\Gamma}(G)$ theory?

- For general G, our theory naturally **generalizes the affine quiver** theory by '**fractionalization**': $N = \alpha_\Gamma \ell + m$, ℓ D3-branes with 'extra charge' m/α_Γ.
    - Is there a **string-theoretic/holographic realization** for such configuration?
- One particular example of holographic dual: A_{2N} type Class-S with 4 twisted minimal punctures realizes the $\hat{D}_4(SU(2N+1))$ theory.



[Beem, Peelaers]
[Kang, Lawrie, Lee, Sacchi, JS]

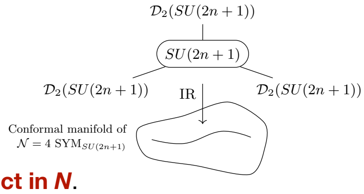
- Holographic dual for the (untwisted) class-S theories are known.

[Gaiotto, Maldacena][Nishinaka]

- Any good reason for $a = c$ from gravity perspective?

Conclusion

Summary & future direction



- We have found an $\mathcal{N}$=1 non-Lagrangian theory that flows in the IR to a point in the same conformal manifolds as $\mathcal{N}$=4 SYM theory of G=SU(2n+1). **Maximal SUSY enhancement**.
- We have a 'landscape' of **genuinely** $\mathcal{N}$=1, 2 SCFTs with $a = c$, **exact in *N***.
    - The Schur index of $\mathcal{N}$=2 SCFTs $\hat{\Gamma}(G)$ is identical to that of $\mathcal{N}$=4 SYM upon rescaling:
    $$I_{\hat{\Gamma}(G)}(q) = I_G^{\mathcal{N}=4}(q^{\alpha_\Gamma}, q^{\alpha_\Gamma/2-1})$$
    - Some of the $\mathcal{N}$=1 SCFTs constructed out of gauging multiple D$_p$[G] theories flow to $\mathcal{N}$=4 SYM upon suitable deformation.
    - **Holographic interpretation**? The term $R^2_{\mu\nu\rho\sigma}$ "forbidden" without any symmetry in SUGRA EFT.
    - Any (generalized) **symmetry constraint** hidden in the landscape of a=c theories?

Thank you!

Non-invertible symmetries and disk partition functions

Satoshi Yamaguchi

ABSTRACT. Recently, the concept of symmetry has been generalized, and what was not traditionally called symmetry is now being used similarly as symmetry. In this talk, we discuss a class of such generalized symmetries, called non-invertible symmetries, from the viewpoint of the lattice field theories. In particular, we construct topological defects in four-dimensional Z_2 lattice gauge theory, including the Kramers-Wannier-Wegner (KWW) duality defect; the KWW duality defect is an example of non-invertible symmetries. Also, we consider the system with a boundary and discuss the relations between the disk partition functions derived from the non-invertible symmetry.

(S. Yamaguchi) Osaka University

Open string Witten indices of 2d $\mathcal{N} = (2, 2)$ GLSMs

Yutaka Yoshida

Abstract. In our previous work, we have derived a supersymmetric localization formula for indices of 2d $\mathcal{N} = (2, 2)$ gauged linear sigma models (GLSMs) on $I \times S^1$. In this talk, we consider the localization formula in the Landau-Ginzburg(LG) phase and discuss BPS boundary conditions which reproduce cylinder amplitudes with Recknagel-Schomerus boundary states in Gepner models.

(Y. Yoshida) Meiji Gakuin University