Osaka Central Advanced Mathematical Institute (OCAMI)
Osaka Metropolitan University
MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics

OCAMI Reports Vol. 5 (2023)
doi: 10.24544/omu.20230727-001

Integrable Systems and Quantum Groups

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March 4–8, 2023

Abstract

The aim of this research project is to seek new developments in areas where integrable systems and quantum groups intersect such as quantum (super)group, quantum symmetric pair, crystal base, and symmetric function.

2020 *Mathematics Subject Classification.*
05E05, 05E10, 20G42, 16T25, 17B05, 17B10, 17B20, 17B37, 17B80, 81R12

Key words and Phrases.
integrable systems, Yang-Baxter equation, reflection equation, quantum group, Lie superalgebra, crystal base, combinatorics, symmetric function

OCAMI. Integrable Systems and Quantum Groups. OCAMI Reports. Vol. 5, Osaka Central Advanced Mathematical Institute, Osaka Metropolitan University. 2023, 165 pp. doi: 10.24544/omu.20230727-001

Preface

This is the proceedings of the international conference "Integrable Systems and Quantum Groups" held at Osaka Metropolitan University, Sugimoto Campus, General Education Building, Room 810, during March 4th–8th, 2023, in honor of Masato Okado's 60th birthday. The conference was held as a part of OCAMI Joint Usage/Research Project.

One of the central problems in integrable systems is to solve the Yang-Baxter equation, which describes collisions of particles in statistical mechanics. Quantum group (also known as quantized enveloping algebra), which is a purely mathematical object, was invented to attack the problem in physics above. As a result, studies of quantum group and related areas such as representation theory, (quantum) Lie superalgebra, quantum symmetric pair, crystal base, orthogonal polynomial, and symmetric function, have provided remarkable results relevant to integrable systems.

The aim of the conference was to seek new developments in branches of mathematics and physics above. It is quite difficult to become deeply familiar with all of these fields, which have been developing at a remarkable pace in recent years. Hence, for new progress, we need to bring together experts in both integrable systems and quantum groups to exchange state-of-the-art information.

We invited seven experts of integrable systems or/and quantum group from both home and abroad to the conference as speakers. The talks were broadcasted via Zoom. Some of them are available on OCAMI's YouTube channel (`https://www.youtube.com/@ocami_math4918/videos`).

During the conference, we had approximately 20–30 participants in person and 30–40 online for each day. There were lively discussions among participants.

We are grateful to the participants of the conference for their contribution. The conference was supported by JSPS KAKENHI Grant Numbers JP18K03250, JP20K14286, and JP21H04993.

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Contents

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Isomorphisms among quantum Grothendieck rings and their applications

Ryo. Fujita (jt. w/ D. Hernandez, S.-j. Oh, H. Oya)

Set up $\mathfrak{g}$: simple Lie alg / $\mathbb{C}$, type A~G

$\rightsquigarrow$ $\mathscr{C} := \underbrace{U_q(L\mathfrak{g})}_{\text{quantum loop algebra}/\mathbb{C}}\text{-mod}_{f.d.}$: rigid monoidal category

$q \in \mathbb{C}^\times$ generic

$\mathscr{D}^{\pm 1}$: left/right duality. $\mathscr{D}^2 \neq id$.

Grothendieck ring.

$K(\mathscr{C}) \xhookrightarrow{\chi_q} \mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1} \mid i \in I : \text{Coxeter node}, a \in \mathbb{C}^\times : \text{spectral param}]$ — χ_q: Frenkel-Reshetikhin's q-character

$K(\mathscr{C}) \cup \mathrm{Irr}\,\mathscr{C} \xleftrightarrow{1:1} \{\text{monomials in } Y_{i,a}\}$

$L(m) \longleftarrow m$ [Chari - Pressley]

$\chi_q(L(m)) = m + \underline{\text{lower terms}}$

Problem Compute ↗ lower terms — at this moment, no known closed formula.

$\rightsquigarrow$ Kazhdan - Lusztig type approach.

Quantum Grothendieck ring. [Nakajima, Varagnolo-Vasserot $\mathfrak{g}$: ADE; Hernandez $\mathfrak{g}$: general]

$K(\mathscr{C}) \cong \mathrm{Im}\,\chi_q = \bigcap_{i \in I} \mathrm{Ker}\,\underbrace{S_i}_{\text{screening op}} \subset \mathcal{Y}$

$\uparrow t=1$ $\qquad\qquad\qquad\qquad$ $\uparrow t=1$

$K_t(\mathscr{C}) := \bigcap_{i \in I} \mathrm{Ker}\,S_{i,t} \xhookrightarrow[\mathbb{Z}[t^{\pm 1}]\text{-subalg.}]{} \mathcal{Y}_t$ quantum torus $/\mathbb{Z}[t^{\pm 1}]$

$\cup$

$\exists \{M_t(m)\}$ standard basis $\quad M_t(Y_{i_1,a_1} \cdots Y_{i_\ell,a_\ell}) = F_t(Y_{i_1,a_1}) \cdots F_t(Y_{i_\ell,a_\ell})$

$F_t(Y_{i,a}) \xrightarrow{t=1} \chi_q(L(Y_{i,a}))$

$\rightsquigarrow$ $\exists \{L_t(m)\}$ canonical basis $\longleftarrow$ computed from $\{M_t(m)\}$ by inductive algorithm.

"simple (q,t)-characters"

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Thm [N, VV] $\mathfrak{g}$: ADE

(KL) $L_t(m)|_{t=1} = \chi_q(L(m))$ $\forall m$

(P1) $L_t(m) \in \mathcal{Y}_t$ has non-neg. coeff. $\forall m$.

(P2) $\{L_t(m)\}$ has non-neg str. const.

Conj [H] (KL), (P1), (P2) are true also for $\mathfrak{g}$: BCFG.

Main results [FHOO]

- (P1) & (P2) hold for $\forall \mathfrak{g}$.
- (KL) holds if (i) $\mathfrak{g}$ is of type B ($\forall m$)

  or (ii) $L(m)$ is "reachable" ($\forall \mathfrak{g}$)

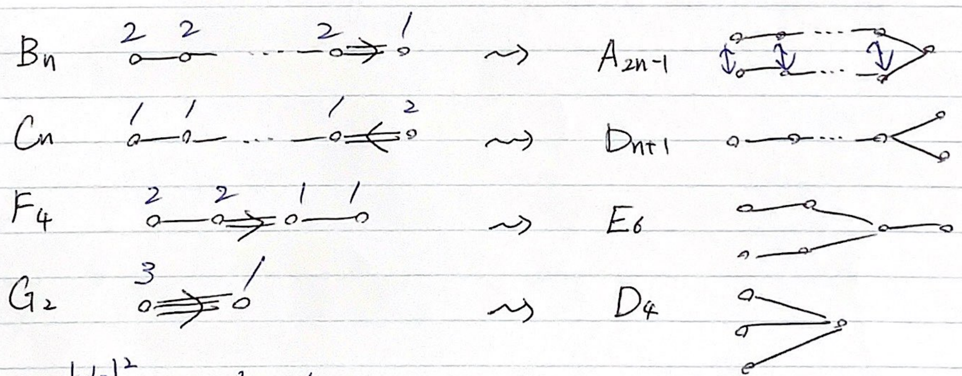
  ("reachable": in a cluster monoidal categorification) e.g. KR-modules.

Key Fact. $\mathfrak{g}$: BCFG.

$\exists \mathbb{Z}[t^{\pm 1}]$-alg isom $\Psi : K_t(\mathscr{C}_\mathfrak{g}) \xrightarrow{\sim} K_t(\mathscr{C}_{\tilde{\mathfrak{g}}})$

respecting the canonical bases. ⇝ (P2) ✓

where $\mathfrak{g}$ "unfolding" ⇝ $\tilde{\mathfrak{g}}$



$d_i = \frac{|\alpha_i|^2}{2} \in \{1, r\}$

r : lacing number.

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Construction of Ψ.

① Reduction to $\mathscr{C}_{\mathbb{Z}}$. "monoidal skeleton"

Def. Fix $\varepsilon : I \to \{0,1\}$ s.t. $\varepsilon_i \equiv \varepsilon_j + \min(d_i, d_j) \mod 2$. if $c_{ij} < 0$.

$I_{\mathbb{Z}} := \{ (i,k) \in I \times \mathbb{Z} \mid k \equiv \varepsilon_i \}$.

$\mathscr{C}_{\mathbb{Z}} \subset \mathscr{C}$ Serre subcat "supported on $I_{\mathbb{Z}}$"

i.e. $\mathrm{Irr}\, \mathscr{C}_{\mathbb{Z}} \overset{1:1}{\longleftrightarrow} \{\text{monom. in } Y_{i,q^k} \ (i,k) \in I_{\mathbb{Z}}\}$.

$\rightsquigarrow$ $\mathscr{C}_{\mathbb{Z}}$ is closed under $\otimes$ & $\mathscr{D}^{\pm 1}$ $\left(\mathscr{D}^{\pm 1} L(Y_{i,a}) = L(Y_{i^*, aq^{\pm r h^\vee}})\right)$

$K_t(\mathscr{C}) \cong \bigotimes_{a \in \mathbb{C}^\times / q^{2\mathbb{Z}}} K_t(\sigma_a^* \mathscr{C}_{\mathbb{Z}})$ $\quad$ (σ_a^*: spectral param. shifts.) $\quad$ $w_0 \alpha_i = -\alpha_{i^*}$ $\quad$ ($h^\vee$: dual Coxeter.)

Notation $J \subset I \times \mathbb{Z}$.

$\rightsquigarrow$ $\mathscr{C}_J \subset \mathscr{C}_{\mathbb{Z}}$ Serre subcat supported on $J \cap I_{\mathbb{Z}}$.

($\mathscr{C}_J$:) closed under $\otimes$ if J is "convex"

② Th'm (Hernandez-Leclerc / FHOO) $\quad \mathfrak{g}$: ADE / BCFG

(i) $K_t(\mathscr{C}_{I \times (-rh^\vee, 0]}) \overset{\Psi}{\simeq} A_t[\widetilde{N}]$: quantum coordinate ring

($\widetilde{N}$: max'l unipotent subgp of $\widetilde{G}$)

$\{L_t(m)\} \longleftrightarrow$ dual canonical basis

(ii)

$$\begin{array}{ccc} \circlearrowright \mathscr{D}^{\pm 1} & \longleftrightarrow & [\pm 1] \circlearrowright \\ K_t(\mathscr{C}_{\mathbb{Z}})_{loc} & \overset{\sim}{\longrightarrow} & \mathrm{Hall}(D^b \mathrm{Rep}\, \widetilde{Q}) \\ \cup & & \cup \\ K_t(\mathscr{C}_{I \times (-rh^\vee, 0]})_{loc} \overset{\Psi}{\simeq} U_t^+(\widetilde{\mathfrak{g}}) & \simeq & \mathrm{Hall}(\mathrm{Rep}\, \widetilde{Q}) \end{array}$$

$(\ )_{loc} = - \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Q}(t)$

$\widetilde{Q}$: quiver of type $\widetilde{\mathfrak{g}}$

[Ringel]

$\rightsquigarrow \exists\, \Psi : K_t(\mathscr{C}_{\mathbb{Z}})_{loc} \simeq K_t(\widetilde{\mathscr{C}}_{\mathbb{Z}})_{loc}$

respecting the canonical bases

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<u>Rem</u> $\mathscr{C}_{I\times(-rh^\vee,0]} \overset{\text{generalize}}{\rightsquigarrow} \mathscr{C}_Q$

where $Q = (\tilde{\Delta}, \sigma, \xi)$: "Q-datum" for $\mathfrak{g}$

- $\tilde{\Delta}$: Coxeter graph for $\tilde{\mathfrak{g}}$
- σ: $\tilde{\Delta} \circlearrowleft \sigma$ s.t. $\Delta = \tilde{\Delta}/\sigma$
- ξ: "height function" $\xi : I \to \mathbb{Z}$.

$\mathscr{C}_Q := \mathscr{C}_J$ with $J := (f\times \mathrm{id})\left(\{(i,k)\in \tilde{I}_{\mathbb{Z}} \mid \xi_{i^*} - rh^\vee < k \leq \xi_i\}\right)$

$f : \tilde{I} \xrightarrow{/\sigma} I$

Thm (ii) $\rightsquigarrow$ $\Phi_Q : K_t(\mathscr{C}_Q) \simeq A_t[\tilde{N}]$

$\{L_t(m)\}$ $\leftrightarrow$ dual canon. basis

$\{M_t(m)\}$ $\leftrightarrow$ dual PBW arising from Q

quantum T-system $\leftrightarrow$ derterminantal identity. —//—

$\rightsquigarrow$ $\Psi = \Psi_{Q,Q'}$ ~~dede~~ depends on Q-data Q for $\mathfrak{g}$, Q' for $\tilde{\mathfrak{g}}$

<u>Categorification</u>.

- When $(\mathfrak{g}, \tilde{\mathfrak{g}}) = (B_n, A_{2n-1})$

  $\Psi|_{t=1}$ is categorified via generalized Schur-Weyl duality [Kang-Kashiwara-Kim-Oh]

  $$\mathscr{C}_{\tilde{\mathfrak{g}}} \longleftarrow \boxed{\mathrm{Rep}(\text{aff Hecke alg of } GL)} \longrightarrow \mathscr{C}_{\mathfrak{g}}$$

  $$\mathrm{Irr} \overset{1:1}{\longleftrightarrow} \mathrm{Irr}$$

  $\rightsquigarrow$ (KL) for type B. ✓

- For any Q, $\Phi_Q|_{t=1}$ is also categorified.

  $$\mathscr{C}_Q \simeq \mathrm{Rep}(\text{KLR alg of type } \tilde{\mathfrak{g}}).$$

KOKUYO

A BRIEF INTRODUCTION TO QUANTUM SYMMETRIC PAIRS

STEFAN KOLB

Abstract. The present notes are an extended version of an introductory talk on quantum symmetric pairs given at the OCAMI conference 'Integrable Systems and Quantum Groups' held at Osaka City University from 4-8 March 2023 in honor of Masato Okado's 60th birthday.

1. **Introduction.** A Lie algebra $\mathfrak{g}$ together with a Lie algebra automorphism $\theta : \mathfrak{g} \to \mathfrak{g}$ such that $\theta^2 = \mathrm{id}_{\mathfrak{g}}$ is called symmetric. If $(\mathfrak{g}, \theta)$ is symmetric then we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and the -1 eigenspace of θ, respectively. Here $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ while $\mathfrak{p}$ is a $\mathfrak{k}$-module. Hence the universal enveloping algebra $U(\mathfrak{k})$ is a Hopf subalgebra of $U(\mathfrak{g})$. We refer to the pair $(\mathfrak{g}, \mathfrak{k})$ as a symmetric pair. If $\mathfrak{g}$ is a complex semisimple Lie algebra then $\mathfrak{k}$ is reductive and we can think of the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ as an infinitesimal realization of a compact Riemannian symmetric space.

Throughout these notes we assume that $\mathfrak{g}$ is a symmetrizable Kac-Moody algebra. Hence there exists a Drinfeld-Jimbo quantized enveloping algebra $U_q(\mathfrak{g})$. However, even if both $\mathfrak{g}$ and $\mathfrak{k}$ are complex simple Lie algebras there is in general no Hopf algebra embedding of $U_q(\mathfrak{k})$ into $U_q(\mathfrak{g})$, see [Bra94]. For $\mathfrak{g}$ of finite type this problem was first addressed in the early nineties by the groups around T. Koornwinder in Amsterdam and M. Noumi in Kobe, see [Nou96], [Dij96], [NS95], with the aim to construct quantum group analogs of compact symmetric spaces. In the late nineties, G. Letzter independently developed a comprehensive theory of quantum symmetric pairs of finite type [Let99], [Let02]. Letzter's approach can be formulated as follows:

Goal: Given $(\mathfrak{g}, \theta)$, find all subalgebras $\mathcal{B} \subset U_q(\mathfrak{g})$ with the following properties:

- L1) $\mathcal{B}$ is a right coideal of $U_q(\mathfrak{g})$, that is $\varDelta(\mathcal{B}) \subset \mathcal{B} \otimes U_q(\mathfrak{g})$, where $\varDelta$ denotes the coproduct of $U_q(\mathfrak{g})$.
- L2) The non-restricted specialization of $\mathcal{B}$ coincides with $U(\mathfrak{k})$.
- L3) The subalgebra $\mathcal{B} \subset U_q(\mathfrak{g})$ is maximal with respect to properties 1) and 2).

We call subalgebras $\mathcal{B} \subseteq U_q(\mathfrak{g})$ with the above properties *quantum symmetric pair coideal subalgebras* (QSP coideal subalgebras), and we refer to $(U_q(\mathfrak{g}), \mathcal{B})$ as a *quantum symmetric pair.* For finite-dimensional $\mathfrak{g}$, Letzter constructed and classified all QSP coideal subalgebras of $U_q(\mathfrak{g})$, see [Let99], [Let02]. Her constructions were extended to the Kac-Moody case in [Kol14].

The theory of quantum symmetric pairs has seen an explosion of activity since the appearance of the preprint versions of the the papers [BW18] and [ES18] in October 2013. It turned out that many constructions for Drinfeld-Jimbo quantum

2020 *Mathematics Subject Classification.* 17B37, 17B67.

Key words and phrases. Kac-Moody algebras, involutions, symmetric pairs, quantum groups, coideal subalgebras.

groups allow analogs for quantum symmetric pairs. H. Bao and W. Wang refer to QSP coideal subalgebras as ıquantum groups and to the program of finding quantum symmetric pair analogs of results for $U_q(\mathfrak{g})$ as the ı-program. Over the past decade, W. Wang, his collaborators, and others have made fantastic progress. Constructions which have been addressed in the ı-program, at least partially, include the classification of representations, canonical and crystal bases, the universal R-matrix, Lusztig's braid group action on modules and on $U_q(\mathfrak{g})$, Hall algebra interpretations of $U_q(\mathfrak{g})$, the Drinfeld-Kohno theorem, categorification, Drinfeld's second realization and more. There would be too many papers to cite for the present short set of notes. Instead we refer the reader to W. Wang's survey article in the proceedings of the ICM 2022, [Wan21], and references therein.

The aim of the present notes is to give a brief account of the construction of QSP coideal subalgebras and of their fundamental algebraic properties. To this end we revisit the paper [Kol14] which was built on Letzter's work [Let99], [Let02]. We attempt to provide explanations and proofs but refer to the literature for more technical arguments. We hope that this will provide the novice reader with an easy entry point into the world of quantum symmetric pairs.

Even foundational aspects of the theory of quantum symmetric pairs are still in flow. In the present notes we modify or amend the constructions in [Kol14] in several ways, which we list in the following for the expert reader:

I) In the present notes we mostly work in the setting of generalized Satake diagrams proposed in [RV20]. This is a minor technical generalization of the setting of Satake diagrams (or admissible pairs) considered in [Kol14] and does not affect the proofs in the quantum group setting. Generalized Satake diagrams provide additional examples of QSP coideal subalgebras, which are no longer related to involutive Lie algebra automorphisms. An underlying classical theory was developed in [RV22].

II) Proposition 5.1 offers an alternative proof of the coideal property for QSP coideal subalgebras. This proof relies on the description of the coproduct of Lusztig's braid group operators in terms of quasi R-matrices. The original proofs in Letzter's work and in [Kol14] rely on the interplay between Lusztig's braid group automorphisms and the adjoint action of $U_q(\mathfrak{g})$ on itself.

III) The QSP coideal subalgebras $\mathcal{B}_{\mathbf{c},\mathbf{s}}$ as defined in [Kol14] depend on two families of parameters $\mathbf{c} \in \mathcal{C}, \mathbf{s} \in \mathcal{S}$ for explicitly described parameter sets $\mathcal{C}$, $\mathcal{S}$. In [Kol14], following [Let99], [Let02], the QSP coideal subalgebras $\mathcal{B}_{\mathbf{c},\mathbf{s}}$ were introduced in one go, in terms of generators inside $U_q(\mathfrak{g})$. It seems more natural to first introduce the standard QSP coideal subalgebra $\mathcal{B}_{\mathbf{c}} = \mathcal{B}_{\mathbf{c},\mathbf{0}}$. The additional parameters $\mathbf{s}$ can then be added by a uniform procedure which works for any right coideal subalgebra C of a Hopf algebra H over a field $\mathbb{K}$ with coproduct $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for $h \in H$. Namely, if $\chi : C \to \mathbb{K}$ is a character, that is a one-dimensional representation, then $C_\chi = \{\chi(c_{(1)})c_{(2)} \mid c \in C\}$ is a right coideal subalgebra of H. As a right H-comodule algebra, C_χ is a homomorphic image of C, see Section 10 for details. For quantum symmetric pairs, this perspective immediately implies that $\mathcal{B}_{\mathbf{c}}$ and $\mathcal{B}_{\mathbf{c},\mathbf{s}}$ are isomorphic as right $U_q(\mathfrak{g})$-comodule algebras. Moreover, this construction suggests a detailed analysis of the characters of $\mathcal{B}_{\mathbf{c}}$ which we indicate at the end of Section 10. It turns out that there are non-standard QSP coideal subalgebras for slightly more parameters than considered in [Let02] and [Kol14]. This phenomenon had already been observed elsewhere, see e.g. [BB10], [RV20], however, the general perspective of twisting by a character allows a uniform treatment of these examples. Many properties of quantum symmetric pairs are easier to prove in the standard case $\mathbf{s} = \mathbf{0}$, and twisting by a character often supports translation into properties of $\mathcal{B}_{\mathbf{c},\mathbf{s}}$.

IV) In [Kol14], building on Letzter's work [Let99], [Let02], we proved several desirable properties of the QSP coideal subalgebra $\mathcal{B}_{\mathbf{c}}$ under the condition $\mathbf{c} \in \mathcal{C}$. These properties include triangular decompositions of $\mathcal{B}_{\mathbf{c}}$ and $U_q(\mathfrak{g})$, in particular a q-analog of the Iwasawa decomposition, the specialization property L2), and the fact that $(U_q(\mathfrak{g}), \mathcal{B}_{\mathbf{c}})$ is a quantum homogeneous space in the sense of, say, [Krä12]. In Theorems 7.2 and 8.2 of the present notes we show that each of these properties is indeed equivalent to the property $\mathbf{c} \in \mathcal{C}$. This underscores the importance of the choice of the parameter set $\mathcal{C}$ for the parameters $\mathbf{c}$.

The talk underlying the present notes was originally planned as part of a three-hour lecture series. A second talk covered the $*$-product interpretation of quantum symmetric pairs, quasi K-matrices and defining relations along the lines of [KY21]. A third talk on universal K-matrices and braided module categories unfortunately had to be cancelled. I hope to extend the present notes to include these topics at some point in the future. The present notes already lay some of the necessary groundwork.

Acknowledgements. I owe much gratitude to the organizers of the OCAMI conference 'Integrable Systems and Quantum Groups', and to H. Watanabe in particular, for the generous invitation and for their patience when I failed to deliver to deadline.

2. **Satake Diagrams.** Letzter's theory is based on the combinatorial description of involutive automorphisms $\theta : \mathfrak{g} \to \mathfrak{g}$ in terms of Satake diagrams. Let I be an index set and let $(a_{ij})_{i,j\in I}$ be the generalized Cartan matrix for $\mathfrak{g}$. Let $\Pi = \{\alpha_i \,|\, i \in I\}$ be a set of simple roots, $Q = \mathbb{Z}\Pi$ the root lattice with positive cone $Q^+ = \mathbb{N}_0\Pi$, and let W be the Weyl group with simple reflections $\{\sigma_i \,|\, i \in I\}$. A Satake diagram for $\mathfrak{g}$ is a pair (X, τ) where $X \subset I$ is a subset of finite type and $\tau : I \to I$ is a diagram automorphism with $\tau(X) = X$ such that the following three properties are satisfied:

S1) $\tau^2 = \mathrm{id}_I$;
S2) $\tau|_X = -w_X$, that is $\alpha_{\tau(i)} = -w_X(\alpha_i)$ for all $i \in X$;
S3) If $i \in I \setminus X$ and $\tau(i) = i$ then $\alpha_i(\rho_X^\vee) \in \mathbb{Z}$.

Here w_X denotes the longest element in the parabolic subgroup $W_X \subset W$ and $\rho_X^\vee$ is the half-sum of the positive coroots corresponding to X. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and let e_i, f_i, h_i for $i \in I$ be the Chevalley generators of $\mathfrak{g}$. Let $\omega : \mathfrak{g} \to \mathfrak{g}$ be the Chevalley involution defined by

$$\omega|_{\mathfrak{h}} = -\mathrm{id}_{\mathfrak{h}}, \qquad \omega(e_i) = -f_i, \qquad \omega(f_i) = -e_i, \qquad \text{for all } i \in I. \tag{2.1}$$

For $i \in I$ define an automorphism $\mathrm{Ad}(\sigma_i)$ of $\mathfrak{g}$ by

$$\mathrm{Ad}(\sigma_i) = \exp(\mathrm{ad}(e_i))\exp(\mathrm{ad}(-f_i))\exp(\mathrm{ad}(e_i)). \tag{2.2}$$

The map $\sigma_i \mapsto \mathrm{Ad}(\sigma_i)$ defines a braid group action on $\mathfrak{g}$. Hence, for any $w \in W$ we obtain a well-defined automorphism $\mathrm{Ad}(w)$ of $\mathfrak{g}$. Let $s = s(X, \tau) : Q \to \{\pm 1\}$ be a group homomorphism such that $s(\alpha_j) = 1$ if $j \in X$ or $\tau(j) = j$, and $s(\alpha_j) =$

$(-1)^{\alpha_j(2\rho_X^\vee)}s(\alpha_{\tau(j)})$ if $j \notin X$ and $\tau(j) \neq j$. Define an automorphism $\mathrm{Ad}(s) : \mathfrak{g} \to \mathfrak{g}$ by $\mathrm{Ad}(s)(x) = s(\beta)x$ for all x in the root space $\mathfrak{g}_\beta$.

A Lie algebra automorphism $\varphi : \mathfrak{g} \to \mathfrak{g}$ is said to be of the second kind if the standard Borel subalgebra $\mathfrak{b}^+ \subset \mathfrak{g}$ satisfies $\dim(\varphi(\mathfrak{b}^+) \cap \mathfrak{b}^+) < \infty$. For example, the Chevalley involution given by (2.1) is of the second kind. The following theorem provides the main conceptual idea behind the construction of QSP coideal subalgebras in terms of Satake diagrams. Any diagram automorphism τ can be lifted to a Lie algebra automorphism of $\mathfrak{g}$, see [KW92, 4.23].

Theorem 2.1. ([KW92], see also [Kol14, Theorem 2.7]) *The map*

$$(X,\tau) \mapsto \theta(X,\tau) := \mathrm{Ad}(s(X,\tau)) \circ \mathrm{Ad}(w_X) \circ \tau \circ \omega \tag{2.3}$$

defines a bijection between the set of Satake diagrams (up to the action by diagram automorphisms) and the set of involutive Lie algebra automorphisms of the second kind of $\mathfrak{g}$ *(up to conjugation by automorphisms of* $\mathfrak{g}$*).*

The involutive Lie algebra automorphism $\theta = \theta(X,\tau)$ defined by (2.3) maps the Cartan subalgebra $\mathfrak{h}$ to itself and the restriction to $\mathfrak{h}$ can by expressed in terms of the Weyl group action as

$$\theta|_{\mathfrak{h}} = -w_X \circ \tau.$$

Hence, θ induces a map on $\mathfrak{h}^*$ which in the following we also write as $\theta = -w_X \circ \tau$.

For any subset $X \subset I$ of finite type let $\mathfrak{g}_X \subset \mathfrak{g}$ be the semisimple Lie subalgebra algebra generated by $\{e_i, f_i, h_i \,|\, i \in X\}$. If (X,τ) is a Satake diagram and $\theta = \theta(X,\tau)$ then $\theta(x) = x$ for all $x \in \mathfrak{g}_X$. Moreover, one checks that the Lie subalgebra $\mathfrak{k}$ is generated by $\mathfrak{g}_X$, $\mathfrak{h}^\theta = \mathfrak{h} \cap \mathfrak{k}$ and the elements

$$f_i + \theta(f_i) = f_i - \mathrm{Ad}(s) \circ \mathrm{Ad}(w_X)(e_{\tau(i)}) \quad \text{for all } i \in I \setminus X, \tag{2.4}$$

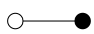
see [Kol14, Lemma 2.8]. In Section 5, we will define the QSP coideal subalgebra $\mathcal{B} \subset U_q(\mathfrak{g})$ corresponding to the Satake diagram (X,τ) as the subalgebra of $U_q(\mathfrak{g})$ generated by suitable quantum group analogs of $\mathfrak{g}_X$, $\mathfrak{h}^\theta$ and the elements in Equation (2.4).

For finite-dimensional or affine $\mathfrak{g}$ the information of a Satake diagram can be encoded in the Dynkin diagram of $\mathfrak{g}$. The nodes corresponding to X are colored back and the diagram automorphism τ is indicated by arrows in the diagram. With this convention, a complete list of Satake diagrams for finite-dimensional $\mathfrak{g}$ can be found in [Ara62, pp. 32/33]. The rank of a Satake diagram is the number of τ-orbits in $I \setminus X$. A rank 1 subdiagram of a Satake diagram is the τ-orbit of a connected component of $\{i\} \cup X$ containing i for some $i \in I \setminus X$. The notion of rank 1 subdiagrams makes sense for any pair (X,τ) with $\tau(X) = X$ which satisfies conditions S1) and S2).

It was observed by V. Regelskis and B. Vlaar that, for the purpose of quantum symmetric pairs, condition (S3) in the definition of a Satake diagram can be replaced by the weaker condition

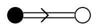
S3') If $\tau(i) = i$ and $a_{ji} = -1$ for $i \in I \setminus X$, $j \in X$, then $\theta(\alpha_i) \neq -\alpha_i - \alpha_j$,

see [RV20]. The condition S3') is equivalent to (X,τ) not having a rank 1 subdiagram of the following form:



As explained in [RV20, Section 4], the construction of quantum symmetric pairs and much of their theory remain valid for generalized Satake diagrams.

Remark 2.2. Every Satake diagram is a generalized Satake diagram, but the converse does not hold. Indeed, even in finite type, the diagram



is a generalized Satake diagram but does not satisfy condition S3).

3. **Quantum group preliminaries.** By construction $(\mathfrak{h},\Pi,\Pi^\vee)$ with $\Pi^\vee=\{h_i\,|\,i\in I\}$ is a minimal realization of the symmetrizable, generalized Cartan matrix A. We extend $\Pi^\vee$ to a basis $\Pi^\vee_{\mathrm{ext}}$ of $\mathfrak{h}$ such that $\alpha_i(d)\in\mathbb{Z}$ for all $i\in I$, $d\in\Pi^\vee_{\mathrm{ext}}\setminus\Pi^\vee$ and we set $Q^\vee_{\mathrm{ext}}=\mathbb{Z}\Pi^\vee_{\mathrm{ext}}$. Define the weight lattice by $P=\{\lambda\in\mathfrak{h}^*\,|\,\lambda(Q^\vee_{\mathrm{ext}})\in\mathbb{Z}\}$. In this situation the abelian groups $Y=Q^\vee_{\mathrm{ext}}$ and $X=P$ together with the embeddings $I\to Y$, $i\mapsto h_i$ and $I\to X$, $i\mapsto\alpha_i$ form an X-regular and Y-regular root datum in the sense of [Lus94, Section 2.2].

Let $D=\mathrm{diag}(\epsilon_i\,|,i\in I)$ be a diagonalizing matrix for A. There exists a non-degenerate, symmetric bilinear form on $\mathfrak{h}$ such that $(h_i,h)=\alpha_i(h)/\epsilon_i$ for all $h\in\mathfrak{h}$, $i\in I$ and $(d',d'')=0$ for all $d',d''\in\Pi^\vee_{\mathrm{ext}}\setminus\Pi^\vee$. This pairing induces a pairing on $\mathfrak{h}^*$ which we denote by the same symbol.

In the present notes we work over the field of rational functions $\mathbb{K}(q)$ where $\mathbb{K}$ is a field of characteristic 0. We define the quantized enveloping algebra $U_q(\mathfrak{g})$ as the associative $\mathbb{K}(q)$-algebra generated by elements E_i,F_i,K_h for all $i\in I$, $h\in Q^\vee_{\mathrm{ext}}$ and relations given [Lus94, 3.1.1]. In particular, the generators E_i,F_i satisfy the quantum Serre relations

$$S_{ij}(E_i,E_j)=0=S_{ij}(F_i,F_j)$$

for all $i,j\in I$, where

$$S_{ij}(x,y)=\sum_{\ell=0}^{1-a_{ij}}(-1)^\ell\begin{bmatrix}1-a_{ij}\\ \ell\end{bmatrix}_{q_i}x^{1-a_{ij}-\ell}yx^\ell \tag{3.1}$$

with $q_i=q^{\epsilon_i}$ denotes the (non-commutative) quantum Serre polynomial [Lus94, Corollary 33.1.5]. We will use the notation $K_i=K_{\epsilon_ih_i}$ for all $i\in I$. With this notation, $U_q(\mathfrak{g})$ is a Hopf algebra with coproduct Δ given by

$$\Delta(E_i)=E_i\otimes 1+K_i\otimes E_i,\quad \Delta(F_i)=F_i\otimes K_i^{-1}+1\otimes F_i,\quad \Delta(K_h)=K_h\otimes K_h$$

for all $i\in I$, $h\in Q^\vee_{\mathrm{ext}}$. Let $U=U_q(\mathfrak{g}')$ be the Hopf subalgebra of $U_q(\mathfrak{g})$ generated by the elements $E_i,F_i,K_i^{\pm1}$ for all $i\in I$. As usual, let U^+, U^- and U^0 be the subalgebras of $U_q(\mathfrak{g})$ generated by the elements of the sets $\{E_i\,|\,i\in I\}$, $\{F_i\,|\,i\in I\}$ and $\{K_h\,|\,h\in Q^\vee_{\mathrm{ext}}\}$, respectively, and define $U^{\ge}=U^+U^0$, $U^{\le}=U^-U^0$. We also write $U^{0'}$ for the subalgebra of U^0 generated by $\{K_i\,|\,i\in I\}$. For any U^0-module M and any $\lambda\in P$ we write $M_\lambda=\{m\in M\,|\,K_hm=q^{\lambda(h)}m \text{ for all } h\in Q^\vee_{\mathrm{ext}}\}$. This notation can be applied in particular to U^+,U^- and $U^{\ge},U^{\le}$ under the left adjoint action of U^0. For any subset $X\subseteq I$ of finite type, define $U_q(\mathfrak{g}_X)\subset U$ to be the Hopf subalgebra of U generated by $E_i,F_i,K_i^{\pm1}$ for $i\in X$. Moreover, we write U_X^+, U_X^- and U_X^0 to denote the subalgebras of $U_q(\mathfrak{g}_X)$ generated by the elements of the sets $\{E_i\,|\,i\in X\}$, $\{F_i\,|\,i\in X\}$ and $\{K_j^{\pm1}\,|\,j\in X\}$, respectively.

By [Lus94, Chapter 1] there exists a unique $\mathbb{K}(q)$-bilinear pairing $\langle\,,\,\rangle:U^{\le}\otimes U^{\ge}\to\mathbb{K}(q)$ such that for all $x,x'\in U^{\ge}$, $y,y'\in U^{\le}$ and $g,h\in Q^\vee_{\mathrm{ext}}$ the following relations hold

$$\langle y, xx'\rangle = \langle \Delta(y), x'\otimes x\rangle\,, \qquad \langle yy', x\rangle = \langle y\otimes y', \Delta(x)\rangle\,,$$
$$\langle K_g, K_h\rangle = q^{-(g,h)}, \qquad \langle F_i, E_j\rangle = \delta_{ij}\frac{-1}{q_i - q_i^{-1}},$$
$$\langle K_h, E_i\rangle = 0, \qquad \langle F_i, K_h\rangle = 0.$$

Here we follow the conventions used in the finite case in [Jan96, 6.12]. The restriction of the pairing $\langle\, ,\, \rangle$ to $U^-_{-\mu}\otimes U^+_\nu$ vanishes if $\mu\neq\nu$ and is non-degenerate if $\mu=\nu$. For any $\mu\in Q^+$ let $\{F_{\mu,j}\}\subset U^-_{-\mu}$ and $\{E_{\mu,j}\}\subset U^+_\mu$ be dual bases with respect to the pairing $\langle\, ,\, \rangle$ and define $\Theta_\mu = \sum_j F_{\mu,j}\otimes E_{\mu,j}$. For simplicity, we usually suppress that summation and write formally $\Theta_\mu = F_\mu\otimes E_\mu$. The quasi R-matrix for $U_q(\mathfrak{g})$ is defined by

$$\Theta = \sum_{\mu\in Q^+} F_\mu\otimes E_\mu, \tag{3.2}$$

see [Lus94, 4.1.2]. For any $\mu=\sum_{i\in I} n_i\alpha_i \in Q$ we write $K_\mu = \prod_{i\in I} K_i^{n_i}$. With this notation we can use the properties of the skew-pairing $\langle\, ,\, \rangle$ to determine the coproducts

$$(\varDelta\otimes \mathrm{id})(\Theta_\mu) = \sum_{\lambda+\nu=\mu} F_\lambda\otimes F_\nu K_\lambda^{-1}\otimes E_\nu E_\lambda$$
$$(\mathrm{id}\otimes\varDelta)(\Theta_\mu) = \sum_{\lambda+\nu=\mu} F_\lambda F_\nu\otimes E_\lambda K_\nu\otimes E_\nu$$

for all μ, see [Lus94, 4.2.2].

4. **Completions of $U_q(\mathfrak{g})$.** The quasi R-matrix for $U_q(\mathfrak{g})$ defined by (3.2) belongs to a larger algebra $\mathscr{U}_0^{(2)}$ which contains $U_q(\mathfrak{g})\otimes U_q(\mathfrak{g})$ as a subalgebra. To define $\mathscr{U}_0^{(2)}$ let $\mathcal{O}_{\mathrm{int}}$ denote the category of integrable $U_q(\mathfrak{g})$-modules in category $\mathcal{O}$, see [BK19, Section 3.1] for our conventions. The category $\mathcal{O}_{\mathrm{int}}$ is semisimple, simple objects in $\mathcal{O}_{\mathrm{int}}$ are irreducible highest weight modules with dominant integral highest weights. If $\mathfrak{g}$ is finite-dimensional then $\mathcal{O}_{\mathrm{int}}$ coincides with the category of finite-dimensional $U_q(\mathfrak{g})$-modules of type 1.

Let $\mathcal{F}or : \mathcal{O}_{\mathrm{int}}\to \mathcal{V}ect$ be the forgetful functor into the category of $\mathbb{K}(q)$-vector spaces and define $\mathscr{U} = \mathrm{End}(\mathcal{F}or)$. Elements of $\mathscr{U}$ are families $(f_M)_{M\in Ob(\mathcal{O}_{\mathrm{int}})}$ of vector space endomorphisms $f_M : M\to M$ such that for any $U_q(\mathfrak{g})$-module homomorphism $\varphi : M\to N$ the relation $\varphi\circ f_M = f_N\circ\varphi$ holds. Multiplication by elements of $U_q(\mathfrak{g})$ gives us such a family of vector space endomorphisms, and hence $U_q(\mathfrak{g})$ may be considered as a subalgebra of $\mathscr{U}$.

Example 4.1. *For any map $\xi : P\to \mathbb{K}(q)$ and $M\in Ob(\mathcal{O}_{\mathrm{int}})$ define a linear map $\xi_M : M\to M$ by $\xi_M(m) = \xi(\lambda)m$ for all $m\in M_\lambda$, $\lambda\in P$. The family $(\xi_M)_{M\in Ob(\mathcal{O}_{\mathrm{int}})}$ defines an element in $\mathscr{U}$ which we also denote by ξ.*

Example 4.2. *For any $i\in I$ and $M\in Ob(\mathcal{O}_{\mathrm{int}})$ let $T_{i,M} : M\to M$ be the linear automorphism denoted by $T'_{i,M}$ in* [Lus94, 5.2]. *The family $T_i = (T_{i,M})_{M\in Ob(\mathcal{O}_{\mathrm{int}})}$ defines an invertible element in $\mathscr{U}$. By* [Lus94, 39.4.3] *the elements $T_i\in\mathscr{U}$ for $i\in I$ satisfy the braid relations of W.*

Moreover, conjugation by T_i leaves the subalgebra $U_q(\mathfrak{g}) \subset \mathscr{U}$ invariant. Hence there exist algebra automorphisms $T_i^U : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ such that

$$T_{i,M}(um) = T_i^U(u)T_{i,M}(m) \qquad \text{for all } M \in Ob(\mathcal{O}_{\text{int}}), m \in M, u \in U_q(\mathfrak{g}),$$

see [Lus94, 37.1.2]. *The automorphism T_i^U is a quantum group analog of the action* $\mathrm{Ad}(\sigma_i)$ *defined by Equation* (2.2). *By construction the algebra automorphisms T_i^U also satisfy the braid relations of W. In particular, for each element $w \in W$ there exists a uniquely determined element $T_w = (T_{w,M})_{M\in Ob(\mathcal{O}_{\text{int}})} \in \mathscr{U}$ and a uniquely determined algebra automorphism $T_w^U : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$, and T_w^U coincides with conjugation by T_w. Following common practice, we omit the superscript U from now on and use the same symbol for the braid group action on modules in $\mathcal{O}_{\text{int}}$ and on $U_q(\mathfrak{g})$.*

The algebra $\mathscr{U}$ is no Hopf algebra. To define a larger algebra containing $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$, consider the forgetful functor $\mathcal{F}or^{(2)} : \mathcal{O}_{\text{int}} \times \mathcal{O}_{\text{int}} \to \mathcal{V}ect$ given on objects by $(M, N) \mapsto M \otimes N$ and define $\mathscr{U}_0^{(2)} = \mathrm{End}(\mathcal{F}or^{(2)})$. Elements of $\mathscr{U}_0^{(2)}$ are families $(f_{M_1,M_2})_{M_1,M_2\in Ob(\mathcal{O}_{\text{int}})}$ of linear maps $f_{M_1,M_2} : M_1 \otimes M_2 \to M_1 \otimes M_2$ such that for any two $U_q(\mathfrak{g})$-module homomorphism $\varphi_{1/2} : M_{1/2} \to N_{1/2}$ the relation $(\varphi_1 \otimes \varphi_2) \circ f_{M_1,M_2} = f_{N_1,N_2} \circ (\varphi_1 \otimes \varphi_2)$ holds.

Example 4.3. *Any infinite sum $\Phi = \sum_{\mu\in Q^+} b_\mu \otimes u_\mu$ with $u_\mu \in U_\mu^+$ and $b_\mu \in U_q(\mathfrak{g})$ defines an element $\mathscr{U}_0^{(2)}$. Indeed, the element Φ has a well-defined action on $M_1 \otimes M_2$ for $M_1, M_2 \in Ob(\mathcal{O}_{\text{int}})$ as only finitely many terms survive on the second tensor factor. In particular, we can consider the quasi R-matrix Θ defined by* (3.2) *as an element of $\mathscr{U}_0^{(2)}$.*

We can now define an algebra homomorphism

$$\varDelta : \mathscr{U} \to \mathscr{U}_0^{(2)}, \qquad \varDelta((f_M)_{M\in Ob(\mathcal{O}_{\text{int}})}) = (f_{M\otimes N})_{M,N\in Ob(\mathcal{O}_{\text{int}})}.$$

This algebra homomorphism restricts to the usual coproduct on $U_q(\mathfrak{g}) \subset \mathscr{U}$. Let $X \subseteq I$ be a subset of finite type. Recall that $w_X \in W$ denotes the longest element of the parabolic subgroup corresponding to X. As discussed above, we have a corresponding braid group operator $T_{w_X} \in \mathscr{U}$. The coproduct $\varDelta(T_{w_X}) \in \mathscr{U}_0^{(2)}$ can be expressed in terms of the quasi R-matrix by the formula

$$\varDelta(T_{w_X}) = (T_{w_X} \otimes T_{w_X}) \circ \Theta_X^{-1} \tag{4.1}$$

where Θ_X denotes the quasi R-matrix of $U_q(\mathfrak{g}_X)$, see for example [Lus94, Proposition 5.3.4], [CP94, Lemma 8.3.11], [BK19, Lemma 3.8].

5. **Construction of QSP coideal subalgebras.** Let (X, τ) be a generalized Satake diagram and let $\mathbf{c} = (c_i)_{i\in I\setminus X} \in \mathbb{K}(q)^{I\setminus X}$ be a family of parameters. Recall from the comments below Theorem 2.1 that we write $\theta = -w_X\tau : \mathfrak{h}^* \to \mathfrak{h}^*$. With this notation we set $Q^\theta = \{\alpha \in Q \,|\, \theta(\alpha) = \alpha\}$. Define $U_\theta^{0\prime} = \mathbb{K}(q)\langle K_\alpha \,|\, \alpha \in Q^\theta\rangle$ and observe that

$$U_\theta^{0\prime} = \mathbb{K}(q)\langle K_i K_{\tau(i)}^{-1}, K_j \,|\, i \in I \setminus X,\ j \in X\rangle.$$

Define $\mathcal{B}_\mathbf{c} = \mathcal{B}_\mathbf{c}(X, \tau) \subset U_q(\mathfrak{g}')$ to be the subalgebra generated by $U_q(\mathfrak{g}_X)$, $U_\theta^{0\prime}$ and the elements

$$B_i = F_i - c_i T_{w_X}(E_{\tau(i)})K_i^{-1} \qquad \text{for all } i \in I \setminus X. \tag{5.1}$$

Observe that $U_q(\mathfrak{g}_X)$ and $U_\theta^{0'}$ are quantum group analogs of $\mathfrak{g}_X$ and $\mathfrak{k}\cap\mathfrak{g}'\cap\mathfrak{h}$, respectively. Hence $\mathcal{B}_{\mathbf{c}}$ may be considered as a quantum group analog of $U(\mathfrak{k}')$ for $\mathfrak{k}' := \mathfrak{g}'\cap\mathfrak{k}$. In the following we will show that the subalgebra $\mathcal{B}_{\mathbf{c}} \subset U_q(\mathfrak{g}')$ satisfies the desired properties L1) and L2) formulated in Section 1, for a suitable choice of parameters $\mathbf{c}$. The coideal property holds independently of the choice of parameters.

Proposition 5.1. *The subalgebra $\mathcal{B}_{\mathbf{c}}$ is a right coideal of $U_q(\mathfrak{g}')$, that is*

$$\varDelta(\mathcal{B}_{\mathbf{c}}) \subset \mathcal{B}_{\mathbf{c}} \otimes U_q(\mathfrak{g}').$$

Proof. As $U_q(\mathfrak{g}_X)$ and $U_\theta^{0'}$ are Hopf subalgebras of $U_q(\mathfrak{g}')$ it suffices to check that the elements B_i defined by (5.1) satisfy $\varDelta(B_i) \in \mathcal{B}_{\mathbf{c}} \otimes U_q(\mathfrak{g}')$ for all $i \in I\setminus X$. To this end consider T_{w_X} as an element of the algebra $\mathscr{U}$ discussed in Section 4. In $\mathscr{U}$ we can hence write

$$B_i = F_i - c_i T_{w_X} E_{\tau(i)} T_{w_X}^{-1} K_i^{-1}.$$

The coproduct formulas for $U_q(\mathfrak{g})$ and (4.1) hence give us

$$\begin{aligned}\varDelta(B_i) &= F_i \otimes K_i^{-1} + 1 \otimes F_i \\ &\quad - c_i(T_{w_X}\otimes T_{w_X})\Theta_X^{-1}(E_{\tau(i)}\otimes 1 + K_{\tau(i)}\otimes E_{\tau(i)})\Theta_X(T_{w_X}\otimes T_{w_X})^{-1}(K_i^{-1}\otimes K_i^{-1})\end{aligned}$$

in $\mathscr{U}_0^{(2)}$. Similar to (3.2), we write formally $\Theta_X = \sum_{Q_X^+} F_{X,\mu}\otimes E_{X,\mu}$ with $F_{X,\mu} \in U_X^-$ and $E_{X,\mu} \in U_X^+$. As Θ_X commutes with $E_{\tau(i)}\otimes 1$ in $\mathscr{U}_0^{(2)}$ we obtain

$$\begin{aligned}\varDelta(B_i) &= B_i \otimes K_i^{-1} + 1 \otimes F_i \\ &\quad - c_i(T_{w_X}\otimes T_{w_X})\Theta_X^{-1}(K_{\tau(i)}\otimes E_{\tau(i)})\Theta_X(T_{w_X}\otimes T_{w_X})^{-1}(K_i^{-1}\otimes K_i^{-1}).\end{aligned} \tag{5.2}$$

The above formula implies that $\Theta_X^{-1}(K_{\tau(i)}\otimes E_{\tau(i)})\Theta_X \in U_q(\mathfrak{g}')\otimes U_q(\mathfrak{g}')$. Given the specific form of Θ_X we hence obtain

$$\Theta_X^{-1}(K_{\tau(i)}\otimes E_{\tau(i)})\Theta_X \in U_X^- K_{\tau(i)} \otimes U^+.$$

As $T_{w_X}(U_X^- K_{\tau(i)}) \subset U_X^+ U_X^0 K_{\tau(i)}$, we obtain

$$\varDelta(B_i) - B_i\otimes K_i^{-1} - 1\otimes F_i \in U_X^+ U_\theta^{0'} \otimes U_q(\mathfrak{g}').$$

This implies $\varDelta(B_i) \in \mathcal{B}_{\mathbf{c}} \otimes U_q(\mathfrak{g}_X)$ and concludes the proof of the proposition. □

Remark 5.2. The element $T_{w_X}(E_{\tau(i)})$ can be expressed in terms of the left-adjoint action of $U_q(\mathfrak{g}_X)$ on $E_{\tau(i)}$. This allows an alternative proof of the coideal property for $\mathcal{B}_{\mathbf{c}}$, see [Kol14, Proposition 5.2] .

To simplify notation define $\mathcal{H} = \mathcal{H}(X,\tau) = U_q(\mathfrak{g}_X)U_\theta^{0'}$ and $\mathcal{H}^{\geq} = U_X^+ U_\theta^{0'}$. We call $\mathcal{H}(X,\tau)$ the partial Levi factor corresponding to the generalized Satake diagram (X,τ). As noted previously, $\mathcal{H}(X,\tau)$ is a Hopf subalgebra of $U_q(\mathfrak{g}')$.

It is convenient to set $B_i = F_i$ for $i \in X$. With this notation the generators of $\mathcal{B}_{\mathbf{c}}$ satisfy the relations

$$E_j B_i - B_i E_j = \delta_{ij}\frac{K_j - K_j^{-1}}{q_j - q_j^{-1}} \qquad \text{for all } i\in I, j\in X, \tag{5.3}$$

$$K_\beta B_i = q^{-(\beta,\alpha_i)} B_i K_\beta \qquad \text{for all } i \in I,\ \beta\in Q^\theta. \tag{5.4}$$

For any multi-index $J = (j_1, \dots, j_m) \in I^m$ we write $F_J = F_{j_1} \dots F_{j_m}$ and $B_J = B_{j_1} \dots B_{j_m}$. The relations (5.3) and (5.4) imply that

$$\mathcal{B}_{\mathbf{c}} = \sum_J \mathcal{H}^{\geq} B_J = \sum_J B_J \mathcal{H}^{\geq}$$

where we sum over all multi-indices J of any length. Let $\mathcal{J} \subset \bigcup_{\ell \in \mathbb{N}_0} I^\ell$ be a subset such that $\{F_J \,|\, J \in \mathcal{J}\}$ is a linear basis of U^-.

Define a set of nonzero parameters $\mathcal{C} \subset (\mathbb{K}(q)^\times)^{I \setminus X}$ by

$$\mathcal{C} = \{\mathbf{c} \in (\mathbb{K}(q)^\times)^{I \setminus X} \,|\, c_i = c_{\tau(i)} \text{ for all } i \in I \setminus X \text{ with } (\alpha_i, \theta(\alpha_i)) = 0\}. \tag{5.5}$$

We will see in Theorem 7.2 that the coideal subalgebras $\mathcal{B}_{\mathbf{c}}$ show good behaviour if and only if $\mathbf{c} \in \mathcal{C}$. For example, we will see that in this case $\{B_J \,|\, J \in \mathcal{J}\}$ is a left and right $\mathcal{H}^{\geq}$-module basis of $\mathcal{B}_{\mathbf{c}}$.

Definition 5.3. *Let (X, τ) be a generalized Satake diagram and $\mathbf{c} \in \mathcal{C}$. Then the subalgebra $\mathcal{B}_{\mathbf{c}}$ is called a standard quantum symmetric pair coideal subalgebra (QSP coideal subalgebra) of $U_q(\mathfrak{g}')$.*

We will discuss *non-standard* QSP coideal subalgebras in Section 10.

6. **Triangular decompositions of $U_q(\mathfrak{g}')$.** Recall that the algebra $U = U_q(\mathfrak{g}')$ has a triangular decomposition

$$U^- \otimes U^{0'} \otimes U^+ \cong U \tag{6.1}$$

in the sense that the multiplication map from the left to the right is a linear isomorphism. We recall some related tensor product decompositions. For any subset $X \subseteq I$ of finite type let $\mathcal{L}_X = \mathbb{K}(q)\langle F_j, E_j, K_i^{\pm 1} \,|\, i \in I, j \in X\rangle$ denote the corresponding Levi factor, and let $\mathcal{P}_X^+ = \mathbb{K}(q)\langle F_j, E_j, K_i^{\pm 1} \,|\, i \in I, j \in X\rangle$ and $\mathcal{P}_X^- = \mathbb{K}(q)\langle F_i, E_j, K_i^{\pm 1} \,|\, i \in I, j \in X\rangle$ be the corresponding positive and negative standard parabolic subalgebras of U, respectively. Let ad_l and ad_r denote the left and right adjoint action of U on itself, defined in Sweedler notation by $\mathrm{ad}_l(u)x = u_{(1)} x S(u_{(2)})$ and $\mathrm{ad}_r(u)(x) = S(u_{(1)}) x u_{(2)}$. Let $\mathcal{R}_X^+ \subset U^+$ be the subalgebra generated by the subspaces $\mathrm{ad}_l(\mathcal{L}_X)(E_i)$ for $i \in I \setminus X$, and similarly, let $\mathcal{R}_X^- \subset U^-$ be the subalgebra generated by the subspaces $\mathrm{ad}_r(\mathcal{L}_X)(F_i)$ for $i \in I \setminus X$. The standard parabolic subalgebras $\mathcal{P}_X^\pm$ are Radford biproducts of $\mathcal{L}_X$ and $\mathcal{R}_X^\pm$, [Rad85]. Moreover, $\mathcal{R}_X^\pm$ can be described in terms of Lusztig's braid group action. The following Lemma is well-known, see for example [KY21, 2.2] for a detailed proof of the statements about $\mathcal{R}_X^-$.

Lemma 6.1. *Let $X \subset I$ be a subset of finite type. Then*

$$\mathcal{R}_X^+ = U^+ \cap T_{w_X}(U^+), \qquad \mathcal{R}_X^- = U^- \cap T_{w_X}(U^-)$$

and the multiplications maps $\mathcal{L}_X \otimes \mathcal{R}_X^\pm \to \mathcal{P}_X^\pm$ are linear isomorphisms.

Comparing the triangular decomposition (6.1) with the above lemma, we obtain linear isomorphisms

$$\mathcal{R}_X^- \otimes U_X^- \cong U^-, \qquad U_X^+ \otimes \mathcal{R}_X^+ \cong U^+ \tag{6.2}$$

via multiplication, and therefore

$$\mathcal{R}_X^- \otimes \mathcal{L}_X \otimes \mathcal{R}_X^+ \cong U. \tag{6.3}$$

7. **The standard filtration of $\mathcal{B}_{\mathbf{c}}$.** We call the subalgebra $\mathcal{A} = \mathcal{A}(X,\tau) := U^-\mathcal{H}(X,\tau) \subset U$ the partial parabolic subalgebra corresponding to the generalized Satake diagram (X,τ). The triangular decomposition (6.1) for U implies the triangular decomposition

$$U^- \otimes U_\theta^{0\,\prime} \otimes U_X^+ \cong \mathcal{A} \tag{7.1}$$

for the partial parabolic subalgebra $\mathcal{A}$. Let $I_\tau \subset I \setminus X$ denote any fixed set of representatives of all τ-orbits in $I \setminus X$ and define

$$U_\tau^{0\,\prime} = \mathbb{K}(q)[K_i^{\pm 1} \,|\, i \in I_\tau].$$

Multiplication gives a linear isomorphism

$$U_\theta^{0\,\prime} \otimes U_\tau^{0\,\prime} \cong U^{0\,\prime}. \tag{7.2}$$

Hence, by (6.1) and (6.2) we obtain a triangular decomposition

$$\mathcal{A} \otimes U_\tau^{0\,\prime} \otimes \mathcal{R}_X^+ \cong U. \tag{7.3}$$

The algebra $\mathcal{A}$ is $\mathbb{N}_0$-graded via a degree function on the generators given by

$$\deg(u) = 0 \qquad \text{if } u \in \mathcal{H},$$
$$\deg(F_i) = 1 \qquad \text{if } i \in I \setminus X.$$

Let $U^{\mathrm{poly}} = U^{\mathrm{poly}}(X,\tau)$ be the subalgebra of U generated by $\mathcal{A}$ and the elements $\widetilde{E}_i = E_i K_i^{-1}$, K_i^{-1} for all $i \in I \setminus X$. As $T_{w_X}(E_{\tau(i)})K_i^{-1} \in U_X^+ E_{\tau(i)} K_{\tau(i)}^{-1} U_X^+ K_{\tau(i)} K_i^{-1}$ we have $\mathcal{B}_{\mathbf{c}} \subset U^{\mathrm{poly}}$. The triangular decomposition (6.1) of U implies that

$$U_\theta^{0\,\prime} \otimes \mathbb{K}(q)[K_i^{-1} \,|\, i \in I_\tau] \cong U^{\mathrm{poly}} \cap U^{0\,\prime}. \tag{7.4}$$

Recall that we write $K_\alpha = \prod_{i\in I} K_i^{n_i}$ for $\alpha = \sum_{i \in I} n_i \alpha_i \in Q$. The following lemma will be needed to prove the implication 5) $\Rightarrow$ 4) of the main Theorem 7.2 below.

Lemma 7.1. *If $\mathcal{B}_{\mathbf{c}} \cap U^{0\,\prime} \neq U_\theta^{0\,\prime}$ then there exists a nonzero $\alpha \in -\sum_{i \in I_\tau} \mathbb{N}_0 \alpha_i$ with $K_\alpha \in \mathcal{B}_{\mathbf{c}}$.*

Proof. Assume that $\sum_{\alpha \in Q} a_\alpha K_\alpha \in \mathcal{B}_{\mathbf{c}} \cap U^{0\,\prime} \setminus U_\theta^{0\,\prime}$ for some $a_\alpha \in \mathbb{K}(q)$. Then, by the coideal property of $\mathcal{B}_{\mathbf{c}}$, there exists a non-zero $\alpha \in Q \setminus Q^\theta$ such that $K_\alpha \in \mathcal{B}_{\mathbf{c}}$. By the decomposition (7.4), we can write $\alpha = \alpha^\theta + \alpha'$ with $\alpha^\theta \in Q^\theta$ and $\alpha' \in -\sum_{i \in I_\tau} \mathbb{N}_0 \alpha_i \setminus \{0\}$. Multiplication by $K_{-\alpha^\theta}$ shows that $K_{\alpha'} \in \mathcal{B}_{\mathbf{c}}$. $\square$

Define a degree function on the generators of $\mathcal{B}_{\mathbf{c}}$ by

$$\deg(u) = 0 \qquad \text{if } u \in \mathcal{H},$$
$$\deg(B_i) = 1 \qquad \text{if } i \in I \setminus X.$$

This degree function defines a filtration $\mathcal{F}_*$ on the algebra $\mathcal{B}_{\mathbf{c}}$. An element of $\mathcal{B}_{\mathbf{c}}$ belongs to $\mathcal{F}_n\mathcal{B}_{\mathbf{c}}$ if it can be written as a polynomial in the generators, involving at most n of the generators B_i for $i \in I \setminus X$ in each monomial.

Let $p = p(x_i \,|\, i \in I)$ be a homogeneous, non-commutative polynomial of degree m in the variables x_i for $i \in I$ with coefficients in $\mathcal{H}^{\geq}$. Here 'homogeneous of degree m' means that each monomial contains precisely m factors x_i with $i \in I \setminus X$. Let $p(\underline{B})$ denote the element of $\mathcal{B}_{\mathbf{c}}$ obtained by evaluating x_i at B_i for all $i \in I$. Similarly, let $p(\underline{F})$ denote the element of $\mathcal{A}$ obtained by evaluating x_i at F_i for all $i \in I$. The triangular decomposition (6.1) implies that

$$p(\underline{B}) \in \mathcal{F}_{m-1}(\mathcal{B}_{\mathbf{c}}) \quad \Longrightarrow \quad p(\underline{F}) = 0.$$

Hence we obtain a surjective homomorphism of graded algebras

$$\varphi : \mathrm{gr}(\mathcal{B}_{\mathbf{c}}) \to \mathcal{A} \tag{7.5}$$

satisfying $\varphi(B_i) = F_i$ for all $i \in I$ and $\varphi(u) = u$ for all $u \in \mathcal{H}$. We would like to know under which conditions the map φ is an isomorphism. Recall that we write $U = U_q(\mathfrak{g}')$ and recall the definition of the set of multi-indices $\mathcal{J}$ given at the end of Section 5.

Theorem 7.2. *Let (X, τ) be a generalized Satake diagram and $\mathbf{c} = (c_i)_{i \in I \setminus X} \in \mathbb{K}(q)^{I \setminus X}$. The following statements are equivalent:*

1) *The map φ given by* (7.5) *is an isomorphism of algebras.*
2) *The multiplication map $\mathrm{mult}_{\mathbf{c}} : \mathcal{B}_{\mathbf{c}} \otimes U^{0'}_{\tau} \otimes \mathcal{R}^+_X \to U$ is a linear isomorphism.*
3) *The set $\{B_J \mid J \in \mathcal{J}\}$ is a basis of $\mathcal{B}_{\mathbf{c}}$ as a right (or left) $\mathcal{H}^{\geq}$-module.*
4) *$\mathcal{B}_{\mathbf{c}} \cap U^{0'} = U^{0'}_{\theta}$.*
5) *U is a free left $\mathcal{B}_{\mathbf{c}}$-module.*
6) *The coefficients $\mathbf{c} = (c_i)_{i \in I \setminus X}$ satisfy the relation*

$$c_i = c_{\tau(i)} \qquad \text{for all } i \in I \setminus X \text{ with } (\alpha_i, \theta(\alpha_i)) = 0.$$

Proof. **1) ⇔ 2):** Via the triangular decomposition (7.3), the grading of $\mathcal{A}$ induces a filtration of U as a vector space. On the other hand, the filtration on $\mathcal{B}_{\mathbf{c}}$ induces a filtration on $\mathcal{B}_{\mathbf{c}} \otimes U^{0'}_{\tau} \otimes \mathcal{R}^+_X$. The multiplication map $\mathrm{mult}_{\mathbf{c}}$ is filtered with respect to these two filtrations. The associated graded map is $(\varphi \otimes \mathrm{id} \otimes \mathrm{id})$ composed with the multiplication $\mathrm{mult}_{\mathcal{A}} : \mathcal{A} \otimes U^{0'}_{\tau} \otimes \mathcal{R}^+_X \to U$. As $\mathrm{mult}_{\mathcal{A}}$ is an isomorphism by (7.3), we see that $\mathrm{gr}(\mathrm{mult}_{\mathbf{c}})$ is an isomorphism if and only if φ is an isomorphism.
1) ⇔ 3): Consider the subspace $W_{\mathcal{J}} = \sum_{J \in \mathcal{J}} B_J \mathcal{H}^{\geq}$ of $\mathcal{B}_{\mathbf{c}}$. The filtration $\mathcal{F}$ on $\mathcal{B}_{\mathbf{c}}$ induces a filtration on $W_{\mathcal{J}}$ and we have an inclusion

$$\mathrm{gr}(W_{\mathcal{J}}) \xhookrightarrow{i^{\mathrm{gr}}} \mathrm{gr}(\mathcal{B}_{\mathbf{c}})$$

By the triangular decomposition (7.1) the algebra $\mathcal{A}$ is a free right $\mathcal{H}^{\geq}$-module with basis $\{F_J \mid J \in \mathcal{J}\}$. As $\varphi \circ i^{\mathrm{gr}}(B_J) = F_J$ for all $J \in \mathcal{J}$, the map $\varphi \circ i^{\mathrm{gr}}$ is a bijection. Hence i^{gr} is a bijection if and only if φ is a bijection. Moreover, as $\{F_J \mid J \in \mathcal{J}\}$ is a basis of the right $\mathcal{H}^{\geq}$-module $\mathcal{A}$, the set $\{B_J \mid J \in \mathcal{J}\}$ is a basis of the right $\mathcal{H}^{\geq}$-module $\mathcal{B}_{\mathbf{c}}$ if and only if φ is bijective.
2) ⇒ 4): By definition of $\mathcal{B}_{\mathbf{c}}$ we have $U^{0'}_{\theta} \subset \mathcal{B}_{\mathbf{c}}$. If 2) holds, then the decomposition (7.2) implies that $\mathcal{B}_{\mathbf{c}}$ cannot contain any element of $U^{0'} \setminus U^{0'}_{\theta}$.
4) ⇒ 5): The Hopf algebra U is pointed with coradical $U^{0'}$. If 4) holds then $\mathcal{B}_{\mathbf{c}} \cap U^{0'}$ is invariant under the antipode S of U. By [Mas91, Proposition 1.4] this means that U is free as a left (and right) $\mathcal{B}_{\mathbf{c}}$-module.
5) ⇒ 4): Assume that $\mathcal{B}_{\mathbf{c}} \cap U^{0'} \neq U^{0'}_{\theta}$. Lemma 7.1 implies that there exist a nonzero $\alpha \in -\sum_{i \in I_\tau} \mathbb{N}_0 \alpha_i$ with $K_\alpha \in \mathcal{B}_{\mathbf{c}}$. As K_α is not invertible in $\mathcal{B}_{\mathbf{c}} \subset U^{\mathrm{poly}}$, we obtain that $K_\alpha \mathcal{B}_{\mathbf{c}}$ is a proper right submodule of $\mathcal{B}_{\mathbf{c}}$. However, the induced map $K_\alpha \mathcal{B}_{\mathbf{c}} \otimes_{\mathcal{B}_{\mathbf{c}}} U \to \mathcal{B}_{\mathbf{c}} \otimes_{\mathcal{B}_{\mathbf{c}}} U$ is surjective. Hence U cannot be free as a left $\mathcal{B}_{\mathbf{c}}$-module.
4) ⇒ 6): Let $i \in I \setminus X$ such that $\tau(i) \neq i$ and $(\alpha_i, \theta(\alpha_i)) = 0$. By [Kol14,

Lemma 5.3] we have $\theta(\alpha_i) = -\alpha_{\tau(i)}$ and $(\alpha_i, \alpha_{\tau(i)}) = 0$ in this case, and hence $B_i = F_i - c_i E_{\tau(i)} K_i^{-1}$ and $B_{\tau(i)} = F_{\tau(i)} - c_{\tau(i)} E_i K_{\tau(i)}^{-1}$. A direct calculation gives

$$(7.6) \qquad [B_i, B_{\tau(i)}] = c_{\tau(i)} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} K_{\tau(i)}^{-1} - c_i \frac{K_{\tau(i)} - K_{\tau(i)}^{-1}}{q_i - q_i^{-1}} K_i^{-1}.$$

Hence, if $c_i \neq c_{\tau(i)}$ then $K_i^{-1} K_{\tau(i)}^{-1} \in \mathcal{B}_{\mathbf{c}}$. This would be a contradiction to 4).

6) $\Rightarrow$ 2): This is the statement of [Kol14, Proposition 6.3] up to a reordering of factors. □

Remarks 7.3. 1. If we assume $c_i \neq 0$ for all $i \in I \setminus X$, then the theorem states that $\mathcal{B}_{\mathbf{c}}$ has any of the properties 1)–5) if and only if $\mathcal{B}_{\mathbf{c}}$ is a QSP coideal subalgebra as defined in Definition 5.3.

2. The triangular decomposition in part 2) of the theorem is commonly known as the quantum Iwasawa decomposition.

3. The final implication 6) $\Rightarrow$ 2) is the hardest part of the proof. It hinges on a subtle argument involving the evaluation of q-Serre polynomials on the generators B_i for $i \in I$, see [Let02, Section 7] and [Kol14, Corollary 5.17].

4. By Lemma 7.1 and the decomposition 7.4, the subalgebra $U_\theta^{0'}$ is the maximal subalgebra of $U^{0'} \cap \mathcal{B}_{\mathbf{c}}$ which is closed under the antipode S. Hence condition 4) in Theorem 7.2 is equivalent to the statement that $U^{0'} \cap \mathcal{B}_{\mathbf{c}}$ is a Hopf subalgebra of U. As U is pointed with coradical $U^{0'}$, this condition is equivalent to the faithful flatness of U as a left (or right) $\mathcal{B}_{\mathbf{c}}$-module, see [Mas91]. A right coideal subalgebra C of a Hopf algebra H such that H is faithfully flat as a right C-module is commonly called a quantum homogeneous space, see [Krä12]. Statements 4) and 5) of Theorem 7.2 hence express the desirable fact that the pair $(U, \mathcal{B}_{\mathbf{c}})$ is a quantum homogeneous space.

8. **The specialization property.** We briefly recall non-restricted specialization as outlined in [CK90, 1.5]. As in [Kol14, Section 10] we follow the presentation in [HK02]. Let $\mathbf{A} = \mathbb{K}[q]_{(q-1)}$ be the localization of the polynomial ring $\mathbb{K}[q]$ at the prime ideal $(q-1)$. For any $i \in I$ we set $(K_i; 0)_q = \frac{K_i - 1}{q-1}$. The $\mathbf{A}$-form $\mathcal{U}'_{\mathbf{A}}$ of $U = U_q(\mathfrak{g}')$ is the $\mathbf{A}$-subalgebra of U generated by the elements $E_i, F_i, K_i^{\pm 1}$, and $(K_i; 0)_q$ for all $i \in I$. The field $\mathbb{K}$ is an $\mathbf{A}$-module via evaluation at 1. The algebra $\mathcal{U}'_1 = \mathbb{K} \otimes_{\mathbf{A}} \mathcal{U}'_{\mathbf{A}}$ is called the specialization of U at $q = 1$.

For any $x \in \mathcal{U}'_{\mathbf{A}}$ we write $\overline{x}$ to denote its image in $\mathcal{U}'_1$. The following result is well-known.

Theorem 8.1. ([CK90, Proposition 1.5], see also [HK02, Theorem 3.4.9]) *There exists an isomorphism of algebras $\mathcal{U}'_1 \to U(\mathfrak{g}')$ such that $\overline{E_i} \mapsto e_i$, $\overline{F_i} \mapsto f_i$ and $\overline{(K_i; 0)_q} \mapsto \epsilon_i h_i$.*

For any Satake diagram (X, τ) recall the signs $s(\alpha_i)$ in the construction of the involution $\theta(X, \tau)$ in Theorem 2.1. We say that a set of parameters $\mathbf{c} = (c_i) \in \mathbf{A}^{I \setminus X}$ is specializable if $c_i(1) = s(\alpha_{\tau(i)})$. If $\mathbf{c} \in \mathbf{A}^{I \setminus X}$ is specializable then the generators B_i of $\mathcal{B}_{\mathbf{c}}$ belong to $\mathcal{U}'_{\mathbf{A}}$ and satisfy $\overline{B}_i = f_i + \theta(f_i)$ for all $i \in I \setminus X$, see [Kol14, Corollary 10.3].

For any subspace $W \subset U$ we define $\overline{W} = \mathbb{K} \otimes_{\mathbf{A}} (W \cap \mathcal{U}'_{\mathbf{A}}) \subset \mathcal{U}'_1$. We call $\overline{W}$ the specialization of the subspace W. The subalgebra $\mathcal{B}_{\mathbf{c}}$ has the desired specialization if and only if the parameters satisfy the conditions in Theorem 7.2. Indeed, let

(X,τ) be a Satake diagram and write $\mathfrak{k}' = \mathfrak{k}\cap\mathfrak{g}'$ where $\mathfrak{k}$ is the Lie subalgebra fixed under the involution $\theta(X,\tau)$. By [Kol14, Theorem 10.8] we know that condition 6) in Theorem 7.2 implies that $\overline{\mathcal{B}}_{\mathbf{c}} = U(\mathfrak{k}')$. Conversely, if condition 6) in Theorem 7.2 fails, then we have seen in the proof of the implication 4) $\Rightarrow$ 6) of the Theorem 7.2 that $K_i^{-1}K_{\tau(i)}^{-1} \in \mathcal{B}_{\mathbf{c}}$ for some $i \in I\setminus X$ with $i \neq \tau(i)$ and hence $K_i^{-2} \in \mathcal{B}_{\mathbf{c}}$. This implies that $h_i \in \overline{\mathcal{B}}_{\mathbf{c}}$, however, $h_i \notin \mathfrak{k}$. We summarize the discussion in the following Theorem.

Theorem 8.2. *Let (X,τ) be a Satake diagram and $\mathbf{c}\in\mathbf{A}^{I\setminus X}$ specializable. Then $\overline{\mathcal{B}}_{\mathbf{c}} = U(\mathfrak{k}')$ if and only if the equivalent conditions in Theorem 7.2 hold.*

Remark 8.3. It is natural to ask how Theorem 8.2 extends to generalized Satake diagrams and general parameters in $\mathbf{A}^{I\setminus X}$. In [RV20], [RV22] V. Regelskis and B. Vlaar introduced the notions of pseudo-involutions and associated pseudo-fixed-point subalgebras. We expect that the above theorem extends to this setting.

9. **Generators and relations for $\mathcal{B}_{\mathbf{c}}$.** Let (X,τ) be a generalized Satake diagram, $\mathbf{c}\in\mathcal{C}$ and $\mathcal{B}_{\mathbf{c}}$ the corresponding QSP coideal subalgebra. For $i,j\in I$ we can evaluate the quantum Serre polynomial $S_{ij}(x,y)$ defined by (3.1) on the generators B_i, B_j of $\mathcal{B}_{\mathbf{c}}$. By definition, we have $S_{ij}(B_i,B_j)\in\mathcal{F}_{\deg(i,j)}(\mathcal{B}_{\mathbf{c}})$ where

$$\deg(i,j) = \begin{cases} 2-a_{ij} & \text{if } i,j\in I\setminus X,\\ 1-a_{ij} & \text{if } i\in I\setminus X,\ j\in X,\\ 1 & \text{if } i\in X,\ j\in I\setminus X,\\ 0 & \text{if } i,j\in X.\end{cases}$$

By the equivalence 1) $\Leftrightarrow$ 6) of Theorem 7.2 there exist elements $C_{ij}(\mathbf{c})\in\mathcal{F}_{\deg(i,j)-1}(\mathcal{B}_{\mathbf{c}})$ such that

$$S_{ij}(B_i,B_j) = C_{ij}(\mathbf{c}) \qquad \text{for all } i,j\in I,\ i\neq j.. \tag{9.1}$$

Comparison with the defining relations of the partial parabolic subalgebra $\mathcal{A}$ then implies the following result. Recall that we write $\mathcal{H}^{\geq} = U_X^+ U_\theta^{0\,\prime}$.

Theorem 9.1. [Let02, Theorem 7.4], [Kol14, Theorem 7.1] *Let $\mathbf{c}\in\mathcal{C}$. The algebra $\mathcal{B}_{\mathbf{c}}$ is generated over $\mathcal{H}^{\geq}$ by the elements B_i for $i\in I$ subject to the defining relations* (5.3), (5.4) *and* (9.1).

The deformed quantum Serre relations (9.1) can be made explicit, see [KY21] and references therein.

Examples 9.2. *We write down the relations* (9.1) *for three explicit examples.*

(1) *Let $A = \begin{pmatrix} 2 & -1\\ -1 & 2\end{pmatrix}$ with $I=\{1,2\}$, that is $\mathfrak{g}=\mathfrak{sl}_3(\mathbb{C})$, and choose $(X,\tau)=(\emptyset,\mathrm{id})$. In this case, $\mathcal{C}=(\mathbb{K}(q)^\times)^2$ and $\mathcal{B}_{\mathbf{c}}\subset U_q(\mathfrak{sl}_3(\mathbb{C}))$ is the subalgebra generated by the elements $B_i = F_i - c_iE_iK_i^{-1}$ for $i=1,2$. The relations* (9.1) *are given explicitly by*

$$B_i^2B_j - (q+q^{-1})B_iB_jB_i + B_jB_i^2 = -qc_iB_j \tag{9.2}$$

for $\{i,j\}=I$.

(2) *Let $A = \begin{pmatrix} 2 & -2\\ -2 & 2\end{pmatrix}$, that is $\mathfrak{g}=\widehat{\mathfrak{sl}_2}(\mathbb{C})$, and choose $(X,\tau)=(\emptyset,\mathrm{id})$. To account for the affine situation, we write $I=\{0,1\}$. The generators of $\mathcal{B}_{\mathbf{c}}$*

are again given by $B_i = F_i - c_i E_i K_i^{-1}$ *for* $i = 0, 1$, *but the defining relations* (9.1) *now read*

$$B_i^3 B_j - [3]_q B_i^2 B_j B_i + [3]_q B_i B_j B_i^2 - B_j B_i^2 = q(q+q^{-1})^2 c_i (B_j B_i - B_i B_j)$$

for $\{i,j\} = \{0,1\}$ *where* $[3]_q = q^2 + 1 + q^{-2}$, *see* [Kol14, Example 7.6].

(3) *Let* $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ *as before but now choose* $(X,\tau) = (\emptyset, (01))$ *for* $I = \{0,1\}$. *The generators of* $\mathcal{B}_{\mathbf{c}}$ *are given by* $B_i = F_i - c_i E_j K_i^{-1}$ *for* $\{i,j\} = I$, *and the defining relations* (9.1) *read*

$$B_i^3 B_j - [3]_q B_i^2 B_j B_i + [3]_q B_i B_j B_i^2 - B_j B_i^2 = c_i q^{-1}(1-q^6)(1+q^2) B_i^2 K_j K_i^{-1} + c_j q^{-1}(1-q^{-6})(1+q^{-2}) B_i^2 K_i K_j^{-1}$$

again for $\{i,j\} = I$.

10. **Non-standard QSP coideal subalgebras.** Let k be a field. For any unital k-algebra A we write $\widehat{A}$ to denote the set of unital k-algebra homomorphisms $\chi : A \to k$. We refer to elements of $\widehat{A}$ as characters of A. If H is a Hopf algebra over k then $\widehat{H}$ is a group. If $C \subset H$ is a right coideal subalgebra then the group $\widehat{H}$ acts on $\widehat{C}$ from the right via

$$\chi \triangleleft \mu(c) = \chi(c_{(1)})\mu(c_{(2)}) \qquad \text{for all } \chi \in \widehat{C},\ \mu \in \widehat{H} \text{ and } c \in C.$$

Moreover, for any $\chi \in \widehat{C}$ the set

$$C_\chi = \{\chi(c_{(1)})c_{(2)} \,|\, c \in C\}$$

is a right coideal subalgebra of H, and the map $\rho_\chi : C \to C_\chi$ defined by

$$\rho_\chi(c) = \chi(c_{(1)})c_{(2)} \qquad \text{for all } c \in C \tag{10.1}$$

is a surjective homomorphism of right H-comodule algebras. Taking the perspective of quantum homogeneous spaces, we refer to the right coideal subalgebra $C_\chi \subset H$ as the *shift of basepoint* of C by the character χ.

We return to the specific setting of these notes. Let (X,τ) be a generalized Satake diagram and $\mathbf{c} \in \mathcal{C}$. For any $\chi \in \widehat{\mathcal{B}_{\mathbf{c}}}$ there exists a character $\mu \in \widehat{U}$ such that $(\chi \triangleleft \mu)|_{U_q(\mathfrak{g}_X)} = \varepsilon|_{U_q(\mathfrak{g}_X)}$. In the following we hence restrict to characters $\chi \in \widehat{\mathcal{B}_{\mathbf{c}}}$ with $\chi|_{U_q(\mathfrak{g}_X)} = \varepsilon|_{U_q(\mathfrak{g}_X)}$. Define a subset $I_{ns} \subset I$ by

$$I_{ns} = \{i \in I \setminus X \,|\, \tau(i) = i \text{ and } \alpha_i(h_j) = 0\ \forall j \in X\}.$$

Proposition 10.1. *Let* $\mathbf{c} \in \mathcal{C}$ *and* $\chi \in \widehat{\mathcal{B}_{\mathbf{c}}}$ *a character such that* $\chi(u) = \varepsilon(u)$ *for all* $u \in U_q(\mathfrak{g}_X)$. *For any* $i \in I \setminus X$ *define* $t_i = \chi(K_i K_{\tau(i)}^{-1})$ *and* $s_i = \chi(B_i)$. *Then the following hold:*

(1) *If* $\tau(i) = i$ *then* $t_i = 1$.
(2) *If* $\tau(i) \neq i$ *and* $(\alpha_i, \theta(\alpha_i)) = 0$ *then* $t_i = \pm 1$.
(3) *If* $i \notin I_{ns}$ *then* $s_i = 0$.

Proof. If $\tau(i) = i$ there is nothing to prove. If $\tau(i) \neq i$ and $(\alpha_i, \theta(\alpha_i)) = 0$ then the condition $\mathbf{c} \in \mathcal{C}$, Equation (5.4) and Equation (7.6) imply that

$$K_i K_{\tau(i)}^{-1} B_i = q_i^2 B_i K_i K_{\tau(i)}^{-1}, \quad K_i K_{\tau(i)}^{-1} B_{\tau(i)} = q_i^{-2} B_{\tau(i)} K_i K_{\tau(i)}^{-1},$$

$$B_i B_{\tau(i)} - B_{\tau(i)} B_i = c_i \frac{K_i K_{\tau(i)}^{-1} - K_i^{-1} K_{\tau(i)}}{q_i - q_i^{-1}}.$$

Hence the subalgebra of $\mathcal{B}_{\mathbf{c}}$ generated by $B_i, B_{\tau(i)}$ and $(K_iK_{\tau(i)}^{-1})^{\pm 1}$ is isomorphic to $U_{q_i}(\mathfrak{sl}_2(\mathbb{C}))$. Hence $K_iK_{\tau(i)}^{-1}$ acts as ± 1 in any one-dimensional representation of $\mathcal{B}_{\mathbf{c}}$. Finally, if $\tau(i) \neq i$ then $K_iK_{\tau(i)}^{-1}B_i = q_i^{-2+a_{i\tau(i)}}B_iK_iK_{\tau(i)}^{-1}$. Hence, any character χ of $\mathcal{B}_{\mathbf{c}}$ satisfies $\chi(B_i) = 0$ in this case. Similarly, if $\alpha_i(h_j) \neq 0$ then $(\alpha_i, \alpha_j) \neq 1$ and hence the relation $K_jB_i = q^{-(\alpha_i,\alpha_j)}B_iK_j$ implies that $\chi(B_i) = 0$. □

Examples 10.2. (1) *Consider Example 9.2.(1). In this case $I_{ns} = I$. Assume that $\chi \in \widehat{\mathcal{B}_{\mathbf{c}}}$ and write $\chi(B_i) = s_i$ for $i = 1, 2$. If $s_j \neq 0$ then the relation (9.2) implies that $(2 - (q + q^{-1}))s_i^2 = -qc_i$ where $\{i, j\} = \{1, 2\}$.*

(2) *Consider Example 9.2.(2). In this case there exist $\chi \in \widehat{\mathcal{B}_{\mathbf{c}}}$ with $\chi(B_i) = s_i$ for all $s_0, s_1 \in \mathbb{K}(q)$.*

(3) *Consider Example 9.2.(3). By Proposition 10.1.(3) any $\chi \in \widehat{\mathcal{B}_{\mathbf{c}}}$ satisfies $\chi(B_i) = 0$ for $i = 0, 1$. However, there exists a unique $\chi \in \widehat{\mathcal{B}_{\mathbf{c}}}$ with $\chi(K_0K_1^{-1}) = t_0$ for any $t_0 \in \mathbb{K}(q)^\times$.*

Any $\chi \in \widehat{\mathcal{B}_{\mathbf{c}}}$ with $\chi|_{U_q(\mathfrak{g}_X)} = \varepsilon|_{U_q(\mathfrak{g}_X)}$ is uniquely determined by two parameter families $\mathbf{s} = (s_i)_{i\in I\setminus X} \in \mathbb{K}(q)^{I\setminus X}$ and $\mathbf{t} = (t_i)_{i\in I\setminus X} \in (\mathbb{K}(q)^\times)^{I\setminus X}$ defined by

$$\chi(B_i) = s_i, \qquad \chi(K_iK_{\tau(i)}^{-1}) = t_i.$$

For $\mathbf{s} \in \mathbb{K}(q)^{I\setminus X}$ and $\mathbf{t} \in (\mathbb{K}(q)^\times)^{I\setminus X}$ we denote the correspond character by $\chi^{\mathbf{c}}_{\mathbf{s},\mathbf{t}}$, if it exists. In this case we define $\rho^{\mathbf{c}}_{\mathbf{s},\mathbf{t}} = \rho_{\chi^{\mathbf{c}}_{\mathbf{s},\mathbf{t}}}$ to be the corresponding map defined by (10.1).

Definition 10.3. *Let $\mathbf{c} \in \mathcal{C}$ and assume that $\chi^{\mathbf{c}}_{\mathbf{s},\mathbf{1}} \in \widehat{\mathcal{B}_{\mathbf{c}}}$ exists for some non-vanishing $\mathbf{s} \in \mathbb{K}(q)^{I\setminus X}$ and $\mathbf{1} = (1, 1, \ldots, 1)$. Then we call $\mathcal{B}_{\mathbf{c},\mathbf{s}} := (\mathcal{B}_{\mathbf{c}})_{\chi^{\mathbf{c}}_{\mathbf{s},\mathbf{1}}} = \rho^{\mathbf{c}}_{\mathbf{s},\mathbf{1}}(\mathcal{B}_{\mathbf{c}})$ a* non-standard *quantum symmetric pair coideal subalgebra.*

It follows from the coproduct formula (5.2) that the non-standard QSP coideal subalgebra $\mathcal{B}_{\mathbf{c},\mathbf{s}}$ is generated by $\mathcal{H}^{\geq} = U_q(\mathfrak{g}_X)U^{0'}_\theta$ and the elements

$$B_i = F_i - c_iT_{w_X}(E_{\tau(i)})K_i^{-1} + s_iK_i^{-1}.$$

for all $i \in I \setminus X$.

Proposition 10.4. *Let $\mathbf{c} \in \mathcal{C}$ and $\chi = \chi^{\mathbf{c}}_{\mathbf{s},\mathbf{t}} \in \widehat{\mathcal{B}_{\mathbf{c}}}$ for some $\mathbf{s} \in \mathbb{K}(q)^{I\setminus X}$, $\mathbf{t} \in (\mathbb{K}(q)^\times)^{I\setminus X}$. Then $(\mathcal{B}_{\mathbf{c}})_{\chi^{\mathbf{c}}_{\mathbf{s},\mathbf{t}}} = \mathcal{B}_{\mathbf{c}',\mathbf{s}}$ with $\mathbf{c}' = (c'_i) \in \mathcal{C}$ defined by $c'_i = c_it_i^{-1}$.*

Proof. By Equation (5.2) we have

$$\rho^{\mathbf{c}}_{\mathbf{s},\mathbf{t}}(B_i) = F_i - c_it_{\tau(i)}T_{w_X}(E_{\tau(i)})K_i^{-1} + s_iK_i^{-1}. \tag{10.2}$$

Moreover, by Proposition 10.1.(2), the element $\mathbf{c}' = (c'_i)$ given by $c'_i = c_it_i^{-1}$ lies in the parameter set $\mathcal{C}$. As $t_{\tau(i)} = t_i^{-1}$ we get $(\mathcal{B}_{\mathbf{c}})_{\chi^{\mathbf{c}}_{\mathbf{s},\mathbf{t}}} = \mathcal{B}_{\mathbf{c}',\mathbf{s}}$. □

By the above proposition, the additional parameter family $\mathbf{t}$ does not produce additional coideal subalgebras and may hence be ignored.

Proposition 10.5. *Let $\mathbf{c} \in \mathcal{C}$ and $\mathbf{s} \in \mathbb{K}(q)^{I\setminus X}$ such that there exists a character $\chi^{\mathbf{c}}_{\mathbf{s},\mathbf{1}} \in \widehat{\mathcal{B}_{\mathbf{c}}}$. The map $\rho^{\mathbf{c}}_{\mathbf{s},\mathbf{1}} : \mathcal{B}_{\mathbf{c}} \to \mathcal{B}_{\mathbf{c},\mathbf{s}}$ is an isomorphism of right U-comodule algebras.*

Proof. By construction, the map $\rho^{\mathbf{c}}_{\mathbf{s},\mathbf{1}}$ is a surjective homomorphism of right U-comodule algebras. Assume that $b \in \mathcal{B}_{\mathbf{c}} \setminus \{0\}$ lies in the kernel of $\rho^{\mathbf{c}}_{\mathbf{s},\mathbf{1}}$. By Theorem 7.2 we can write $b = \sum_{J\in\mathcal{J}} B_J a_J$ for uniquely determined elements $a_J \in \mathcal{H}^{\geq}$. Choose $J = (j_1, \dots, j_{\ell(J)}) \in \mathcal{J}$ of maximal length $\ell = \ell(J)$ such that $a_J \neq 0$. Then the explicit form of $\rho^{\mathbf{c}}_{\mathbf{s},\mathbf{1}}$ in (10.2) and the triangular decomposition (6.1) imply that $\sum_{\substack{J\in\mathcal{J}\\ \ell(J)=\ell}} F_J a_J = 0$, in contradiction to the linear independence of the set $\{F_J \mid J \in \mathcal{J}\}$ over $\mathcal{H}^{\geq}$. □

We would like to know all $\mathbf{s} \in \mathbb{K}(q)^{I\setminus X}$ for which there exists a character $\chi^{\mathbf{c}}_{\mathbf{s}} = \chi^{\mathbf{c}}_{\mathbf{s},\mathbf{1}} \in \widehat{\mathcal{B}_{\mathbf{c}}}$. By Proposition 10.1, any such $\mathbf{s} = (s_i)_{i\in I\setminus X}$ needs to satisfy the condition

$$s_i \neq 0 \quad \Longrightarrow \quad i \in I_{ns}. \tag{10.3}$$

However, as example 10.2.(1) illustrates, condition (10.3) is not sufficient for the existence of a character $\chi^{\mathbf{c}}_{\mathbf{s},\mathbf{1}} \in \widehat{\mathcal{B}_{\mathbf{c}}}$. Consider the set

$$\mathcal{S} = \{\mathbf{s} \in \mathbb{K}(q)^{I\setminus X} \mid s_i \neq 0 \Rightarrow (i \in I_{ns} \text{ and } a_{ji} \in -2\mathbb{N}_0 \,\forall j \in I_{ns} \setminus \{i\})\}.$$

The following proposition can be deduced from [KY21, Thm. 1.2 and Prop. 4.4].

Proposition 10.6. *Let* $\mathbf{c} \in \mathcal{C}$ *and* $\mathbf{s} \in \mathcal{S}$. *Then there exists a character* $\chi^{\mathbf{c}}_{\mathbf{s},\mathbf{1}} \in \widehat{\mathcal{B}_{\mathbf{c}}}$.

Remark 10.7. The character χ in Example 10.2.(2) is of the form described in Proposition 10.6. However, Example 10.2.(1) shows that not all characters in $\widehat{\mathcal{B}_{\mathbf{c}}}$ are of the form described in the proposition.

A careful analysis of the defining relations in [KY21] shows that if $\chi^{\mathbf{c}}_{\mathbf{s},\mathbf{1}} \in \widehat{\mathcal{B}_{\mathbf{c}}}$ and $s_i \neq 0$ for $i \in I_{ns}$ with odd a_{ji} for some $j \in I_{ns}$ then $\frac{s_j^2}{c_j}$ must satisfy a certain algebraic equation related to the q-Serre relations. This algebraic equation can be explicitly described in terms of continuous q-Hermite polynomials. The family of all algebraic equations in $\frac{s_j^2}{c_j}$ obtained in this way for all $i, j \in I_{ns}$ provides a necessary and sufficient condition for the existence of the character $\chi^{\mathbf{c}}_{\mathbf{s},\mathbf{1}}$. It would be interesting to find a simple description of these algebraic equations. See [RV20, End of Section 1.1] for a related conjecture.

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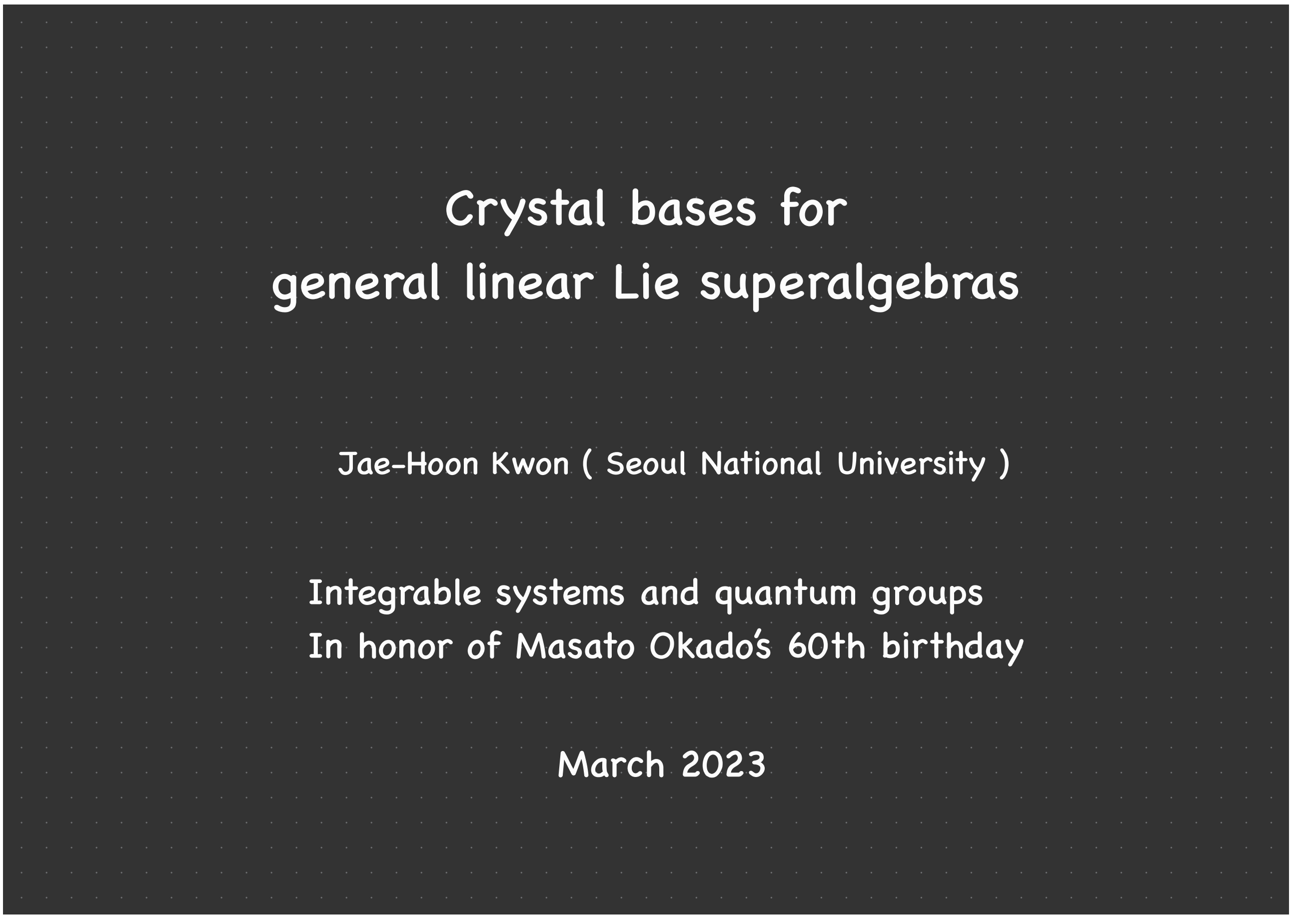
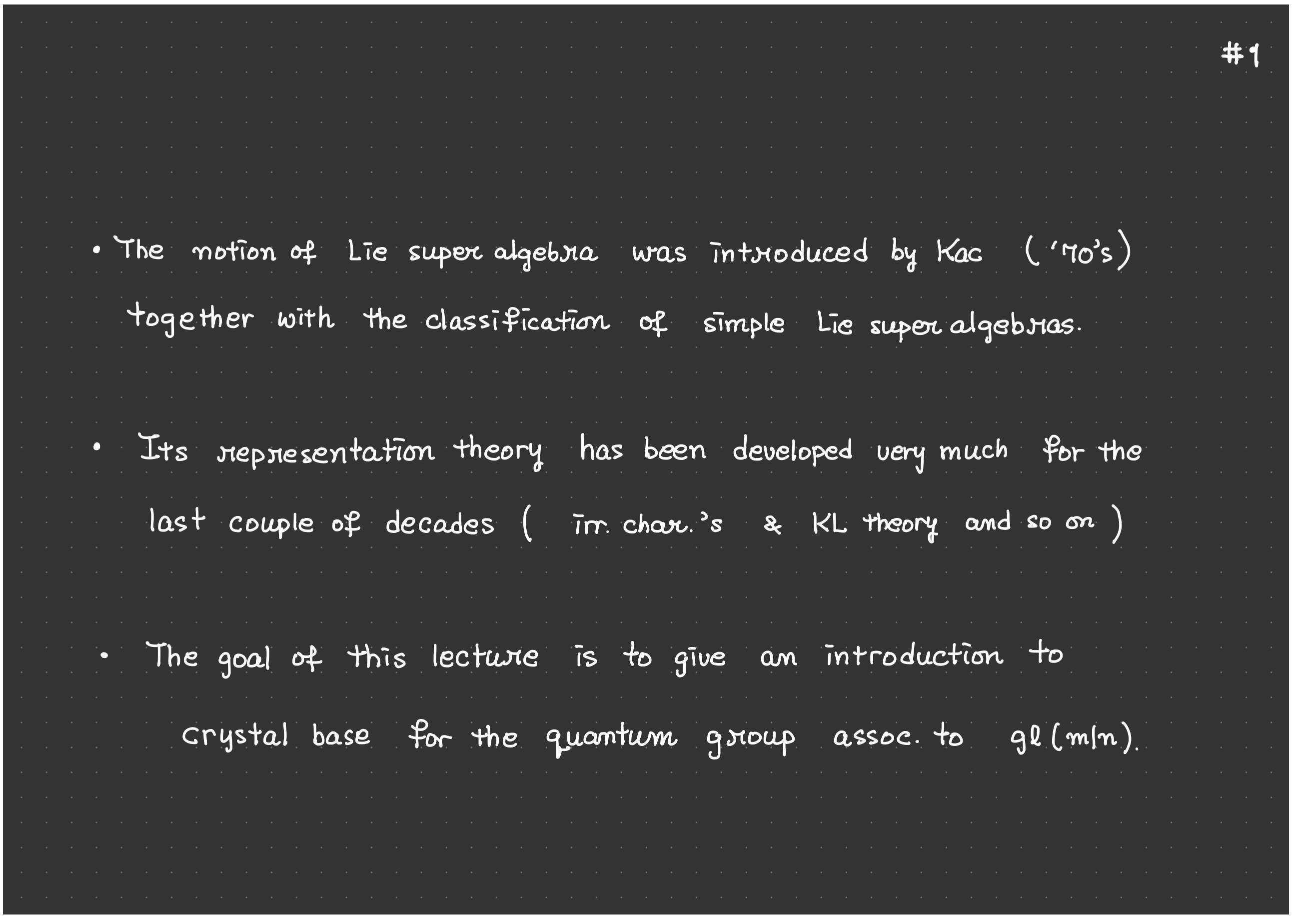


Crystal bases for
general linear Lie superalgebras
Jae-Hoon Kwon (Seoul National University)
Integrable systems and quantum groups
In honor of Masato Okado's 60th birthday
March 2023



#1
• The notion of Lie super algebra was introduced by Kac ('70's) together with the classification of simple Lie super algebras.
• Its representation theory has been developed very much for the last couple of decades (irr. char.'s & KL theory and so on)
• The goal of this lecture is to give an introduction to crystal base for the quantum group assoc. to gl(m|n).

#2

1. Quantum super algebra $U_q(\mathfrak{gl}(m|n))$

(Crystal base of)

2. Polynomial representation $V(\lambda)$

3. Kac module $K(\lambda)$

4. The negative half of $U_q(\mathfrak{gl}(m|n))$

#3

1. Quantum superalgebra

- Assume that the base field $= \mathbb{C}$

  A super space $=$ a $\mathbb{Z}_2$-graded space $V = \underset{\text{even}}{V_0} \oplus \underset{\text{odd}}{V_1}$

  $\mathfrak{gl}(V) := \mathrm{End}_{\mathbb{C}}(V)$

  : a super space $\mathfrak{gl}(V)_\varepsilon \ni f : V_k \longrightarrow V_{k+\varepsilon}$

  a Lie superalgebra w.r.t. $[f, g] = f \circ g - (-1)^{|f||g|} g \circ f$

  called a general linear Lie super algebra

#4

- $m, n \geq 0$ $\quad I(m|n) = \{ \underbrace{1 < 2 < \cdots < m}_{\text{even}} < \underbrace{m+1 < \cdots < m+n}_{\text{odd}} \}$

$$\mathbb{C}^{m|n} = \underbrace{\mathbb{C}^{m|0}}_{\text{even}} \oplus \underbrace{\mathbb{C}^{0|n}}_{\text{odd}} = \bigoplus_{I(m|n)} \mathbb{C} v_a$$

- $\mathfrak{gl}(m|n) := \mathfrak{gl}(\mathbb{C}^{m|n})$

$= $ the set of $(m+n) \times (m+n)$ matrices $\quad \begin{array}{c} \\ m \\ n \end{array} \begin{bmatrix} \overbrace{A}^{m} & \overbrace{B}^{n} \\ C & D \end{bmatrix}$

A, D : even

B, C : odd

$$\mathfrak{gl}(m|n)_0 \cong \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$$

$\mathfrak{h}$ = span of E_{aa} (Cartan subalg)

#5

- $(X, Y) := \mathrm{str}(XY)$: non-deg inv super symm. bilinear form

$\{ \delta_a \mid a \in I(m|n) \}$: a basis of $\mathfrak{h}^*$ dual to $\{ E_{aa} \}$

$$(\delta_a | \delta_b) = \begin{cases} 1 & a = b \leq m \\ -1 & a = b > m \\ 0 & \text{otherwise} \end{cases}$$

(induced bilinear form)

- $\Phi^+ = \{ \delta_a - \delta_b \mid a \neq b,\ a < b \}$: the set of positive roots

$\Phi_0^+ = \{ \delta_a - \delta_b \mid a < b \leq m,\ m < a < b \}$ even

$\Phi_1^+ = \{ \delta_a - \delta_b \mid a \leq m < b \}$ odd (isotropic)

#6

$\Delta = \{ \delta_a - \delta_{a+1} \mid 1 \leq a < m+n \}$: the set of simple roots

$(\alpha|\alpha) > 0$ even $\quad (\alpha|\alpha) = 0$ odd $\quad (\alpha|\alpha) < 0$ even

$\ell = m+n$

- $\mathcal{U}(\mathfrak{gl}(m|n))$: the enveloping algebra of $\mathfrak{gl}(m|n)$

$$\mathcal{U}(\mathfrak{gl}(m|n)^+) \cong \mathcal{U}(\mathfrak{gl}(m|n)^+_0) \otimes \mathcal{U}(\mathfrak{gl}(m|n)^+_1)$$

$$\mathcal{U}(\mathfrak{gl}(m|n)^+_1) \cong \Lambda(\mathfrak{gl}(m|n)^+_1) \quad \text{as a } \mathbb{C}\text{-alg.}$$

#7

- q : indeterminate , $\Bbbk = \mathbb{Q}(q)$

$\underline{\mathcal{U}_q(\mathfrak{gl}(m|n))}$: the q-analogue of $\mathcal{U}(\mathfrak{gl}(m|n))$ introduced by Yamane (99)

generators : $E_\alpha, F_\alpha, K_a, K_a^{-1}$ $\quad (\alpha \in \Delta , \ a \in I(m|n))$

relations :

$K_a^{\pm 1}$: commutative $\qquad K_a E_\alpha K_a^{-1} = q^{(\alpha, \epsilon_a)} E_\alpha \qquad K_a F_\alpha K_a^{-1} = q^{-(\alpha, \epsilon_a)} F_\alpha$

$$E_\alpha F_\beta - (-1)^{|\alpha||\beta|} F_\beta E_\alpha = \delta_{\alpha\beta} \cdot \mathrm{sgn}(\alpha|\alpha) \frac{K_\alpha - K_\alpha^{-1}}{q - q^{-1}} \qquad (K_\alpha = K_a K_{a+1}^{-1})$$

usual Serre relations for E_α, F_α $\quad (\alpha|\alpha) > 0, < 0$

+ odd Serre relation for E_α, F_α $\quad (\alpha|\alpha) = 0$

#8

Rmk

① $(\alpha|\alpha) > 0$ $\quad \langle E_\alpha, F_\alpha, K_\alpha^{\pm 1} \rangle \cong \mathcal{U}_q(sl_2)$

$\quad\ < 0$ $\quad \langle E_\alpha, F_\alpha, K_\alpha^{\pm 1} \rangle \cong \mathcal{U}_{q^{-1}}(sl_2)$

② $\mathcal{U}_q(gl(m|n))$: $\mathbb{Z}_2$-graded $\quad \deg(E_\alpha) = \deg(F_\alpha) = 1 \quad (\alpha|\alpha) = 0$

③ Instead of super representations, we consider a repn of

$$\mathcal{U}_q(gl(m|n))[\sigma] = \mathcal{U}_q(gl(m|n)) \oplus \mathcal{U}_q(gl(m|n))\sigma$$

$$\sigma^2 = 1.$$

$$\sigma K_\alpha^{\pm 1} = K_\alpha^{\pm 1}\sigma \qquad \sigma X_\alpha = (-1)^{|\alpha|} X_\alpha \sigma \quad (X = E, F)$$

#9

V : a $\mathcal{U}_q(gl(m|n))[\sigma]$-module if

- $V = V_0 \oplus V_1$: $\mathbb{Z}_2$-graded $\mathcal{U}_q(gl(m|n))$-module
- $\sigma V_\varepsilon = (-1)^\varepsilon V$

(= a super representation of $\mathcal{U}_q(gl(m|n))$)

Hopf algebra structure of $\mathcal{U}_q(gl(m|n))[\sigma]$

$$\Delta K_\alpha^{\pm 1} = K_\alpha^{\pm 1} \otimes K_\alpha^{\pm 1}$$

$$\Delta E_\alpha = E_\alpha \otimes K_\alpha^{-1} + \sigma \otimes E_\alpha, \quad \Delta F_\alpha = F_\alpha \otimes 1 + \sigma K_\alpha \otimes F_\alpha$$

#10

- $\mathcal{P} = \bigoplus_a \mathbb{Z}\,\delta_a$
- A weight space of V with $\lambda \in \mathcal{P}$

$$V_\lambda = \{ v \mid K_\mu v = q^{(\mu|\lambda)} v \text{ for } \mu \in \mathcal{P} \}$$

We assume V has a wt. space decomposition

<u>Rmk</u> We may consider another version of QSA

(due to Kuniba-Okado-Sergeev '15)

$$q_a = \begin{cases} q & (1 \leq a \leq m) \\ -q^{-1} & (m < a \leq m+n) \end{cases}$$

#11

$$\mathfrak{q}(\lambda, \mu) = \prod_a q_a^{\lambda_a \mu_a} \qquad \text{for } \lambda = \sum_a \lambda_a \delta_a, \quad \mu = \sum_a \mu_a \delta_a$$

defining relations
weight space : $q^{(\lambda|\mu)} \rightsquigarrow \mathfrak{q}(\lambda, \mu)$

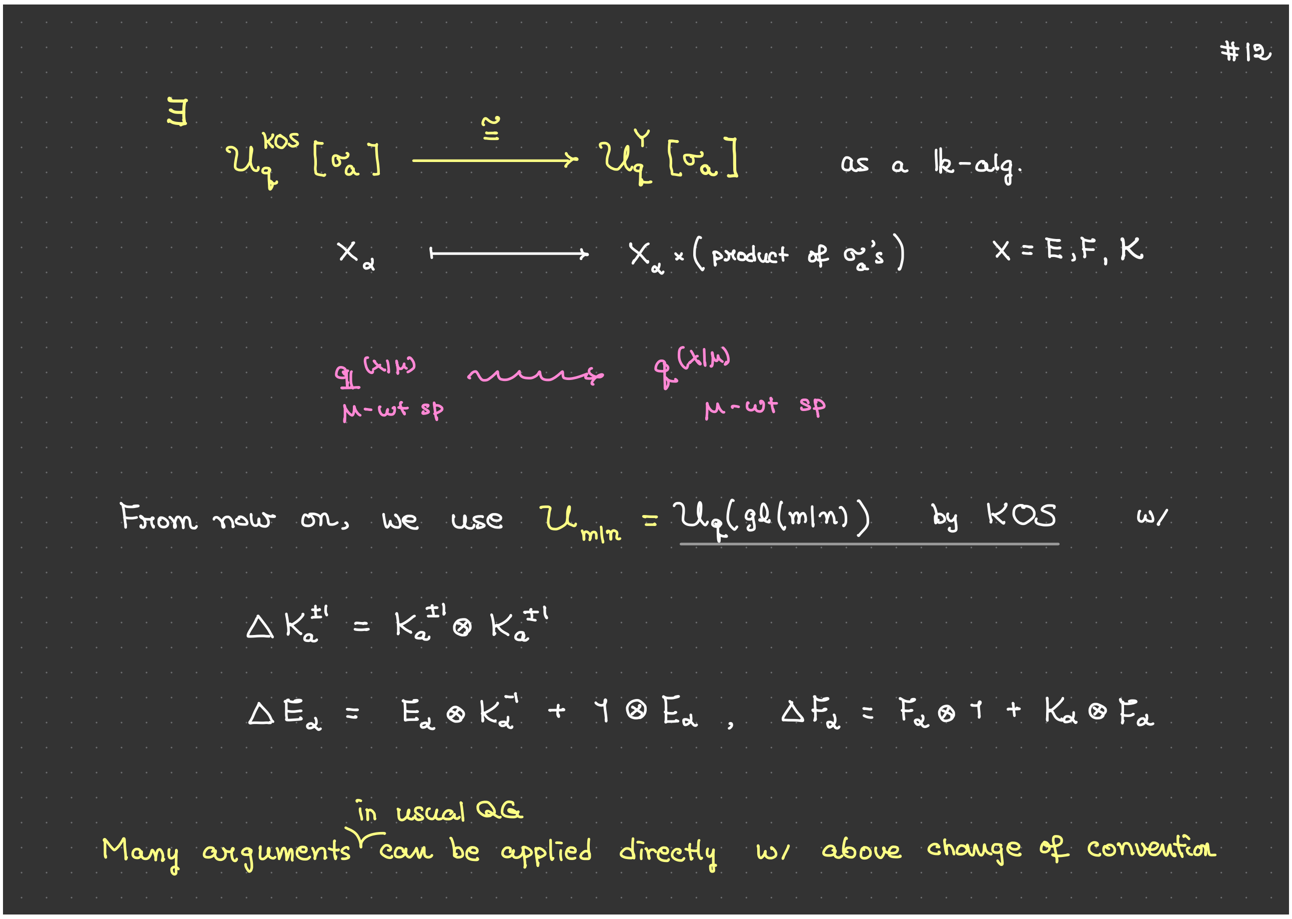
$q \rightsquigarrow -q^{-1}$ (on odd space)

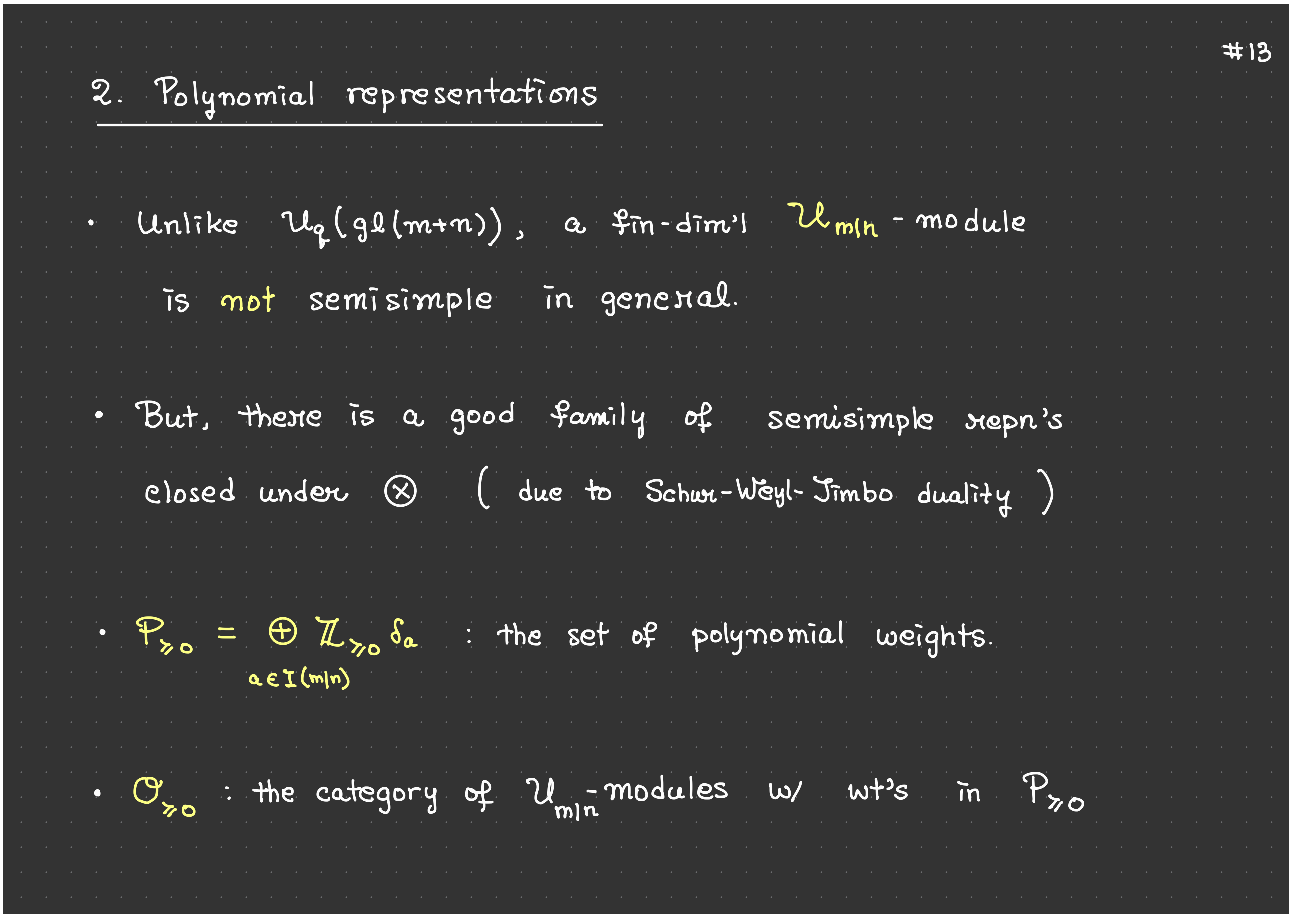
It is almost isomorphic to $U_q(\mathfrak{gl}(m|n))$ in the sense;

$$U_q(\mathfrak{gl}(m|n))[\sigma_a] = \langle U_q(\mathfrak{gl}(m|n)), \sigma_a \rangle$$

σ_a $(a \in I(m|n))$: $\sigma_a \sigma_b = \sigma_b \sigma_a \quad \sigma_a^2 = 1$

$$\sigma_a K_b = K_b \sigma_a \qquad \sigma_a X_\alpha = (\delta_a|\delta_a)^{(\delta_a|\alpha)} X_\alpha \sigma_a \qquad (X = E, F)$$


#12
$\exists$ $\mathcal{U}_q^{KOS}[\sigma_a^\pm] \xrightarrow{\cong} \mathcal{U}_q^{Y}[\sigma_a^\pm]$ as a $\Bbbk$-alg.
$X_\alpha \longmapsto X_\alpha \times$ (product of $\sigma_a^\pm$'s) $X = E, F, K$
$q_{\mathbb{I}}^{(\lambda|\mu)}$ $\rightsquigarrow$ $q_{\mathfrak{b}}^{(\lambda|\mu)}$
μ-wt sp μ-wt sp
From now on, we use $\mathcal{U}_{m|n} = \mathcal{U}_q(\mathfrak{gl}(m|n))$ by KOS w/
$\Delta K_a^{\pm 1} = K_a^{\pm 1} \otimes K_a^{\pm 1}$
$\Delta E_\alpha = E_\alpha \otimes K_\alpha^{-1} + 1 \otimes E_\alpha$, $\Delta F_\alpha = F_\alpha \otimes 1 + K_\alpha \otimes F_\alpha$
Many arguments in usual QG can be applied directly w/ above change of convention


#13
2. Polynomial representations
• Unlike $\mathcal{U}_q(\mathfrak{gl}(m+n))$, a fin-dim'l $\mathcal{U}_{m|n}$-module
is not semisimple in general.
• But, there is a good family of semisimple repn's
closed under $\otimes$ (due to Schur-Weyl-Jimbo duality)
• $P_{\geq 0} = \bigoplus_{a \in I(m|n)} \mathbb{Z}_{\geq 0}\, \delta_a$: the set of polynomial weights.
• $\mathcal{O}_{\geq 0}$: the category of $\mathcal{U}_{m|n}$-modules w/ wt's in $P_{\geq 0}$

#14

- $V_{m|n} = \bigoplus_{a \in I(m|n)} \Bbbk\, v_a$: the natural representation of $\mathcal{U}_{m|n}$

$$v_a \underset{E_\alpha}{\overset{F_\alpha}{\rightleftarrows}} v_{a+1} \qquad (\alpha = \delta_a - \delta_{a+1})$$

$$\delta_a \qquad\qquad \delta_{a+1}$$

$V_{m|n}^{\otimes d} \in \mathcal{O}_{\geq 0}$

Moreover, $\exists$ analog of Jimbo's duality on $V_{m|n}^{\otimes d}$

$\mathcal{U}_{m|n} \curvearrowright V_{m|n}^{\otimes d} \curvearrowleft \mathcal{H}_d$: Heck alg. of type A_{d-1}

$\rightsquigarrow$ semisimple

#15

- $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$: a partition w/ $\lambda_{m+1} \leq n$ $(\in \mathcal{P}_{m|n})$

$$\Lambda_\lambda = \lambda_1 \delta_1 + \cdots + \lambda_m \delta_m + \mu_1 \delta_{m+1} + \cdots + \mu_n \delta_{m+n}$$

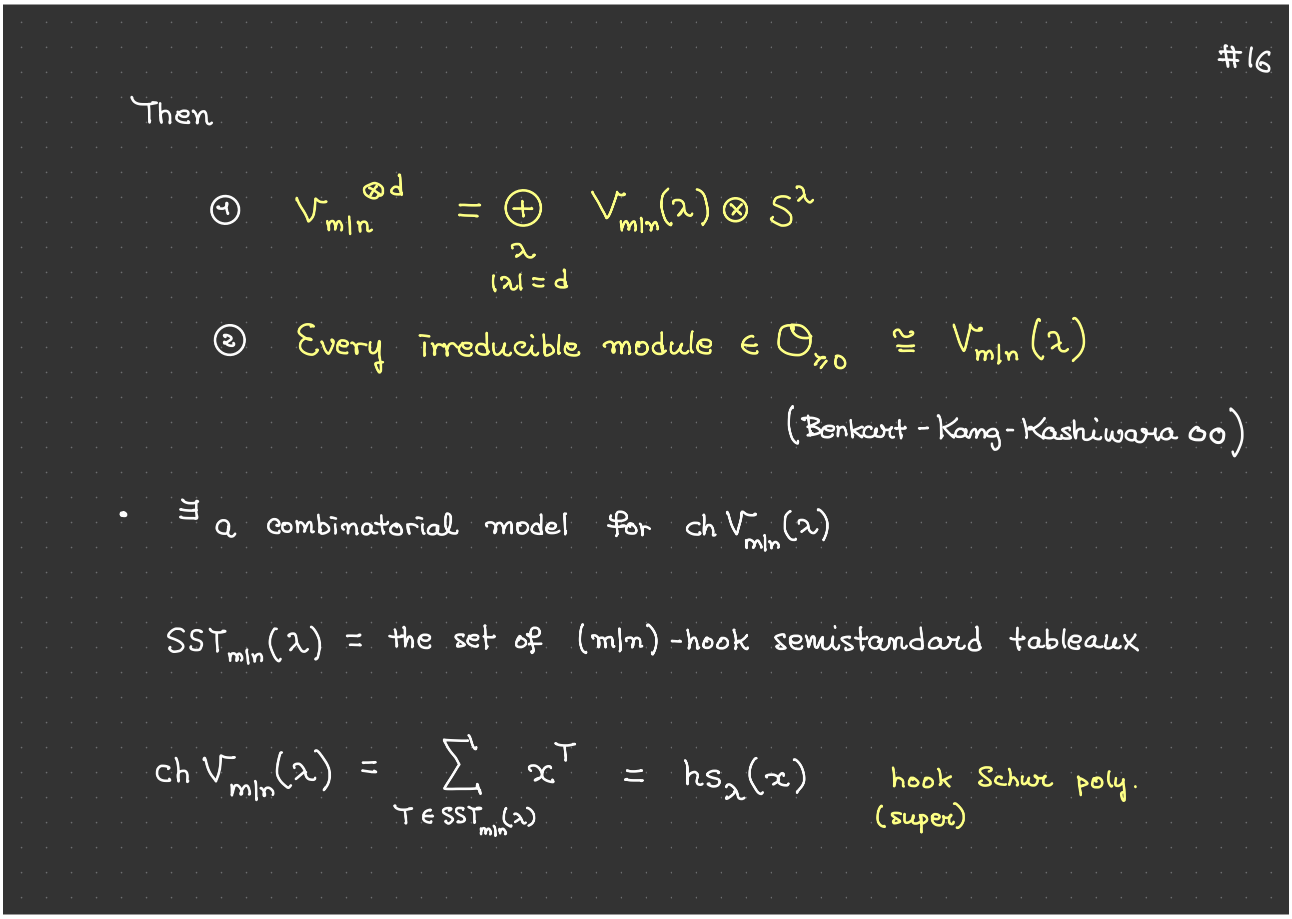
where $\mu = (\lambda_{m+1} \geq \lambda_{m+2} \geq \cdots)'$

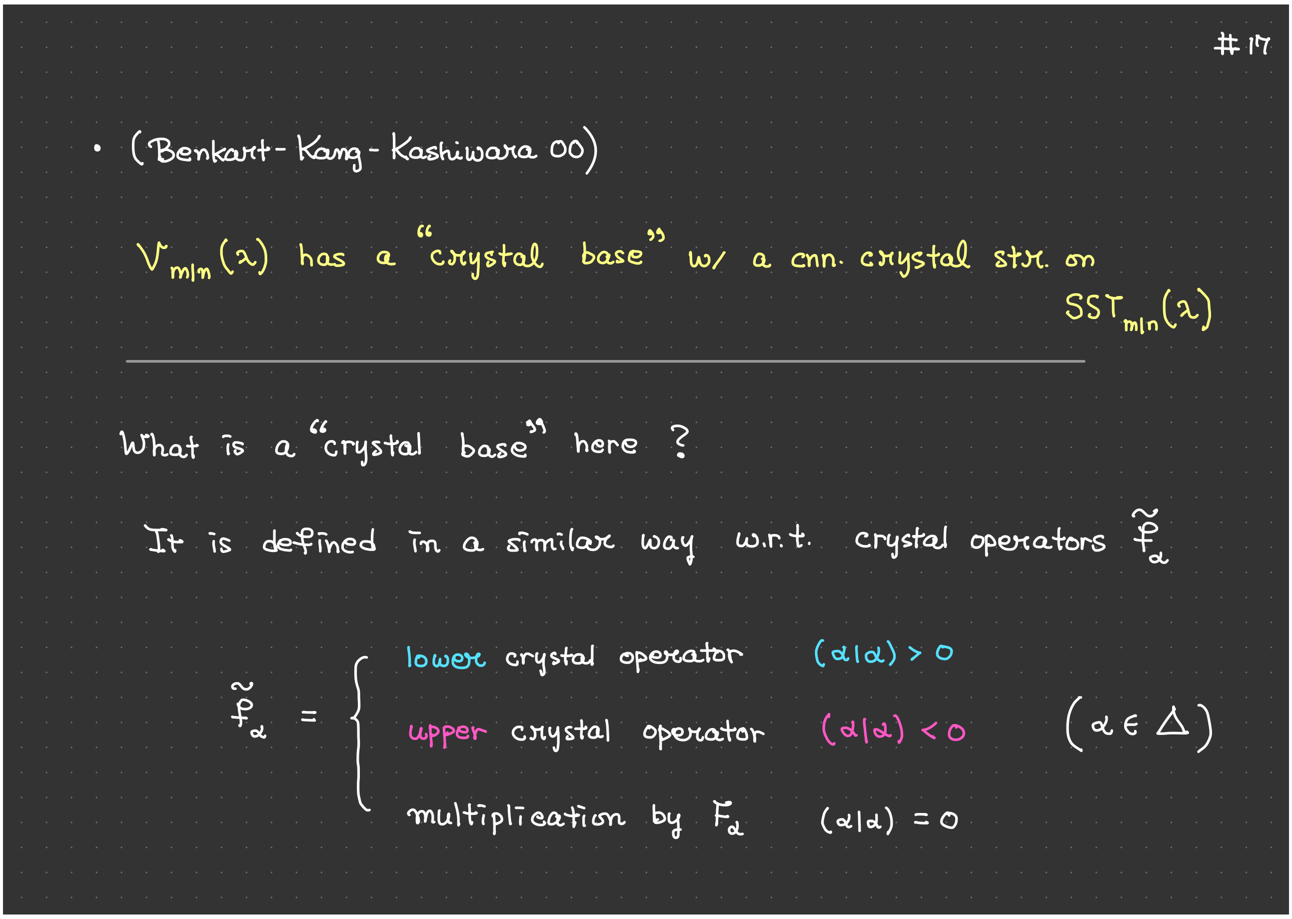
1	1	1	1	1	1
2	2	2	2	2	
3	4	5	6	7	
3	4	5	6		
3	4				
3					
3					

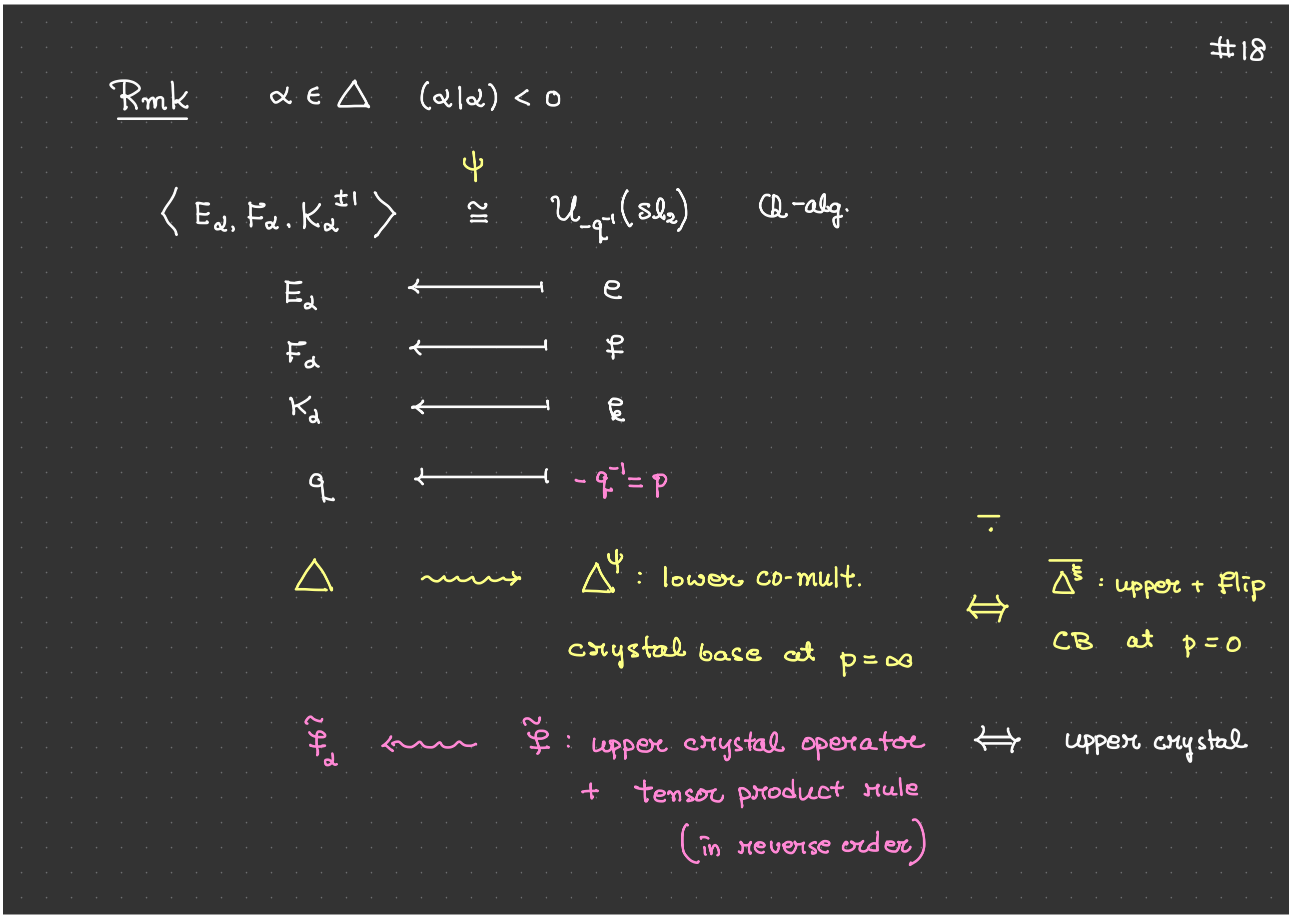
$m = 2$

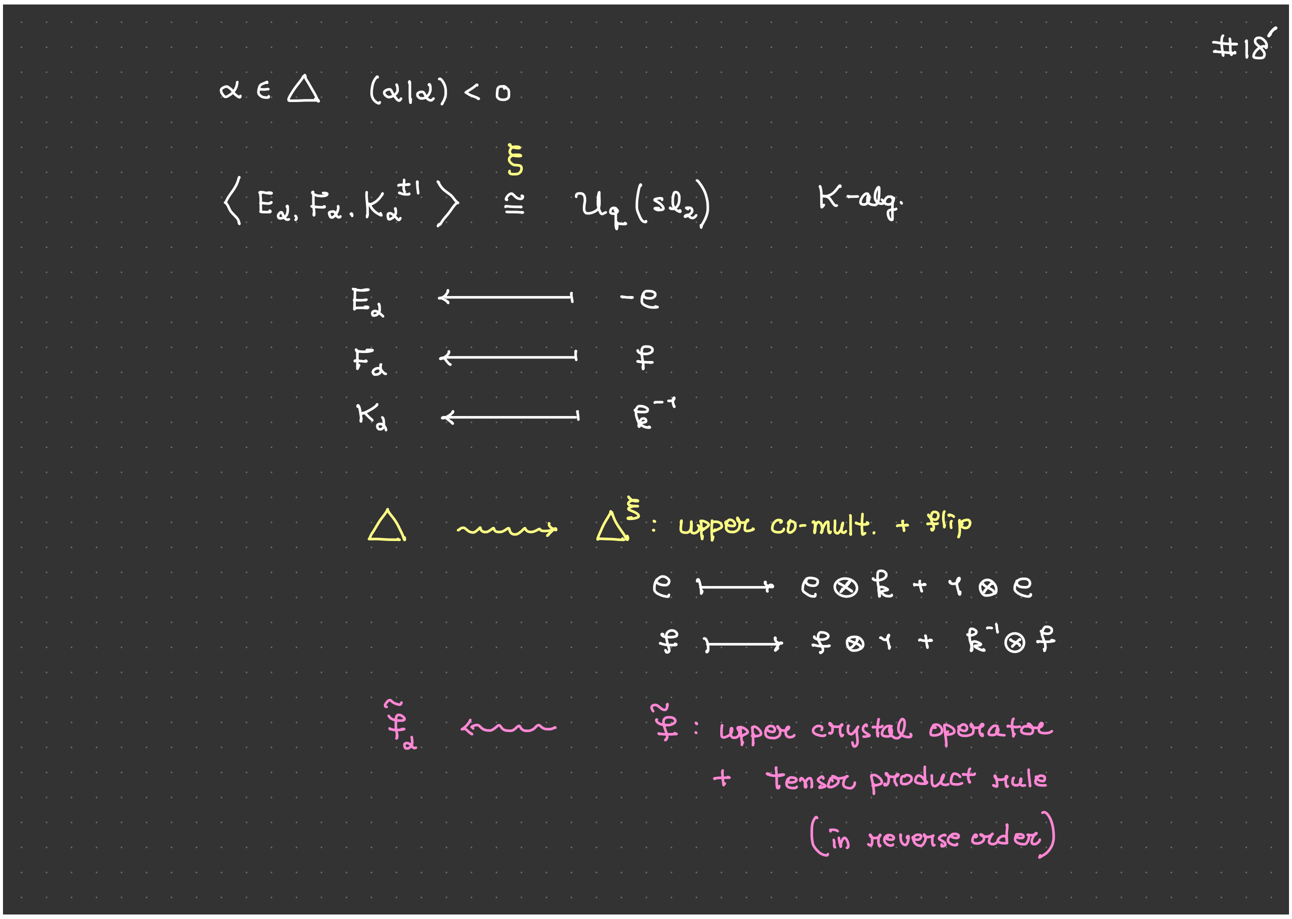
$n = 5$

- $V_{m|n}(\lambda)$: the irreducible h.w. module w/ h.w. Λ_λ


#16
Then
① $V_{m|n}^{\otimes d} = \bigoplus_{\lambda, |\lambda| = d} V_{m|n}(\lambda) \otimes S^{\lambda}$
② Every irreducible module $\in \mathcal{O}_{\geq 0} \cong V_{m|n}(\lambda)$
(Benkart - Kang - Kashiwara 00)
• $\exists$ a combinatorial model for $\mathrm{ch}\, V_{m|n}(\lambda)$
$SST_{m|n}(\lambda)$ = the set of $(m|n)$-hook semistandard tableaux
$\mathrm{ch}\, V_{m|n}(\lambda) = \sum_{T \in SST_{m|n}(\lambda)} x^{T} = hs_{\lambda}(x)$ hook Schur poly. (super)


#17
• (Benkart - Kang - Kashiwara 00)
$V_{m|n}(\lambda)$ has a "crystal base" w/ a cnn. crystal str. on $SST_{m|n}(\lambda)$
What is a "crystal base" here ?
It is defined in a similar way w.r.t. crystal operators $\tilde{f}_{\alpha}$
$\tilde{f}_{\alpha}$ = lower crystal operator $(\alpha|\alpha) > 0$
upper crystal operator $(\alpha|\alpha) < 0$
multiplication by F_{α} $(\alpha|\alpha) = 0$
$(\alpha \in \Delta)$


#18
Rmk α ∈ Δ (α|α) < 0
⟨ E_α, F_α, K_α^{±1} ⟩ ≅^ψ U_{-q^{-1}}(sl_2) Q-alg.
E_α ⟵ e
F_α ⟵ f
K_α ⟵ k
q ⟵ -q^{-1} = p
Δ ⇝ Δ^ψ : lower co-mult.
crystal base at p = ∞
⟺
Δ̄^ξ : upper + flip
CB at p = 0
f̃_α ⇜ f̃ : upper crystal operator
+ tensor product rule
(in reverse order)
⟺ upper crystal


#18′
α ∈ Δ (α|α) < 0
⟨ E_α, F_α, K_α^{±1} ⟩ ≅^ξ U_q(sl_2) K-alg.
E_α ⟵ -e
F_α ⟵ f
K_α ⟵ k^{-1}
Δ ⇝ Δ^ξ : upper co-mult. + flip
e ⟼ e ⊗ k + 1 ⊗ e
f ⟼ f ⊗ 1 + k^{-1} ⊗ f
f̃_α ⇜ f̃ : upper crystal operator
+ tensor product rule
(in reverse order)

#19

• $V \in \mathcal{O}_{\geqslant 0}$ $\quad (\mathcal{L}, \mathcal{B})$: a crystal base of V if

① $\mathcal{L}$: A_0-lattice of V + wt sp. decomp.

② $\mathcal{B} = \mathbb{B} \cup (-\mathbb{B})$ $\quad \mathbb{B} \subset \mathcal{L}/q\mathcal{L}$: $\mathbb{Q}$-basis. + wt. sp. decomp.

③ $\tilde{x}_\alpha \mathcal{L} \subset \mathcal{L}$, $\quad \tilde{x}_\alpha \mathcal{B} \subset \mathcal{B} \cup \{0\}$ $\quad x = e, f$ $\quad \alpha \in \Delta$

where $A_0 = \{ h \in K \mid \text{regular at } q = 0 \}$

• (BKK) $(\mathcal{L}_i, \mathcal{B}_i)$: a crystal base of $V_i \in \mathcal{O}_{\geqslant 0}$

$\Rightarrow$ $(\mathcal{L}_1 \otimes \mathcal{L}_2, \mathcal{B}_1 \otimes \mathcal{B}_2)$: a crystal base of $V_1 \otimes V_2$

+ explicit description of $\tilde{f}_\alpha$

#20

for $\alpha \in \Delta$ w/ $(\alpha|\alpha) = 0$

$$\tilde{f}_\alpha(b_1 \otimes b_2) = \begin{cases} \tilde{f}_\alpha b_1 \otimes b_2 & \text{if } (\mathrm{wt}(b_1)|\alpha) \neq 0 \\ b_1 \otimes \tilde{f}_\alpha b_2 & \text{if } (\mathrm{wt}(b_1)|\alpha) = 0 \end{cases}$$

<u>Rmk</u> $\quad V_{m|n}(\lambda) \quad \lambda \in \mathcal{P}_{m|n}$

① $\mathcal{L}_{m|n}(\lambda) = \sum_{\beta_1, \cdots, \beta_r} A_0 \tilde{x}_{\beta_1} \cdots \tilde{x}_{\beta_r} v_\lambda \quad (r \geqslant 0,\ \beta_i \in \Delta,\ x = e, f)$

$\mathcal{B}_{m|n}(\lambda) = \{ \pm \tilde{x}_{\beta_1} \cdots \tilde{x}_{\beta_r} v_\lambda \pmod{q \mathcal{L}_{m|n}(\lambda)} \} \setminus \{0\}$

: a crystal base.

#21

② $B_{m|n}(\lambda)$ can be realized as a subgraph of $B_{m|n}^{\otimes|\lambda|}$

where $B_{m|n}$: crystal of $V_{m|n}$

$$\Rightarrow B_{m|n}(\lambda) \cong SST_{m|n}(\lambda) \subset B_{m|n}^{\otimes|\lambda|}$$

③ $B_{m|n}(\lambda)$ may have an element b s.t

$b \neq v_\lambda$ but $\tilde{e}_\alpha v_\lambda = 0$ for all $\alpha \in \Delta$

④ Unlike $B_m(\lambda)$, $\exists$ no natural crystal embedding

$B_{m|n}(\lambda) \longrightarrow B_{m|n}(\mu)$ for $\lambda, \mu \in P_{m|n}$

which yields an inverse limit.

#21-1

Example

$m = 3, \ n = 4$

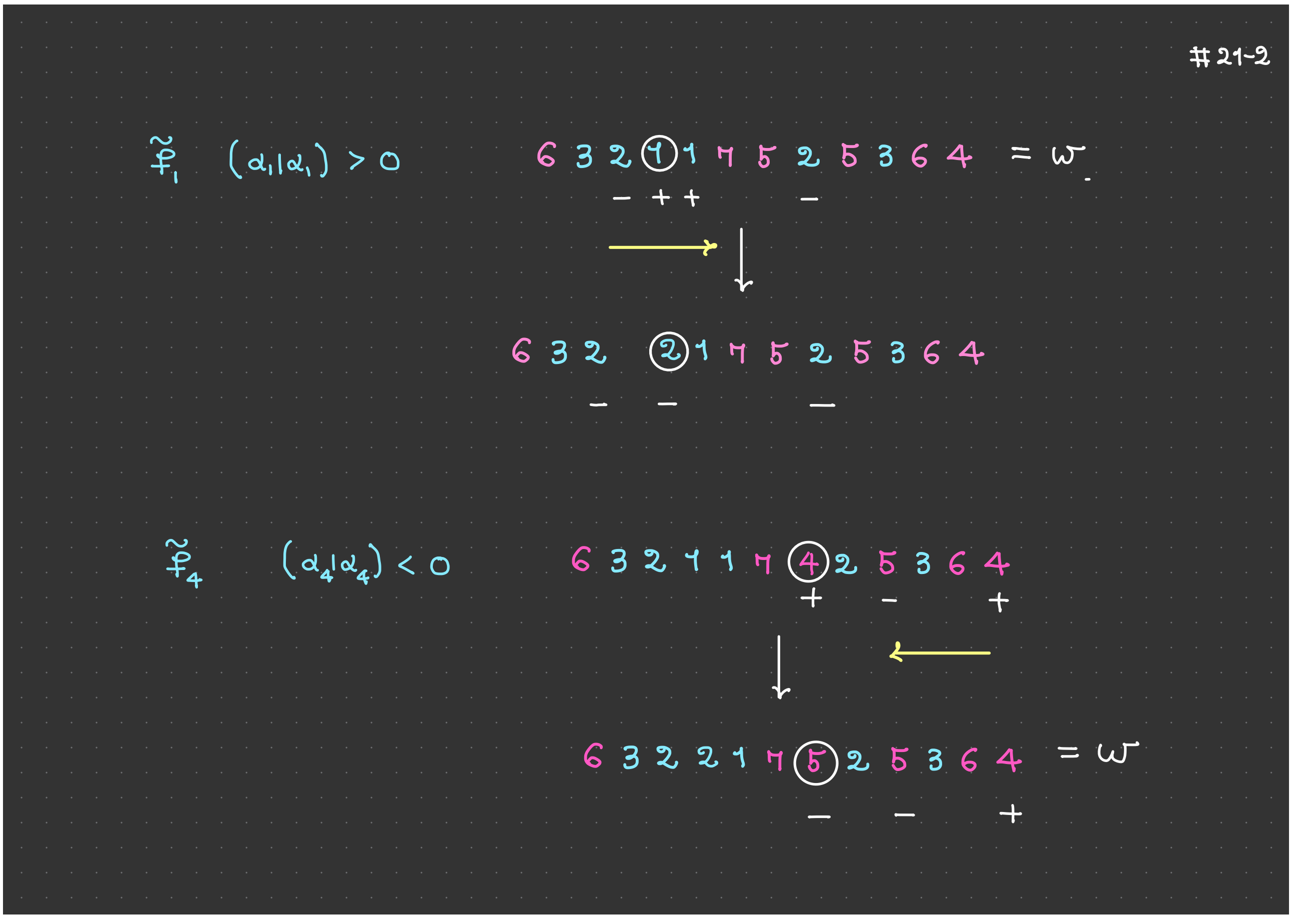
$$B_{3|4} : \ 1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{3} 4 \xrightarrow{4} 5 \xrightarrow{5} 6 \xrightarrow{6} 7.$$

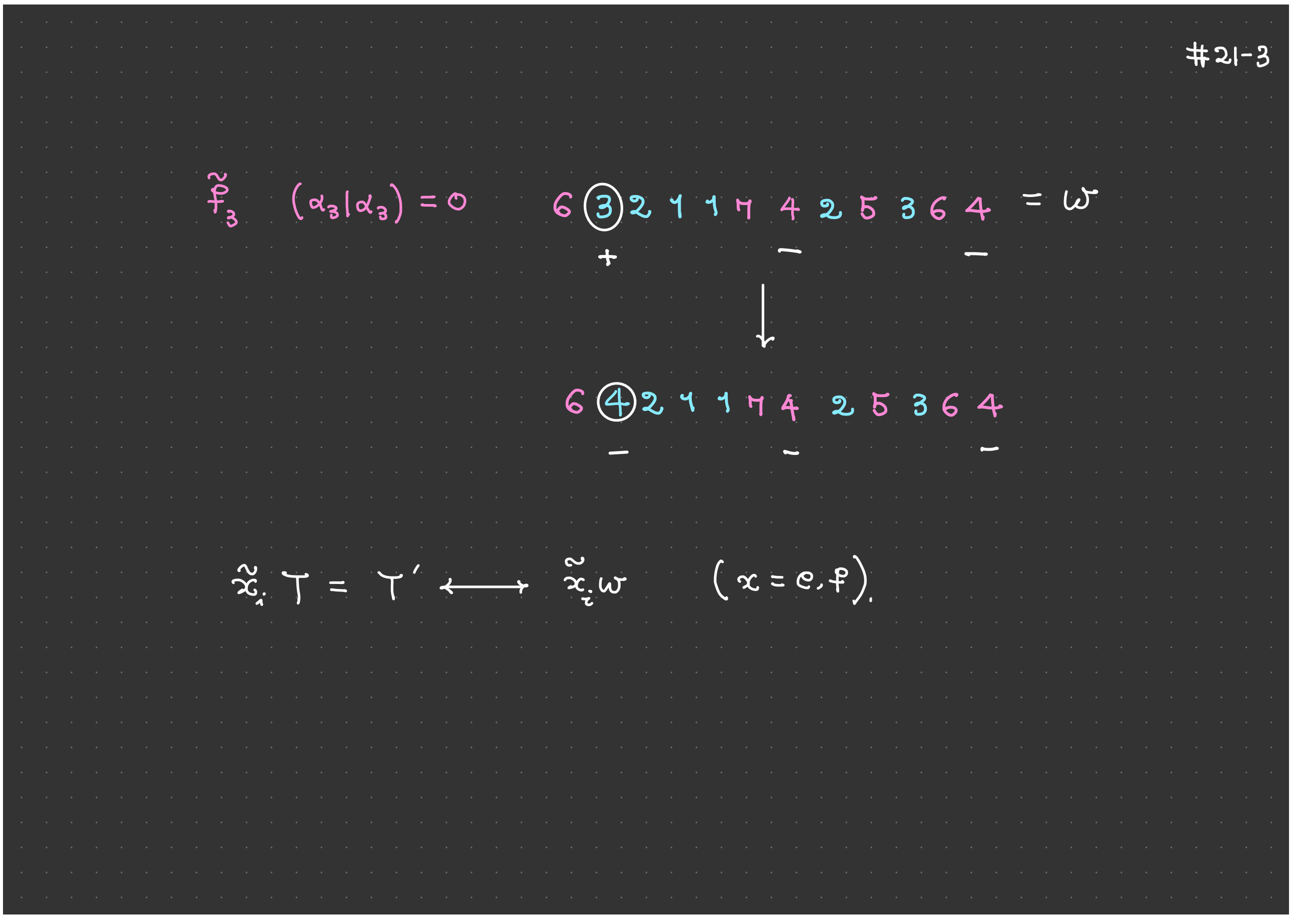
$T =$

1	1	2	3	6
2	5	7		
3	5			
4	6			

$\in SST_{3|4}(5,3,2,2)$

$\xrightarrow{\text{row reading}}$ $w = 6\ 3\ 2\ 1\ 1\ 7\ 5\ 2\ 5\ 3\ 6\ 4 \in B_{3|4}^{\otimes 12}$


#21-2
$\tilde{f}_1$ $(\alpha_1|\alpha_1) > 0$
6 3 2 (1) 1 7 5 2 5 3 6 4 = w
− + + −
6 3 2 (2) 1 7 5 2 5 3 6 4
− − −
$\tilde{f}_4$ $(\alpha_4|\alpha_4) < 0$
6 3 2 1 1 7 (4) 2 5 3 6 4
+ − +
6 3 2 2 1 7 (5) 2 5 3 6 4 = w
− − +


#21-3
$\tilde{f}_3$ $(\alpha_3|\alpha_3) = 0$
6 (3) 2 1 1 7 4 2 5 3 6 4 = w
+ − −
6 (4) 2 1 1 7 4 2 5 3 6 4
− − −
$\tilde{x}_i T = T' \longleftrightarrow \tilde{x}_i w$ $(x = e, f)$.

\# 22

Applications / problem

① (non-standard Borel)

One can consider $\mathcal{U}_{m|n}$ w.r.t. a non-std Borel

$1 < 2 < 3 < 4 < 5 < 6$ standard

odd even

$1 < 2 < 3 < 4 < 5 < 6$

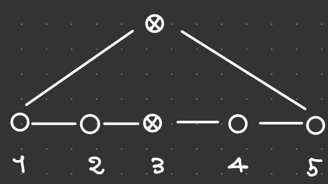
The (non-standard) crystal structure on $B_{m|n}^{\otimes d}$

has a connection with quasi-symmetric functions. (K 09)

\# 23

② (affine case)

One can define the QSA of affine type A



$\exists$ Kirillov-Reshetikhin type module $W^{r,s}$ with a crystal base

for $(r^s) = (\underbrace{r, \cdots, r}_{s}) \in \mathcal{P}_{m|n}$ (K-Okado 21)

and $B^{r,s} \cong SST_{m|n}((r^s))$ as a crystal of finite type

#24

③ ∃ other crystal realization?

(∃ combinatorial model e.g. LS-path model ?)

④ ∃ global crystal basis (canonical basis) of $V_{m|n}(\lambda)$?

#1

3. Kac modules.

- ∃ crystal base of a Verma module for $\mathcal{U}_{m|n}$?
- ∃ natural inverse system on $\{ B_{m|n}(\lambda) \mid \lambda \in P_{m|n} \}$?

No presentation for $V_{m|n}(\lambda)$ is known. so far.

⇝ We do not know a natural partial order on $P_{m|n}$

- How to take a limit of $B_{m|n}(\lambda)$?

#2

- In repn theory of $\mathfrak{gl}(m|n)$,

$\exists$ an important family of fin-dim indecomp h.w. modules

$\approx$ a parabolic Verma module w.r.t $\mathfrak{gl}(m|n)_0$

- 

$E_i := E_{\alpha_i}$ $\quad F_i = F_{\alpha_i}$

$\mathcal{U}_{m,n} = \langle E_i, F_i, K_a^{\pm 1} \mid i \neq m \rangle$

#3

- $\mathcal{P}^+ = \{ \lambda = \sum \lambda_a \delta_a \in \mathcal{P} \mid \lambda_1 \geq \cdots \geq \lambda_m, \ \lambda_{m+1} \geq \cdots \geq \lambda_{m+n} \}$

$\lambda \in \mathcal{P}^+ \qquad \lambda_+ = \sum_{1 \leq a \leq m} \lambda_a \delta_a \qquad \lambda^- = \sum_{m < a \leq m+n} \lambda_a \delta_a$

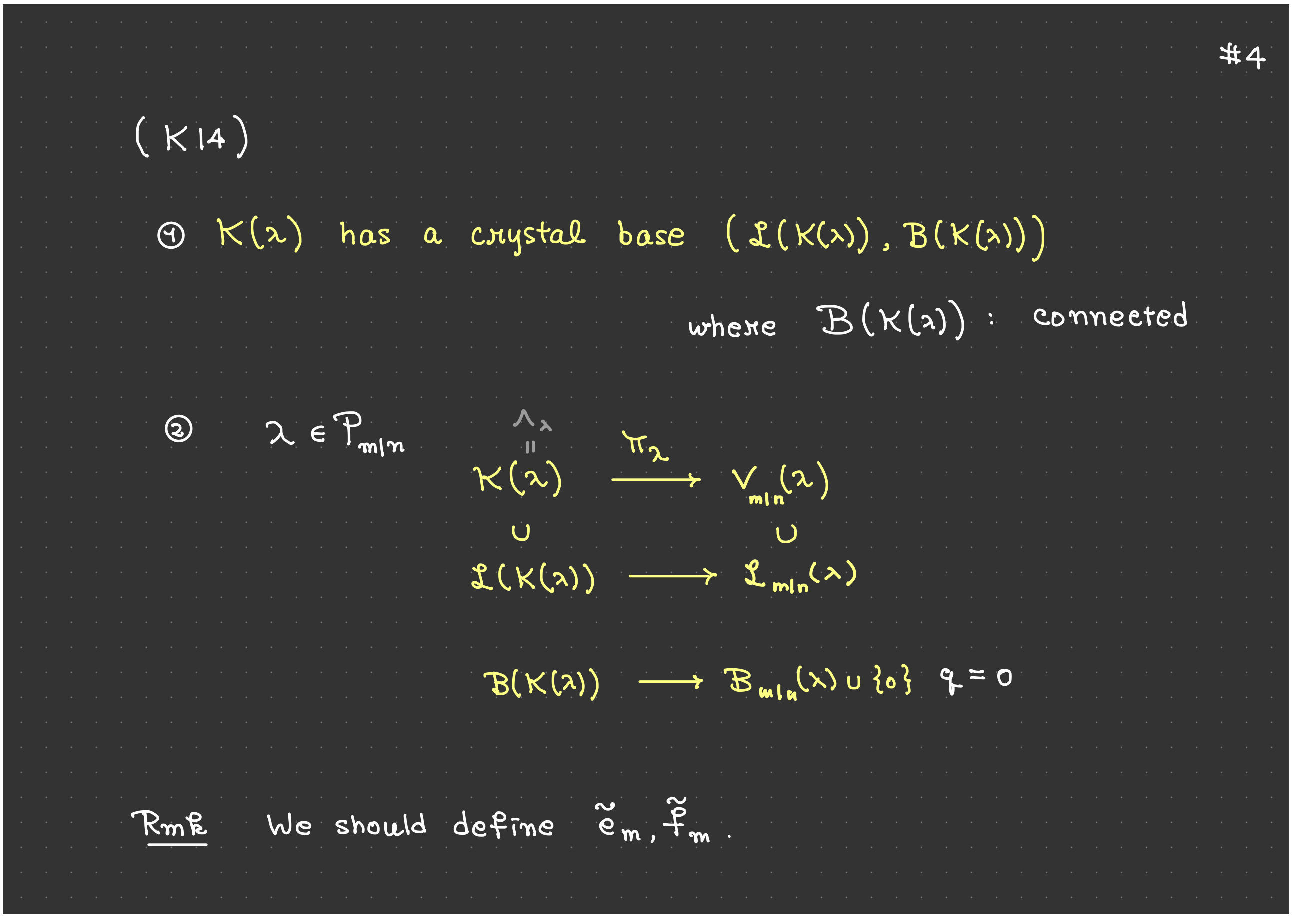
$V_{m,n}(\lambda) := V_{m|0}(\lambda^+) \otimes V_{0|n}(\lambda^-) \curvearrowleft \mathcal{P}$ (E_m : trivially)

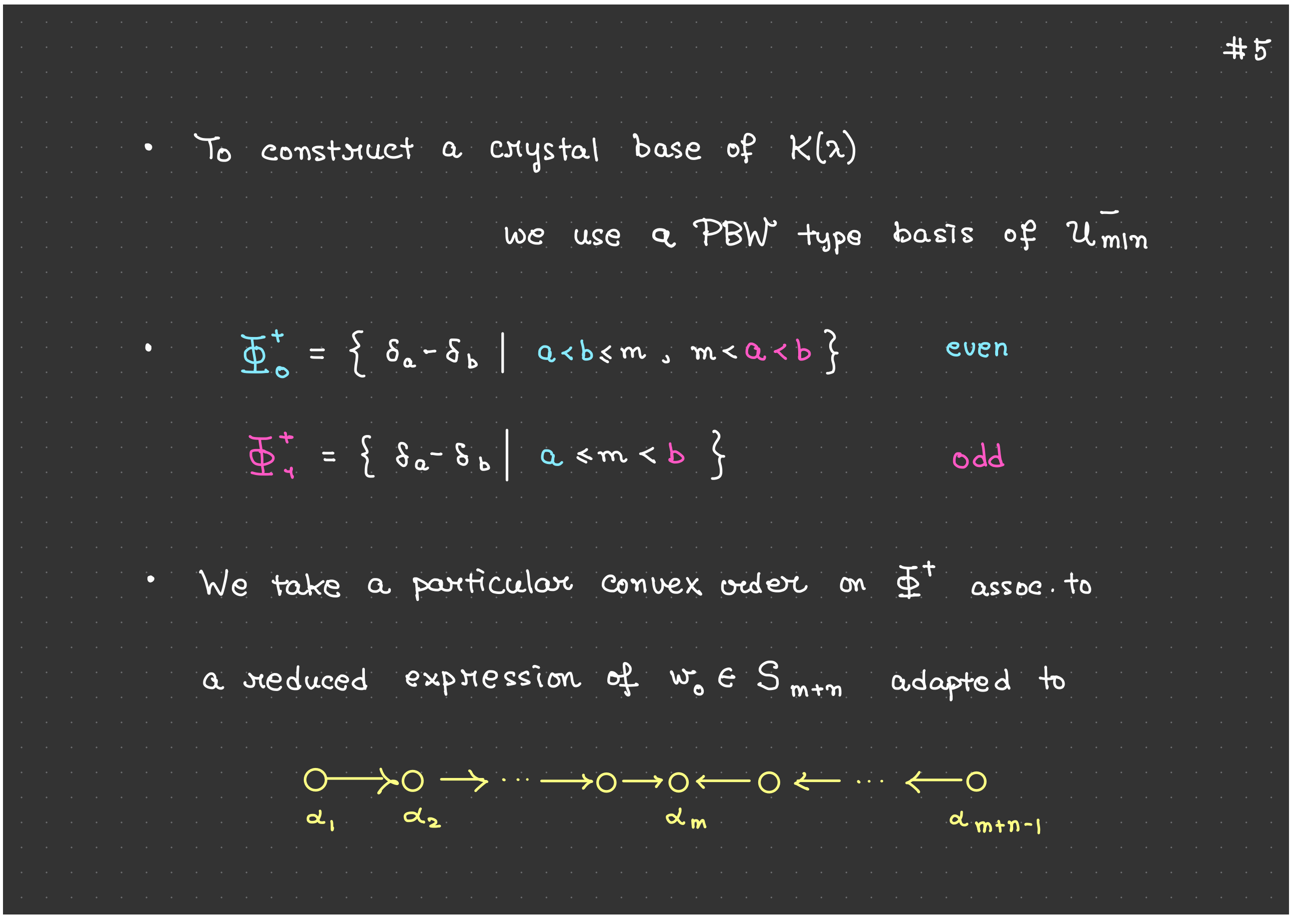
$\mathcal{P} = \langle \mathcal{U}_{m,n}, E_m \rangle$

$K(\lambda) := \mathcal{U}_{m|n} \otimes_{\mathcal{P}} V_{m,n}(\lambda)$

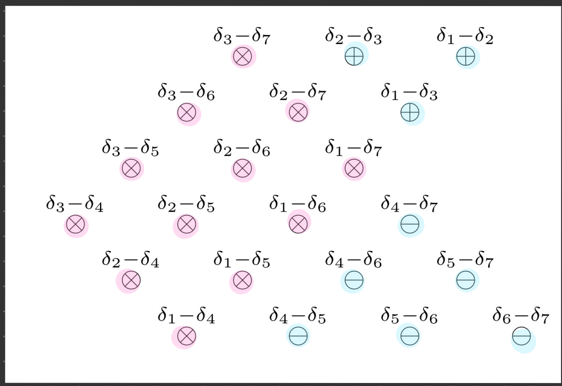
: indecomposable h.w. module w/ h.w. λ

$V_{m|n}(\lambda) =$ the max. quotient of $K(\lambda)$


#4
(K14)
① $K(\lambda)$ has a crystal base $(\mathcal{L}(K(\lambda)), B(K(\lambda)))$
where $B(K(\lambda))$: connected
② $\lambda \in P_{m|n}$
$\hat{\lambda}$
$K(\lambda) \xrightarrow{\pi_\lambda} V_{m|n}(\lambda)$
$\cup$ $\cup$
$\mathcal{L}(K(\lambda)) \longrightarrow \mathcal{L}_{m|n}(\lambda)$
$B(K(\lambda)) \longrightarrow B_{m|n}(\lambda) \cup \{0\}$ $q=0$
Rmk We should define $\tilde{e}_m, \tilde{f}_m$.


#5
• To construct a crystal base of $K(\lambda)$
we use a PBW type basis of $\mathcal{U}^-_{m|n}$
• $\Phi^+_0 = \{ \delta_a - \delta_b \mid a < b \leq m,\ m < a < b \}$ even
$\Phi^+_1 = \{ \delta_a - \delta_b \mid a \leq m < b \}$ odd
• We take a particular convex order on Φ^+ assoc. to
a reduced expression of $w_0 \in S_{m+n}$ adapted to
α_1
α_2
α_m
α_{m+n-1}

#6

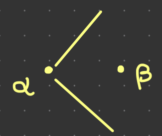


$m = 3$

$n = 4$

$\oplus : (\beta|\beta) > 0 \qquad \ominus : (\beta|\beta) < 0 \qquad \otimes : (\beta|\beta) = 0$

$\alpha < \beta \iff$



#7

- $\beta \in \Phi^+$, define a root vector F_β by

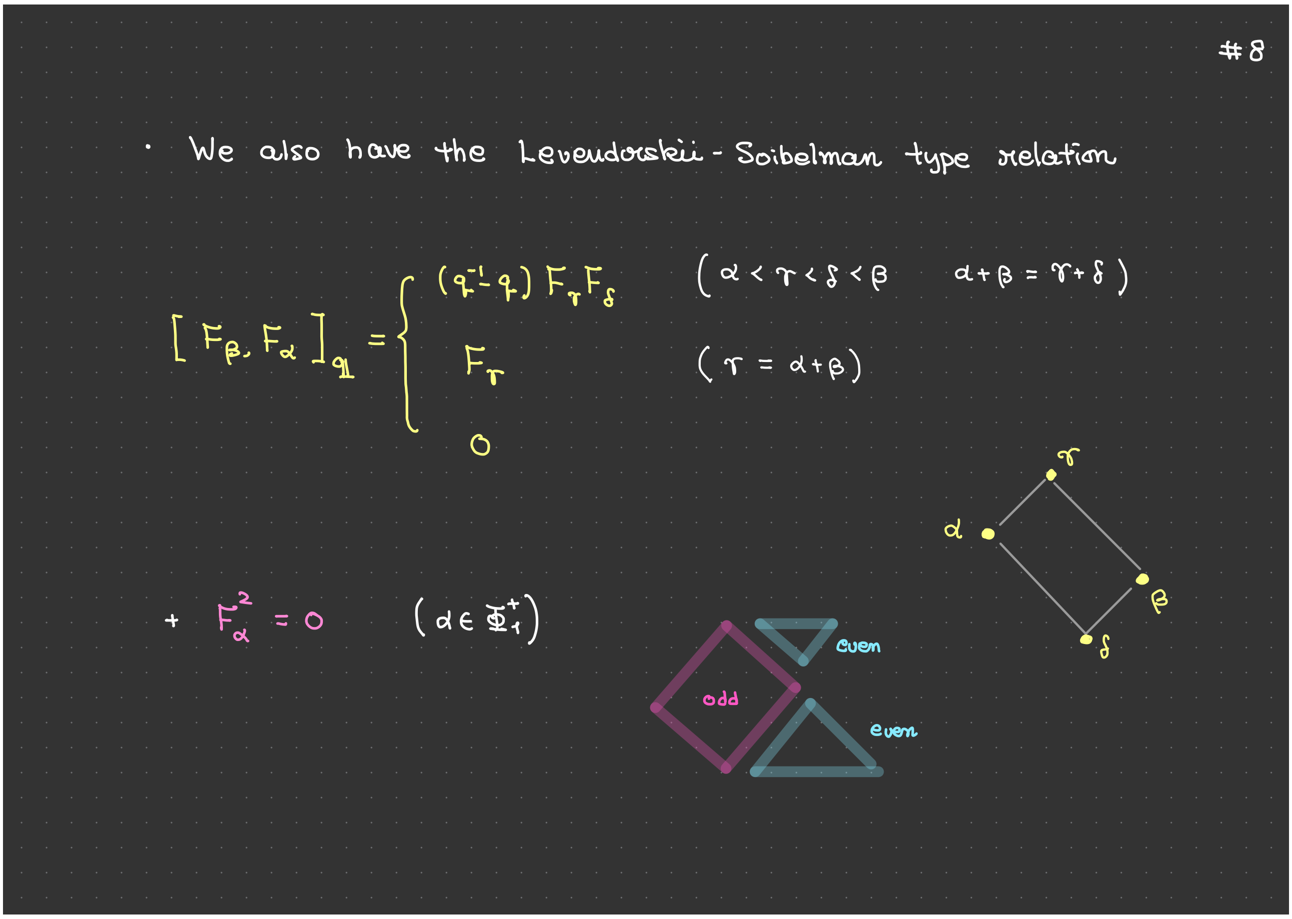
using Lusztig's transf. T_i's $(i \neq m)$ if $\beta \in \Phi^+_0$

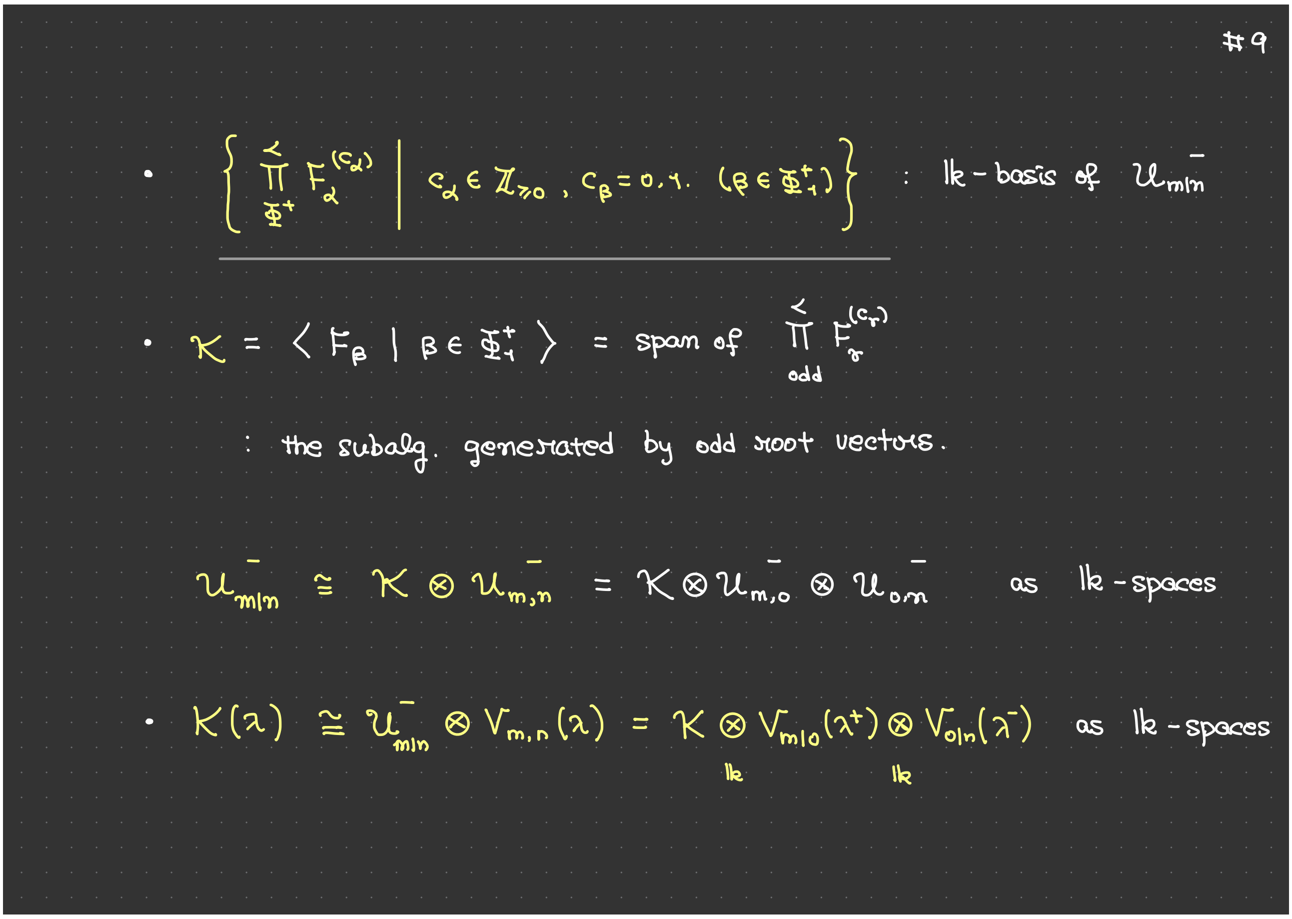
applying q-adjoint $ad_q(F_i)$'s $(i \neq m)$ to F_m if $\beta \in \Phi^+_1$

$$F_\beta = ad_q(F_j) \circ \cdots \circ ad_q(F_{m+1}) \circ ad_q(F_i) \circ \cdots \circ ad_q(F_{m-1})(F_m)$$

$$(\beta = \alpha_i + \cdots + \alpha_m + \cdots + \alpha_j)$$

where $ad_q(x)(y) = [x,y]_{\mathfrak{q}} = xy - \mathfrak{q}(|x|,|y|)\, yx$


#8
• We also have the Levendorskii-Soibelman type relation
$[F_\beta, F_\alpha]_q = \begin{cases} (q^{-1}-q) F_\gamma F_\delta & (\alpha < \gamma < \delta < \beta \quad \alpha+\beta = \gamma+\delta) \\ F_\gamma & (\gamma = \alpha+\beta) \\ 0 & \end{cases}$
+ $F_\alpha^2 = 0$ $(\alpha \in \Phi_1^+)$
γ
α
β
δ
even
odd
even


#9
• $\left\{ \prod^{<}_{\Phi^+} F_\alpha^{(c_\alpha)} \;\middle|\; c_\alpha \in \mathbb{Z}_{\geq 0},\ c_\beta = 0, 1.\ (\beta \in \Phi_1^+) \right\}$: $\Bbbk$-basis of $\mathcal{U}^-_{m|n}$
• $\mathcal{K} = \langle F_\beta \mid \beta \in \Phi_1^+ \rangle$ = span of $\prod^{<}_{\text{odd}} F_\gamma^{(c_\gamma)}$
: the subalg. generated by odd root vectors.
$\mathcal{U}^-_{m|n} \cong \mathcal{K} \otimes \mathcal{U}^-_{m,n} = \mathcal{K} \otimes \mathcal{U}^-_{m,0} \otimes \mathcal{U}^-_{0,n}$ as $\Bbbk$-spaces
• $K(\lambda) \cong \mathcal{U}^-_{m|n} \otimes V_{m,n}(\lambda) = \mathcal{K} \otimes_{\Bbbk} V_{m|0}(\lambda^+) \otimes_{\Bbbk} V_{0|n}(\lambda^-)$ as $\Bbbk$-spaces

#10

- $K(\lambda)$: fin-dimensional &

$$\operatorname{ch} K(\lambda) = \operatorname{ch} K \cdot \operatorname{ch} V_{m,n}(\lambda) = \prod_{\substack{1\le i\le m \\ 1\le j\le n}} (1 + u_i^{-1} v_j)\, s_{\lambda^+}(u)\, s_{(\lambda^-)^t}(v)$$

which can be viewed as a q-deformed Kac-module

- Define $\tilde{e}_i, \tilde{f}_i$ on $K(\lambda)$ by

  - lower crystal operator for $1 \le i < m$ $\quad ((\alpha_i|\alpha_i) > 0)$
  - upper crystal operator for $m < i \le m+n-1$ $\quad ((\alpha_i|\alpha_i) < 0)$
  - f_m (multiplication) where $\tilde{e}_m = e_m'$ (left derivation)

#11

Sketch of proof) (Existence)

$$K \xrightarrow[\phi]{\cong} \Lambda^q(\mathbb{k}^m \otimes \mathbb{k}^n) \;:\; \text{q-deformed exterior alg. gen. by } v_i \otimes w_j$$

$$\begin{array}{ccc} \cup\!\!\!\!\mid & & \cup\!\!\!\!\mid \\ F_\beta & \longmapsto & v_i \otimes w_j \quad (\beta = \delta_i - \delta_j) \end{array}$$

$\exists$ an action of $\underline{\mathcal{U}_q(\mathfrak{gl}_m) \otimes \mathcal{U}_p(\mathfrak{gl}_n)}$ on $\Lambda^q(\mathbb{k}^m \otimes \mathbb{k}^n)$ $\quad p = -q^{-1}$

$$\text{S}\!\!\parallel \quad \mathcal{U}_{m,n}$$

$\mathcal{U}_{m,n}$-module K induced by $\phi \cong K(0) = K \otimes V_{m|n}(0)$

#12

$\Rightarrow$ $K(\lambda) \cong \Lambda^{q}(\Bbbk^{m} \otimes \Bbbk^{n}) \otimes V_{m|n}(\lambda)$ as $\mathcal{U}_{m,n}$-module

$$\mathcal{L}(K) := \bigoplus_{(c_\beta)} A_0 \prod_{\Phi_1^+} F_\beta^{c_\beta} \qquad B(K) := \left\{ \pm \prod_{\Phi_1^+} F_\beta^{c_\beta} \pmod{q\mathcal{L}(K)} \right\}$$

$\Rightarrow$ $\mathcal{L}(K(\lambda)) := \mathcal{L}(K) \otimes \mathcal{L}(V_{m|n}(\lambda))$

$B(K(\lambda)) := B(K) \times B(V_{m|n}(\lambda))$

forms a crystal base of $K(\lambda)$ as a $\mathcal{U}_{m,n}$-module

#13

Finally, one can check $\tilde{x}_m \mathcal{L}(K(\lambda)) \subset \mathcal{L}(K(\lambda))$ $(x = e, f)$

$\tilde{f}_m \prod_{\Phi_1^+} F_\beta^{c_\beta}$ is given by
$$\begin{cases} c_{\alpha_m} \longrightarrow c_{\alpha_m} + 1 & (c_{\alpha_m} = 0) \\ 0 & (c_{\alpha_m} = 1) \end{cases}$$

$B(K) \overset{1-1}{\longleftrightarrow} \mathcal{P}(\Phi_1^+)$: power set

We may regard

$$B(K(\lambda)) = \mathcal{P}(\Phi_1^+) \times B(V_{m|0}(\lambda^+)) \times B(V_{0|n}(\lambda^-))$$

#14

$\mathcal{B}(K(\lambda))$ can be described explicitly since

$\mathcal{U}_{m,n}$-crystal structure is well-known + tensor product rule

This implies the following

① the connectedness of $\mathcal{B}(K(\lambda))$

② $\mathcal{L}(K(\lambda)) = \sum A_0 \tilde{x}_{i_1} \cdots \tilde{x}_{i_r} v_\lambda$ $\quad (i_1, \cdots, i_r,\ x = e, f)$

$$\mathcal{B}(K(\lambda)) = \{ \pm \tilde{x}_{i_1} \cdots \tilde{x}_{i_r} v_\lambda \ (\mathrm{mod}\ q\mathcal{L}(K(\lambda)) \} \setminus \{0\}$$

③ Uniqueness of a crystal base of $K(\lambda)$

#15

(Compatibility with $V_{m|n}(\lambda)$ for $\lambda \in \mathcal{P}_{m|n}$)

- $K(\lambda)$: irreducible $\iff$ λ : typical

  i.e. $(\lambda + \rho_{m|n} | \beta) = 0$ for all $\beta \in \Phi_1^+$

  where $\rho_{m|n} = \frac{1}{2} \sum_{\Phi_0^+} \alpha - \frac{1}{2} \sum_{\Phi_1^+} \beta$

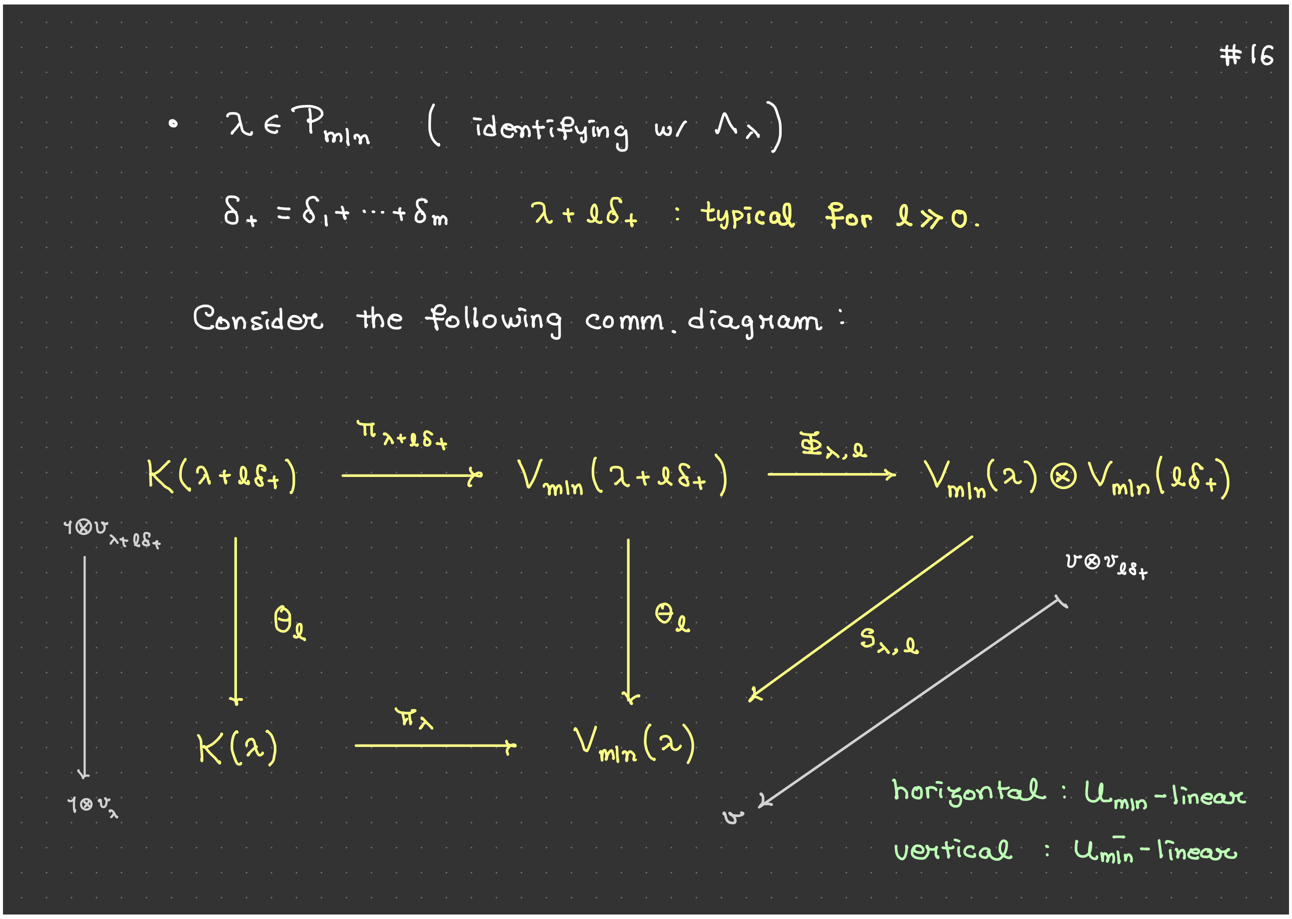
(It follows from the fact at $q = 1$ due to Kac)

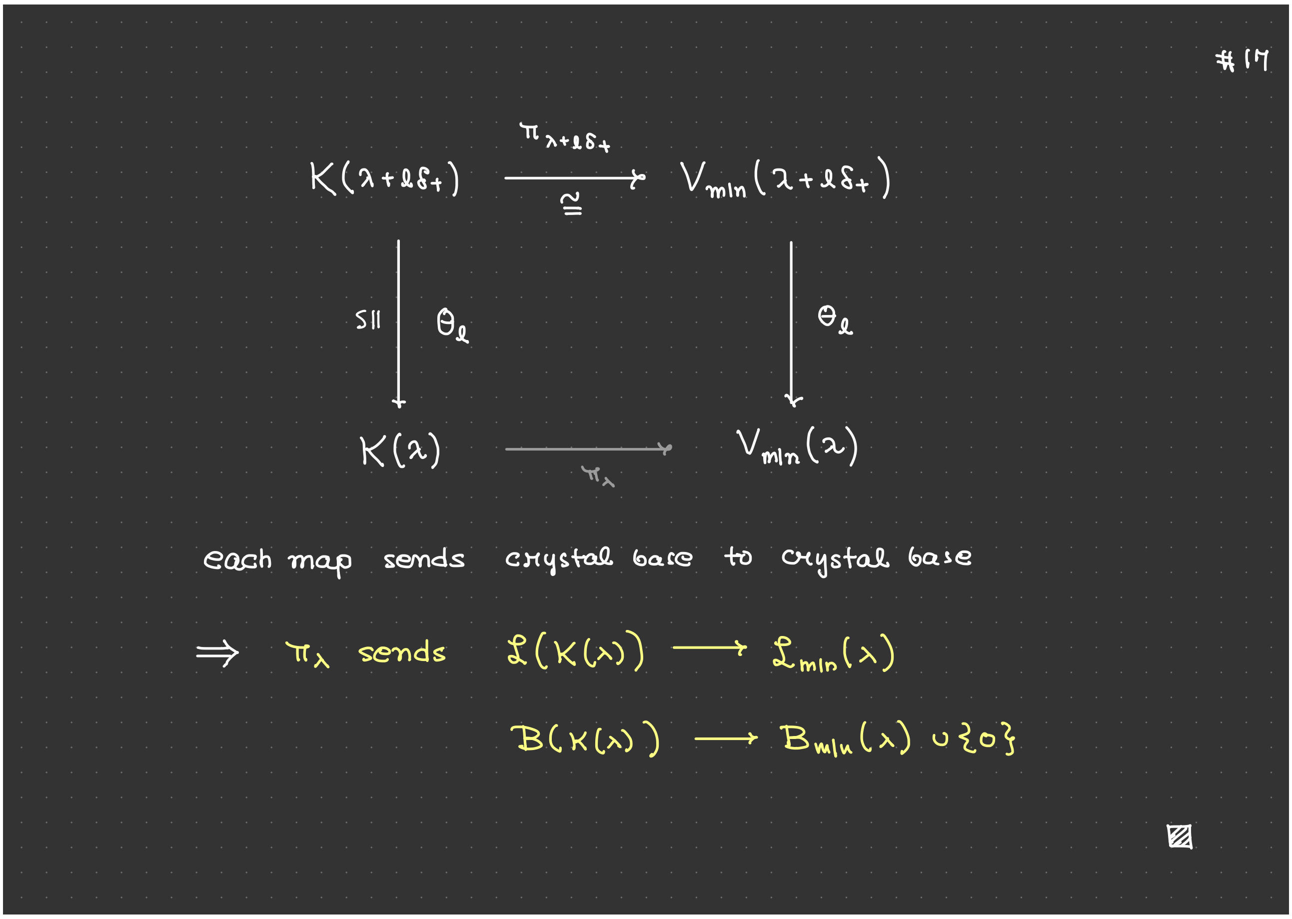
In particular,

$$\lambda \in \mathcal{P}_{m|n} \qquad \Lambda_\lambda : \text{typical} \iff (n^m) \subset \lambda$$

$$\iff K(\lambda) = V_{m|n}(\lambda)$$

$$K(\lambda) = K(\Lambda_\lambda)$$


#16
• λ ∈ P_{m|n} (identifying w/ Λ_λ)
δ_+ = δ_1 + ⋯ + δ_m   λ + ℓδ_+ : typical for ℓ ≫ 0.
Consider the following comm. diagram :
K(λ+ℓδ_+)
π_{λ+ℓδ_+}
V_{m|n}(λ+ℓδ_+)
Φ_{λ,ℓ}
V_{m|n}(λ) ⊗ V_{m|n}(ℓδ_+)
1⊗v_{λ+ℓδ_+}
1⊗v_λ
θ_ℓ
θ_ℓ
S_{λ,ℓ}
v⊗v_{ℓδ_+}
v
K(λ)
π_λ
V_{m|n}(λ)
horizontal : U_{m|n}-linear
vertical : U_{m|n}^{-}-linear


#17
K(λ+ℓδ_+)
π_{λ+ℓδ_+}
≅
V_{m|n}(λ+ℓδ_+)
≅
θ_ℓ
θ_ℓ
K(λ)
π_λ
V_{m|n}(λ)
each map sends crystal base to crystal base
⇒ π_λ sends ℒ(K(λ)) ⟶ ℒ_{m|n}(λ)
B(K(λ)) ⟶ B_{m|n}(λ) ∪ {0}

\# 18

Rmk

① We have a combinatorial description of crystal embedding

$$\begin{array}{ccccc} \mathcal{B}_{m|n}(\lambda) & \longrightarrow & \mathcal{P}(\Phi_1^+) \times \mathcal{B}_{m|0}(\lambda^+) & \times & \mathcal{B}_{0|n}(\lambda^-) \\ \| & & \| & & \| \\ SST_{m|n}(\lambda) & & SST_{m|0}(\lambda_{\leq m}) & & SST_{0|n}(\lambda_{>m}) \end{array}$$

$$T = (T^{\leq m}, T^{>m}) \longmapsto (S, T', T^{>m})$$

$$T^{\leq m} = (T_0^{\leq m}, T_1^{\leq m}) \quad \text{(Sagan-Stanley's skew RSK)}$$

pair of skew SST's in $\{m^\vee, \cdots, 1^\vee\}$, $\{m+1, \cdots, m+n-1\}$ of same inner shape

\# 19

② **Crystal structure of $\mathcal{B}(K(\lambda))$**

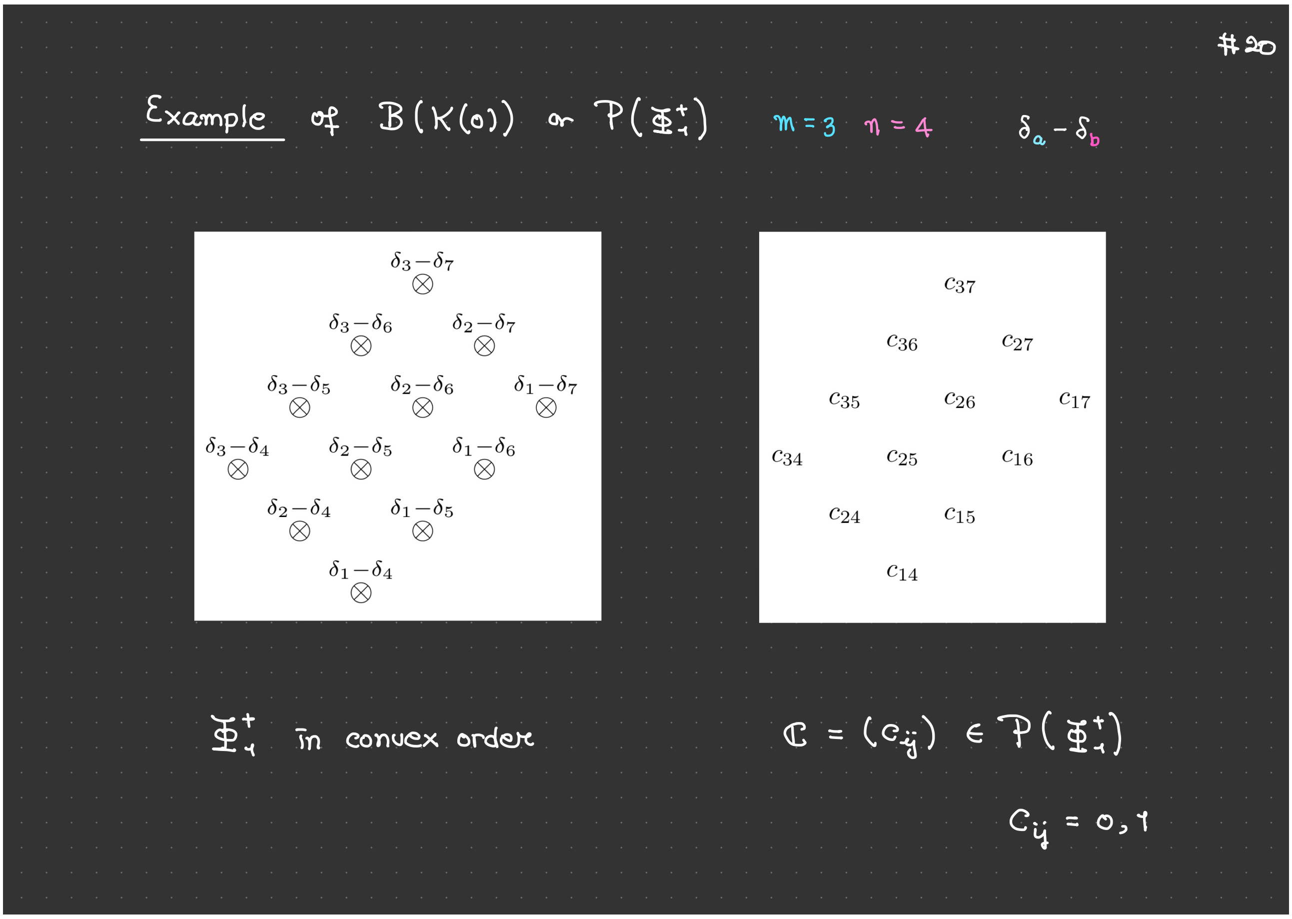
$$\mathcal{B}(K(\lambda)) = \mathcal{P}(\Phi_1^+) \times \mathcal{B}_{m|0}(\lambda^+) \times \mathcal{B}_{0|n}(\lambda^-)$$

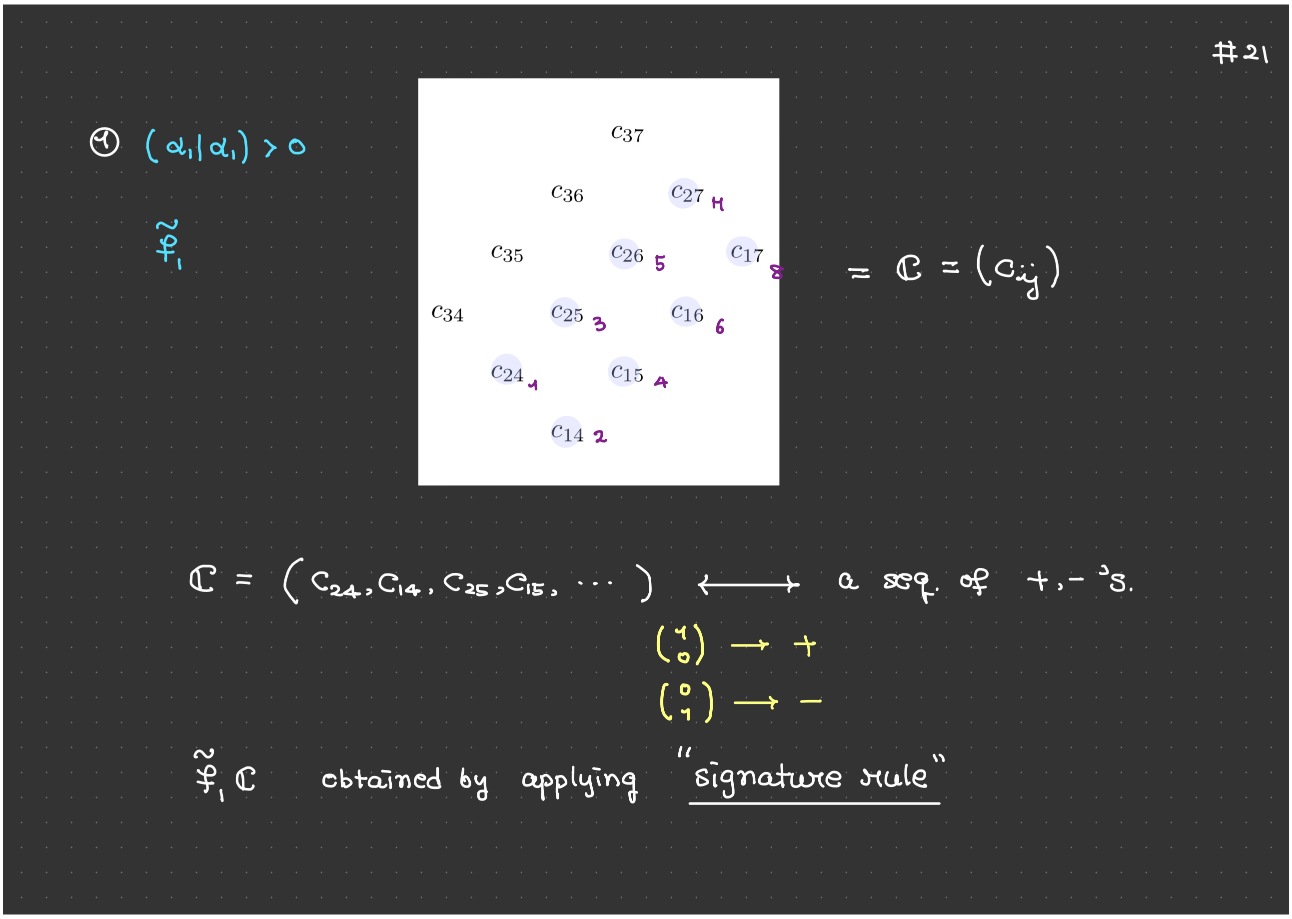
$$\cong \mathcal{P}(\Phi_1^+) \overset{\rightarrow}{\otimes} \mathcal{B}_{m|0}(\lambda^+) \times \mathcal{B}_{0|n}(\lambda^-) \quad \text{as } \mathcal{U}_{m|0}\text{-crystal}$$

$$\cong \mathcal{P}(\Phi_1^+) \times \mathcal{B}_{m|0}(\lambda^+) \overset{\leftarrow}{\otimes} \mathcal{B}_{0|n}(\lambda^-) \quad \text{as } \mathcal{U}_{0|n}\text{-crystal}$$

$$\cong \mathcal{P}(\Phi_1^+) \times \mathcal{B}_{m|0}(\lambda^+) \times \mathcal{B}_{0|n}(\lambda^-) \quad \text{for } \tilde{e}_m, \tilde{f}_m$$

So it suffices to consider $\mathcal{B}(K(0)) \cong \mathcal{P}(\Phi_1^+)$

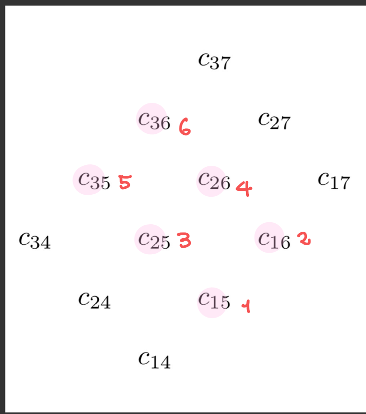

#20
Example of B(K(0)) on P(Φ₄⁺)
m = 3 n = 4
δa − δb
δ3 − δ7
δ3 − δ6 δ2 − δ7
δ3 − δ5 δ2 − δ6 δ1 − δ7
δ3 − δ4 δ2 − δ5 δ1 − δ6
δ2 − δ4 δ1 − δ5
δ1 − δ4
c37
c36 c27
c35 c26 c17
c34 c25 c16
c24 c15
c14
Φ₄⁺ in convex order
C = (cij) ∈ P(Φ₄⁺)
cij = 0, 1


#21
① (α1|α1) > 0
f̃1
c37
c36 c27 7
c35 c26 5 c17 8
c34 c25 3 c16 6
c24 1 c15 4
c14 2
= C = (cij)
C = (c24, c14, c25, c15, ···) ⟷ a seq. of +, −'s.
(1 0) → +
(0 1) → −
f̃1 C obtained by applying "signature rule"

#22

② $(\alpha_5|\alpha_5) < 0$

$\tilde{f}_5$



$= \mathbb{C} = (c_{ij})$

$\mathbb{C} = (c_{15}, c_{16}, c_{25}, c_{26}, \cdots) \longleftrightarrow$ a seq. of $+,-$'s.

$\binom{1}{0} \longrightarrow +$

$\binom{0}{1} \longrightarrow -$

$\tilde{f}_5\mathbb{C}$ obtained by applying "signature rule"

#23

Example of embedding $B_{m|n}(\lambda) \longrightarrow B(K(\lambda))$

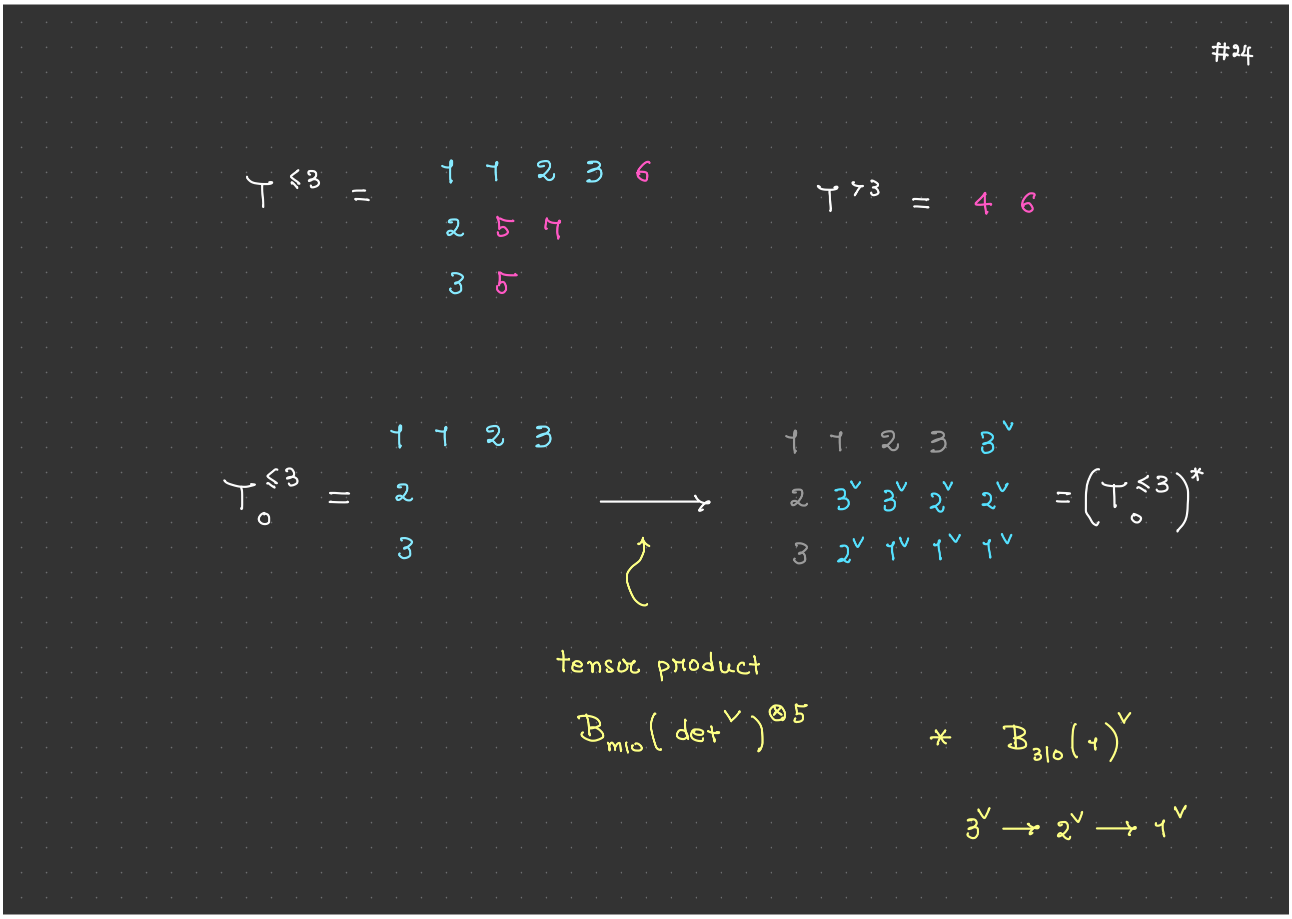
$m = 3,\ n = 4,\ \lambda = (5,3,2,2) = \underbrace{5\delta_1 + 3\delta_2 + 2\delta_3}_{\lambda^+} + \underbrace{\delta_4 + \delta_5}_{\lambda^-}$

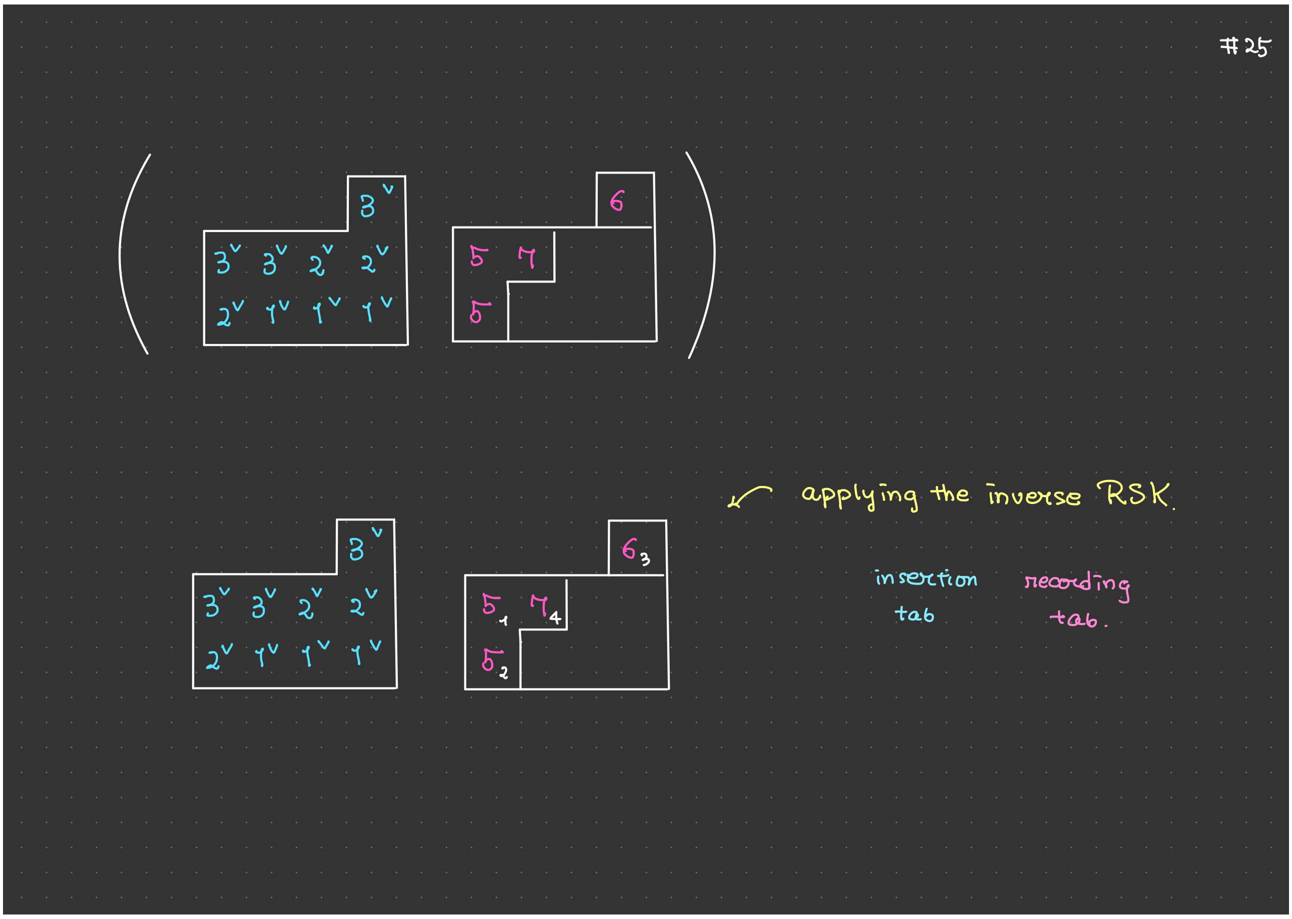
$T =$

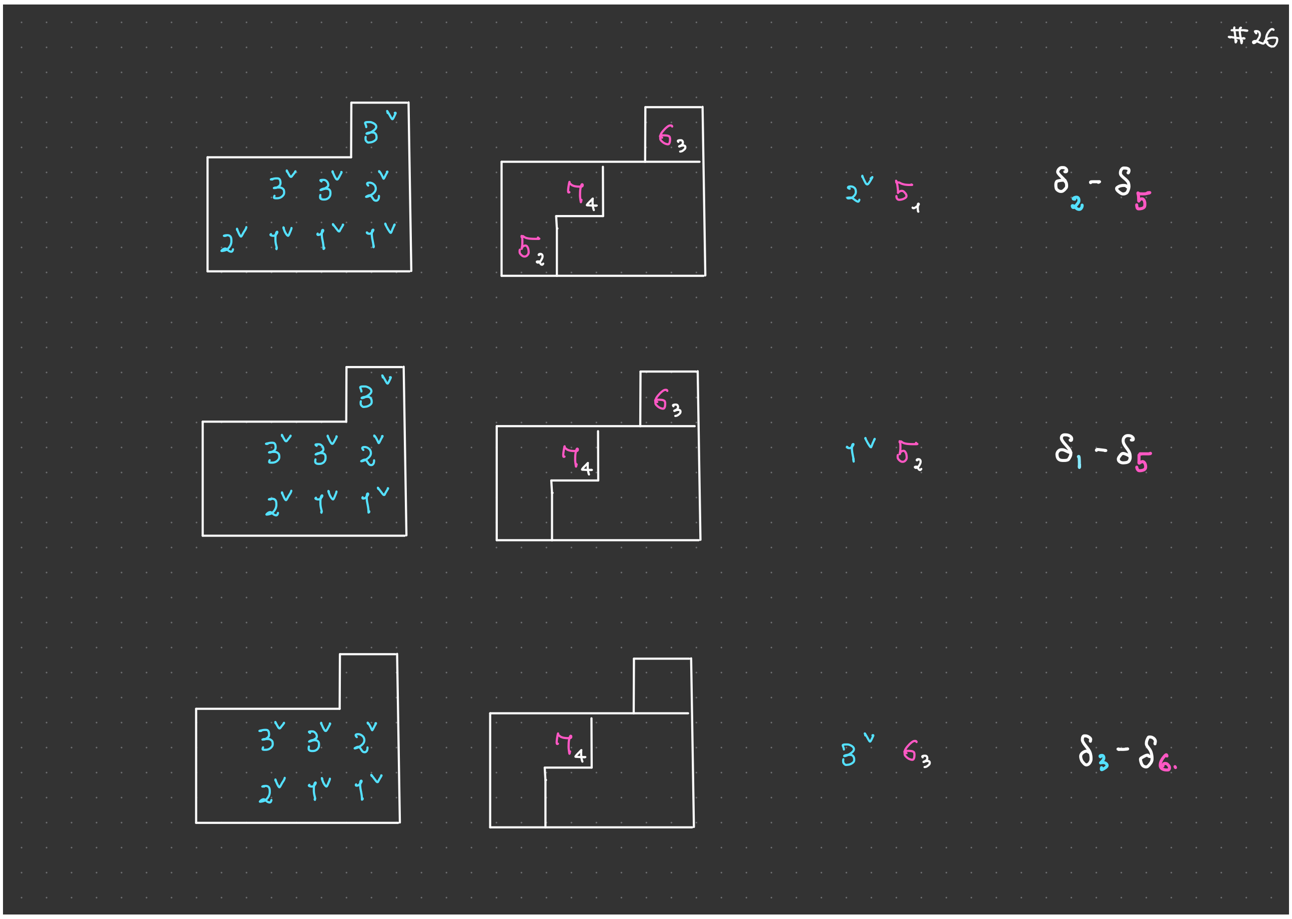
1	*1*	*2*	*3*	6
2	5	7		
3	5			
4	6			

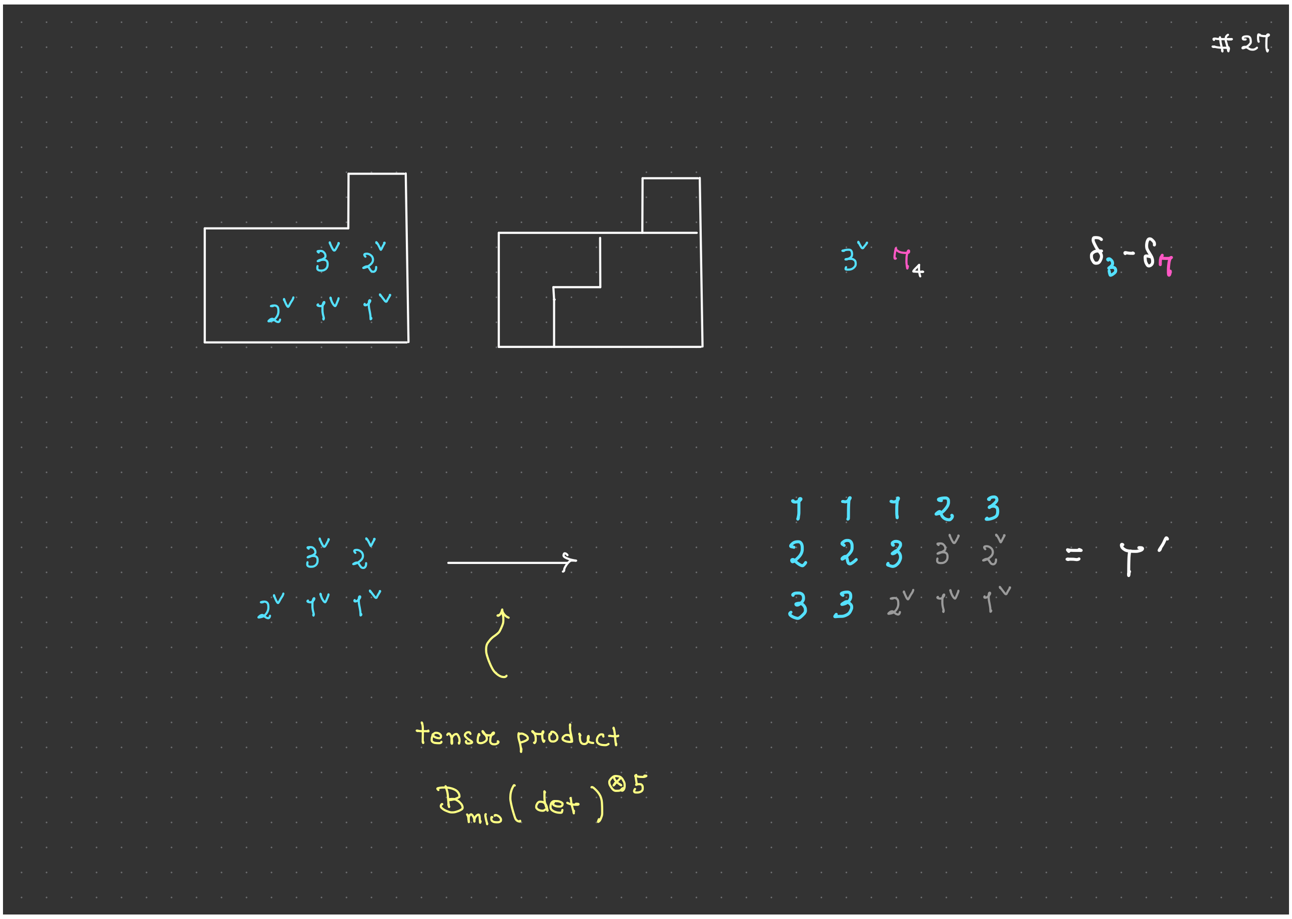
$\in SST_{3|4}(5,3,2,2)$

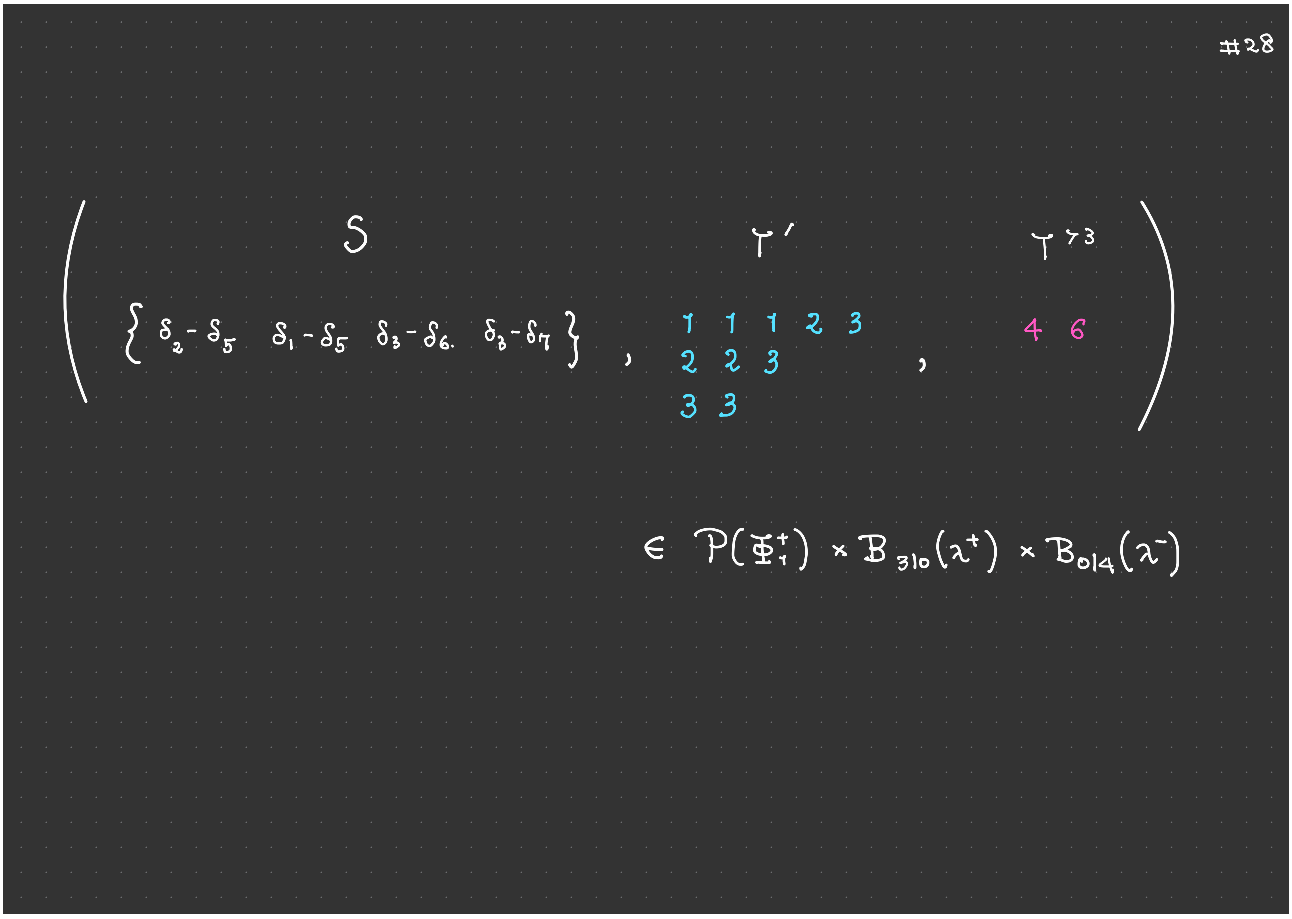
$\longrightarrow (S, T', T^{>3}) \in \mathcal{P}(\Phi_1^+) \times B_{3|0}(\lambda^+) \times B_{0|4}(\lambda^-)$

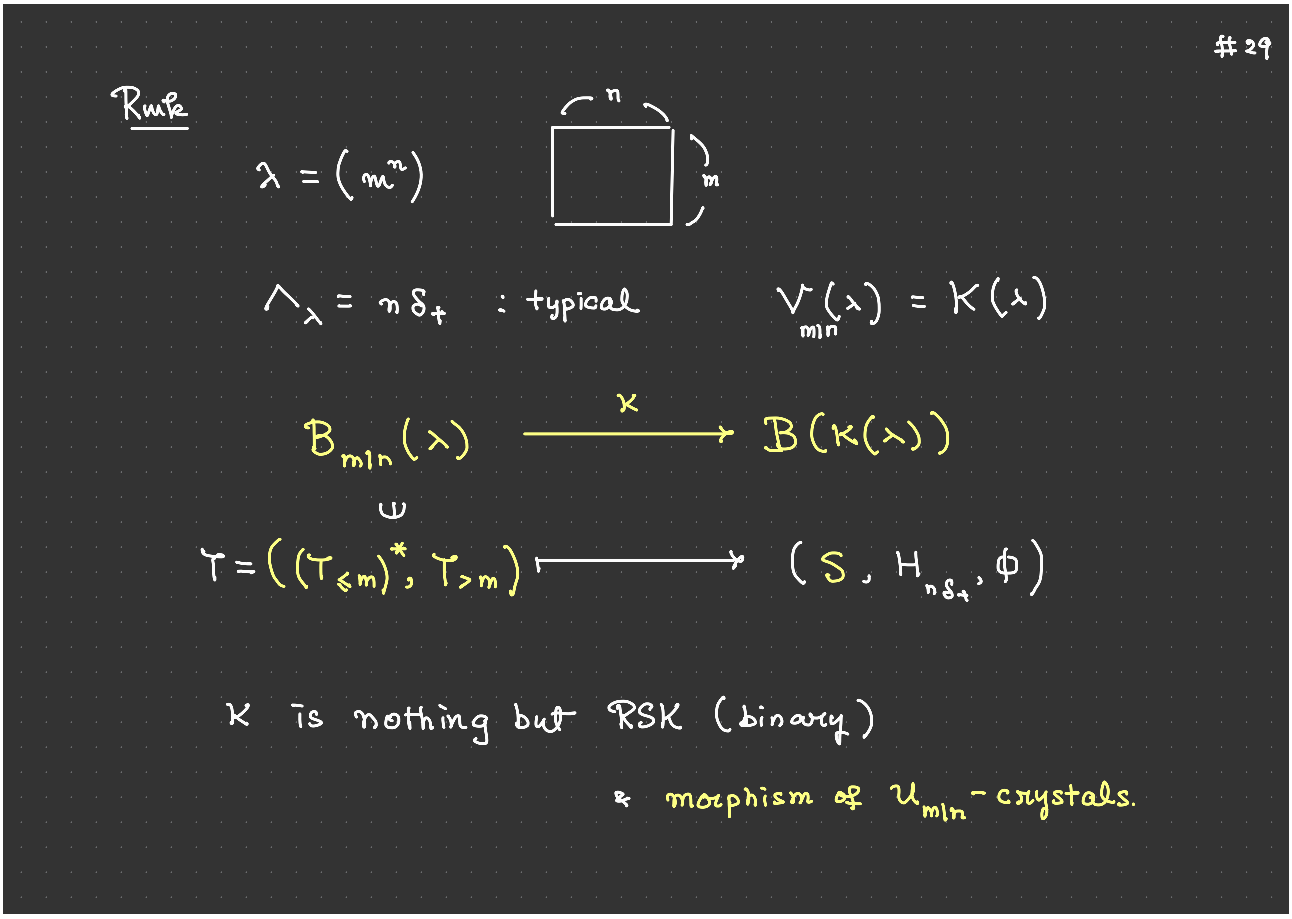

#24
tensor product
3∨ → 2∨ → 1∨


#25
applying the inverse RSK.
insertion tab
recording tab.


#26
$\delta_2 - \delta_5$
$\delta_1 - \delta_5$
$\delta_3 - \delta_6$


#27
$\delta_3 - \delta_7$
tensor product
$B_{m10}(\det)^{\otimes 5}$


#28
S
T′
T^{>3}
{δ₂−δ₅ δ₁−δ₅ δ₃−δ₆ δ₃−δ₇}
1 1 1 2 3
2 2 3
3 3
4 6
∈ P(Φ₁⁺) × B₃₁₀(λ⁺) × B₀₁₄(λ⁻)


#29
Rmk
λ = (mⁿ)
n
m
Λ_λ = nδ₊ : typical
V_{m|n}(λ) = K(λ)
B_{m|n}(λ)
κ
B(K(λ))
T = ((T_{≤m})*, T_{>m})
(S, H_{nδ₊}, φ)
κ is nothing but RSK (binary)
& morphism of U_{m|n}-crystals.

#1

4 The negative part of $\mathcal{U}_{m|n}$

- Now, we can take a limit of

$$B(K(\lambda)) \cong \mathcal{P}(\Phi_1^+) \times B_{m|0}(\lambda^+) \times B_{0|n}(\lambda^-)$$

$$\longrightarrow \mathcal{P}(\Phi_1^+) \times B_{m|0}(\infty) \times B_{0|n}(\infty) \qquad (\lambda^\pm \to \infty)$$

- We will

① describe the crystal structure of the limit (& cnn. components)

② Construct a crystal base of $\mathcal{U}_{m|n}^-$ w/ the above crystal.

③ compatibility w/ crystal base of $K(\lambda)$.

#2

- Recall

$$\lim_{\lambda \to \infty} B_{m|0}(\lambda) = B_{m|0}(\infty) \qquad (\lambda : \mathfrak{gl}_m\text{-dominant})$$

$\lambda < \mu \iff \mu - \lambda$: dominant integral

$$B_{m|0}(\lambda) \xrightarrow{\Theta_{\lambda,\mu}} B_{m|0}(\mu)$$

$$X v_\lambda \longmapsto X v_\mu \qquad (X : \text{prod. of } \tilde{f}_i\text{'s})$$

: well-defined directed system of embedding of crystals

whose limit is iso. to crystal $B_{m|0}(\infty)$ of $\mathcal{U}_{m|0}^-$.

#3

- $\lambda < \mu \overset{\text{def}}{\iff} \lambda^{\pm} < \mu^{\pm} \qquad (\lambda, \mu \in P^+)$

$b = (S, b_+, b_-) \in B(K(\lambda)) \quad \left(\cong P(\Phi_1^+) \times B_{m|0}(\lambda^+) \overleftarrow{\otimes} B_{0|n}(\lambda^-) \right)$

$b = Y(S_0, X v_{\lambda^+}, v_{\lambda^-})$ for some

$X = \prod_{i<m} \tilde{f}_i$'s , $Y = \prod_{j \geq m} \tilde{f}_j$'s. s.t. $S_0 \otimes v_{\lambda^-}$: $\mathcal{U}_{0|n}$ - maximal

Define

$$\begin{array}{ccc} B(K(\lambda)) & \xrightarrow{\Theta_{\lambda,\mu}} & B(K(\mu)) \\ b = (S, b_+, b_-) & \longmapsto & Y(S_0, X v_{\mu^+}, v_{\mu^-}) \end{array}$$

#4

- $\Theta_{\lambda,\mu}$ is

① injective (w/ limit $P(\Phi_1^+) \times B_{m|0}(\infty) \times B_{0|n}(\infty)$ as a set)

② an embedding of $\mathcal{U}_{m,n}$ - crystals (i.e. for $\mathfrak{gl}_m \oplus \mathfrak{gl}_n$)

③ "locally" an embedding of $\mathcal{U}_{m|n}$ - crystal i.e.

$\forall b = (b_v) \in P(\Phi_1^+) \times B_{m|0}(\infty) \times B_{0|n}(\infty)$ (as a limit)

$\exists \lambda \in P^+$ such that

$$b_\lambda \xrightarrow{m} b'_\lambda \iff b_\mu \xrightarrow{m} b'_\mu \quad \text{for all } \mu > \lambda$$

#5

- The limit of the directed system $\left(\{\Theta_{\lambda,\mu}\}, \{B(K(\lambda))\} \right)$ has a well-defined abstract $\mathcal{U}_{m|n}$-crystal structure

$$P(\Phi_1^+) \times B_{m|0}(\infty) \times B_{0|n}(\infty)$$

$$\cong P(\Phi_1^+) \times B_{m|0}(\infty) \times B_{0|n}(\infty) \qquad \text{for } \tilde{e}_m, \tilde{f}_m$$

$$\cong P(\Phi_1^+) \otimes B_{m|0}(\infty) \times B_{0|n}(\infty) \qquad \text{as a } \mathcal{U}_{m|0}\text{-crystal}$$

(hence $\mathcal{U}_{m|1}$)

$$\cong P(\Phi_1^+) \times B_{m|0}(\infty) \times B_{0|n}(\infty) \qquad \text{as a } \mathcal{U}_{0|n}\text{-crystal}$$

where the crystal operators for $\mathcal{U}_{m|1}$, $\mathcal{U}_{0|n}$ commute.

#6

- $B_{m|n}(\infty) := P(\Phi_1^+) \times B_{m|0}(\infty) \times B_{0|n}(\infty)$ $\quad$ $\mathcal{U}_{m|n}$-crystal

- $B_{m|n}(\infty)$ is NOT connected in general.

- To describe a connected component of $B_{m|n}(\infty)$, recall

For λ : dominant int. wt for $\mathfrak{gl}_m$, we have

$$B_m(\lambda) \otimes B_m(\infty) \cong \coprod_{b \in B_m(\lambda)} B_m(\infty) \otimes T_{\mathrm{wt}(b)}$$

(a crystal version of Verma filtration $V(\lambda) \otimes M(0)$)

#7

pf.) (Kashiwara)

$B_q = B_q(\mathfrak{gl}_m)$: the alg. of q-bosons. assoc. to $\mathfrak{gl}_m$

$= \langle e_i', f_i \mid i = 1, \cdots, m-1 \rangle$ (possibly + Cartan part)

① $\exists!$ simple B_q-module V_∞ (up to h.w)

$V_\infty \cong U_q(\mathfrak{gl}_m)^- = U_q^-$ ↺ e_i' : q-derivation, f_i : multiplication

with a crystal base $\cong (\mathcal{L}_m(\infty), \mathcal{B}_m(\infty))$

#8

② M : fin-dim $U_q(\mathfrak{gl}_m)$-module

$\exists$ B_q-module structure on $M \otimes V_\infty$ where the action is given by a U_q-comodule str. on B_q

$$\Delta : B_q \longrightarrow U_q \otimes B_q$$

$$e_i' \longmapsto (q^{-1}-q)\, k_i e_i \otimes 1 + k_i \otimes e_i'$$

$$f_i \longmapsto f_i \otimes 1 + k_i \otimes f_i$$

& $M \otimes V_\infty$ is a semisimple B_q-module $\cong \bigoplus V_\infty$

#9

③ $(\mathcal{L}, \mathcal{B})$: crystal base of M as a U_q-module

$(\mathcal{L} \otimes \mathcal{L}(\infty), \mathcal{B} \otimes \mathcal{B}(\infty))$: a crystal base of $M \otimes V$

$\cong \bigoplus (\mathcal{L}(\infty), \mathcal{B}(\infty))$

$\tilde{e}_i, \tilde{f}_i$ act on $\mathcal{B} \otimes \mathcal{B}(\infty)$ following the tensor product rule

④ By ①~③

$$B_m(\lambda) \otimes B_m(\infty) \xrightarrow{\ \cong\ } \coprod_{B_m(\lambda)} B_m(\infty) \otimes T_{wt(b)}$$

$$\underline{u_\lambda \otimes b} \longrightarrow u_\infty \otimes t_{wt(b^*)}$$

maximal

$\varepsilon_i(b) \leqslant (\lambda|\alpha_i)$

☒

#10

<u>Thm</u> (Jang-K-Uruno 22)

① Each cnn. component $\cong B_{m|1}(\infty) \times B_{0|n}(\infty)$

② $B_{m|n}(\infty) \cong B_{m|1}(\infty) \times B_{0|n}(\infty)^{\oplus 2^{m(n-1)}}$

In particular, $B_{m|n}(\infty)$: connected $\iff n = 1$

<u>Rmk</u>

$\tilde{e}_i, \tilde{f}_i \curvearrowright B_{m|1}(\infty) \times B_{0|n}(\infty) \curvearrowleft \tilde{e}_i, \tilde{f}_i$ $(i \geqslant m+1)$

$(1 \leqslant i \leqslant m)$

#11

pf) $\mathcal{P}(\Phi_1^+) \cong B(K_{m|n}(0)) \cong B(K_{m|1}(0)) \times \mathcal{E}$

as $U_{m|1}$ - crystals

$$\mathcal{E} = \{ (c_\beta) \mid c_\beta = 0 \text{ for } \beta : \text{odd root of } \mathfrak{gl}_{m|1} \} \subset \mathcal{P}(\Phi_1^+)$$

$$\overset{1-1}{\longleftrightarrow} M_{m\times(n-1)}(\mathbb{Z}_2)$$

$$\mathcal{E} \cong \bigsqcup_{\substack{\ell(\lambda)\le m \\ \ell(\lambda^t)<n}} B_m(\lambda) \times B_{n-1}(\lambda^t) : (\mathfrak{gl}_m, \mathfrak{gl}_{n-1})\text{-bicrystal}$$

via skew RSK.

$$\cong \bigsqcup_{\substack{\ell(\lambda)\le m \\ \ell(\lambda^t)<n}} B_{m|0}(\lambda)^{\oplus m_\lambda} \qquad m_\lambda = |B_{0|n-1}(\lambda^t)|$$

#12

$B_{m|n}(\infty)$

$$= B(K_{m|n}(0)) \times B_{m|0}(\infty) \times B_{0|n}(\infty)$$

$$\cong B(K_{m|1}(0)) \times \mathcal{E} \times B_{m|0}(\infty) \times B_{0|n}(\infty)$$

$$\cong B(K_{m|1}(0)) \times \bigsqcup_{\substack{\ell(\lambda)\le m \\ \ell(\lambda^t)<n}} B_{m|0}(\lambda)^{\oplus m_\lambda} \times B_{m|0}(\infty) \times B_{0|n}(\infty)$$

$$\cong \bigsqcup_{\substack{\ell(\lambda)\le m \\ \ell(\lambda^t)<n}} \left(B(K_{m|1}(0)) \times B_{m|0}(\lambda) \times B_{m|0}(\infty) \times B_{0|n}(\infty) \right)^{\oplus m_\lambda}$$

#13

$$\cong \coprod_{\substack{\ell(\lambda)\leq m \\ \ell(\lambda^t)<n}} \coprod_{B_{m|0}(\lambda)} B(K_{m|1}(0)) \times \left(B_{m|0}(\infty)\otimes T_{wt(b)}\right) \times B_{0|n}(\infty)^{\oplus m_\lambda}$$

$$\cong \coprod_{\substack{\ell(\lambda)\leq m \\ \ell(\lambda^t)<n}} \coprod_{B_{m|0}(\lambda)} \left(B_{m|1}(\infty) \times B_{0|n}(\infty)\right) \otimes T_{wt(b)}^{\oplus m_\lambda}$$

$$\cong B_{m|1}(\infty) \times B_{0|n}(\infty)^{\oplus |\mathcal{C}|} \quad \text{where } |\mathcal{C}| = 2^{m(n-1)}$$

$$\because \mathcal{C} \overset{1-1}{\longleftrightarrow} M_{m\times(n-1)}(\mathbb{Z}_2) \cong \coprod_{\substack{\ell(\lambda)\leq m \\ \ell(\lambda^t)<n}} B_{m|0}(\lambda)^{\oplus m_\lambda}$$

∎

#14

- Now, we want to construct

  a crystal base $(\mathcal{L}(\infty), B(\infty))$ of $\mathcal{U}^-_{m|n}$ such that

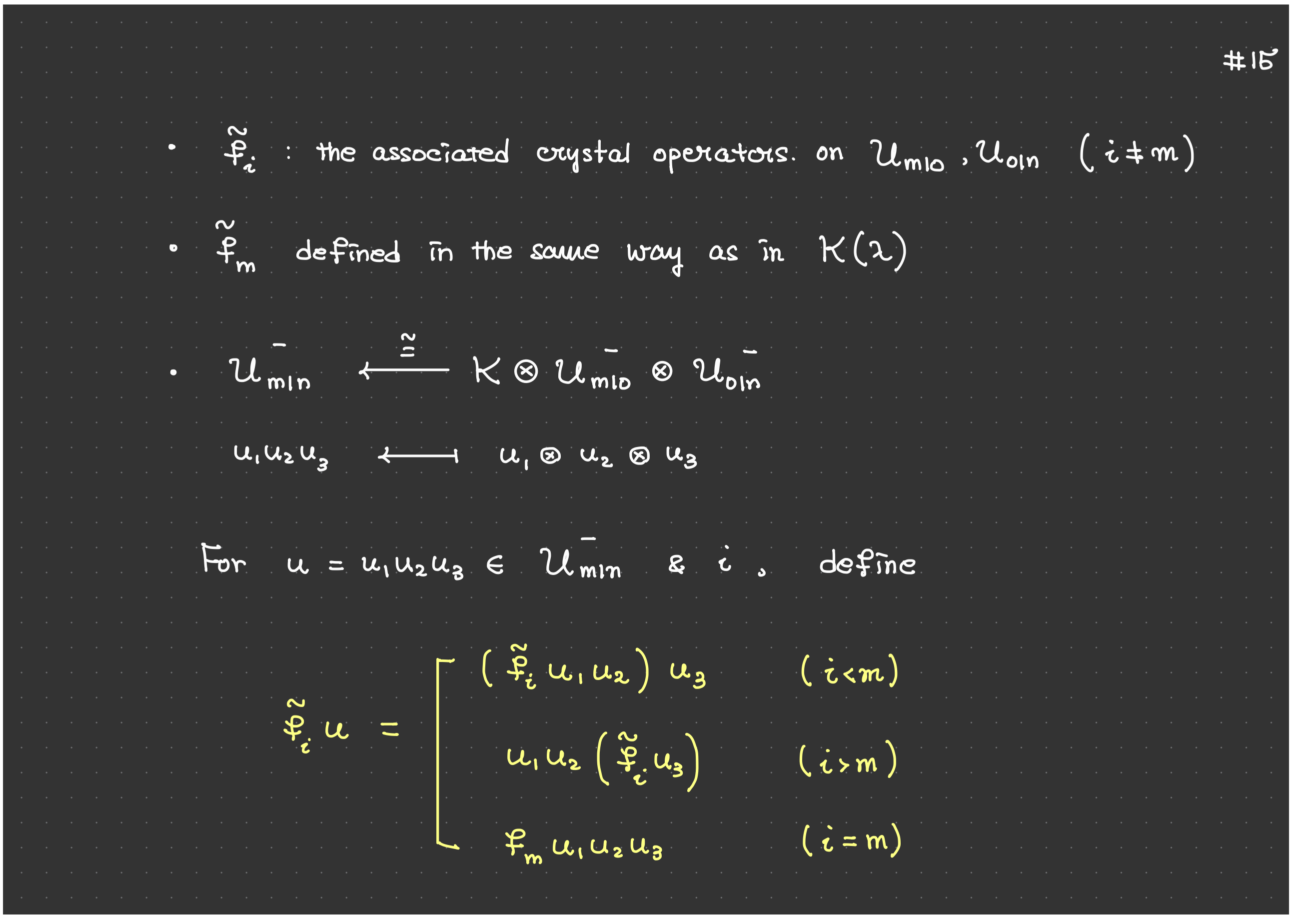
  $B(\infty) = B_{m|n}(\infty)$ & it is compatible w/ $\left(\mathcal{L}(K(\lambda)), B(K(\lambda))\right)$

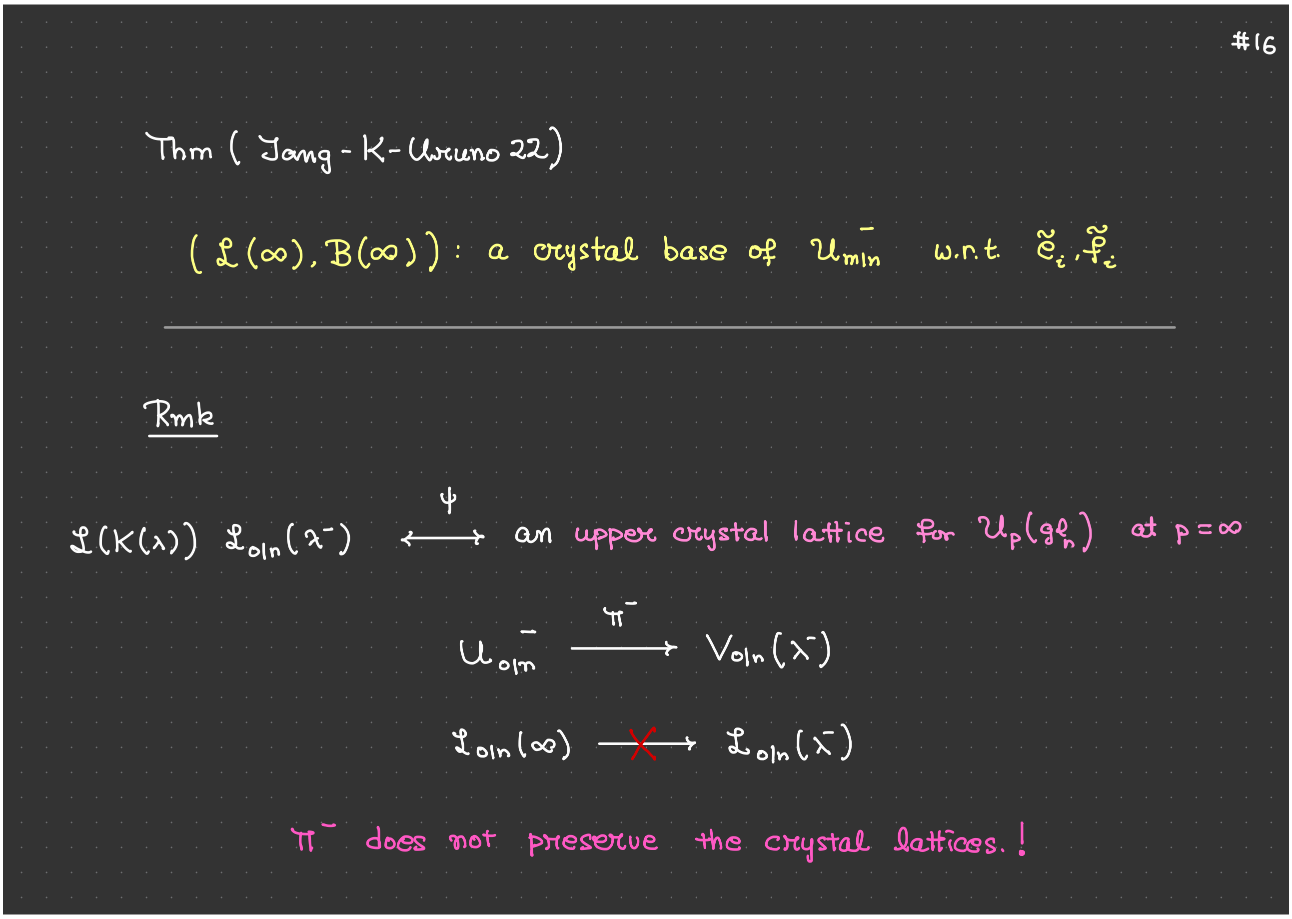
- One may take

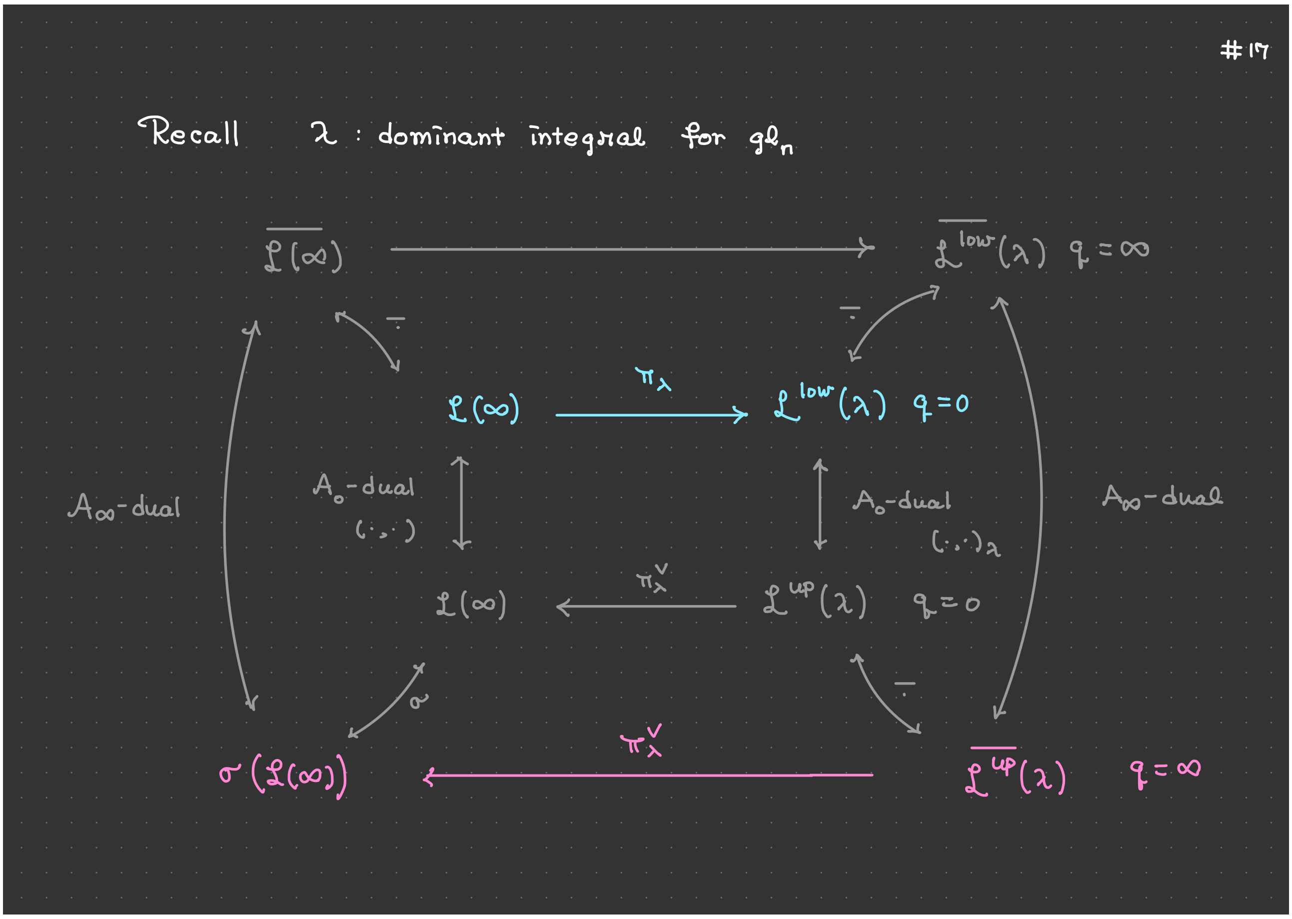
$$\mathcal{L}(\infty) = \mathcal{L}(K(0))\cdot \mathcal{L}_{m|0}(\infty)\cdot \mathcal{L}_{0|n}(\infty)$$

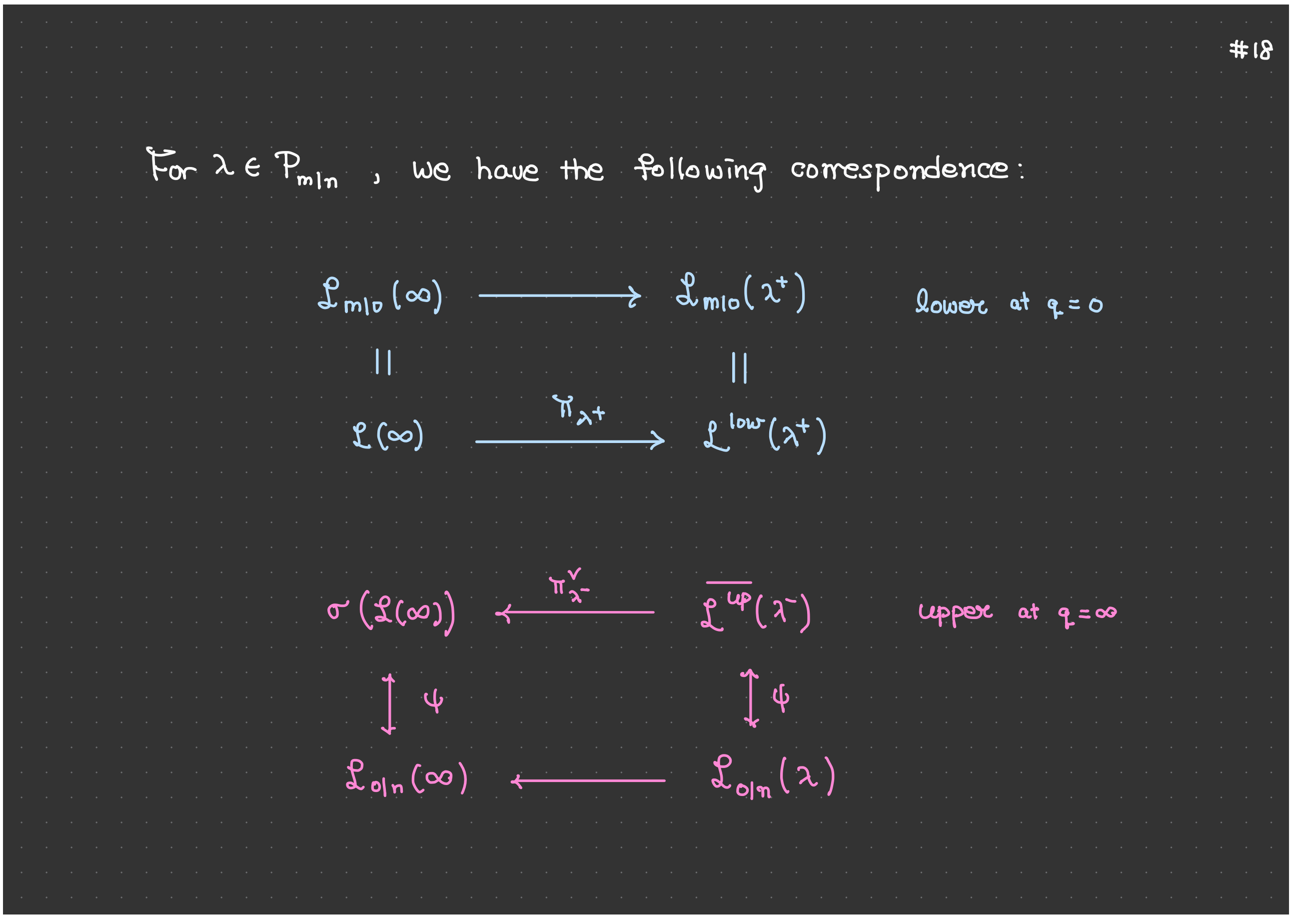
where $\mathcal{L}_{m|0}(\infty)$: the crystal lattice of $\mathcal{U}^-_{m|0} \cong \mathcal{U}_q(\mathfrak{gl}_m)$ at $q=0$

$\mathcal{L}_{0|n}(\infty)$: " " " $\mathcal{U}^-_{0|n} \overset{\psi}{\cong} \mathcal{U}_p(\mathfrak{gl}_n)$ at $p=\infty$


#15
• $\tilde{f}_i$: the associated crystal operators on $\mathcal{U}_{m|0}, \mathcal{U}_{0|n}$ $(i \neq m)$
• $\tilde{f}_m$ defined in the same way as in $K(\lambda)$
• $\mathcal{U}^-_{m|n} \xleftarrow{\cong} K \otimes \mathcal{U}^-_{m|0} \otimes \mathcal{U}^-_{0|n}$
$u_1 u_2 u_3 \longleftarrow u_1 \otimes u_2 \otimes u_3$
For $u = u_1 u_2 u_3 \in \mathcal{U}^-_{m|n}$ & i, define
$\tilde{f}_i u = \begin{cases} (\tilde{f}_i u_1 u_2) u_3 & (i<m) \\ u_1 u_2 (\tilde{f}_i u_3) & (i>m) \\ f_m u_1 u_2 u_3 & (i=m) \end{cases}$


#16
Thm (Jang - K - Kwon 22)
$(\mathcal{L}(\infty), \mathcal{B}(\infty))$: a crystal base of $\mathcal{U}^-_{m|n}$ w.r.t. $\tilde{e}_i, \tilde{f}_i$
Rmk
$\mathcal{L}(K(\lambda))\ \mathcal{L}_{0|n}(\lambda^-) \xleftrightarrow{\psi}$ an upper crystal lattice for $\mathcal{U}_p(\mathfrak{gl}_n)$ at $p=\infty$
$\mathcal{U}^-_{0|n} \xrightarrow{\pi^-} V_{0|n}(\lambda^-)$
$\mathcal{L}_{0|n}(\infty) \not\longrightarrow \mathcal{L}_{0|n}(\lambda^-)$
π^- does not preserve the crystal lattices!


#17
Recall λ : dominant integral for gl_n
\overline{L(∞)}
\overline{L^{low}(λ)} q = ∞
−
π_λ
L(∞)
L^{low}(λ) q = 0
A_∞-dual
A_0-dual
(·,·)
A_0-dual
(·,·)_λ
A_∞-dual
π_λ^∨
L(∞)
L^{up}(λ) q = 0
σ
π_λ^∨
σ(L(∞))
\overline{L^{up}(λ)} q = ∞


#18
For λ ∈ P_{m|n} , we have the following correspondence:
L_{m|0}(∞)
L_{m|0}(λ^+)
lower at q = 0
||
||
π_{λ^+}
L(∞)
L^{low}(λ^+)
π_{λ^-}^∨
σ(L(∞))
\overline{L^{up}(λ^-)}
upper at q = ∞
ψ
ψ
L_{0|n}(∞)
L_{0|n}(λ)

#19

- $X(\lambda) := \mathcal{U}_{m|n} \otimes_{Q} V_{0|n}(\lambda^-)$ where $Q = \langle \mathcal{U}^{\geq 0}_{m|n}, \mathcal{U}_{0|n} \rangle$

$$\cong K \otimes \mathcal{U}^-_{m|0} \otimes V_{0|n}(\lambda^-) \quad \text{as } \Bbbk\text{-space}$$

- We have

$$\mathcal{U}^-_{m|n} \xrightarrow{\pi_-} X(\lambda) \xrightarrow{\pi_+} K(\lambda)$$

- $X(\lambda)$ has a crystal base $\big(\mathcal{L}(X(\lambda)), \mathcal{B}(X(\lambda))\big)$ where

$$\mathcal{L}(X(\lambda)) = \mathcal{L}(K(0)) \cdot \mathcal{L}_{m|0}(\infty) \cdot \mathcal{L}_{0|n}(\lambda^-)$$

$$\mathcal{B}(X(\lambda)) = \mathcal{P}(\Phi^+_1) \times \mathcal{B}_{m|0}(\infty) \times \mathcal{B}_{0|n}(\lambda^-)$$

#20

- Consider

$$\mathcal{U}^-_{m|n} \xleftarrow[\iota_K \otimes \iota_{\mathcal{U}^-_{m|0}} \otimes \pi^\vee_{\lambda^-}]{\pi^\vee_-} X(\lambda) \xrightarrow{\pi_+} K(\lambda)$$

Then we have

$$\mathcal{U}^-_{m|n} \xleftarrow{\pi^\vee_-} X(\lambda) \xrightarrow{\pi_+} K(\lambda)$$

$$\mathcal{L}(\infty) \longleftarrow \mathcal{L}(X(\lambda)) \longrightarrow \mathcal{L}(K(\lambda))$$

$$\mathcal{B}(\infty) \longleftarrow \mathcal{B}(X(\lambda)) \longrightarrow \mathcal{B}(K(\lambda)) \cup \{0\}$$

#21

Rmk

① ∃ a categorification of $\mathcal{U}^-_{m|1}$ (Khovanov-Sussan 16)

② A (pseudo) canonical basis of $\mathcal{U}^-_{m|n}$ (Clark-Hill-Wang 16)
via quantum shuffle alg

③ A canonical basis of $\mathcal{U}_{m|n}$ (Du, Gu 15)
via quantum Schur superalg.

Questions

① a categorical realization of $B_{m|1}(\infty)$

② ∃ a canonical basis of $\mathcal{U}^-_{m|n}$? (compatible w/ irr. repn's & its crystal)

③ ∃ a categorification of $\mathcal{U}^-_{m|n}$? (of odd Serre relation)

#22

Related works

① $\mathfrak{q}(n)$: queer Lie superalg

- crystal base of an irr. polynomial repn (← Sergeev duality)
(Grantcharov-Jung-Kang-Kashiwara-Kim 15)

- (abstract) crystal $B(-\infty)$ (Salisbury-Scrimshaw 22)

② $\mathfrak{osp}(m|2n)$: orthosymplectic

- crystal base of an irr. q-oscillator repn (K 15)
(←→ integrable h.w. module of classical types)
super duality

- crystal base of a parabolic Verma / U_q^-
(in progress w/ Jang & Uruno)

Relations among the q-characters of simple modules over quantum loop algebras of several Dynkin types

Hironori Oya

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Based on a joint work with
Ryo Fujita, David Hernandez, and Se-jin Oh
arXiv:2304.02562

Integrable Systems and Quantum Groups
–In Honor of Masato Okado's 60th Birthday–

Plan

Quantum loop algebra $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$

Representation theory of the quantum loop algebra $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$

- $\mathfrak{g}$ a fin. dim. simple Lie alg. / $\mathbb{C}$,
  - $C = (c_{ij})_{i,j \in I}$ the Cartan matrix of $\mathfrak{g}$, type $\mathrm{A}_n, \mathrm{B}_n, \dots, \mathrm{G}_2$
  - $D = \mathrm{diag}(d_i)_{i \in I}$ s.t. $d_i \in \mathbb{Z}_{>0}$, $\gcd_{i \in I}(d_i) = 1$ and DC is symmetric.
- $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ the Drinfeld–Jimbo quantum loop alg. / $\mathbb{C}$. $q \in \mathbb{C}^\times, |q^{\mathbb{Z}}| = \infty$.
- $\mathscr{C}_\mathfrak{g} :=$ the category of fin. dim'l reps of $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ of type **1**
  (i.e. the eigenvalues of the actions of $\{k_i \mid i \in I\}$ are of the form q^m, $m \in \mathbb{Z}$).

$\mathscr{C}_\mathfrak{g}$ is an abelian rigid $\otimes$-category, but non-semisimple and non-braided

- Fix a map $\epsilon \colon I \to \{0, 1\}$ (parity function) satisfying

$$\epsilon_i \equiv \epsilon_j + \min\{d_i, d_j\} \mod 2 \quad \text{whenever } c_{ij} < 0.$$

- $\widehat{I} := \{(i, p) \in I \times \mathbb{Z} \mid p \equiv \epsilon_i \bmod 2\}$.
- $\mathscr{C}_\mathfrak{g} \supset \mathscr{C}_{\mathfrak{g},\mathbb{Z}}$ the abelian monodal subcategory "supported on" $\widehat{I}$.

q-character

The q-*character* gives an injective alg. hom. [FR99].

$$\chi_q \colon K(\mathscr{C}_{\mathfrak{g},\mathbb{Z}}) \hookrightarrow \mathbb{Z}[Y_{i,q^p}^{\pm 1} \mid (i,p) \in \widehat{I}] =: \mathcal{Y}_\mathfrak{g}, \ [V] \mapsto \chi_q(V).$$

The simple modules in $\mathscr{C}_{\mathfrak{g},\mathbb{Z}}$ is parametrized by *dominant monomials* in $\mathcal{Y}_\mathfrak{g}$:

ℓ-highest weight theory [CP91, CP95, CP]

$$\begin{array}{ccc} \mathrm{Irr}\, \mathscr{C}_{\mathfrak{g},\mathbb{Z}}/\sim & \overset{\text{bij.}}{\longleftrightarrow} & \mathcal{M}_\mathfrak{g} := \{\text{Monomials in } Y_{i,q^p}\text{'s}, (i,p) \in \widehat{I}\}, \\ \Uparrow & & \Uparrow \\ [L(m)] (= [L^\mathfrak{g}(m)]) & \leftrightarrow & m \end{array}$$

Here we have

$$\chi_q(L(m)) = m + \text{lower terms}.$$

e.g.

- $\mathfrak{g} = \mathfrak{sl}_4, I = \{1, 2, 3\}$

$$\chi_q(L(Y_{1,q^{-5}})) = Y_{1,q^{-5}} + Y_{2,q^{-4}} Y_{1,q^{-3}}^{-1} + Y_{3,q^{-3}} Y_{2,q^{-2}}^{-1} + Y_{3,q^{-1}}^{-1}.$$

- $\mathfrak{g} = \mathfrak{so}_5, I = \{1, 2\}$

$$\chi_q(L(Y_{1,q^{-7}})) = Y_{1,q^{-7}} + Y_{2,q^{-6}} Y_{2,q^{-4}} Y_{1,q^{-3}}^{-1} + Y_{2,q^{-6}} Y_{2,q^{-2}}^{-1} + Y_{1,q^{-5}} Y_{2,q^{-4}}^{-1} Y_{2,q^{-2}}^{-1} + Y_{1,q^{-1}}^{-1}.$$

q-character

The *q-character* gives an injective alg. hom. [FR99].

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The simple modules in $\mathscr{C}_{\mathfrak{g},\mathbb{Z}}$ is parametrized by *dominant monomials* in $\mathcal{Y}_{\mathfrak{g}}$:

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Here we have

$$\chi_q(L(m)) = m + \text{lower terms}.$$

Fundamental problem

For $m \in \mathcal{M}_{\mathfrak{g}}$, $\chi_q(L(m)) = ??$.

Our result (Recall from Fujita-san's talk)

Theorem [FHOO22]

Let $\mathfrak{g}_1, \mathfrak{g}_2$ be simple Lie algebras / $\mathbb{C}$ such that the "unfoldings" of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are the same. Then there exists an isomorphism of $\mathbb{Z}[t^{\pm 1/2}]$-algebras

$$\Psi_{\mathfrak{g}_1,\mathfrak{g}_2} \colon K_t(\mathscr{C}_{\mathfrak{g}_1,\mathbb{Z}}) \xrightarrow{\sim} K_t(\mathscr{C}_{\mathfrak{g}_2,\mathbb{Z}})$$

satisfying

$$\Psi_{\mathfrak{g}_1,\mathfrak{g}_2}\left(\{L_t^{\mathfrak{g}_1}(m) \mid m \in \mathcal{M}_{\mathfrak{g}_1}\}\right) = \{L_t^{\mathfrak{g}_2}(m) \mid m \in \mathcal{M}_{\mathfrak{g}_2}\}.$$

Notation

- $K_t(\mathscr{C}_{\mathfrak{g}_i,\mathbb{Z}})$ the quantum Grothendieck ring of $\mathscr{C}_{\mathfrak{g}_i,\mathbb{Z}}$ in the sense of Hernandez [H04].
- $L_t^{\mathfrak{g}_i}(m)$ the (q,t)-character of $L^{\mathfrak{g}_i}(m)$.

Our result (Recall from Fujita-san's talk)

Theorem [FHOO22]

Let $\mathfrak{g}_1, \mathfrak{g}_2$ be simple Lie algebras / $\mathbb{C}$ such that the "unfoldings" of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are the same. Then there exists an isomorphism of $\mathbb{Z}[t^{\pm 1/2}]$-algebras

$$\Psi_{\mathfrak{g}_1,\mathfrak{g}_2} \colon K_t(\mathscr{C}_{\mathfrak{g}_1,\mathbb{Z}}) \xrightarrow{\sim} K_t(\mathscr{C}_{\mathfrak{g}_2,\mathbb{Z}})$$

satisfying

$$\Psi_{\mathfrak{g}_1,\mathfrak{g}_2}\left(\{L_t^{\mathfrak{g}_1}(m) \mid m \in \mathcal{M}_{\mathfrak{g}_1}\}\right) = \{L_t^{\mathfrak{g}_2}(m) \mid m \in \mathcal{M}_{\mathfrak{g}_2}\}.$$

Remark

Our isomorphism $\Psi_{\mathfrak{g}_1,\mathfrak{g}_2}$ can be constructed according to the choice of *Q-data* $\mathcal{Q}^{(i)} = (\widetilde{\mathfrak{g}}_i, \sigma_i, \xi^{(i)})$ of $\mathfrak{g}_i$ $(i = 1, 2)$. Hence, precisely speaking, we should write $\Psi_{\mathfrak{g}_1,\mathfrak{g}_2}$ as

$$\Psi_{\mathfrak{g}_1,\mathfrak{g}_2}(\mathcal{Q}^{(2)}, \mathcal{Q}^{(2)}).$$

The Q-datum is a generalization of *height function* $\xi \colon I \to \mathbb{Z}$ for simply-laced case [FO21].

Our result (Recall from Fujita-san's talk)

Theorem [FHOO22]

Let $\mathfrak{g}_1, \mathfrak{g}_2$ be simple Lie algebras / $\mathbb{C}$ such that the "unfoldings" of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are the same. Then there exists an isomorphism of $\mathbb{Z}[t^{\pm 1/2}]$-algebras

$$\Psi_{\mathfrak{g}_1,\mathfrak{g}_2} \colon K_t(\mathscr{C}_{\mathfrak{g}_1,\mathbb{Z}}) \xrightarrow{\sim} K_t(\mathscr{C}_{\mathfrak{g}_2,\mathbb{Z}})$$

satisfying

$$\Psi_{\mathfrak{g}_1,\mathfrak{g}_2}\left(\{L_t^{\mathfrak{g}_1}(m) \mid m \in \mathcal{M}_{\mathfrak{g}_1}\}\right) = \{L_t^{\mathfrak{g}_2}(m) \mid m \in \mathcal{M}_{\mathfrak{g}_2}\}.$$

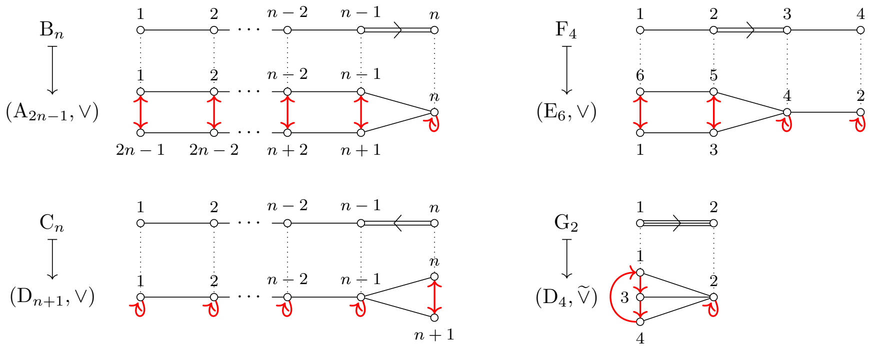
Applications

- Proof of several positivity properties for non-symmetric case.
- Proof of the Kazhdan–Lusztig type conjecture (Hernandez's conjecture) for type B_n.

Folding/Unfolding

The Folding/Unfolding correspondence is given as follows:

$\mathfrak{g}$	$\widetilde{\mathfrak{g}}$	σ
A_n	A_n	id
D_n	D_n	id
$\mathrm{E}_{6,7,8}$	$\mathrm{E}_{6,7,8}$	id
B_n	A_{2n-1}	$\vee$
C_n	D_{n+1}	$\vee$
F_4	E_6	$\vee$
G_2	D_4	$\widetilde{\vee}$



Example of the correspondence under $\Psi_{\mathfrak{g}_1,\mathfrak{g}_2}$

e.g. $\mathfrak{g}_1 = \mathfrak{sl}_4, \mathfrak{g}_2 = \mathfrak{so}_5$ (type $\mathrm{A}_3/\mathrm{B}_2$) :

$$\Psi_{\mathfrak{g}_1,\mathfrak{g}_2}(L_t^{\mathfrak{sl}_4}(Y_{1,q^0})) = L_t^{\mathfrak{so}_5}(Y_{1,q^0}), \quad \Psi_{\mathfrak{g}_1,\mathfrak{g}_2}(L_t^{\mathfrak{sl}_4}(Y_{1,q^{-2}})) = L_t^{\mathfrak{so}_5}(Y_{2,q^{-5}}),$$
$$\Psi_{\mathfrak{g}_1,\mathfrak{g}_2}(L_t^{\mathfrak{sl}_4}(Y_{1,q^{-4}})) = L_t^{\mathfrak{so}_5}(Y_{2,q^{-3}}), \quad \Psi_{\mathfrak{g}_1,\mathfrak{g}_2}(L_t^{\mathfrak{sl}_4}(Y_{2,q^{-1}})) = L_t^{\mathfrak{so}_5}(Y_{2,q^{-1}}),$$
$$\Psi_{\mathfrak{g}_1,\mathfrak{g}_2}(L_t^{\mathfrak{sl}_4}(Y_{2,q^{-3}})) = L_t^{\mathfrak{so}_5}(Y_{2,q^{-5}}Y_{2,q^{-3}}), \quad \Psi_{\mathfrak{g}_1,\mathfrak{g}_2}(L_t^{\mathfrak{sl}_4}(Y_{3,q^{-2}})) = L_t^{\mathfrak{so}_5}(Y_{1,q^{-2}}).$$

This correspondence preserves neither dimension nor degree of ℓ-highest weight.

What to do next

Suppose that
$$\Psi_{\mathfrak{g}_1,\mathfrak{g}_2}(L_t^{\mathfrak{g}_1}(m)) = L_t^{\mathfrak{g}_2}(m').$$

Actually, we can calculate m' from m explicitly (although we need the case-by-case calculation for the explicit computation).
$\leftarrow$ This can be seen as the explicit correspondence between "highest terms".

Question

Can we calculate "lower terms" of $L_t^{\mathfrak{g}_2}(m')$ from those of $L_t^{\mathfrak{g}_1}(m)$?

We will give an answer to this question by looking at

the (quantum) cluster algebra structure

on the quantum Grothendieck rings!

Plan

1. Introduction
2. Brief review of the monoidal categorification of cluster algebras
3. Main result: Substitution formulas

Cluster algebra

Cluster algebra $\mathcal{A}(\Gamma, J)$ is defined associated with a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ without loops and 2-cycles, and a subset $J \subset \Gamma_0$ of its vertex set Γ_0 [FZ02][1].
The input datum (Γ, J) has an information of the "*seed*" of the cluster algebra $\mathcal{A}(\Gamma, J)$.

Let $\mathcal{F} := \mathbb{Q}(z_j \mid j \in \Gamma_0)$. A pair $(\Upsilon, \mathcal{X} = (x_j)_{j\in\Gamma_0})$ is called a **seed** in $\mathcal{F}$ if

1. $\Upsilon = (\Upsilon_0, \Upsilon_1)$ is a quiver without loops and 2-cycles such that $\Upsilon_0 = \Gamma_0$.
2. $\mathcal{X} = (x_j)_{j\in\Gamma_0} \subset \mathcal{F}$ is a Γ_0-tuple of elements of $\mathcal{F}$ which are algebraically independent and $\mathcal{F} = \mathbb{Q}(x_j \mid j \in \Gamma_0)$.

Next, we explain the "*mutation*" of seeds, which is a procedure of producing generators of $\mathcal{A}(\Gamma, J)$.

[1]Here we explain the cluster algebras of *skew-symmetric* type

Mutation of seeds

Let $(\Upsilon, \mathcal{X})$ be a seed in $\mathcal{F}$ and $k \in \Gamma_0$.
The **mutation**

$$\mu_k(\Upsilon, \mathcal{X}) = (\Upsilon', \mathcal{X}' = (x'_j)_{j\in\Gamma_0})$$

of the seed $(\Upsilon, \mathcal{X})$ in direction k is defined as follows:

Definition of Υ'_1:

(i) Add one arrow $j \to \ell$ for each subquiver of the form $j \to k \to \ell$ in Υ_1.
(ii) Reverse the arrows in Υ_1 which are connected with the vertex k.
(iii) Remove all 2-cycles generated as a result of (i) and (ii).

Definition of x'_j:

$$x'_j = \begin{cases} \dfrac{\prod_{\alpha\in\Upsilon_1; s(\alpha)=k} x_{t(\alpha)} + \prod_{\alpha\in\Upsilon_1; t(\alpha)=k} x_{s(\alpha)}}{x_k} & \text{if } j = k, \\ x_j & \text{if } j \neq k. \end{cases}$$

Cluster algebra

The **cluster algebra** $\mathcal{A}(\Gamma, J)$ is a $\mathbb{Z}$-subalgebra of $\mathcal{F}$ generated by the set $\widetilde{\mathcal{X}}$ of **cluster variables** defined as follows:
Denote by

$$(\Gamma, \mathcal{Z}) \overset{\mathrm{mut}}{\sim} (\Upsilon, \mathcal{X})$$

when $(\Upsilon, \mathcal{X})$ is obtained from the initial seed $(\Gamma, \mathcal{Z} = (z_j)_{j \in \Gamma_0})$ of $\mathcal{F}$ by a finite number of mutations in direction indexed by $J (\subset \Gamma_0)$. Then

$$\widetilde{\mathcal{X}} := \bigcup_{(\Upsilon, \mathcal{X}) \overset{\mathrm{mut}}{\sim} (\Gamma, \mathcal{Z})} \mathcal{X}.$$

Remark

$\mu_k(\mu_k(\Upsilon, \mathcal{X})) = (\Upsilon, \mathcal{X})$.

e.g.

$$\mathcal{A}(1 \to 2, \{1\}) = \langle z_1, z_2, \frac{1 + z_2}{z_1} \rangle_{\mathbb{Z}\text{-alg.}} \left(\simeq \mathbb{Z}\left[z_1, \frac{1 + z_2}{z_1} \right] \right)$$

Cluster algebra

The **cluster algebra** $\mathcal{A}(\Gamma, J)$ is a $\mathbb{Z}$-subalgebra of $\mathcal{F}$ generated by the set $\widetilde{\mathcal{X}}$ of **cluster variables** defined as follows:
Denote by

$$(\Gamma, \mathcal{Z}) \overset{\mathrm{mut}}{\sim} (\Upsilon, \mathcal{X})$$

when $(\Upsilon, \mathcal{X})$ is obtained from the initial seed $(\Gamma, \mathcal{Z} = (z_j)_{j \in \Gamma_0})$ of $\mathcal{F}$ by a finite number of mutations in direction indexed by $J (\subset \Gamma_0)$. Then

$$\widetilde{\mathcal{X}} := \bigcup_{(\Upsilon, \mathcal{X}) \overset{\mathrm{mut}}{\sim} (\Gamma, \mathcal{Z})} \mathcal{X}.$$

e.g.

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Remark

$\widetilde{X}$ is an infinite set in general, and $\mathcal{A}(\Gamma, J)$ may not be finitely generated.

Comments on quantum cluster algebras

By definition, cluster algebras are commutative algebras. A *quantum cluster algebra* is a non-commutative deformation of cluster algebras [BZ05].
For the definition of a quantum cluster algebra, we need an additional data

$$\Lambda = (\lambda_{i,j})_{i,j\in\Gamma_0} \in \mathbb{Z}^{\Gamma_0\times\Gamma_0},$$

satisfying certain conditions, which encodes the non-commutativity of variables:

$$Z_iZ_j = t^{\lambda_{ij}}Z_jZ_i. \quad (\lambda_{ij} = -\lambda_{ji}, t\colon \text{ indeterminate})$$

$\rightsquigarrow$

- $\mathcal{T}(\Lambda)$ a quantum torus over $\mathbb{Z}[t^{\pm 1/2}]$ in the variables $\mathcal{Z}_t = (Z_j)_{j\in\Gamma_0}$
- $\mathbb{F}(\mathcal{T}(\Lambda))$ the skew field of fractions of $\mathcal{T}(\Lambda)$

The quantum cluster algebra $\mathcal{A}_t(\Gamma, J, \Lambda)$ is defined as a $\mathbb{Z}[t^{\pm 1/2}]$-subalgebra of $\mathbb{F}(\mathcal{T}(\Lambda))$ generated by the quantum cluster variables.

$$\mathcal{A}_t(\Gamma, J, \Lambda) \subset \mathcal{T}(\Lambda) \quad \text{(Laurent phenomenon)}$$

Monoidal categorification

Monoidal categorification of cluster algebras [HL10]

An abelian monoidal category $\mathscr{C}$ is said to be a **monoidal categorification** of $\mathcal{A}(\Gamma, J)$ if it satisfies the following:

$$\exists\iota\colon \mathcal{A}(\Gamma, J) \xrightarrow[\mathbb{Z}\text{-alg.}]{\sim} K(\mathscr{C}) \text{ satisfying } \iota(\widetilde{\mathcal{M}}) \subset \{[\text{simple objects}]\}.$$

Here

$$\widetilde{\mathcal{M}} := \bigcup_{(\Upsilon,\mathcal{X})\overset{\text{mut}}{\sim}(\Gamma,\mathcal{Z})} \{\text{Monomials in } \mathcal{X}\} \quad (\textbf{cluster monomials}).$$

Many subcategories of $\mathscr{C}_{\mathfrak{g}}$ are known to give examples:

- $\mathscr{C}_\ell$, $\ell \in \mathbb{Z}_{>0}$ [Hernandez–Leclerc '10 –, Nakajima '11, Qin '17,...]
- $\mathscr{C}_{\mathcal{Q}}$ [Hernandez–Leclerc '15 + Kang–Kashiwara–Kim–Oh '18,...]
- $\mathscr{C}_{\leq\xi}$ [Kashiwara–Kim–Oh–Park '21]

$\rightsquigarrow$ It produces an algorithm for calculating q-characters of simple modules which correspond to cluster monomials ("*reachable* simple modules").

Example of monoidal categorification

$\mathfrak{g} = \mathfrak{sl}_2$

In $\mathscr{C}_{\mathfrak{sl}_2,\mathbb{Z}}$, there exists a (non-split) short exact sequence

$$0 \to L(Y_{q^{-1}}Y_q) \to L(Y_{q^{-1}}) \otimes L(Y_q) \to L(1) \to 0.$$

In particular, we have

$$\chi_q(L(Y_{q^{-1}}))\chi_q(L(Y_q)) = \chi_q(L(Y_{q^{-1}}Y_q)) + 1.$$

On the other hand, in $\mathcal{A}(1 \to 2, \{1\})$, we have

$$z_1' z_1 = z_2 + 1$$

Here we write $\mu_1(1 \to 2, (z_1, z_2)) = (1 \leftarrow 2, (z_1', z_2))$.

Example of monoidal categorification

If we write the Serre subcategory of $\mathscr{C}_{\mathfrak{sl}_2,\mathbb{Z}}$ generated by $L(Y_q)$ and $L(Y_{q^{-1}})$ ("supported" on $\{(1,1),(1,-1)\} \subset I \times \mathbb{Z}$) as $\mathscr{C}_1$, then

$$\begin{array}{ccc} \exists\iota\colon \mathcal{A}(1 \to 2, \{1\}) & \xrightarrow[\mathbb{Z}\text{-alg.}]{\sim} & K(\mathscr{C}_1) \\ \Cup & & \Cup \\ z_1 & \mapsto & [L(Y_q)] \\ z_2 & \mapsto & [L(Y_{q^{-1}}Y_q)] \\ z_1' & \mapsto & [L(Y_{q^{-1}})] \end{array}$$

Moreover,

$$L(Y_q)^{\otimes a} \otimes L(Y_{q^{-1}}Y_q)^{\otimes b}, \quad L(Y_{q^{-1}})^{\otimes a} \otimes L(Y_{q^{-1}}Y_q)^{\otimes b} \qquad (a, b \in \mathbb{Z}_{\geq 0})$$

are simple modules.

The relation

$$\chi_q(L(Y_{q^{-1}}))\chi_q(L(Y_q)) = \chi_q(L(Y_{q^{-1}}Y_q)) + 1.$$

is a special case of T*-system*, for *Kirillov–Reshetikhin modules*.

T-system

For each $(i,p) \in \widehat{I}$ and $k \in \mathbb{Z}_{\geq 0}$, set

$$W_{k,p}^{(i)} := L(m_{k,p}^{(i)}), \qquad m_{k,p}^{(i)} := \prod_{s=1}^{k} Y_{i,q^{p+2(s-1)d_i}}.$$

These simple modules are called *Kirillov–Reshetikhin modules* (or *KR modules*).

T-system [N03, H06]

For $(i,p) \in \widehat{I}$, we have the following equality in $K(\mathscr{C}_{\mathfrak{g},\mathbb{Z}})$:

$$\left[W_{k,p}^{(i)}\right]\left[W_{k,p+2d_i}^{(i)}\right] = \left[W_{k+1,p}^{(i)}\right]\left[W_{k-1,p+2d_i}^{(i)}\right] + \left[S_{k,a}^{(i)}\right],$$

where $S_{k,a}^{(i)}$ is also an explicit simple tensor products of Kirillov–Reshetikhin modules.

Remark

There exists a quantum analog of T-system (=T-system for (q,t)-characters of KR-modules) [HL15, FHOO22, FHOO23+].

The subcategory $\mathscr{C}_{\leq\xi}$

Fix a Q-data $\mathcal{Q} = (\widetilde{\mathfrak{g}}, \sigma, \xi)$ of $\mathfrak{g}$.
The essential datum of $\mathcal{Q}$ is the *height function* $\xi \colon \widetilde{I} \to \mathbb{Z}$ ($\widetilde{I}$=index set for $\widetilde{\mathfrak{g}}$).

Moreover, we have $I \overset{\text{identify}}{=\!=} \widetilde{I}/\langle\sigma\rangle$ and $\pi \colon \widetilde{I} \to I$ can. proj.
Set

$$\widehat{I}_{\leq\xi} := \{(\pi(\imath), p) \in \widehat{I} \mid \xi_\imath - p \in 2d_{\pi(\imath)}\mathbb{Z}_{\geq 0}\}.$$

Let $\mathscr{C}_{\leq\xi}$ be a monoidal abelian subcategory of $\mathscr{C}_{\mathfrak{g},\mathbb{Z}}$ "supported" on $\widehat{I}_{\leq\xi}$.
Associated to $\mathscr{C}_{\leq\xi}$, Hernandez–Leclerc [HL16] found the quiver $\Gamma_{\leq\xi}$ which "encodes" the T-system for KR-modules in $\mathscr{C}_{\leq\xi}$, and proved that

There exists a $\mathbb{Z}$-algebra isomorphism

$$\mathcal{A}(\Gamma_{\leq\xi}, (\Gamma_{\leq\xi})_0) \xrightarrow{\sim} K(\mathscr{C}_{\leq\xi})$$

which sends the initial cluster variables to certain KR-modules.

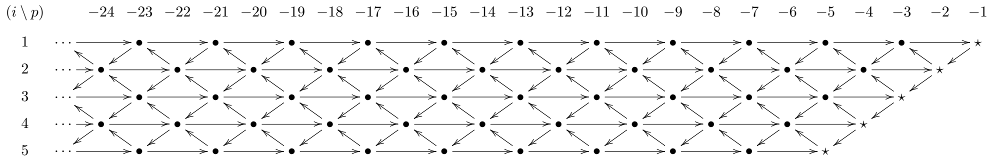
Moreover,

Theorem [KKOP21+]

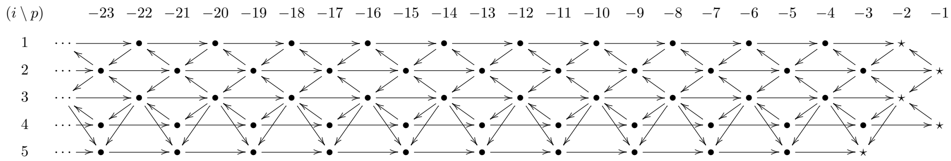
$\mathscr{C}_{\leq\xi}$ is a monoidal categorification of $\mathcal{A}(\Gamma_{\leq\xi}, (\Gamma_{\leq\xi})_0)$.

Examples of $\Gamma_{\leq \xi}$

- Type A_5:

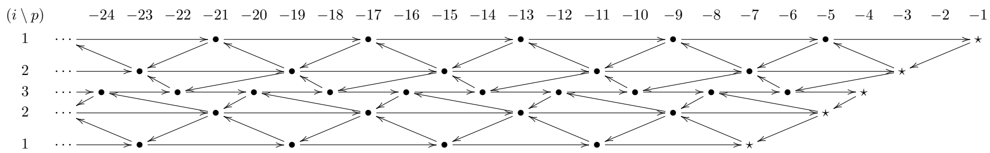


- Type D_5:

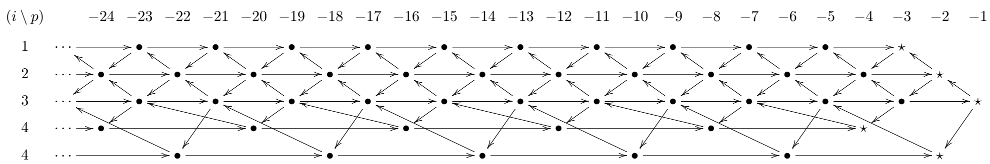


Examples of $\Gamma_{\leq \xi}$

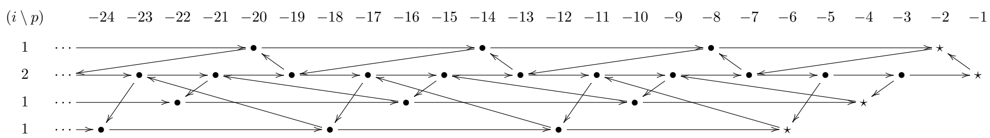
- Type B_3:



- Type C_4:



- Type G_2:



Plan

1. Introduction
2. Brief review of the monoidal categorification of cluster algebras
3. Main result: Substitution formulas

Application of cluster structures

Easy observation

If two seeds $(\Upsilon, \mathcal{X}), (\Upsilon', \mathcal{X}')$ in $\mathcal{F}$ satisfies

$$(\Upsilon, \mathcal{X}) \overset{\mathrm{mut}}{\sim} (\Upsilon', \mathcal{X}'),$$

then there exists a $\mathbb{Z}$-algebra isomorphism

$$\begin{array}{ccc} \mathcal{A}(\Upsilon, \mathcal{X}) & \simeq & \mathcal{A}(\Upsilon', \mathcal{X}') \\ \cup & & \cup \\ \widetilde{\mathcal{M}} & \leftrightarrow & \widetilde{\mathcal{M}}'. \end{array}$$

A parallel statement holds for quantum cluster algebras.

$\rightsquigarrow$ This type of isomorphisms produces a non-trivial isomorphism in our situation!

Main result

Theorem [FHOO23+]

Let

- $\mathfrak{g}_i$ be a simple Lie algebra / $\mathbb{C}$
- $\mathcal{Q}^{(i)} = (\widetilde{\mathfrak{g}}_i, \sigma_i, \xi^{(i)})$ be a Q-datum of $\mathfrak{g}_i$

for $i = 1, 2$. Assume that $\widetilde{\mathfrak{g}}_1 = \widetilde{\mathfrak{g}}_2$. Then

(1) $K_t(\mathscr{C}_{\le\xi^{(i)}}) \simeq \mathcal{A}_t(\Gamma_{\le\xi^{(i)}}, (\Gamma_{\le\xi^{(i)}})_0, \exists\Lambda_{\le\xi^{(i)}})$ which specializes to HL's isom. at $t = 1$,

(2)
$$(\Gamma_{\le\xi^{(1)}}, \Lambda_{\le\xi^{(1)}}) \overset{\text{mut}}{\sim} (\Gamma_{\le\xi^{(2)}}, \Lambda_{\le\xi^{(2)}})$$

(3) The following isomorphism induced from (1) & (2)
$$\begin{aligned} K(\mathscr{C}_{\le\xi^{(1)}}) &\simeq \mathcal{A}_t(\Gamma_{\le\xi^{(1)}}, (\Gamma_{\le\xi^{(1)}})_0, \Lambda_{\le\xi^{(1)}}) \\ &\simeq \mathcal{A}_t(\Gamma_{\le\xi^{(2)}}, (\Gamma_{\le\xi^{(2)}})_0, \Lambda_{\le\xi^{(2)}}) \simeq K(\mathscr{C}_{\le\xi^{(2)}}) \end{aligned}$$
coincides with the $\Psi_{\mathfrak{g}_1,\mathfrak{g}_2}|_{K(\mathscr{C}_{\le\xi^{(1)}})}$.

Substitution formulas

Remark

The mutation sequence required for
$$(\Gamma_{\le\xi^{(1)}}, \Lambda_{\le\xi^{(1)}}) \overset{\text{mut}}{\sim} (\Gamma_{\le\xi^{(2)}}, \Lambda_{\le\xi^{(2)}})$$
is of infinite length. However, it is well-defined since it is "locally finite".

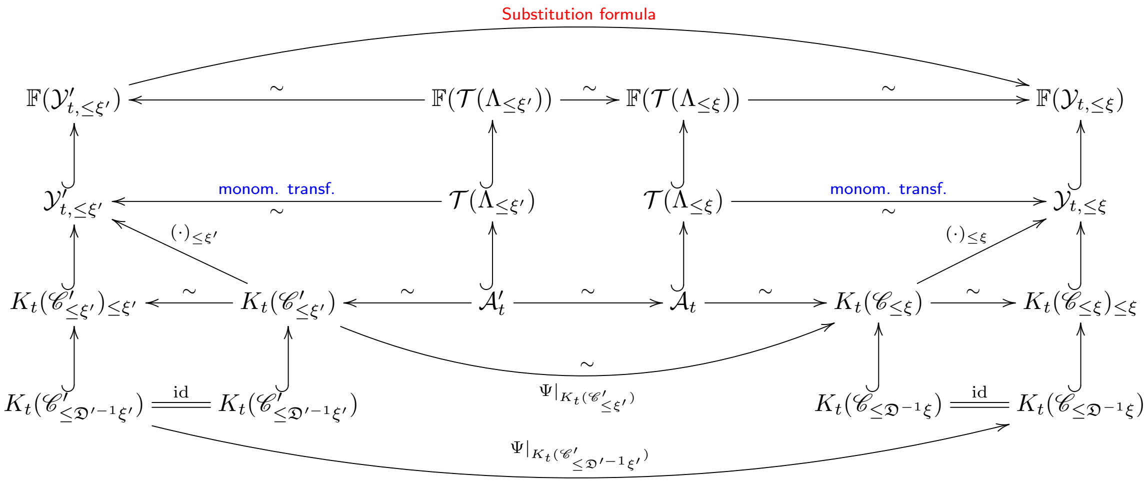
Moreover, by investigating the mutation sequence above, we can obtain the following;

Theorem (Substitution formulas [FHOO23+])

With the assumption above, $\exists$ an explicit birational transformation between the variables Y_{i,q^p}, which makes the (q,t)-characters of simple modules in $\mathscr{C}_{\mathfrak{g}_1,\mathbb{Z}}$ into those in $\mathscr{C}_{\mathfrak{g}_2,\mathbb{Z}}$.

Sketch of the proof

The essential part of the construction of substitution formulas can be explained by the following commutative diagram:



Example of Substitution formulas

Substitution formula from $\mathrm{B}_2^{(1)}$ to $\mathrm{A}_3^{(1)}$ $(t=1)$:

$$Y_{i,p} \mapsto \begin{cases} Y_{1,-3-8m}Y_{1,-1-8m} & \text{if } (i,p)=(1,-3-12m),\\ Y_{1,-5-8m} & \text{if } (i,p)=(1,-7-12m),\\ Y_{1,-7-8m} & \text{if } (i,p)=(1,-11-12m),\\ Y_{2,-8m} & \text{if } (i,p)=(2,-12m),\\ Y_{2,-2-8m}Y_{1,-1-8m}^{-1}+Y_{1,-3-8m} & \text{if } (i,p)=(2,-2-12m),\\ \dfrac{1}{Y_{1,-1-8m}^{-1}+Y_{2,-2-8m}^{-1}Y_{1,-3-8m}} & \text{if } (i,p)=(2,-4-12m),\\ Y_{2,-4-8m} & \text{if } (i,p)=(2,-6-12m),\\ Y_{3,-7-8m}+Y_{2,-6-8m}Y_{3,-5-8m}^{-1} & \text{if } (i,p)=(2,-8-12m),\\ \dfrac{1}{Y_{2,-6-8m}^{-1}Y_{3,-7-8m}+Y_{3,-5-8m}^{-1}} & \text{if } (i,p)=(2,-10-12m),\\ Y_{3,-1-8m} & \text{if } (i,p)=(1,-1-12m),\\ Y_{3,-3-8m} & \text{if } (i,p)=(1,-5-12m),\\ Y_{3,-7-8m}Y_{3,-5-8m} & \text{if } (i,p)=(1,-9-12m). \end{cases}$$

Here $Y_{i,p} := Y_{i,q^p}$.

Example of Substitution formulas

Applying the formula above to

$$\chi_q(L^{\mathfrak{so}_5}(Y_{1,-7})) = Y_{1,-7} + Y_{2,-6}Y_{2,-4}Y_{1,-3}^{-1} + Y_{2,-6}Y_{2,-2}^{-1} + Y_{1,-5}Y_{2,-4}^{-1}Y_{2,-2}^{-1} + Y_{1,-1}^{-1},$$

we obtain

$$\begin{aligned}
&Y_{1,-5} + \frac{Y_{2,-4}}{(Y_{1,-1}^{-1} + Y_{2,-2}^{-1}Y_{1,-3})Y_{1,-3}Y_{1,-1}} + \frac{Y_{2,-4}}{Y_{2,-2}Y_{1,-1}^{-1} + Y_{1,-3}} \\
&+ \frac{Y_{3,-3}(Y_{1,-1}^{-1} + Y_{2,-2}^{-1}Y_{1,-3})}{Y_{2,-2}Y_{1,-1}^{-1} + Y_{1,-3}} + Y_{3,-1}^{-1} \\
&= Y_{1,-5} + Y_{2,-4}\frac{Y_{1,-3}^{-1}Y_{1,-1}^{-1} + Y_{2,-2}^{-1}}{Y_{1,-1}^{-1} + Y_{2,-2}^{-1}Y_{1,-3}} + Y_{3,-3}Y_{2,-2}^{-1} + Y_{3,-1}^{-1} \\
&= Y_{1,-5} + Y_{2,-4}Y_{1,-3}^{-1} + Y_{3,-3}Y_{2,-2}^{-1} + Y_{3,-1}^{-1} = \chi_q(L^{\mathfrak{sl}_4}(Y_{1,-5})).
\end{aligned}$$

Further direction

- Relation with integrable systems?
  - q-characters $\approx$ transfer matrices (Frenkel–Reshetikhin)
  - We have a family of simple modules in $\mathscr{C}_{\mathfrak{g}_1}$ whose q-characters give a solution of T-system of type $\mathfrak{g}_2$.
    ($\sim$ Fermionic type formula?)
  - Extend this story to the category $\mathcal{O}$ for quantum affine Borel algebra?
- Categorical/conceptual understanding of substitution formulas?

Thank you for your attention & Happy Birthday, Okado-sensei!

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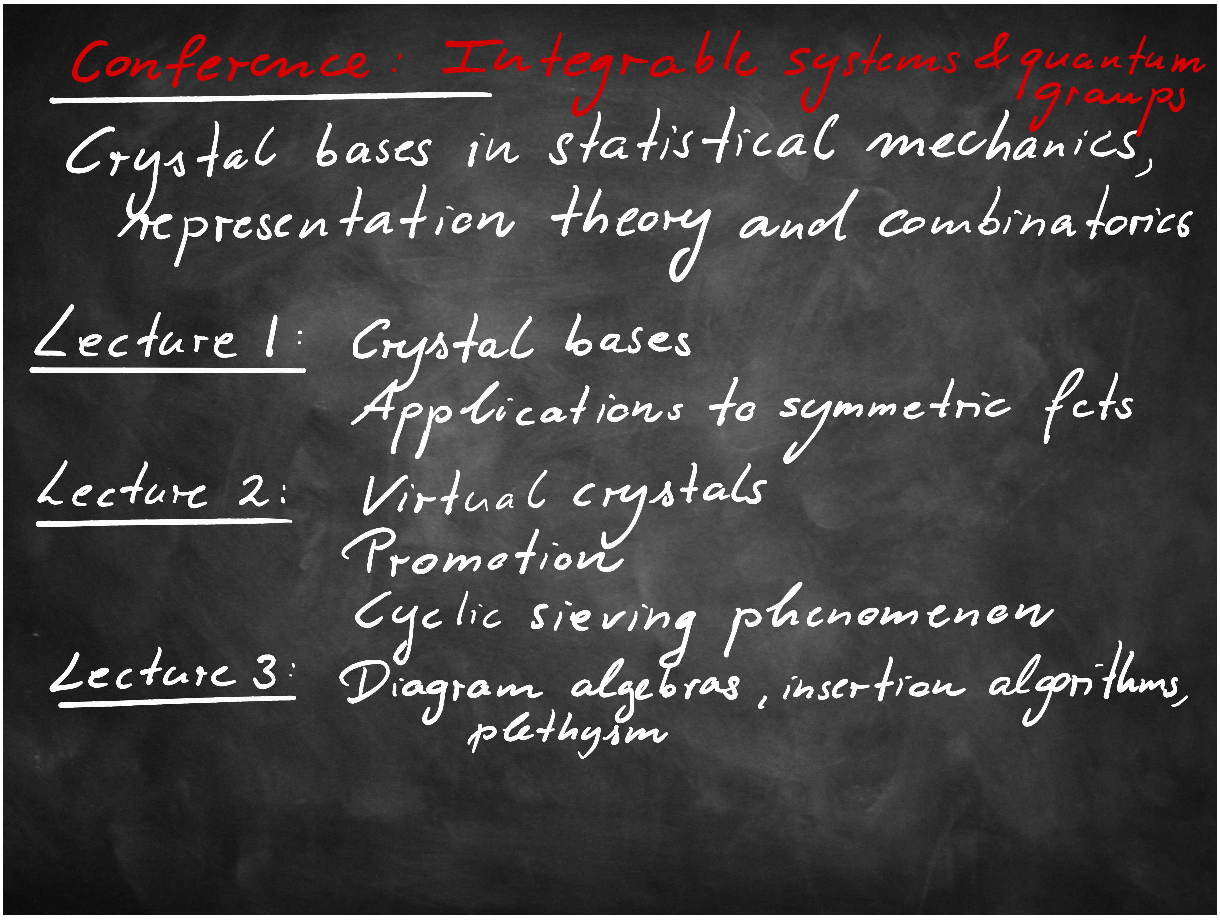
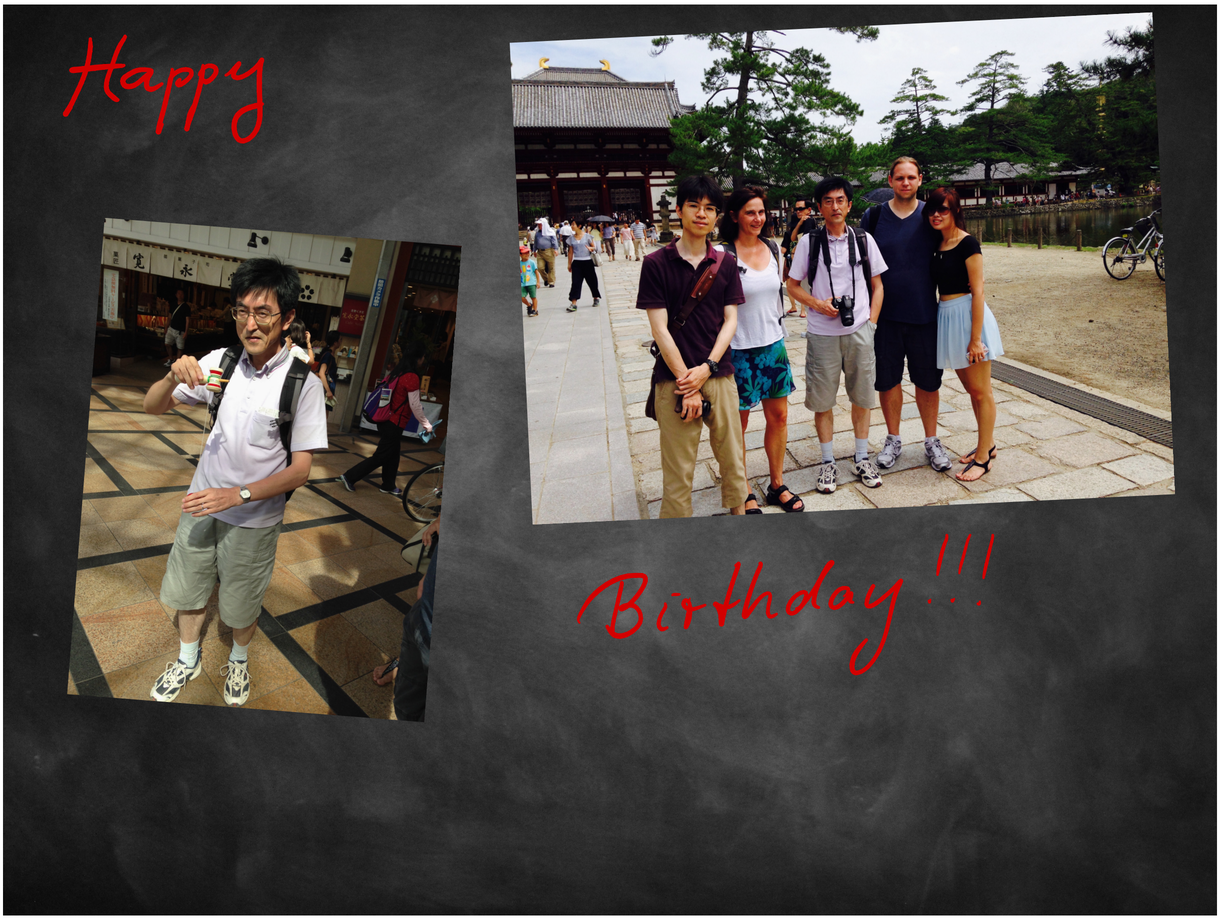
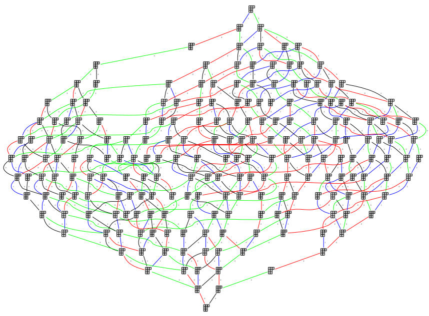

Conference: Integrable systems & quantum groups
Crystal bases in statistical mechanics, representation theory and combinatorics
Lecture 1: Crystal bases
Applications to symmetric fcts
Lecture 2: Virtual crystals
Promotion
Cyclic sieving phenomenon
Lecture 3: Diagram algebras, insertion algorithms, plethysm


Happy
Birthday!!!

Lecture 1: Crystals for stable Grothendieck polynomials

Anne Schilling, University of California at Davis



Osaka, March 5, 2023

Motivation Crystal for Stanley symmetric functions Crystal for Grothendieck polynomials Properties and results

This is based joint work with
Jennifer Morse (2016) & Jennifer Morse, Jianping Pan, Wencin Poh (2020)

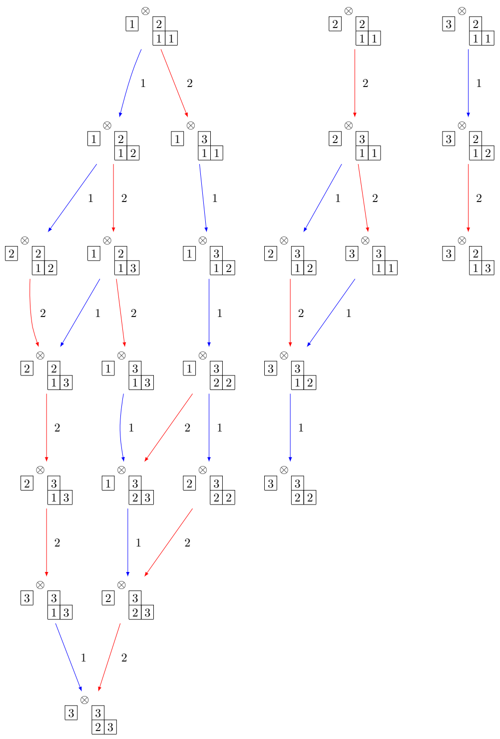
  

Outline

1. Motivation
2. Crystal for Stanley symmetric functions
3. Crystal for Grothendieck polynomials
4. Properties and results

Motivation Crystal for Stanley symmetric functions Crystal for Grothendieck polynomials Properties and results

Crystal graphs



- The generating function

$$\sum_{\text{vertex } b} \mathbf{x}^{\text{weight}(b)}$$

is the character of the crystal.

- The character of each connected component is a Schur function

$$s_\lambda(\mathbf{x}) = \sum_{T \in \mathsf{SSYT}(\lambda)} \mathbf{x}^{\text{weight}(T)}$$

where λ is the weight of the highest element.

Motivation | Crystal for Stanley symmetric functions | Crystal for Grothendieck polynomials | Properties and results

Crystal operators

Action of crystal operators e_i, f_i on words/tableaux:

1. Consider letters i and $i+1$ in row reading word of the tableau
2. Successively "bracket" pairs of the form $(i+1, i)$
3. Left with word of the form $i^r(i+1)^s$

$$e_i(i^r(i+1)^s) = \begin{cases} i^{r+1}(i+1)^{s-1} & \text{if } s > 0 \\ 0 & \text{else} \end{cases}$$

$$f_i(i^r(i+1)^s) = \begin{cases} i^{r-1}(i+1)^{s+1} & \text{if } r > 0 \\ 0 & \text{else} \end{cases}$$

Motivation | Crystal for Stanley symmetric functions | Crystal for Grothendieck polynomials | Properties and results

Outline

Stable Schubert polynomials F_w

- restriction: $\mathfrak{S}_{1_m \times w} \longrightarrow$ Stanley symmetric functions F_w for $w \in S_n$
- for 321-avoiding w,

$$F_w = s_{\nu/\mu} = \sum_\lambda c^\nu_{\lambda\mu} \, s_\lambda$$

- symmetric and Schur positive (Stanley 1984, Edelman, Greene 1987)

$$F_w = \sum_\lambda a_{w\lambda} \, s_\lambda$$

- coefficient of $x_1 x_2 \cdots x_r$ counts reduced words of w

$$S_n = \langle s_1, \dots, s_{n-1} \rangle \qquad s_i s_j = s_j s_i \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \qquad s_i^2 = id$$

$$(3,2,1,4) = s_1 s_2 s_1 = s_2 s_1 s_2 = s_3 s_3 s_1 s_2 s_1$$

Stable Schubert polynomials

$$F_w = \sum_{v^r \cdots v^1 = w} x_1^{\ell(v^1)} \cdots x_r^{\ell(v^r)}$$

Decreasing factorization of w

1. w is the product of permutations $v^r \cdots v^1$
2. each v^i has a strictly decreasing reduced word
3. $\ell(w) = \ell(v^r) + \cdots + \ell(v^1)$

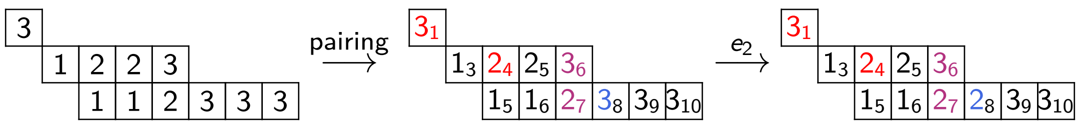
$w = (2,1,4,3) = s_1 s_3 = s_3 s_1$:

$$(s_1)(s_3) \longrightarrow x_1 x_2$$
$$(s_3)(s_1) \longrightarrow x_1 x_2$$
$$()(s_3 s_1) \longrightarrow x_1^2$$
$$(s_3 s_1)() \longrightarrow x_2^2$$

$$F_{(2,1,4,3)} = 2\, x_1 x_2 + x_1^2 + x_2^2$$

Crystal operators on factorizations – residue map

Label cells diagonally



$$\underbrace{(10\ 9\ 8\ 6\ 1)}_{\text{label of 3's}}\ \underbrace{(7\ 5\ 4)}_{\text{label of 2's}} \xrightarrow{\text{pairing}} \underbrace{(10\ 9\ 8\ 6\ 1)}_{\text{label of 3's}}\ \underbrace{(7\ 5\ 4)}_{\text{label of 2's}}$$

$$\xrightarrow{e_2} \underbrace{(10\ 9\ 6\ 1)}_{\text{label of 3's}}\ \underbrace{(8\ 7\ 5\ 4)}_{\text{label of 2's}}$$

operator e_i

from big to small:
pair $x \in$ 3's with smallest $y \in$ 2's that is bigger than x
delete smallest unpaired $z \in$ 3's and add $z - t$ to 2's

$(9\,8\,6\,5\,4\,1)(9\,6\,5\,2\,1) \to (9\,8\,5\,4\,1)(9\,6\,5\,4\,2\,1)$

Crystal Theorem

Definition

Fix $w \in S_n$.

Graph $B(w)$

1. vertices are decreasing factorizations of w
2. edges are imposed and colored by f_i, e_i
3. highest weights are vertices with no unpaired entries

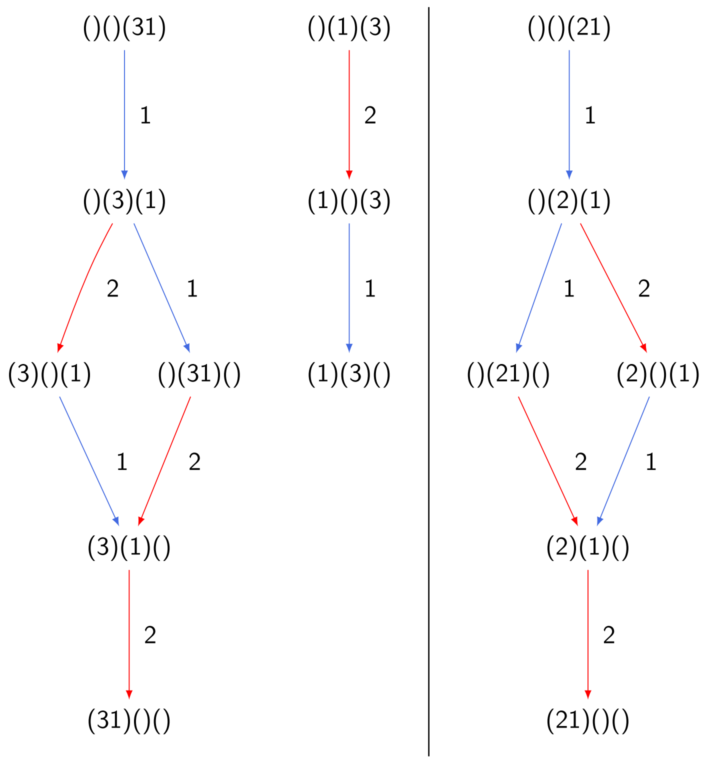
Theorem (with Morse; 2016)

$B(w)$ is a crystal graph of type A_ℓ

Proof

Checking Stembridge local axioms

Examples



Motivation | Crystal for Stanley symmetric functions | Crystal for Grothendieck polynomials | Properties and results

Schur expansion

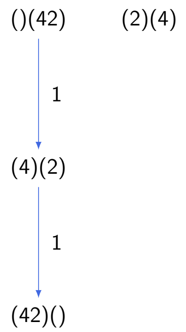
Fix $w \in S_n$

Theorem (with Morse; 2016)

$$F_w = \sum_\lambda a_{w\lambda}\, s_\lambda$$

$a_{w\lambda}$ counts highest weights $v^r \cdots v^1$ of $B(w)$ with $(\ell(v^1), \ldots, \ell(v^r)) = \lambda$

In S_5:



$\Longrightarrow F_{s_2 s_4} = s_2 + s_{11}$

Motivation | Crystal for Stanley symmetric functions | Crystal for Grothendieck polynomials | Properties and results

Edelman-Greene insertion

Theorem (with Morse; 2016)

For any permutation $w \in S_n$, the crystal isomorphism

$$B(w) \cong \bigoplus_\lambda B(\lambda)^{\oplus a_{w\lambda}}$$

is explicitly given by the Edelman-Greene insertion $\varphi^Q_{\mathrm{EG}}(v^\ell \cdots v^1) = Q$:

$$\varphi^Q_{\mathrm{EG}} \circ e_i = e_i \circ \varphi^Q_{\mathrm{EG}}$$

Motivation Crystal for Stanley symmetric functions Crystal for Grothendieck polynomials Properties and results

Outline

Motivation: Schubert Calculus

Polynomial Representatives for Schubert Cells

	Grassmannian $\mathbb{G}_{m,n}$	**Flag Varieties** Fl_n
Cohomology	s_λ	$\mathfrak{S}_w \to F_w$
K**-theory**	$\mathfrak{G}_\lambda$	$\mathfrak{G}_w$

Grassmannian Grothendieck polynomials: $\mathfrak{G}_\lambda$ Lascoux, Schützenberger 1982
Stable Grothendieck polynomials: $\mathfrak{G}_w$ Fomin, Kirillov 1994

Combinatorial Approach?

Combining:

- Crystal structure on decreasing factorizations for F_w (Morse, S. 2016)
- Crystal structure for $\mathfrak{G}_\lambda$ on set-valued tableaux (Monical & Pechenik & Scrimshaw 2018)

0-Hecke Monoid

Definition

0-Hecke monoid $\mathcal{H}_0(n)$:
monoid of all finite words in $[n] := \{1, 2, \ldots, n\}$ such that

$$pp \equiv p, \quad pqp \equiv qpq \quad \text{for all } p, q \in [n]$$
$$pq \equiv qp \quad \text{if } |p - q| > 1$$

Examples

- $2112 \equiv 212 \equiv 121$
- $2121 \equiv 1211 \equiv 121 \equiv 212$
- $31312 \equiv 3132 \equiv 312 \equiv 132$

Decreasing factorizations in $\mathcal{H}_0(n)$

Definition

A decreasing factorization of $w \in \mathcal{H}_0(n)$ into m factors is a product of decreasing factors

$$\mathbf{h} = h^m \dots h^2 h^1$$

such that $\mathbf{h} \equiv w$ in $\mathcal{H}_0(n)$.

$\mathcal{H}_w^m$ = set of decreasing factorizations of w in $\mathcal{H}_0(n)$ with m factors

Example

Decreasing factorizations for $132 \in \mathcal{H}_0(3)$ of length 5 with 3 factors:

(31)(31)(2)	(31)(32)(2)	(31)(1)(32)
(31)(3)(32)	(1)(31)(32)	(3)(31)(32)

Stable Grothendieck polynomials for w

Definition

Stable Grothendieck polynomial (or K-Stanley symmetric function):

$$\mathfrak{G}_w(\mathbf{x}, \beta) = \sum_{h^m \dots h^2 h^1 \in \mathcal{H}_w^m} \beta^{\ell(h^1)+\cdots+\ell(h^m)-\ell(w)} x_1^{\ell(h^1)} \dots x_m^{\ell(h^m)}$$

where $\ell(w)$ is the length of any reduced word of w.

Example

$w = 132 \in \mathcal{H}_0(3)$
Reduced Hecke words $132, 312$
Decreasing factorizations for constant term:
$(31)(2), (1)(32)$ $(3)(1)(2), (1)(3)(2)$

$$\beta^0 : \ (x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2) + 2x_1 x_2 x_3 = s_{21}$$

Schur positivity

Schur positivity (Fomin, Greene 1998)

$$\mathfrak{G}_w(\mathbf{x},\beta) = \sum_{\lambda} \beta^{|\lambda|-\ell(w)} g_w^\lambda s_\lambda(\mathbf{x})$$

$$g_w^\lambda = |\{T \in \mathsf{SSYT}^n(\lambda') \mid \text{column reading of } T \equiv w\}|$$

Example

$$\mathfrak{G}_{132}(\mathbf{x},\beta) = s_{21} + \beta(2s_{211} + s_{22}) + \beta^2(3s_{2111} + 2s_{221}) + \cdots$$

Motivation Crystal for Stanley symmetric functions Crystal for Grothendieck polynomials Properties and results

321-avoiding Hecke words (braid-free)

Definition

$w \in \mathcal{H}_0(n)$ is 321-avoiding if none of the reduced expressions for w contain a consecutive subword of the form $i\ i+1\ i$ for any $i \in [n-1]$.

Examples

- $1321 \equiv 3121 \equiv 3212$ is not 321-avoiding
- $22132 \equiv 2132 \equiv 2312$ is 321-avoiding

Definition

$\mathcal{H}^{m,\star}$ = set of decreasing factorizations into m factors for 321-avoiding w

Example

- $(\ )(1)(21) \in \mathcal{H}^3, \notin \mathcal{H}^{3,\star}$
- $(31)(2) \in \mathcal{H}^{2,\star}$
- $(2)(21)(32) \in \mathcal{H}^{3,\star}$

$\star$-Crystal on $\mathcal{H}^{m,\star}$ (Morse, Pan, Poh, S.)

Bracketing rule on $h^m \dots h^{i+1}\ h^i \dots h^1$

1. Start with the **largest** letter b in h^{i+1}, pair it with the smallest $a \geqslant b$ in h^i. If there is no such a, then b is unpaired.
2. Proceed in decreasing order in h^{i+1}, ignore previously paired letters.

Crystal operator $f_i^\star$, x : largest unpaired letter in h^i

- If $x+1 \in h^i \cap h^{i+1}$, then remove $x+1$ from h^i, add x to h^{i+1}.
- Otherwise, remove x from h^i and add x to h^{i+1}.

Example

- $(1)(32) \xrightarrow{\text{bracket}} (1)(32) \xrightarrow{f_1^\star} (31)(2)$
- $(7532)(621) \xrightarrow{\text{bracket}} (7532)(621) \xrightarrow{f_1^\star} (75321)(61)$

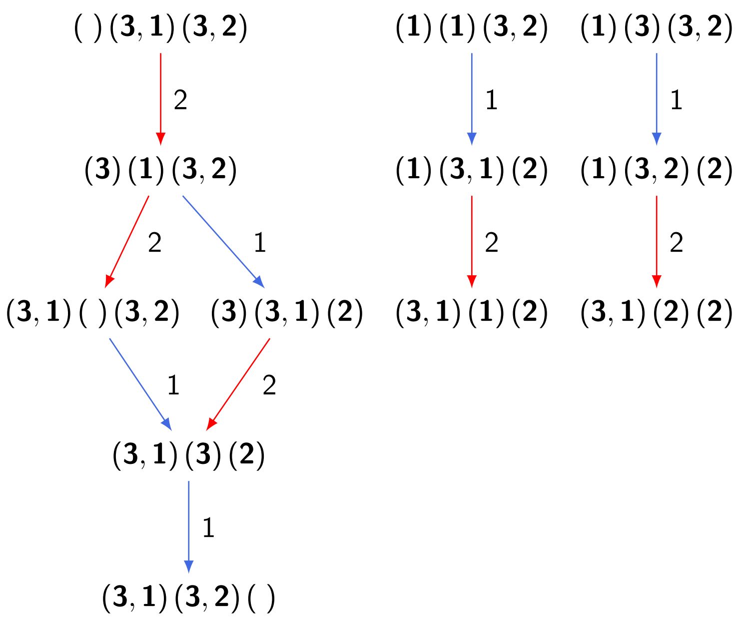
Motivation Crystal for Stanley symmetric functions **Crystal for Grothendieck polynomials** Properties and results

Vertices and edges

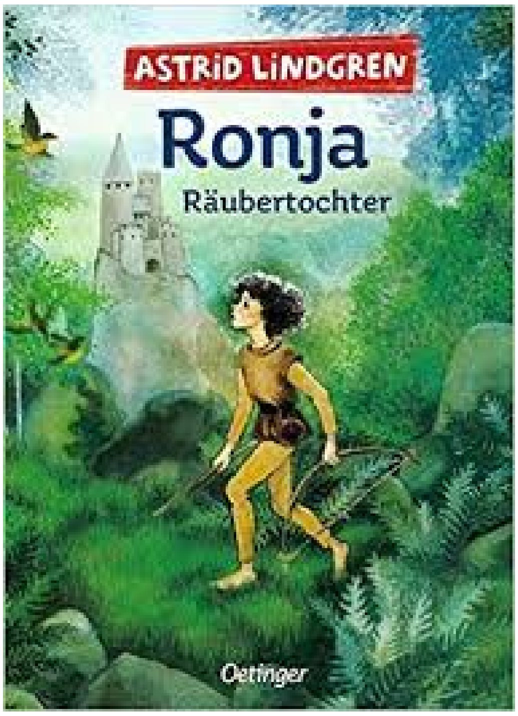
$w = 132, \ \beta^1$

1. $(3,1)(3,2)(\,)$
2. $(3,1)(1)(2)$
3. $(3,1)(2)(2)$
4. $(3,1)(3)(2)$
5. $(1)(3,1)(2)$
6. $(1)(3,2)(2)$
7. $(3)(3,1)(2)$
8. $(3,1)(\,)(3,2)$
9. $(1)(1)(3,2)$
10. $(1)(3)(3,2)$
11. $(3)(1)(3,2)$
12. $(\,)(3,1)(3,2)$

$$\mathfrak{G}_{132}(\mathbf{x},\beta) = s_{21} + \beta(2s_{211} + s_{22}) + \beta^2(3s_{2111} + 2s_{221}) + \cdots$$



Motivation Crystal for Stanley symmetric functions **Crystal for Grothendieck polynomials** Properties and results



Outline

Grothendieck polynomials for skew shapes

$$\mathfrak{G}_{\nu/\lambda}(\mathbf{x};\beta) = \sum_{T \in \mathsf{SVT}(\nu/\lambda)} \beta^{\mathrm{ex}(T)} \mathbf{x}^{\mathrm{wt}(T)} \qquad \text{(Buch 2002)}$$

$\mathsf{SVT}(\nu/\lambda)$ = set of semistandard set-valued tableaux of shape ν/λ
Excess in T is $\mathrm{ex}(T)$

Semistandard set-valued tableaux $\mathsf{SVT}(\nu/\lambda)$

Fill boxes of skew shape ν/λ with nonempty sets. Semistandardness:

C	
A	B

$\max(A) \leqslant \min(B)$, $\max(A) < \min(C)$

Example (Which one is a valid filling?)

✓

34	45	
	12	25

34	35	
	12	456

2	35	
	14	56

Crystal structure on SVT (Monical, Pechenik, Scrimshaw)

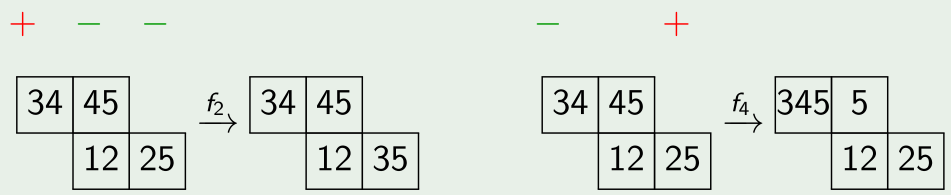
Signature rule

Assign $-$ to every column of T containing an i but not an $i+1$.
Assign $+$ to every column of T containing an $i+1$ but not an i.
Successively pair each $+$ that is adjacent to a $-$.

Crystal operator f_i

- changes the rightmost unpaired i $-$ to $i+1$, except
- if its right neighbor contains both $i, i+1$, then *move* the i over and turn it into $i+1$

Example



Motivation | Crystal for Stanley symmetric functions | Crystal for Grothendieck polynomials | Properties and results

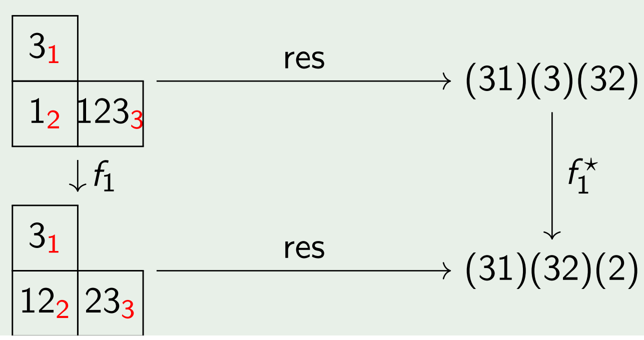
Residue map as a crystal isomorphism

Theorem (Morse, Pan, Poh, S. 2020)

The crystal on skew semistandard set-valued tableaux and the crystal on decreasing factorizations $\mathcal{H}^{m,\star}$ intertwine under the residue map. That is, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{SVT}^m(\lambda/\mu) & \xrightarrow{\text{res}} & \mathcal{H}^{m,\star} \\ \downarrow f_k & & \downarrow f_k^\star \\ \mathrm{SVT}^m(\lambda/\mu) & \xrightarrow{\text{res}} & \mathcal{H}^{m,\star}. \end{array}$$

Example



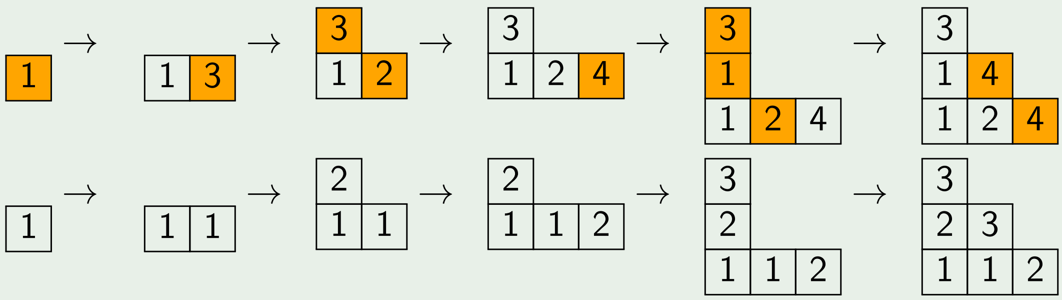
$\star$-Insertion

Insert x into row R of a transpose of a semistandard tableau

1. Try to append x to the right of R (terminate and record)
2. $x \notin R$, bump the minimal $z > x$ (proceed to the next row)
3. $x \in R$, proceed to next row with y minimal such that $[y, x] \subseteq R$

Example

$$\mathbf{h} = (42)(42)(31) = \begin{bmatrix} \mathbf{3} & \mathbf{3} & \mathbf{2} & \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{4} & \mathbf{2} & \mathbf{4} & \mathbf{2} & \mathbf{3} & \mathbf{1} \end{bmatrix}$$



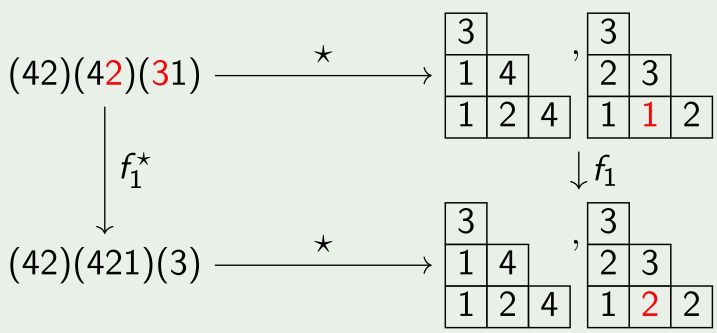
Association with $\star$-crystal

Theorem (Morse, Pan, Poh, S. 2020)

The following diagram commutes:

$$\begin{array}{ccc} \mathcal{H}^{m,\star} & \xrightarrow{Q^\star} & \mathrm{SSYT}^m \\ \downarrow f_i^\star & & \downarrow f_i \\ \mathcal{H}^{m,\star} & \xrightarrow{Q^\star} & \mathrm{SSYT}^m \end{array}$$

Example

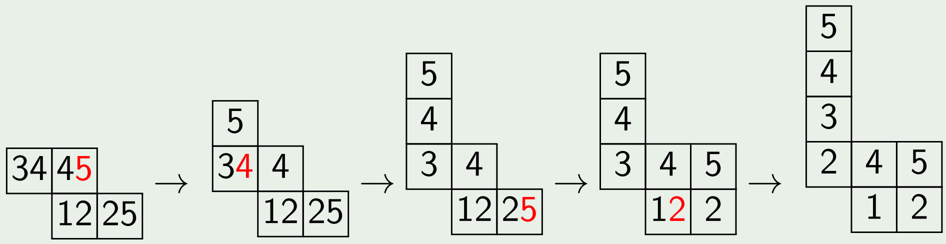


Uncrowding SVT

Uncrowding operator Lenart 2000; Buch 2002; Bandlow, Morse 2012; Patrias 2016; Reiner, Tenner, Yong 2018

- Identify the topmost row in T containing a multicell.
- Let x be the largest letter in that row which lies in a multicell.
- Delete x and perform RSK algorithm into the rows above. Repeat.
- Result is a single-valued skew tableau.

Example



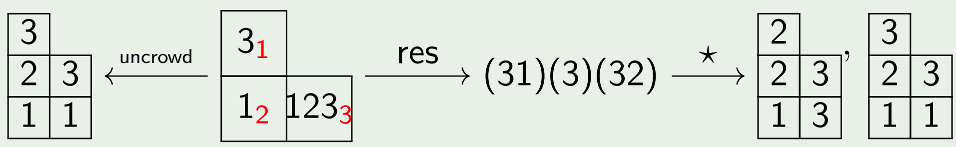
Motivation Crystal for Stanley symmetric functions Crystal for Grothendieck polynomials Properties and results

Connection to uncrowding map

Theorem (Morse, Pan, Poh, S. 2020)

Let $T \in \mathrm{SVT}^m(\lambda)$, $(\tilde{P}, \tilde{Q}) = \mathrm{uncrowd}(T)$, *and* $(P, Q) = \star \circ \mathrm{res}(T)$. *Then* $Q = \tilde{P}$.

Example



Hecke insertion (Buch 2008, Patrias, Pylyavskyy 2016)

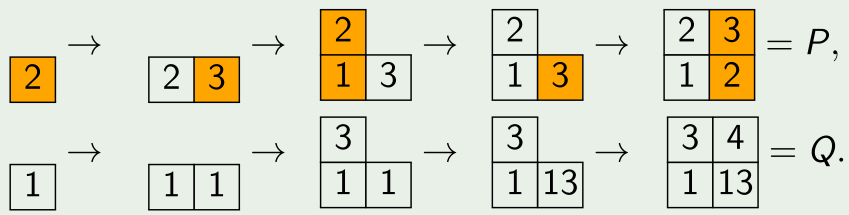
Insert x to row R of an increasing tableau

- Try to append x to the right of R (record and terminate)
- Try to bump the smallest letter that is bigger (proceed to the next row)

$$\mathcal{H}^m \longleftrightarrow (P, Q)$$

Example

$$\mathbf{h} = (2)(31)(\,)(32) = \begin{bmatrix} \mathbf{4} & \mathbf{3} & \mathbf{3} & \mathbf{1} & \mathbf{1} \\ \mathbf{2} & \mathbf{3} & \mathbf{1} & \mathbf{3} & \mathbf{2} \end{bmatrix}.$$



Motivation | Crystal for Stanley symmetric functions | Crystal for Grothendieck polynomials | Properties and results

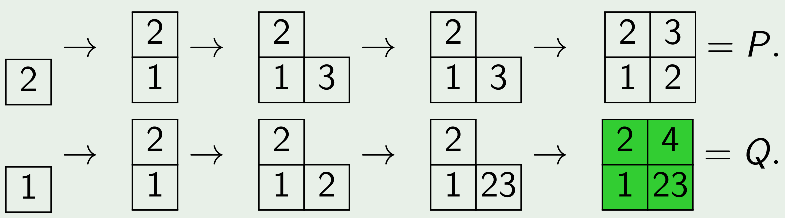
Hecke insertion and the residue map

Theorem (Morse, Pan, Poh, S. 2020)

Let $T \in \mathrm{SVT}(\lambda)$ and $[\mathbf{k}, \mathbf{h}]^t = \mathrm{res}(T)$. Apply Hecke row insertion from the right on $[\mathbf{k}, \mathbf{h}]^t$ to obtain the pair of tableaux (P, Q). Then $Q = T$.

Example

$$T = \begin{array}{|c|c|} \hline 2_{1} & 4_{2} \\ \hline 1_{2} & 23_{3} \\ \hline \end{array} \xrightarrow{\text{res}} (2)(3)(31)(2) = \begin{bmatrix} \mathbf{4} & \mathbf{3} & \mathbf{2} & \mathbf{2} & \mathbf{1} \\ \mathbf{2} & \mathbf{3} & \mathbf{3} & \mathbf{1} & \mathbf{2} \end{bmatrix}$$

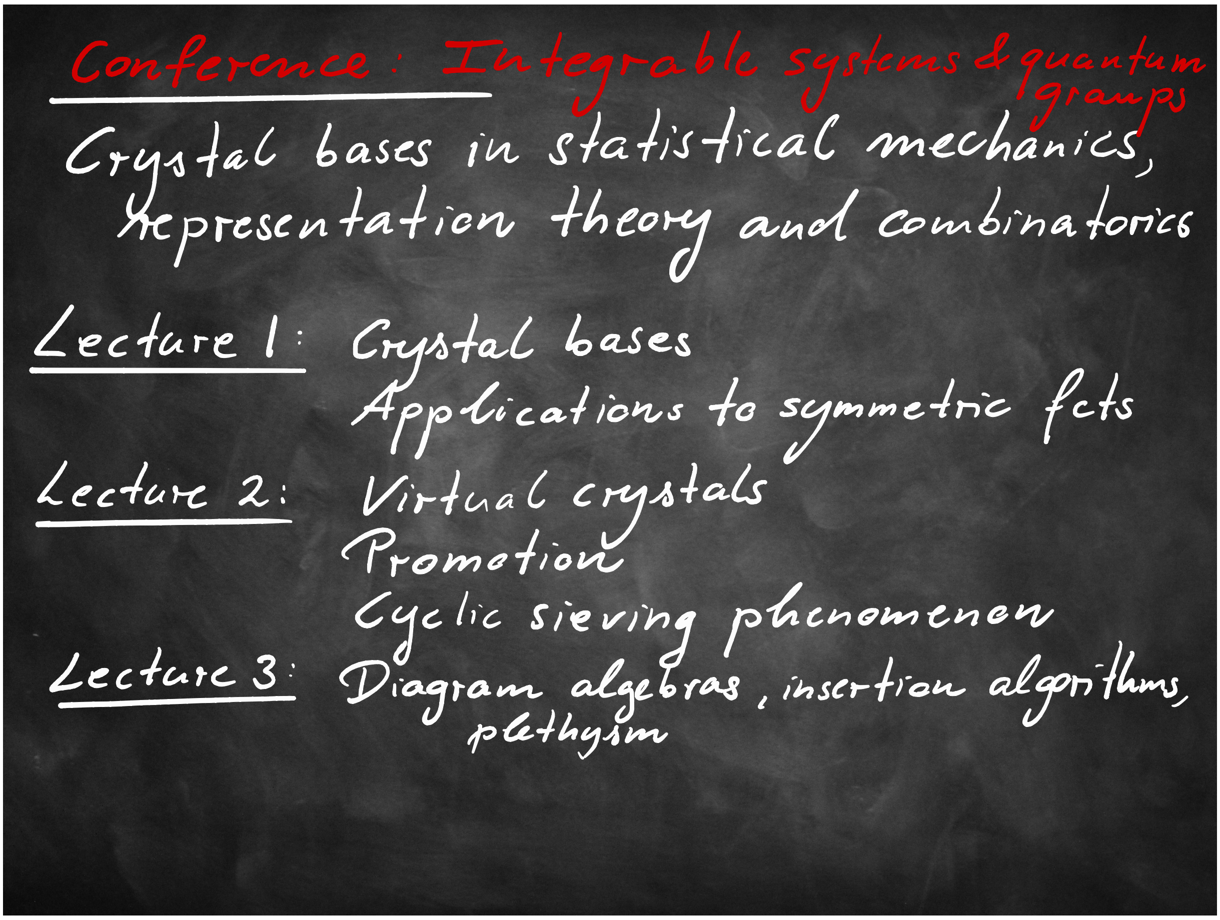


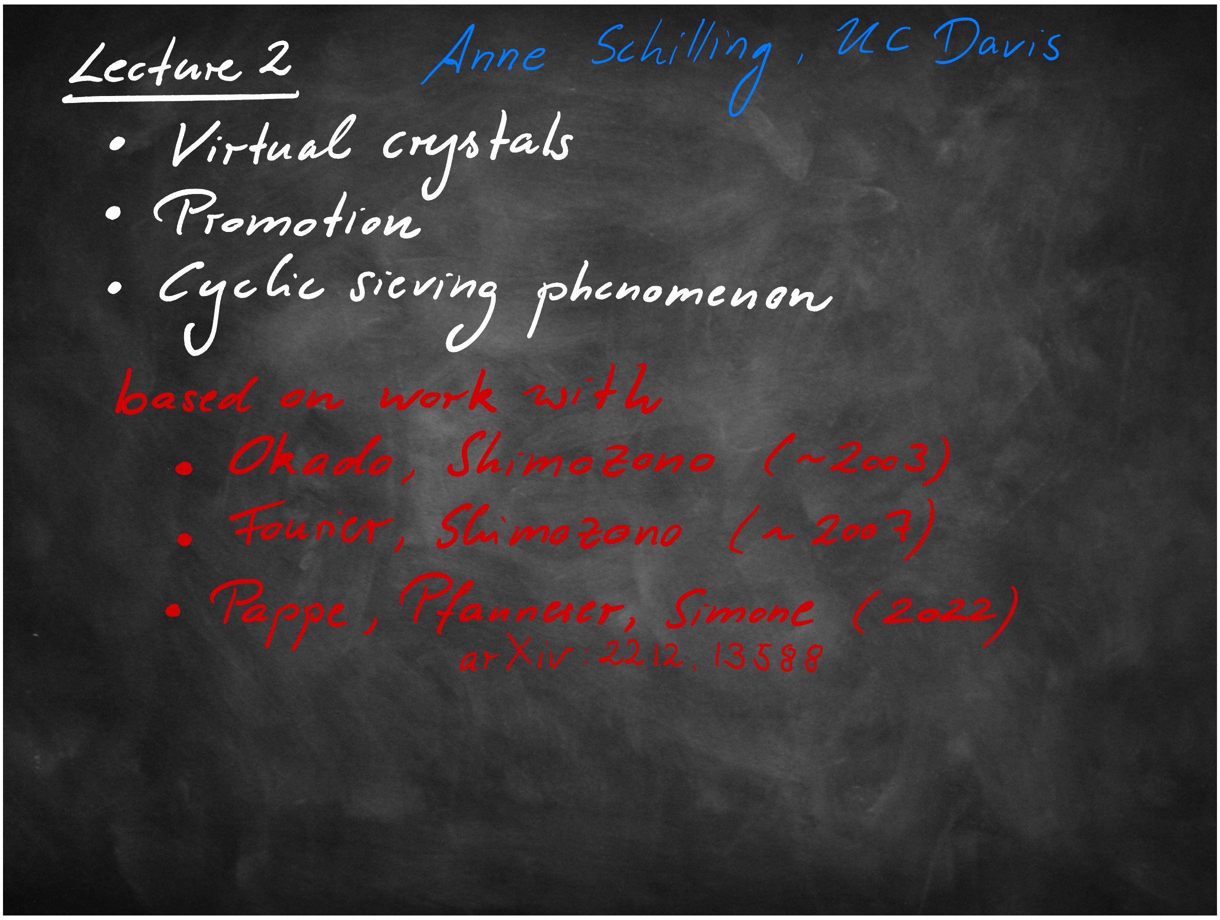
Future Work

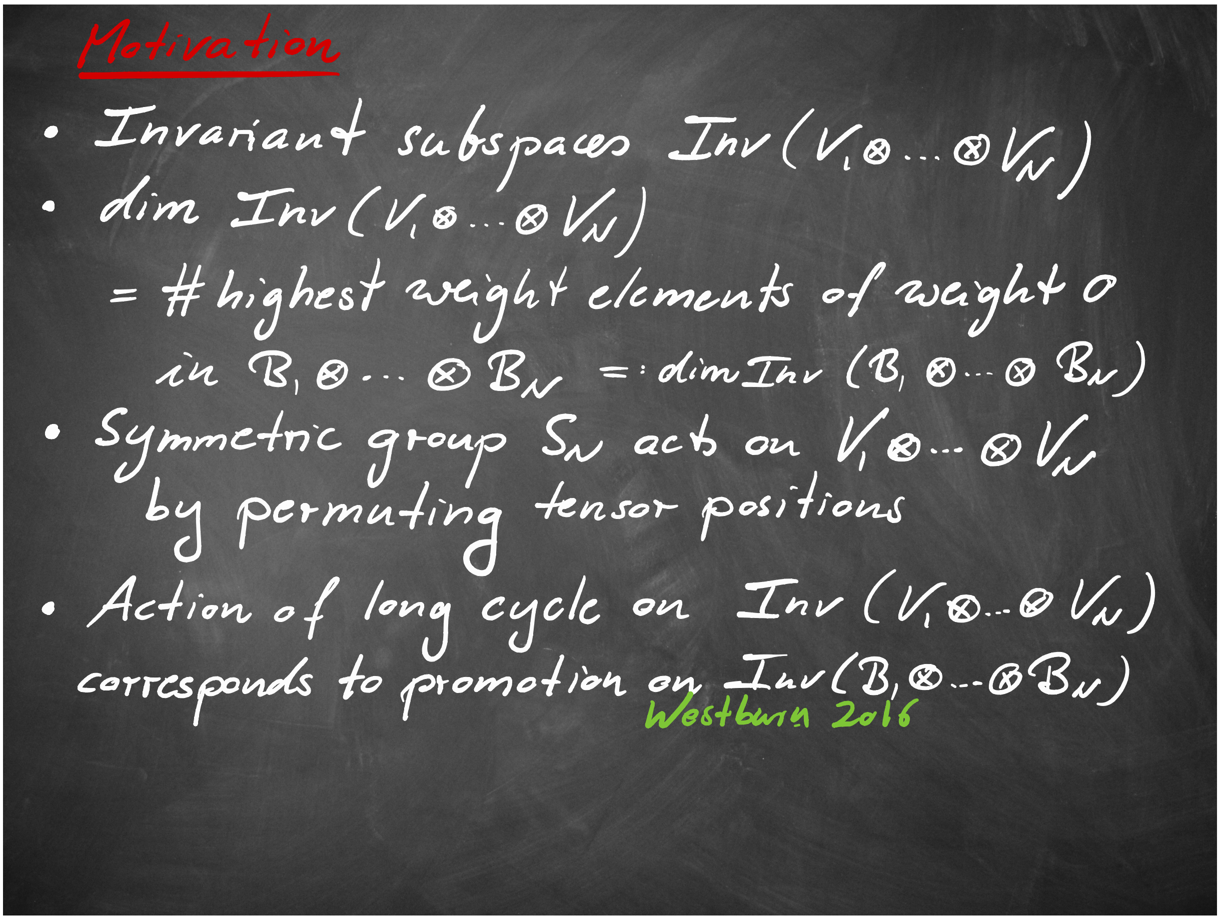
- Crystal structure for the non-321 avoiding case (beyond skew shapes)
- Demazure crystal structure to compute the intersection number?

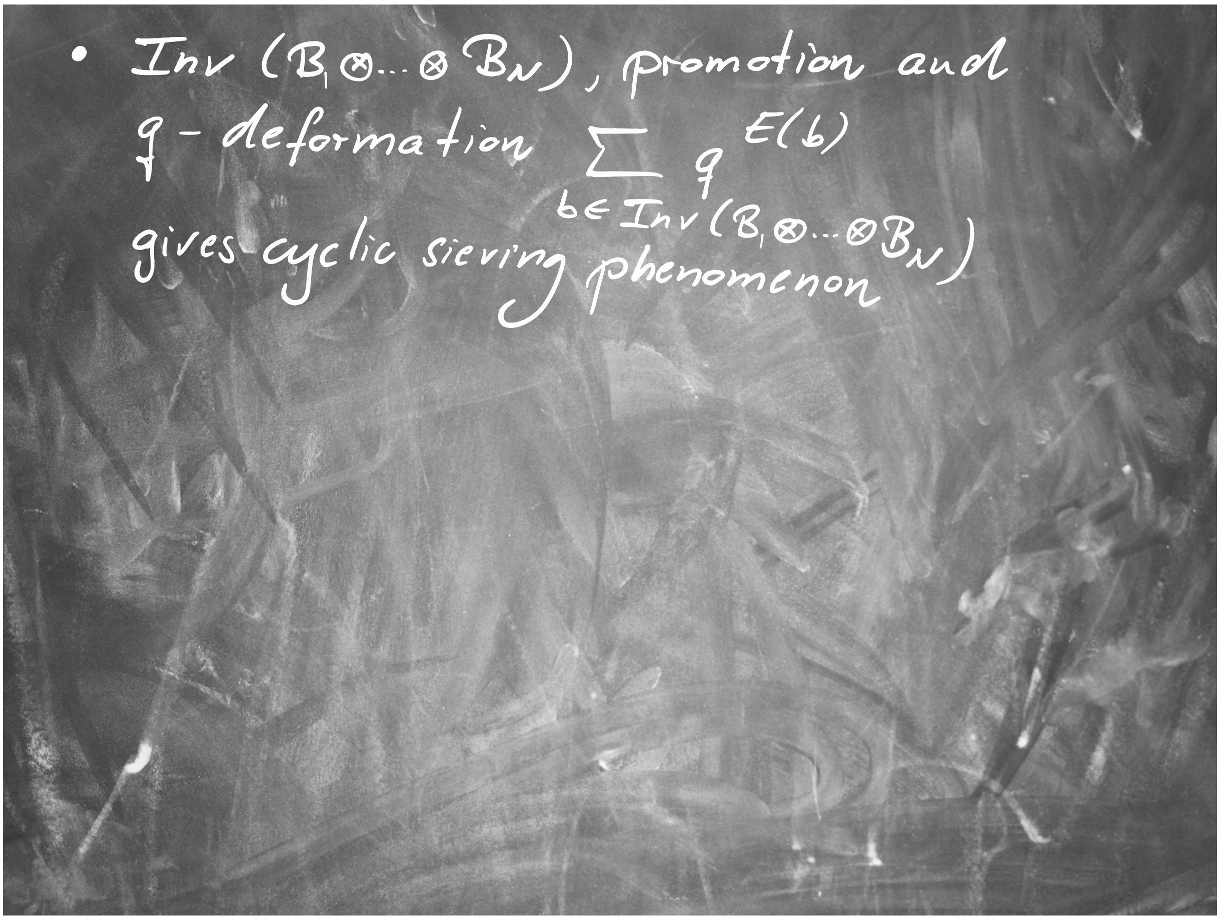
Motivation Crystal for Stanley symmetric functions Crystal for Grothendieck polynomials Properties and results

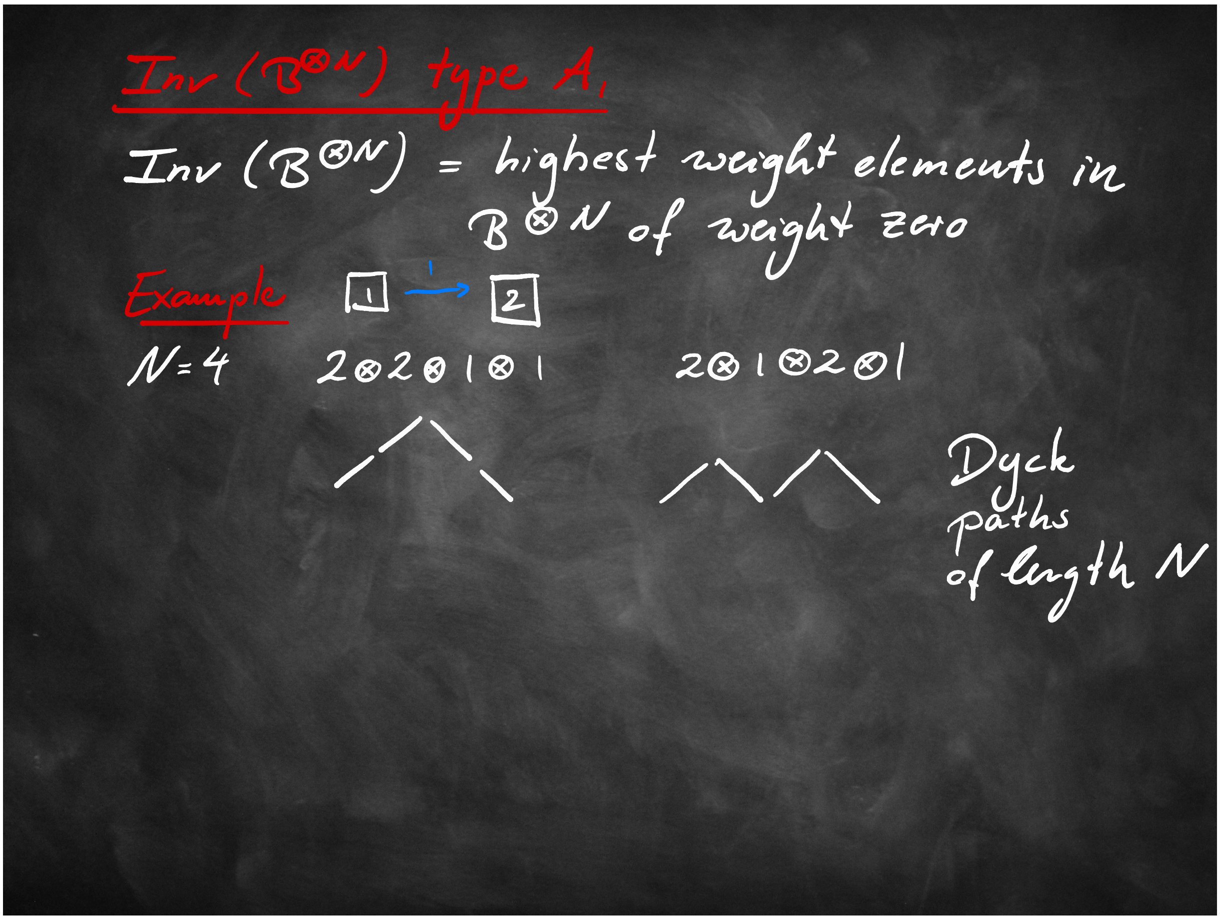
Thank you !

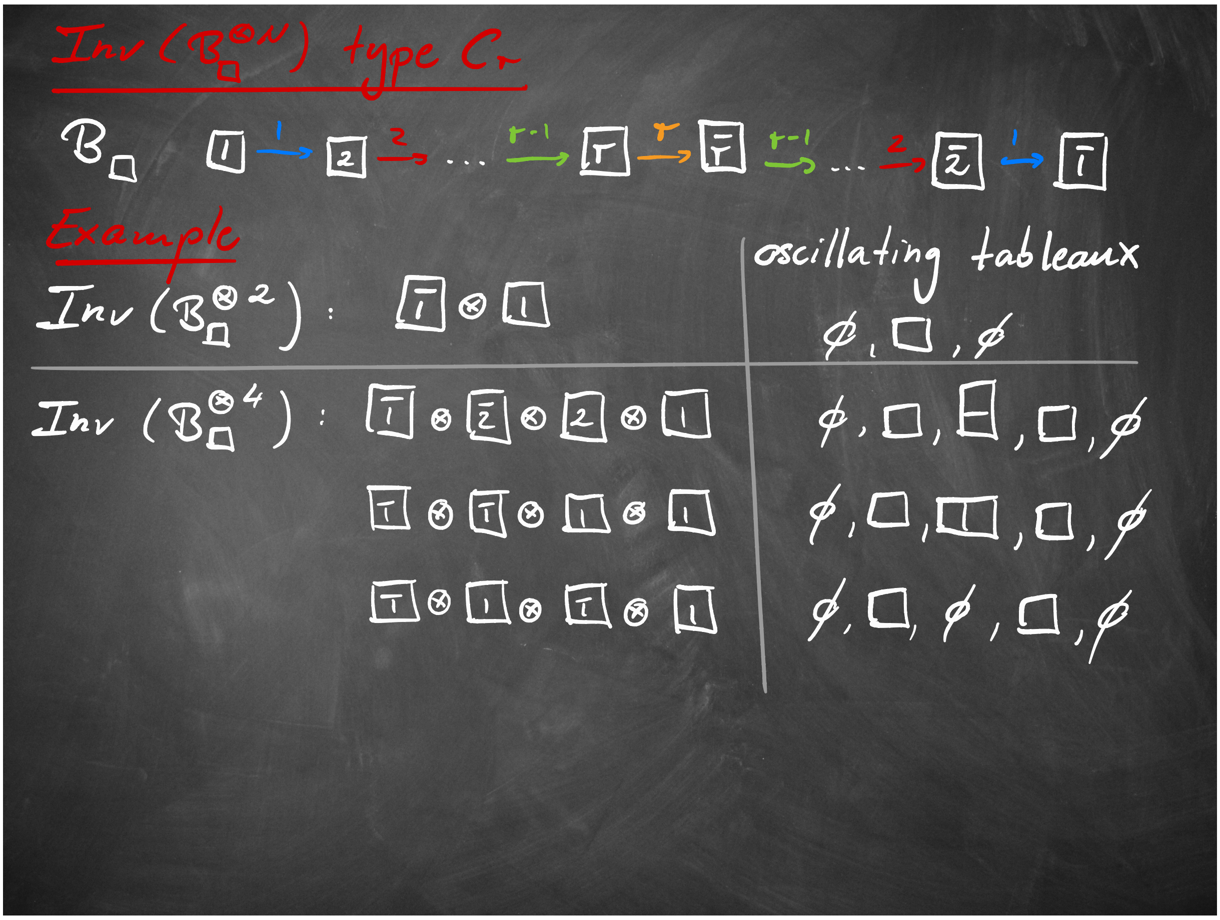

Conference: Integrable systems & quantum groups
Crystal bases in statistical mechanics, representation theory and combinatorics
Lecture 1: Crystal bases
Applications to symmetric fcts
Lecture 2: Virtual crystals
Promotion
Cyclic sieving phenomenon
Lecture 3: Diagram algebras, insertion algorithms, plethysm

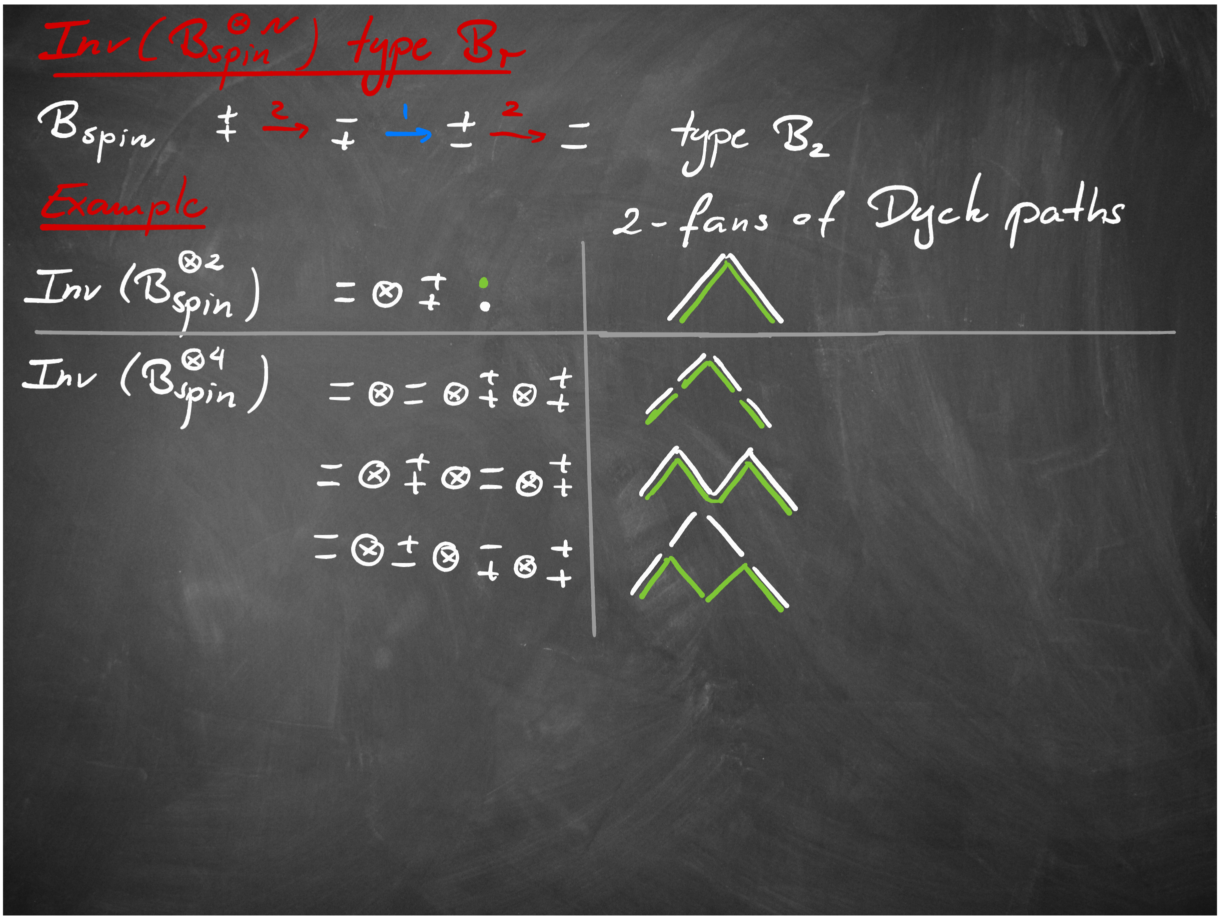

Lecture 2
Anne Schilling, UC Davis
• Virtual crystals
• Promotion
• Cyclic sieving phenomenon
based on work with
• Okado, Shimozono (~2003)
• Fourier, Shimozono (~2007)
• Pappe, Pfannerer, Simone (2022)
arXiv: 2212.13588

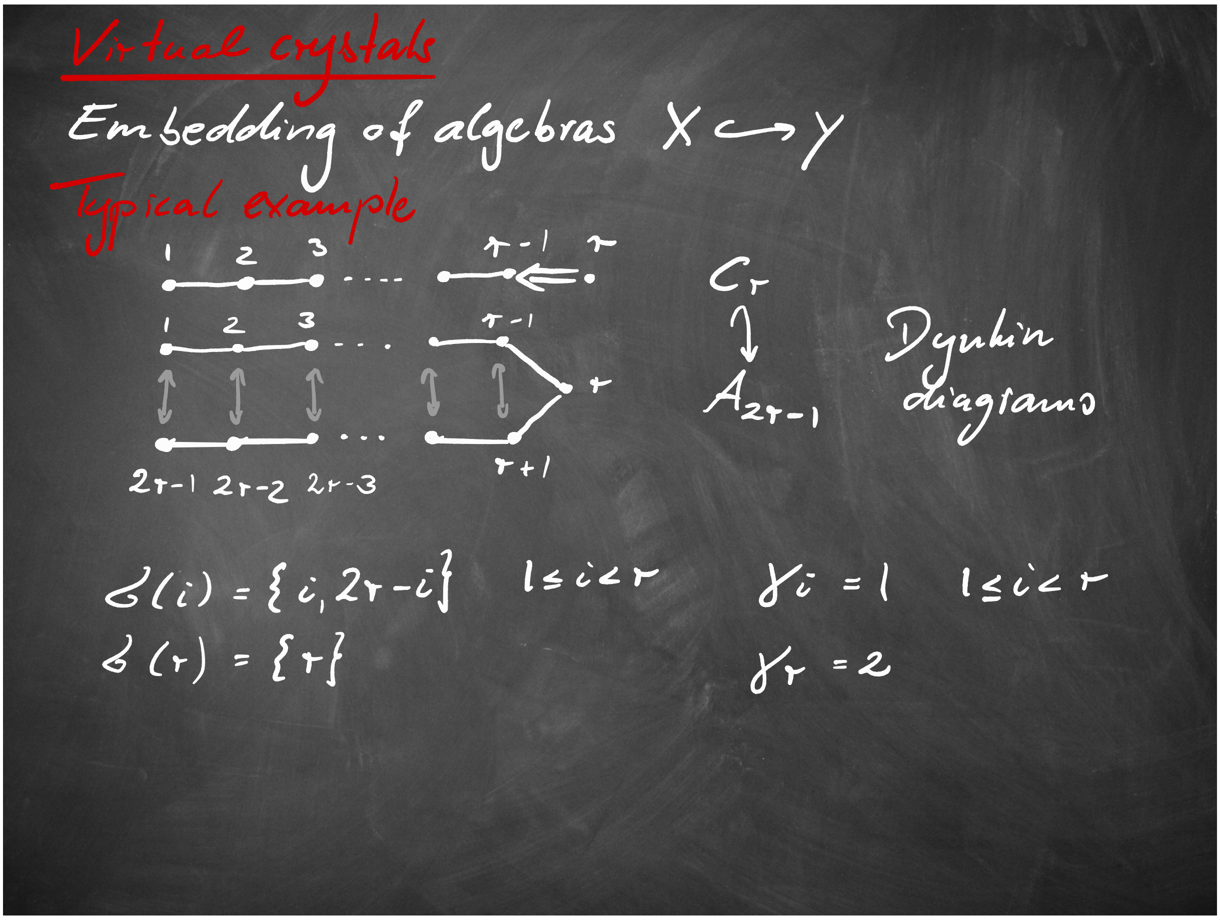

Motivation
• Invariant subspaces $\mathrm{Inv}(V_1\otimes\cdots\otimes V_N)$
• $\dim \mathrm{Inv}(V_1\otimes\cdots\otimes V_N)$
= # highest weight elements of weight 0
in $B_1\otimes\cdots\otimes B_N$ =: $\dim \mathrm{Inv}(B_1\otimes\cdots\otimes B_N)$
• Symmetric group S_N acts on $V_1\otimes\cdots\otimes V_N$
by permuting tensor positions
• Action of long cycle on $\mathrm{Inv}(V_1\otimes\cdots\otimes V_N)$
corresponds to promotion on $\mathrm{Inv}(B_1\otimes\cdots\otimes B_N)$
Westburn 2016

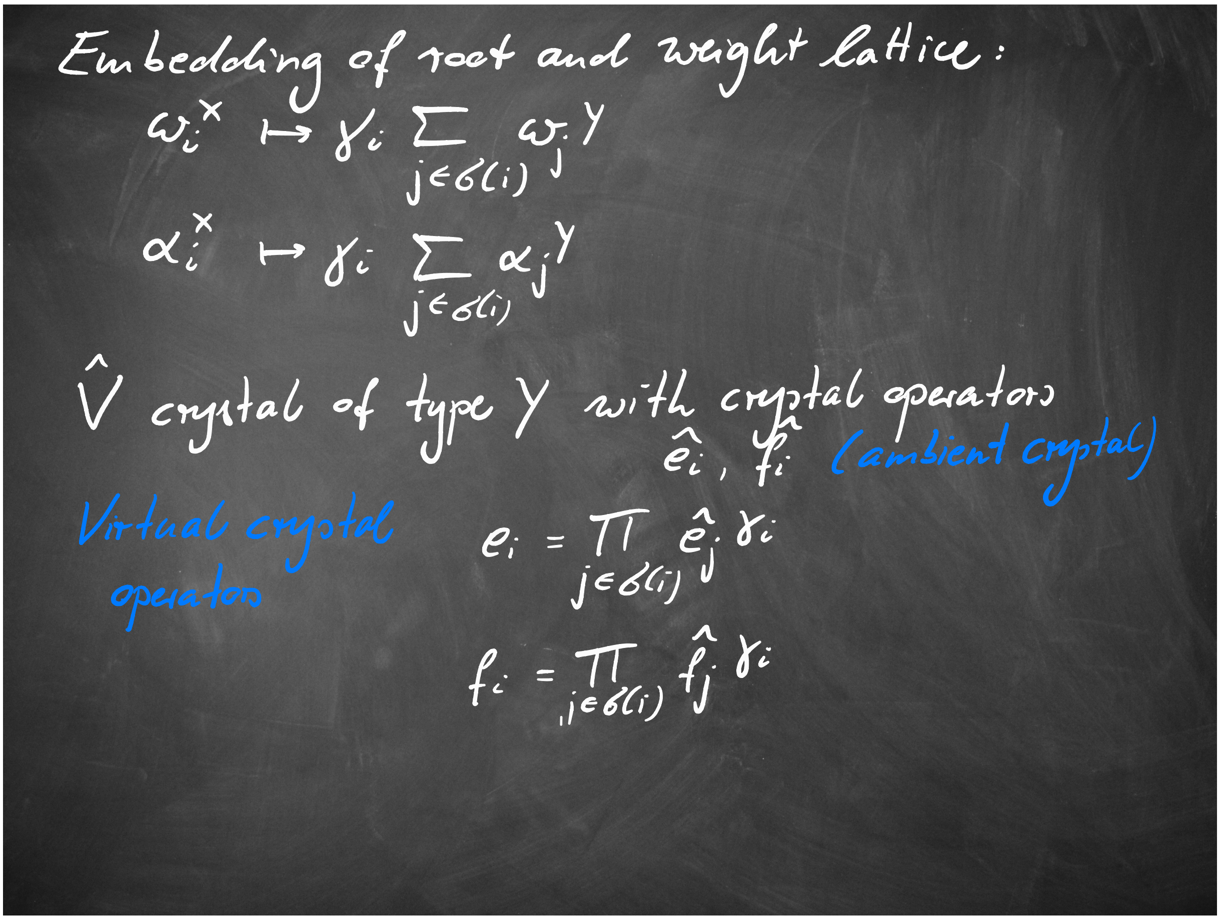

• $\mathrm{Inv}(B_1\otimes\cdots\otimes B_N)$, promotion and
q-deformation $\sum_{b\in \mathrm{Inv}(B_1\otimes\cdots\otimes B_N)} q^{E(b)}$
gives cyclic sieving phenomenon

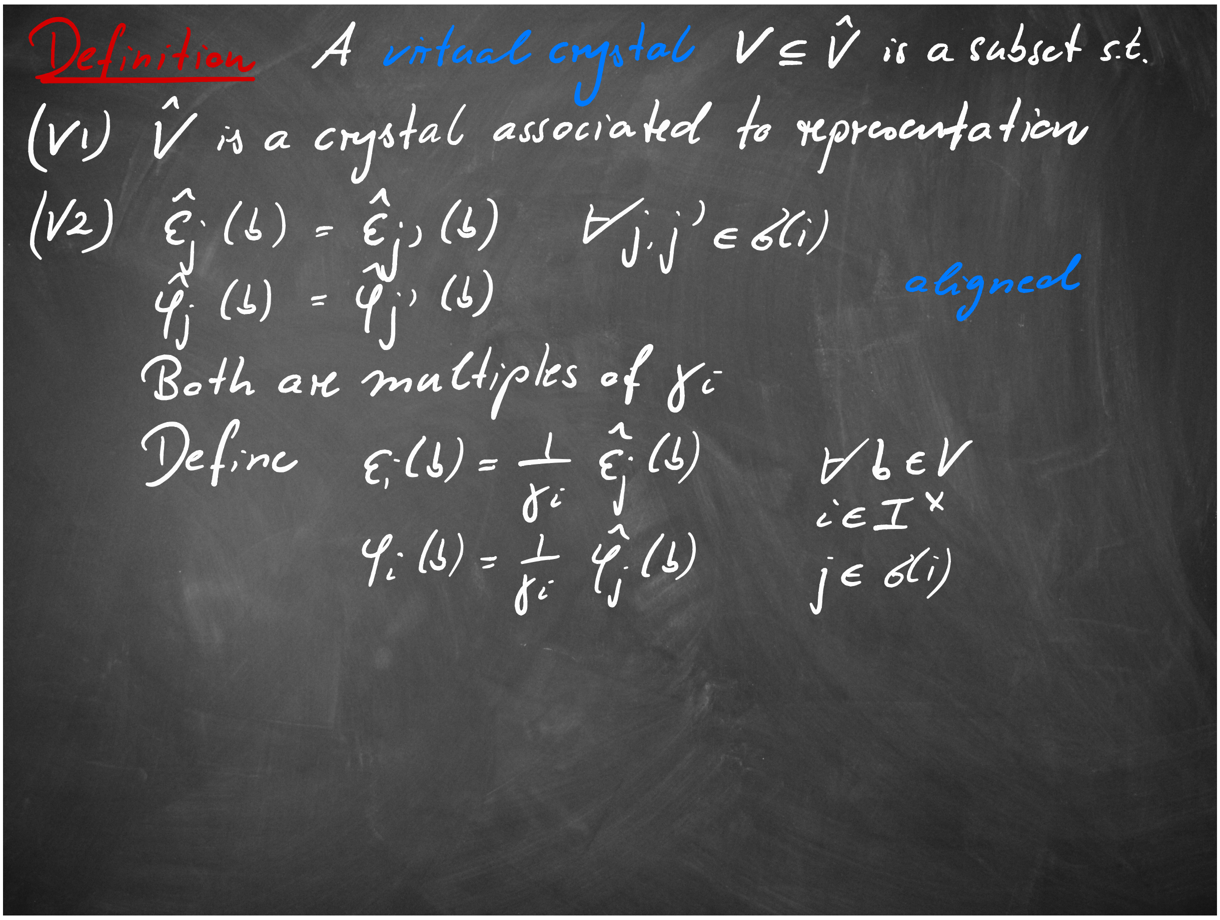

Inv (B^⊗N) type A1
Inv (B^⊗N) = highest weight elements in B^⊗N of weight zero
Example
N=4
2⊗2⊗1⊗1
2⊗1⊗2⊗1
Dyck paths of length N

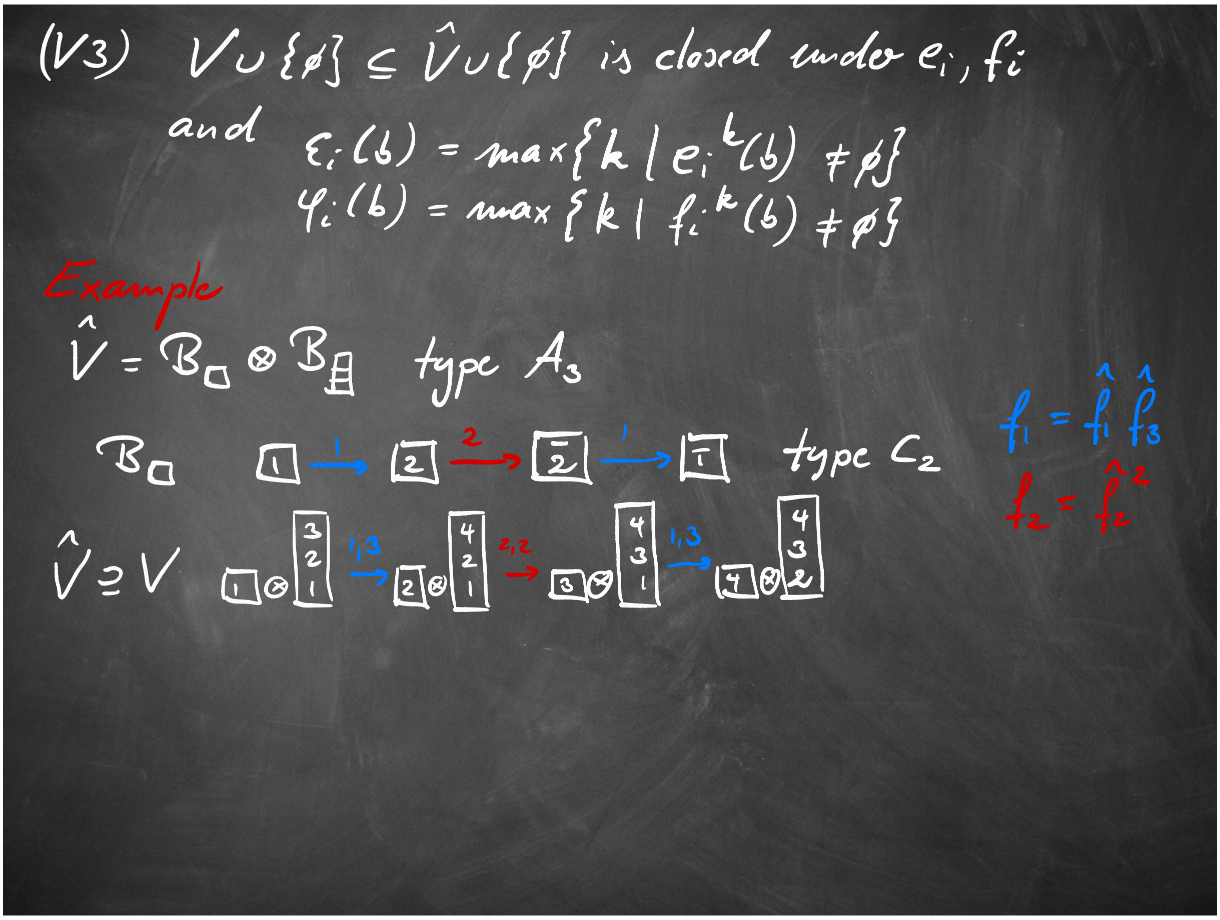

Inv (B□^⊗N) type Cr
B□
Example
Inv (B□^⊗2) :
oscillating tableaux
Inv (B□^⊗4) :

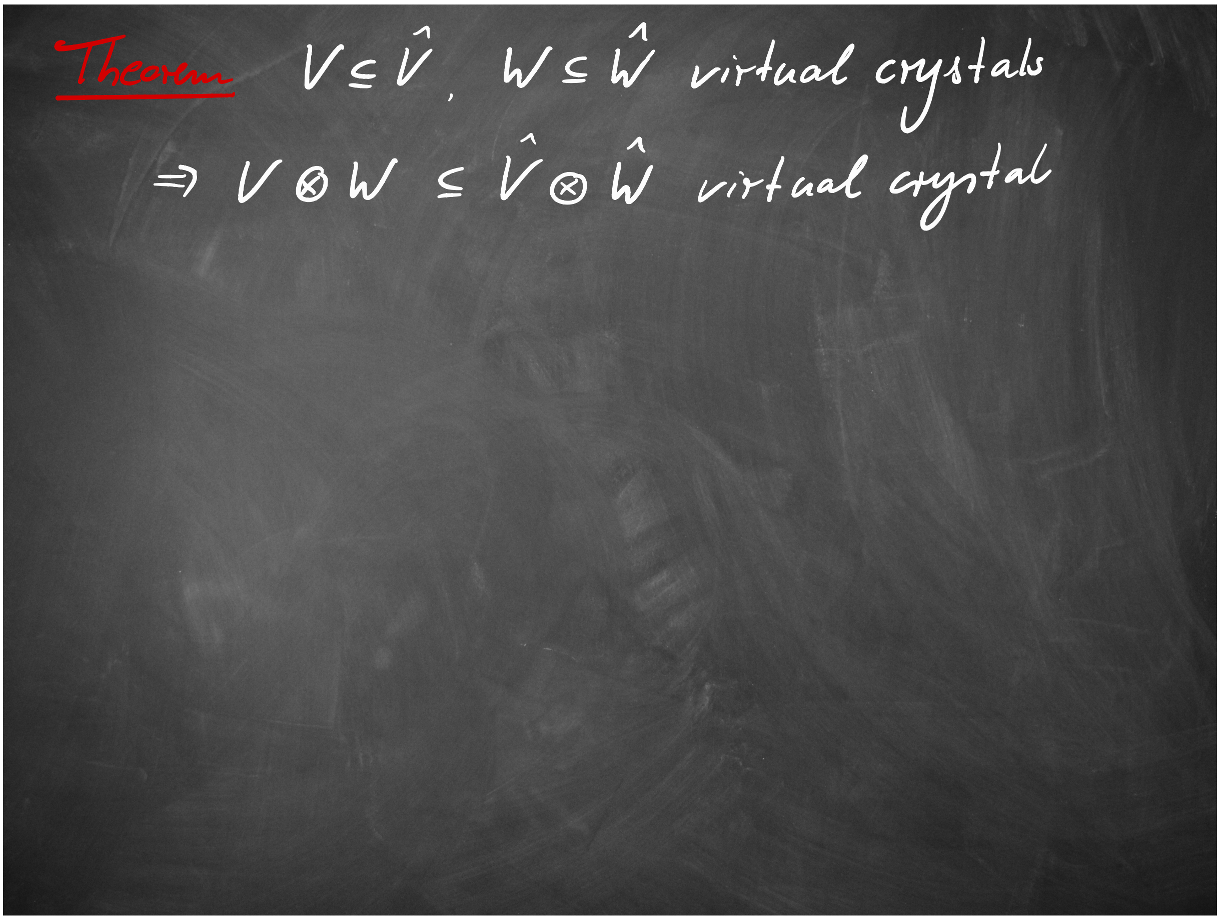

Inv(B_spin^⊗N) type B_r
B_spin
type B_2
Example
2-fans of Dyck paths
Inv(B_spin^⊗2)
Inv(B_spin^⊗4)

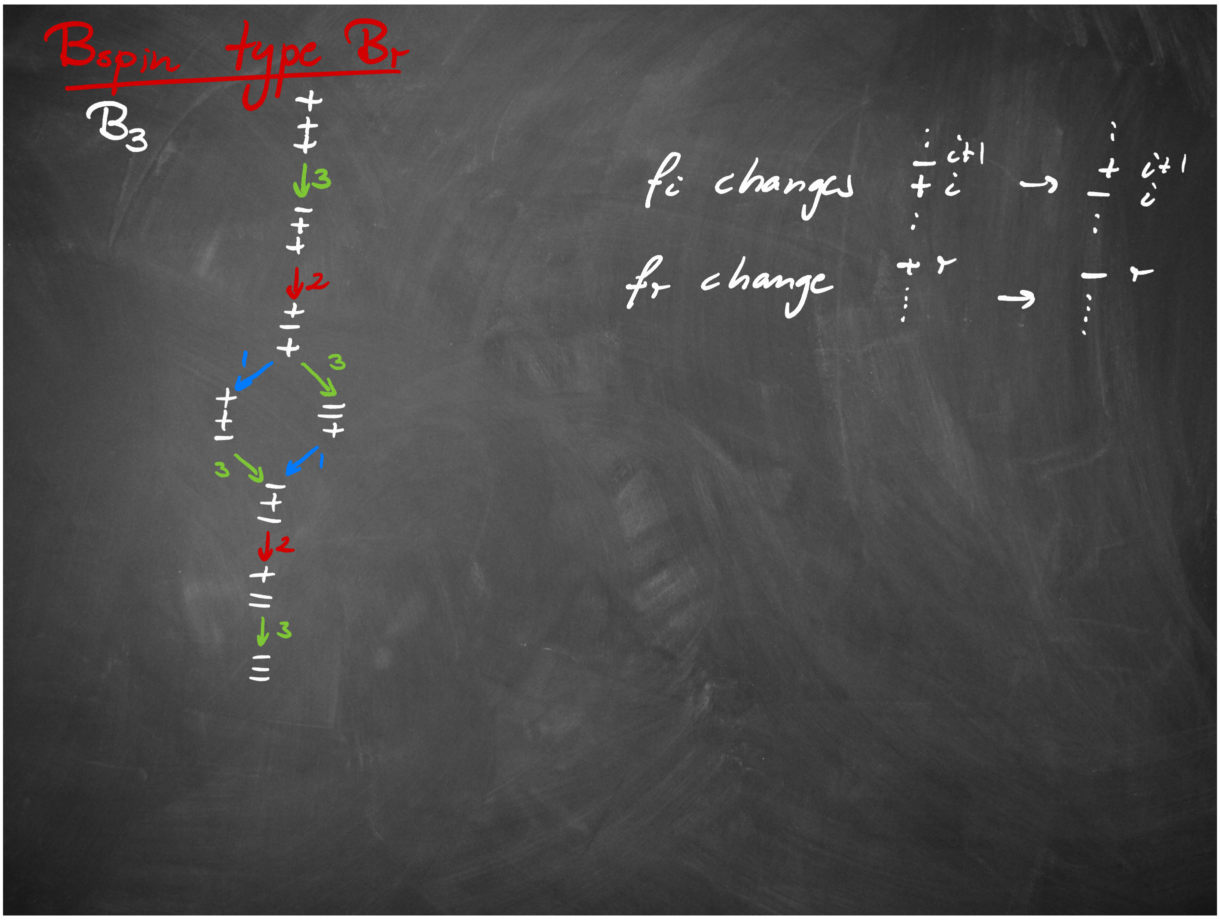

Virtual crystals
Embedding of algebras X ↪ Y
Typical example
C_r
A_{2r-1}
Dynkin diagrams
1 2 3 r-1 r
r+1
2r-1 2r-2 2r-3
σ(i) = {i, 2r−i} 1≤i<r
σ(r) = {r}
γ_i = 1 1≤i<r
γ_r = 2

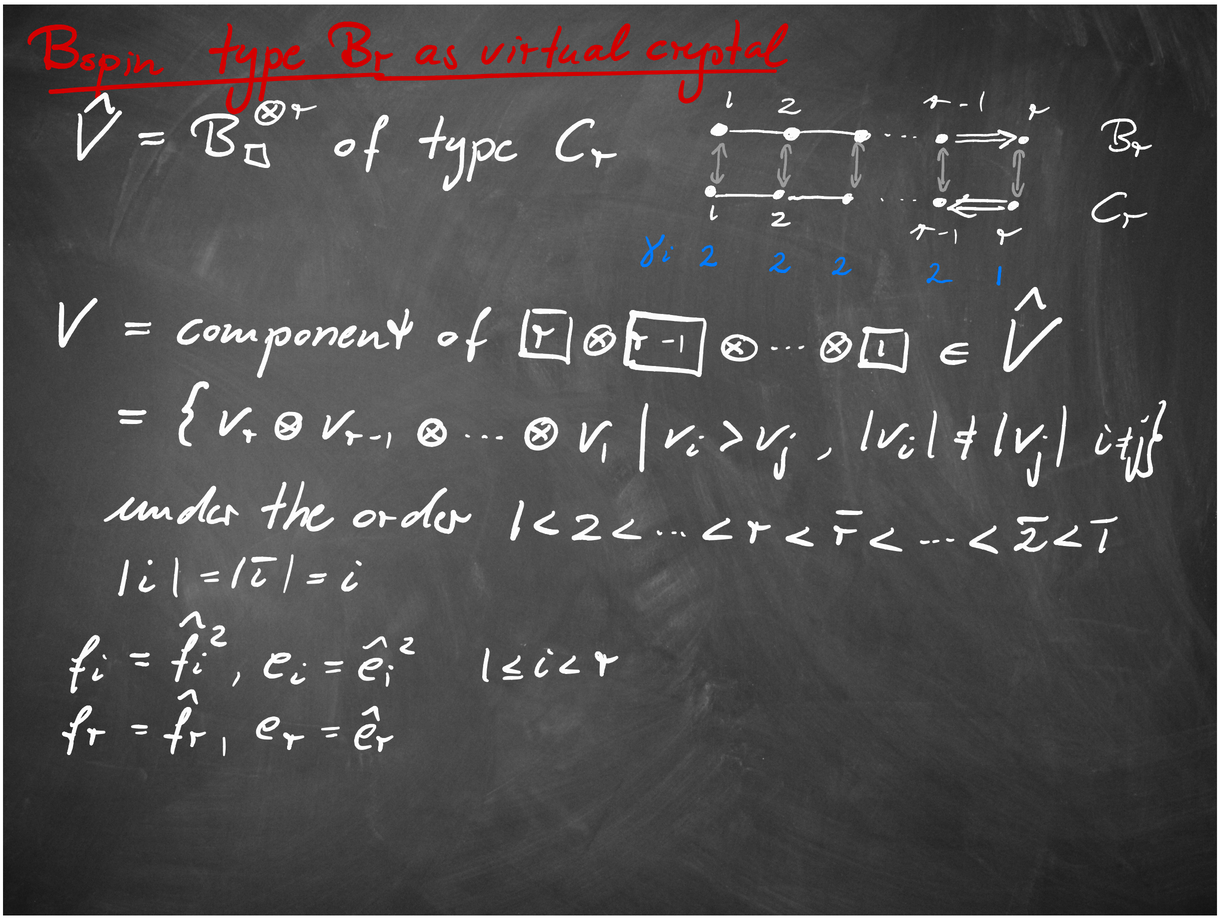

Embedding of root and weight lattice:
$\omega_i^X \mapsto \gamma_i \sum_{j\in\sigma(i)} \omega_j^Y$
$\alpha_i^X \mapsto \gamma_i \sum_{j\in\sigma(i)} \alpha_j^Y$
$\hat{V}$ crystal of type Y with crystal operators $\hat{e}_i, \hat{f}_i$ (ambient crystal)
Virtual crystal operators
$e_i = \prod_{j\in\sigma(i)} \hat{e}_j^{\gamma_i}$
$f_i = \prod_{j\in\sigma(i)} \hat{f}_j^{\gamma_i}$

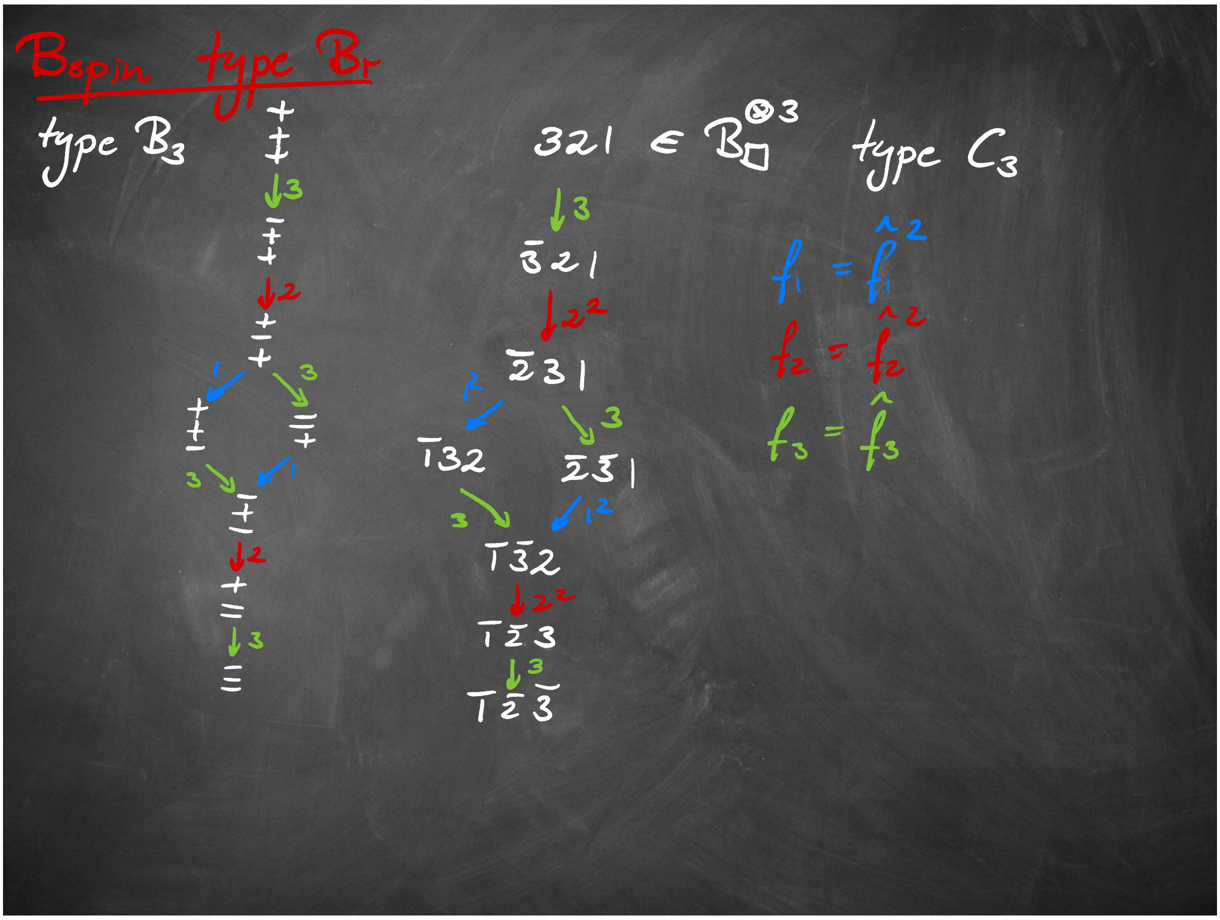

Definition A virtual crystal $V \subseteq \hat{V}$ is a subset s.t.
(V1) $\hat{V}$ is a crystal associated to representation
(V2) $\hat{\varepsilon}_j(b) = \hat{\varepsilon}_{j'}(b)$ $\forall j, j' \in \sigma(i)$
$\hat{\varphi}_j(b) = \hat{\varphi}_{j'}(b)$
aligned
Both are multiples of γ_i
Define $\varepsilon_i(b) = \frac{1}{\gamma_i}\hat{\varepsilon}_j(b)$ $\forall b \in V$
$\varphi_i(b) = \frac{1}{\gamma_i}\hat{\varphi}_j(b)$ $i \in I^X$
$j \in \sigma(i)$


(V3) V ∪ {∅} ⊆ V̂ ∪ {∅} is closed under e_i, f_i
and ε_i(b) = max{k | e_i^k(b) ≠ ∅}
φ_i(b) = max{k | f_i^k(b) ≠ ∅}
Example
V̂ = B□ ⊗ B type A_3
B□ type C_2
V̂ ⊇ V
f_1 = f̂_1 f̂_3
f_2 = f̂_2^2

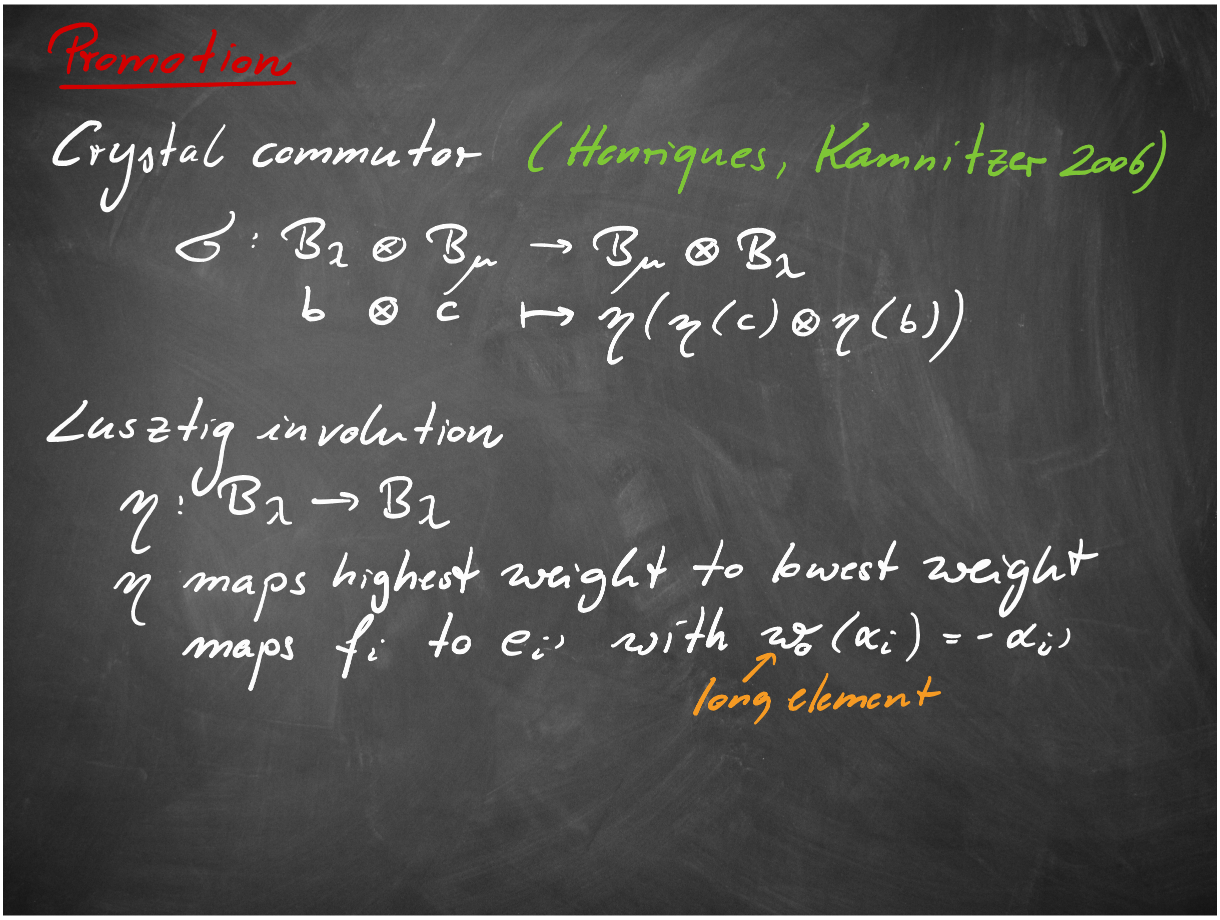

Theorem V ⊆ V̂, W ⊆ Ŵ virtual crystals
⇒ V ⊗ W ⊆ V̂ ⊗ Ŵ virtual crystal


Bspin type Br
B3
fi changes
fr change

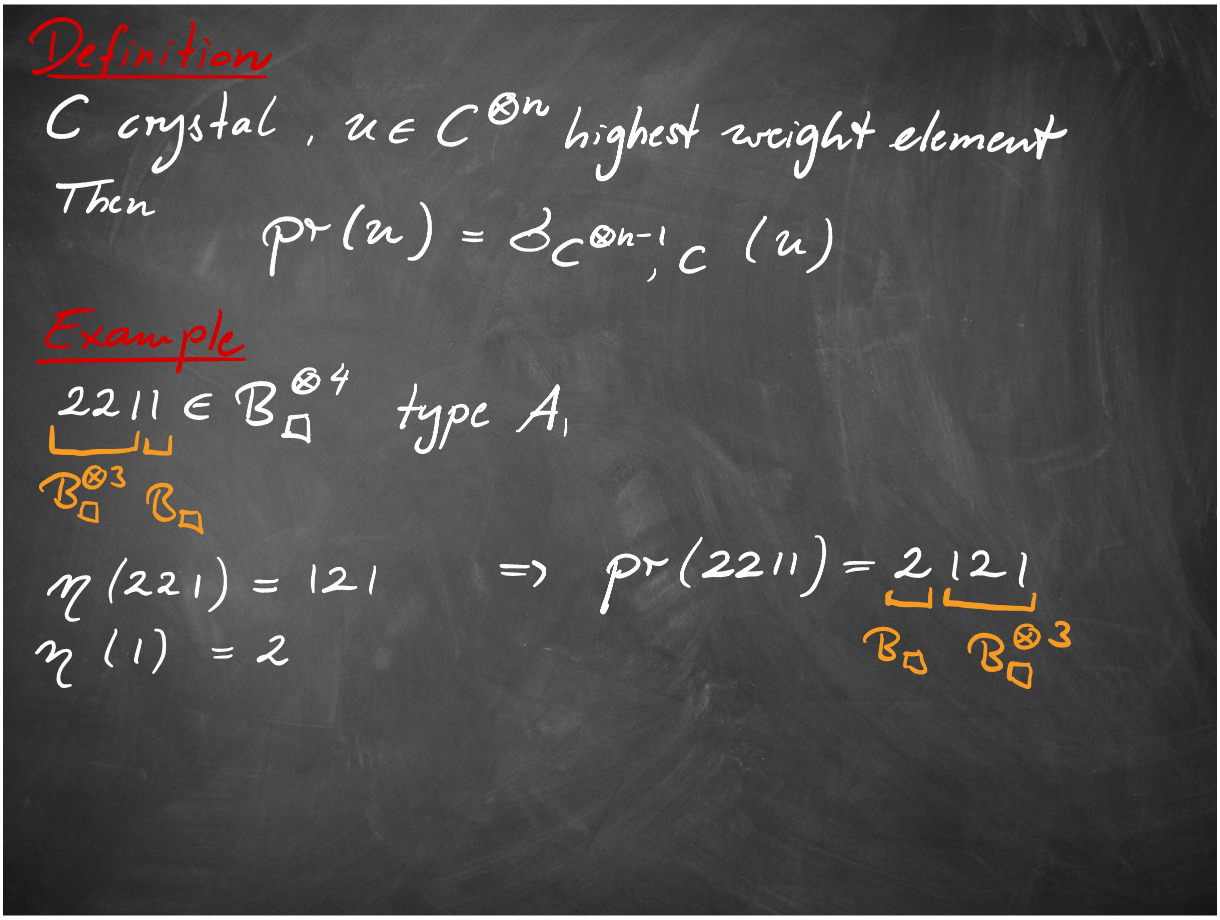

Bspin type Br as virtual crystal
V̂ = B□^⊗r of type Cr
Br
Cr
V = component of r ⊗ r-1 ⊗ ··· ⊗ 1 ∈ V̂
= { vr ⊗ vr-1 ⊗ ··· ⊗ v1 | vi > vj , |vi| ≠ |vj| i≠j }
under the order 1 < 2 < ··· < r < r̄ < ··· < 2̄ < 1̄
|i| = |ī| = i
fi = f̂i², ei = êi²   1 ≤ i < r
fr = f̂r , er = êr

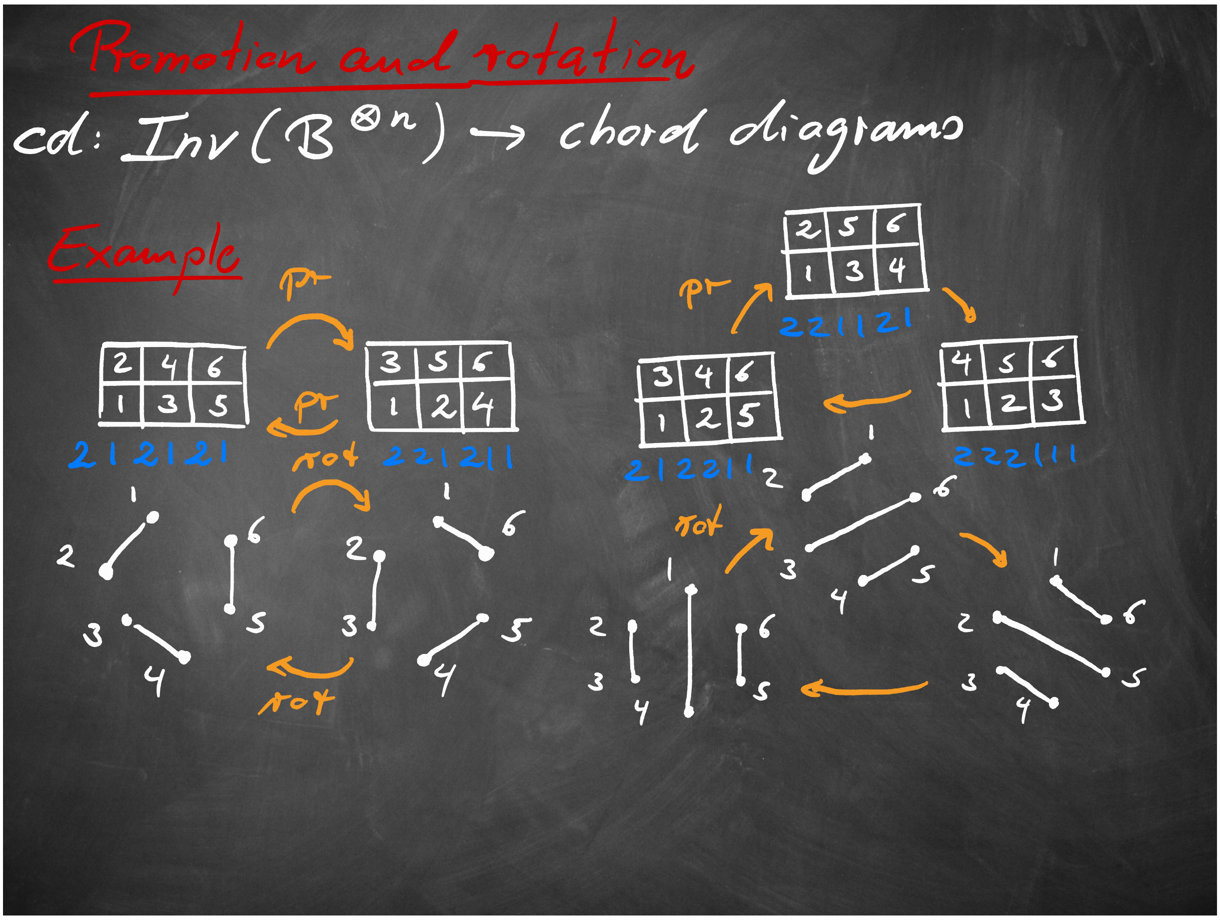


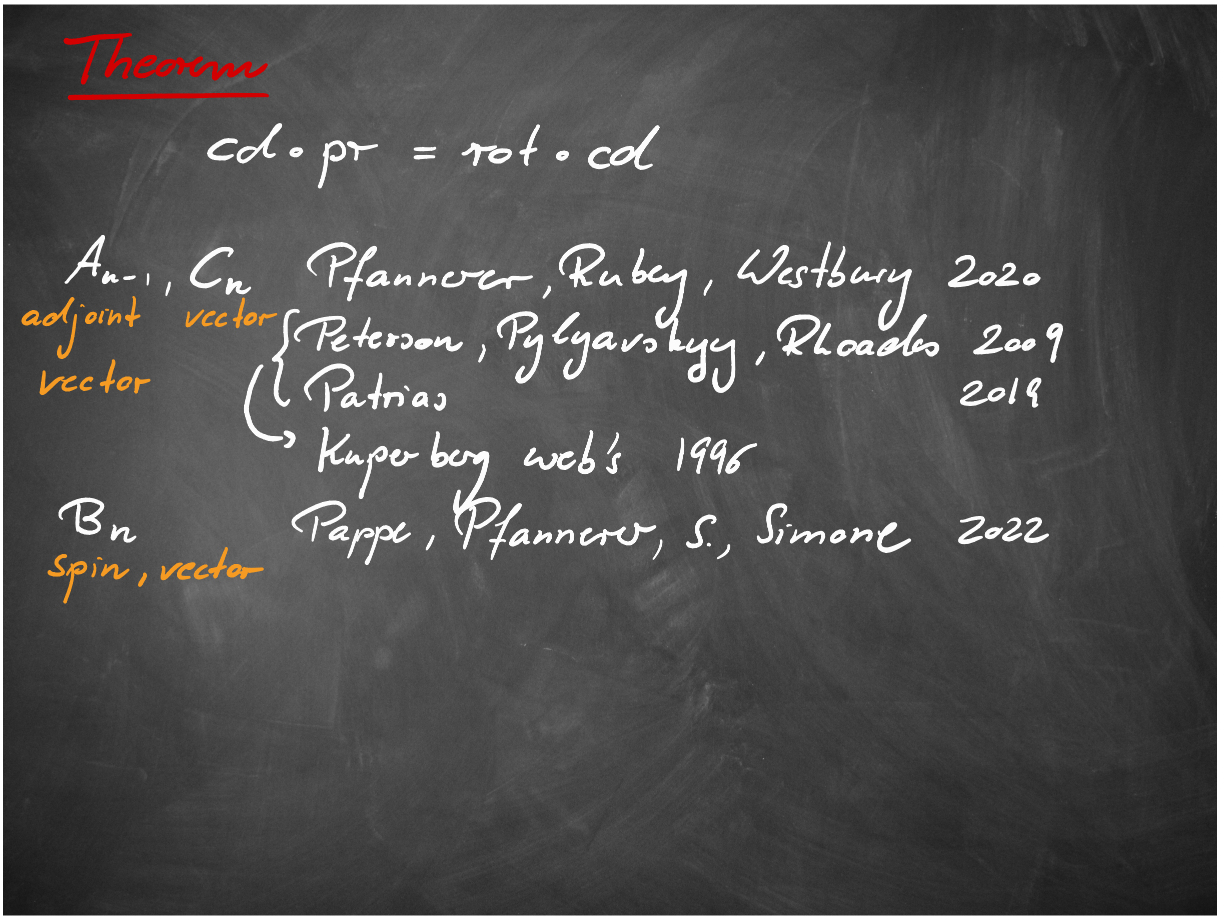
Bspin type Br
type B3
321 ∈ B□⊗3
type C3

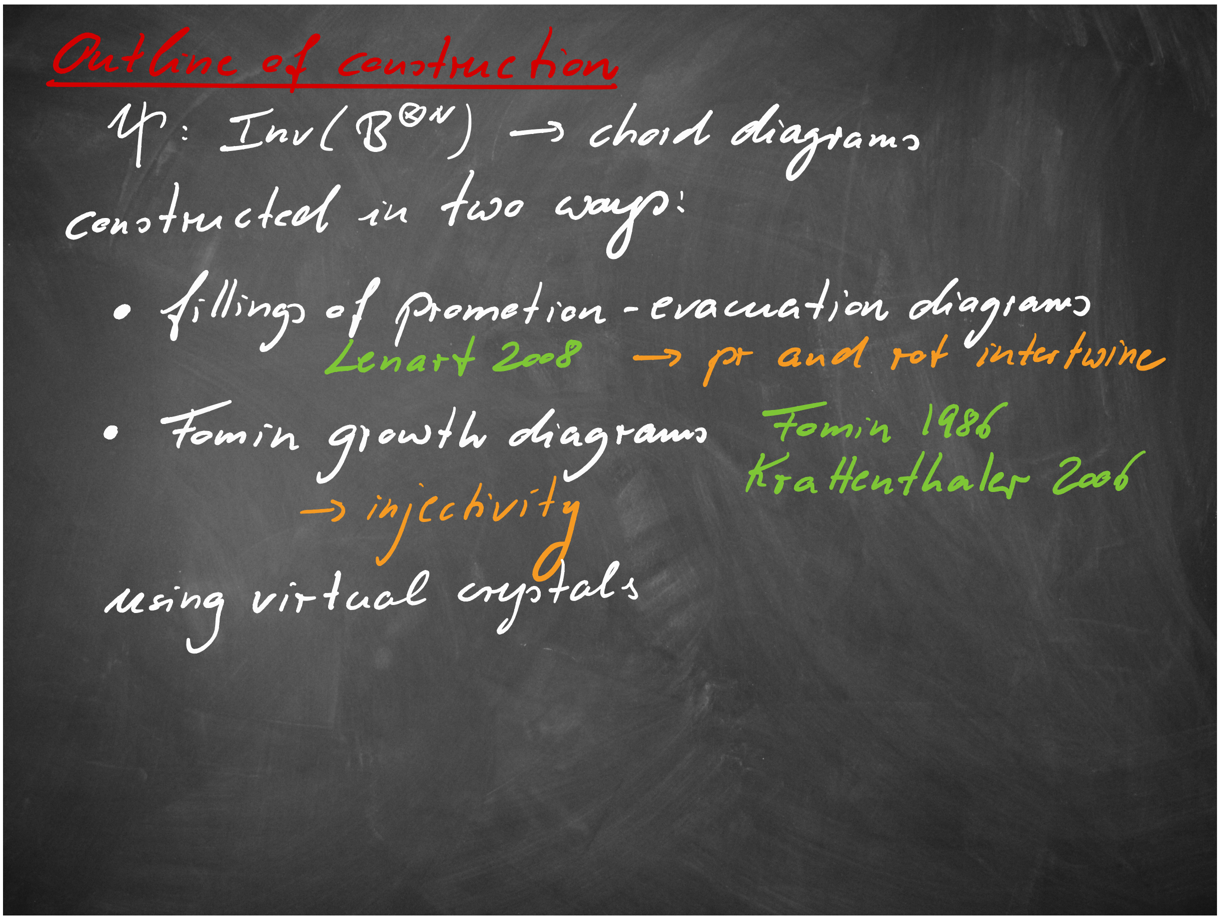


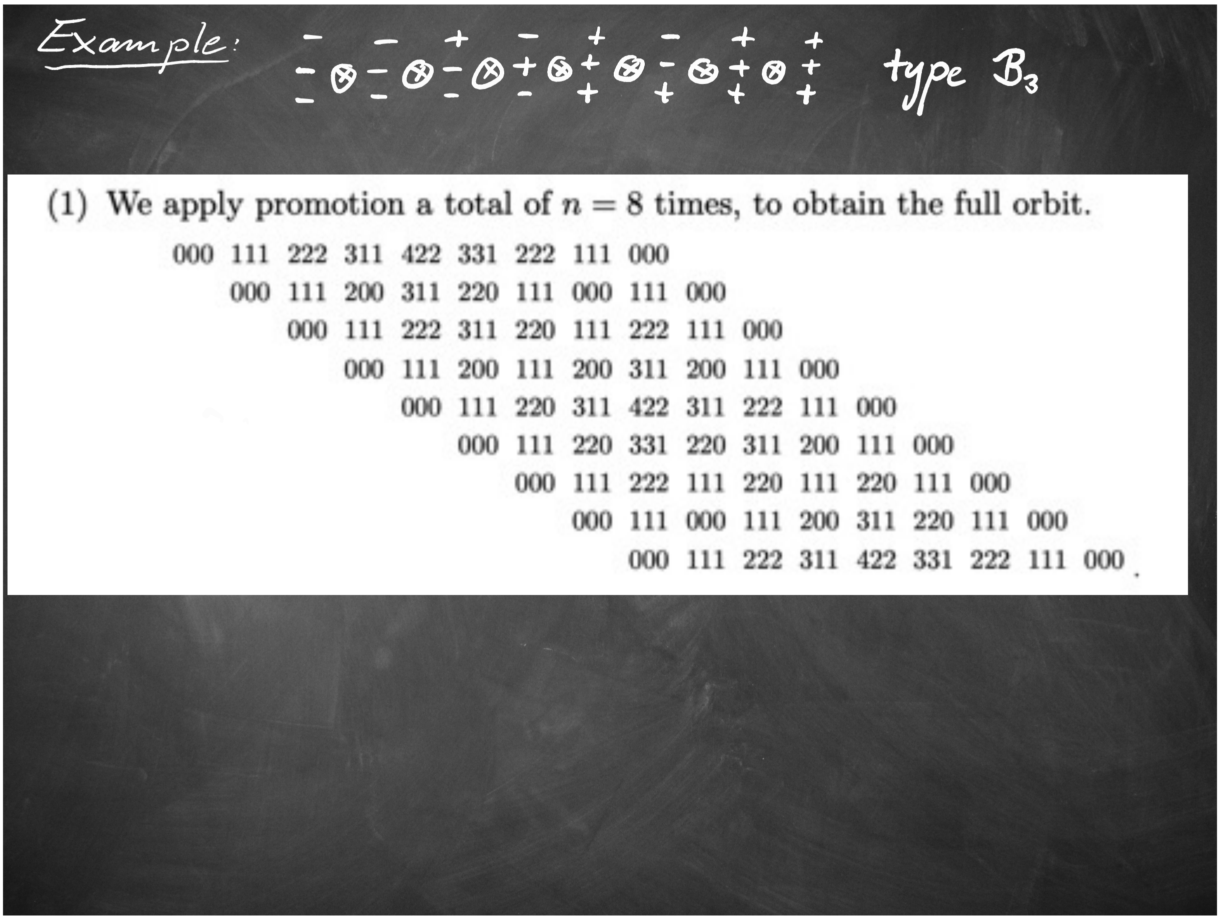
Promotion
Crystal commutor (Henriques, Kamnitzer 2006)
σ : Bλ ⊗ Bμ → Bμ ⊗ Bλ
b ⊗ c ↦ η(η(c) ⊗ η(b))
Lusztig involution
η : Bλ → Bλ
η maps highest weight to lowest weight
maps fi to ei, with w0(αi) = -αi
long element

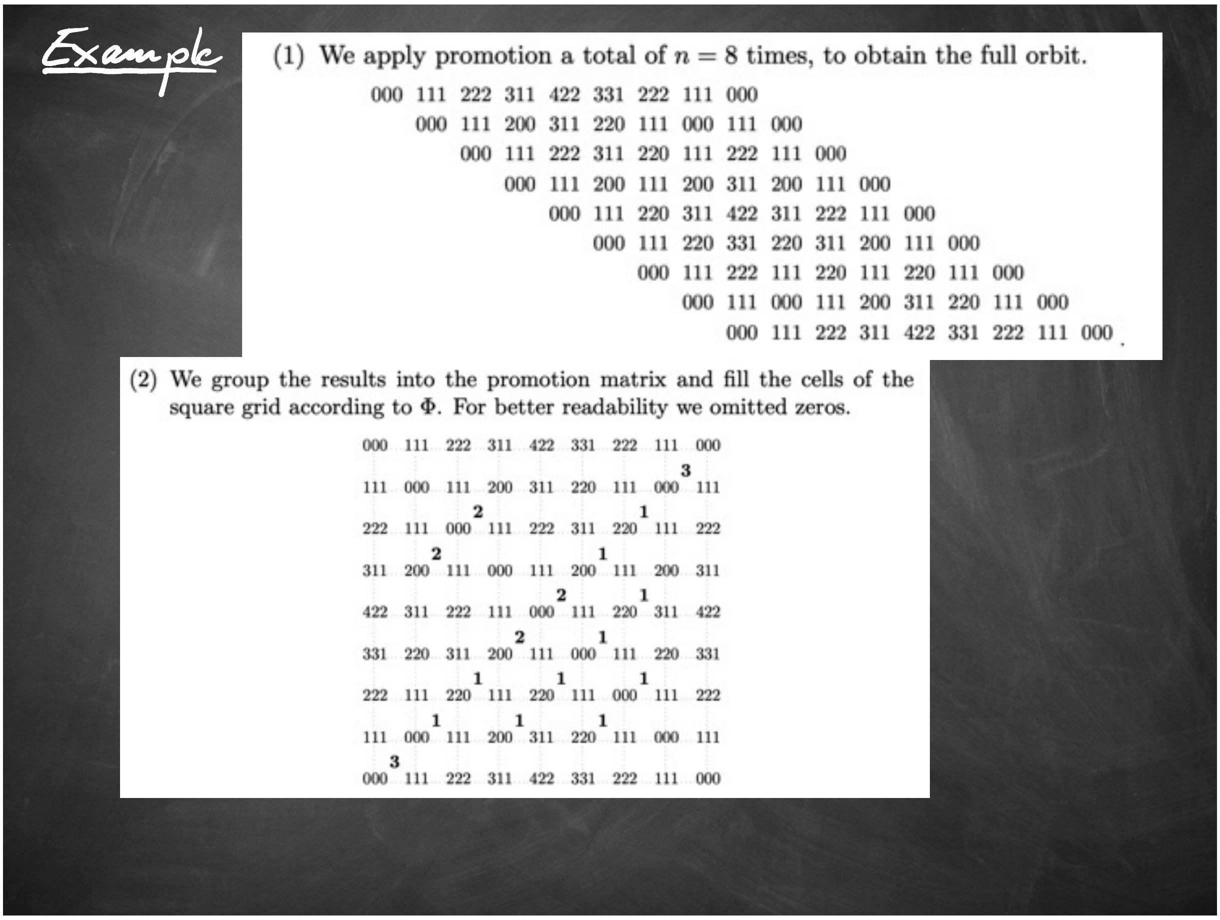

Definition
C crystal, u ∈ C^{⊗n} highest weight element
Then
pr(u) = δ_{C^{⊗n-1}, C}(u)
Example
2211 ∈ B_□^{⊗4} type A_1
B_□^{⊗3} B_□
η(221) = 121
η(1) = 2
⇒ pr(2211) = 2121
B_□ B_□^{⊗3}

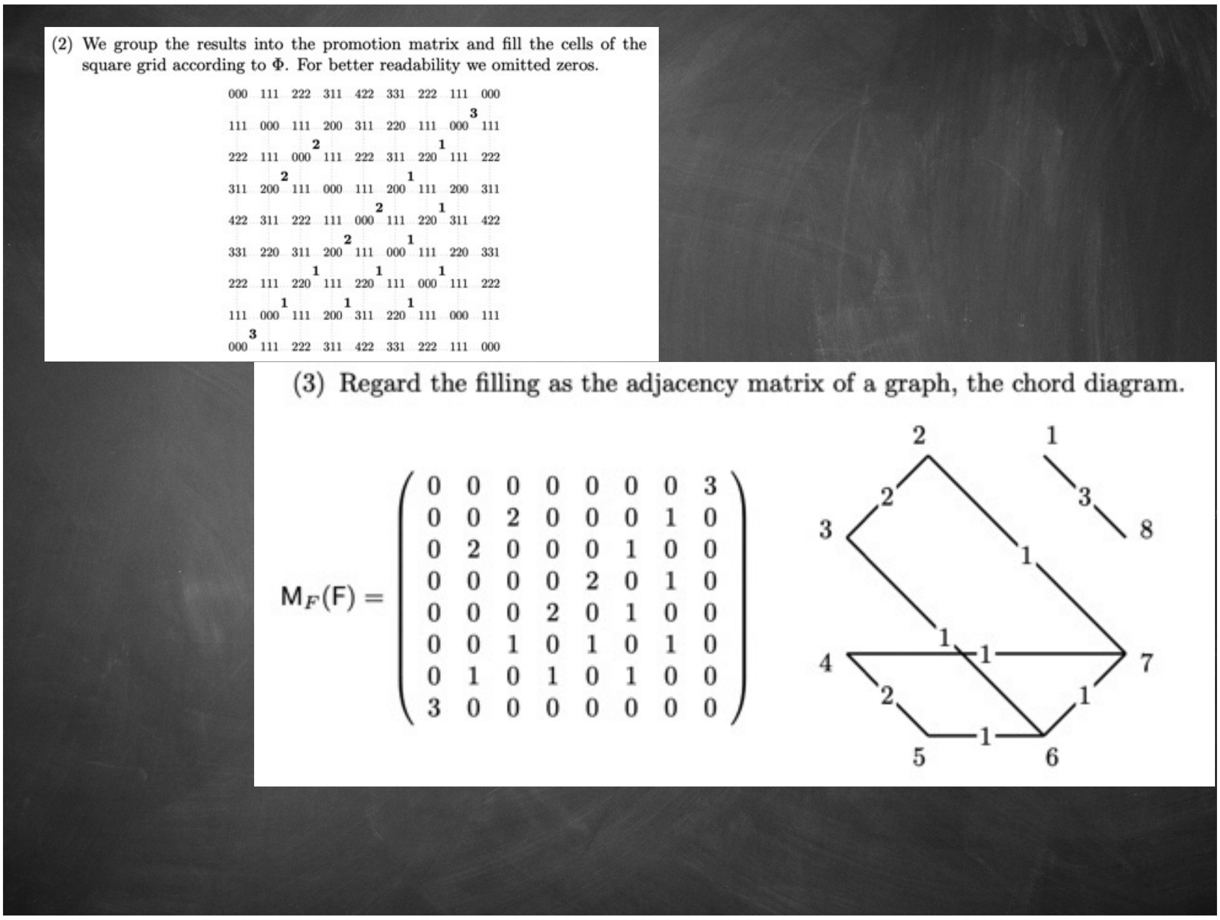

Promotion and rotation
cd: Inv(B^{⊗n}) → chord diagrams
Example
pr
pr
rot
rot
212121
221211
212211
221121
222111
rot
pr


Theorem
cd ∘ pr = rot ∘ cd
A_{n-1}, C_n Pfannerer, Rubey, Westbury 2020
adjoint vector Peterson, Pylyavskyy, Rhoades 2009
vector Patrias 2019
Kuperberg web's 1996
B_n Pappe, Pfannerer, S., Simone 2022
spin, vector


Outline of construction
Ψ : Inv(B^{⊗N}) → chord diagrams
constructed in two ways:
• fillings of promotion-evacuation diagrams
Lenart 2008 → pr and rot intertwine
• Fomin growth diagrams Fomin 1986
Krattenthaler 2006
→ injectivity
using virtual crystals


Example:
type B_3
(1) We apply promotion a total of $n = 8$ times, to obtain the full orbit.
000 111 222 311 422 331 222 111 000
000 111 200 311 220 111 000 111 000
000 111 222 311 220 111 222 111 000
000 111 200 111 200 311 200 111 000
000 111 220 311 422 311 222 111 000
000 111 220 331 220 311 200 111 000
000 111 222 111 220 111 220 111 000
000 111 000 111 200 311 220 111 000
000 111 222 311 422 331 222 111 000 .


Example
(1) We apply promotion a total of $n = 8$ times, to obtain the full orbit.
000 111 222 311 422 331 222 111 000
000 111 200 311 220 111 000 111 000
000 111 222 311 220 111 222 111 000
000 111 200 111 200 311 200 111 000
000 111 220 311 422 311 222 111 000
000 111 220 331 220 311 200 111 000
000 111 222 111 220 111 220 111 000
000 111 000 111 200 311 220 111 000
000 111 222 311 422 331 222 111 000 .
(2) We group the results into the promotion matrix and fill the cells of the square grid according to Φ. For better readability we omitted zeros.
000 111 222 311 422 331 222 111 000
111 000 111 200 311 220 111 000 111
222 111 000 111 222 311 220 111 222
311 200 111 000 111 200 111 200 311
422 311 222 111 000 111 220 311 422
331 220 311 200 111 000 111 220 331
222 111 220 111 220 111 000 111 222
111 000 111 200 311 220 111 000 111
000 111 222 311 422 331 222 111 000

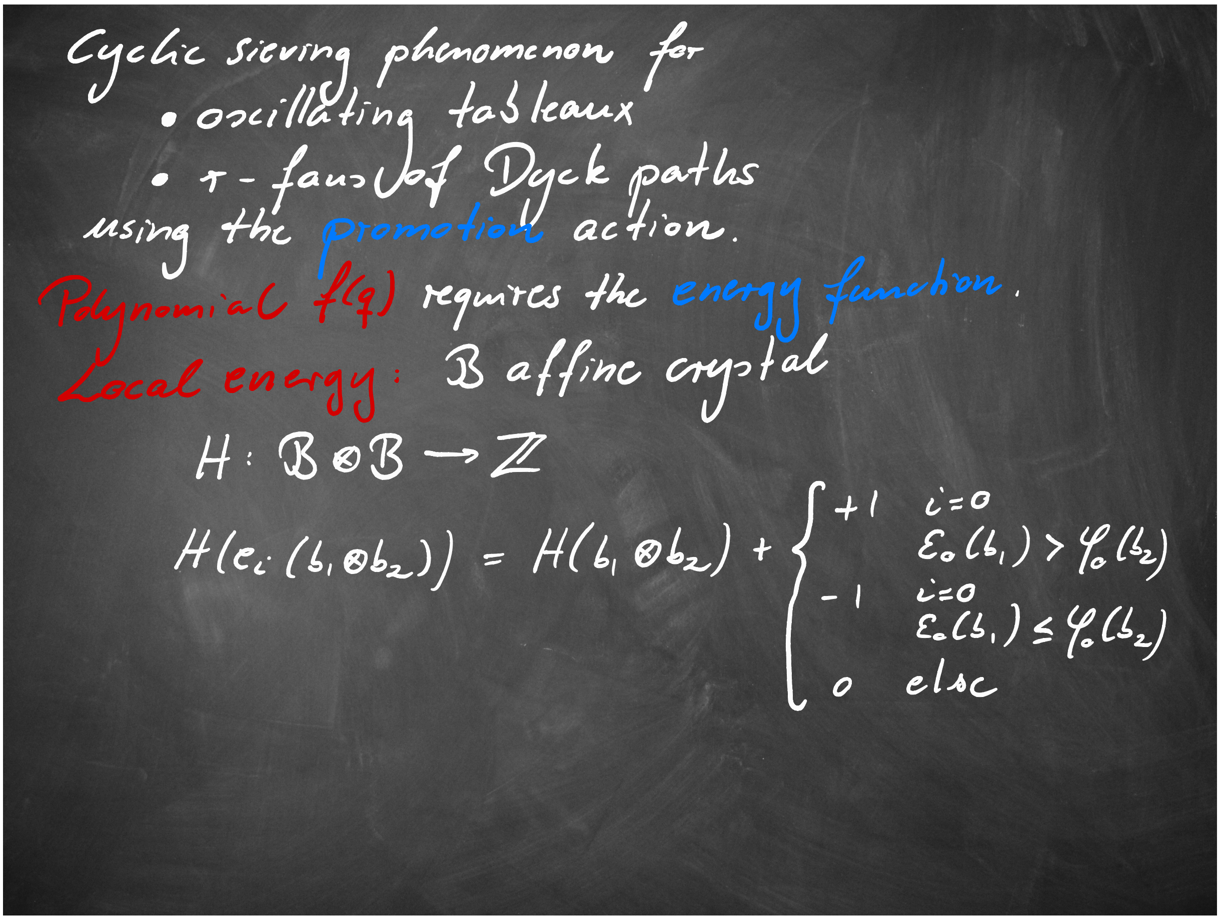


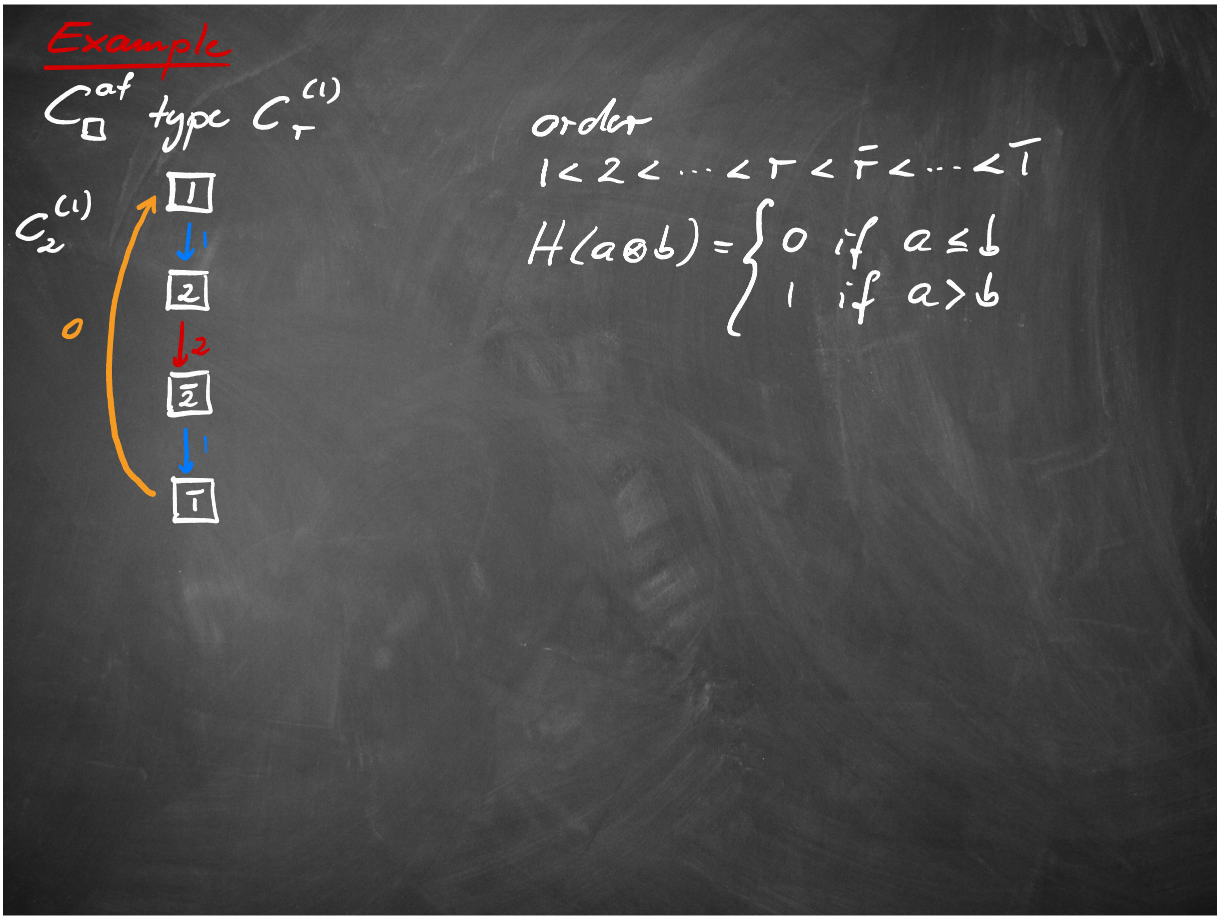
Cyclic sieving phenomenon

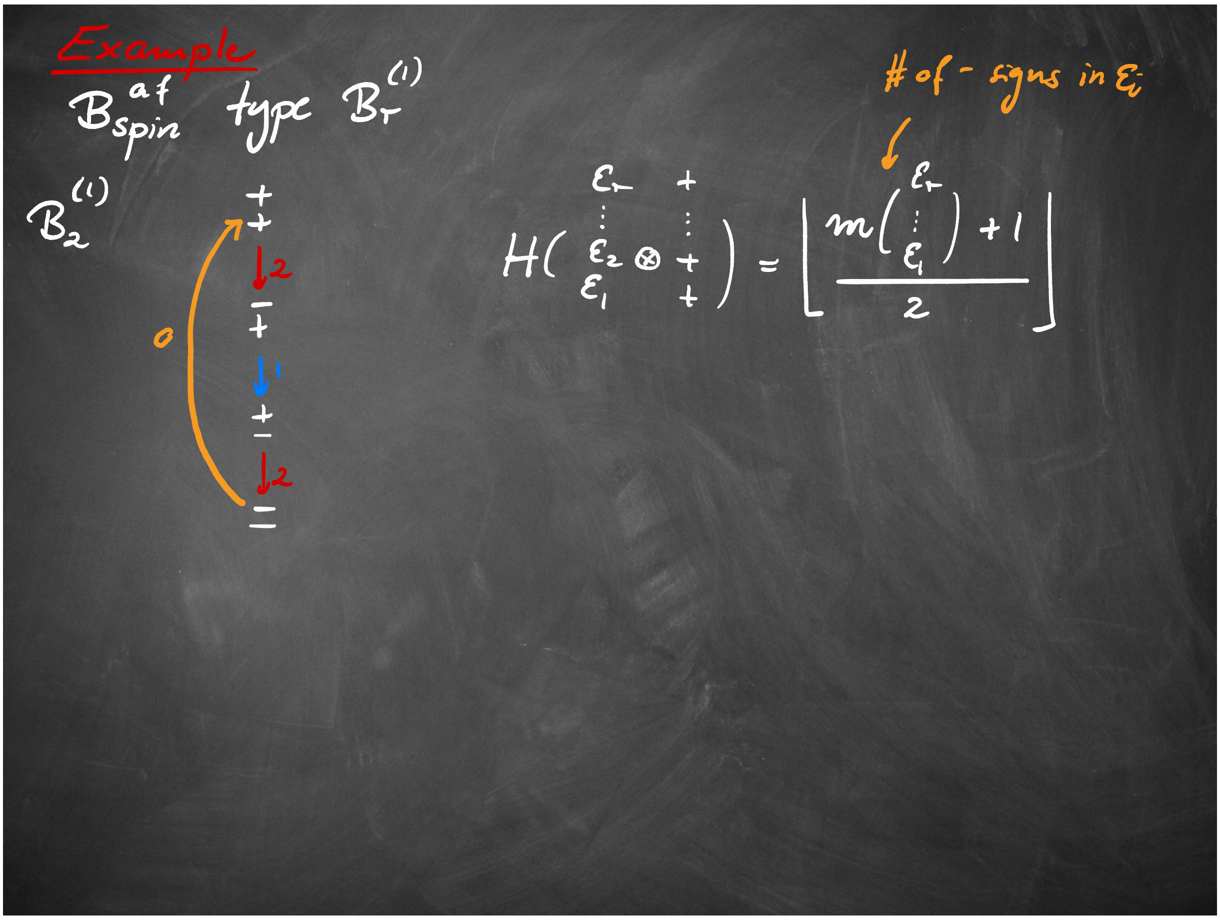
introduced by Reiner, Stanton, White 2004 as generalization of $q=-1$ phenomenon by Stembridge.

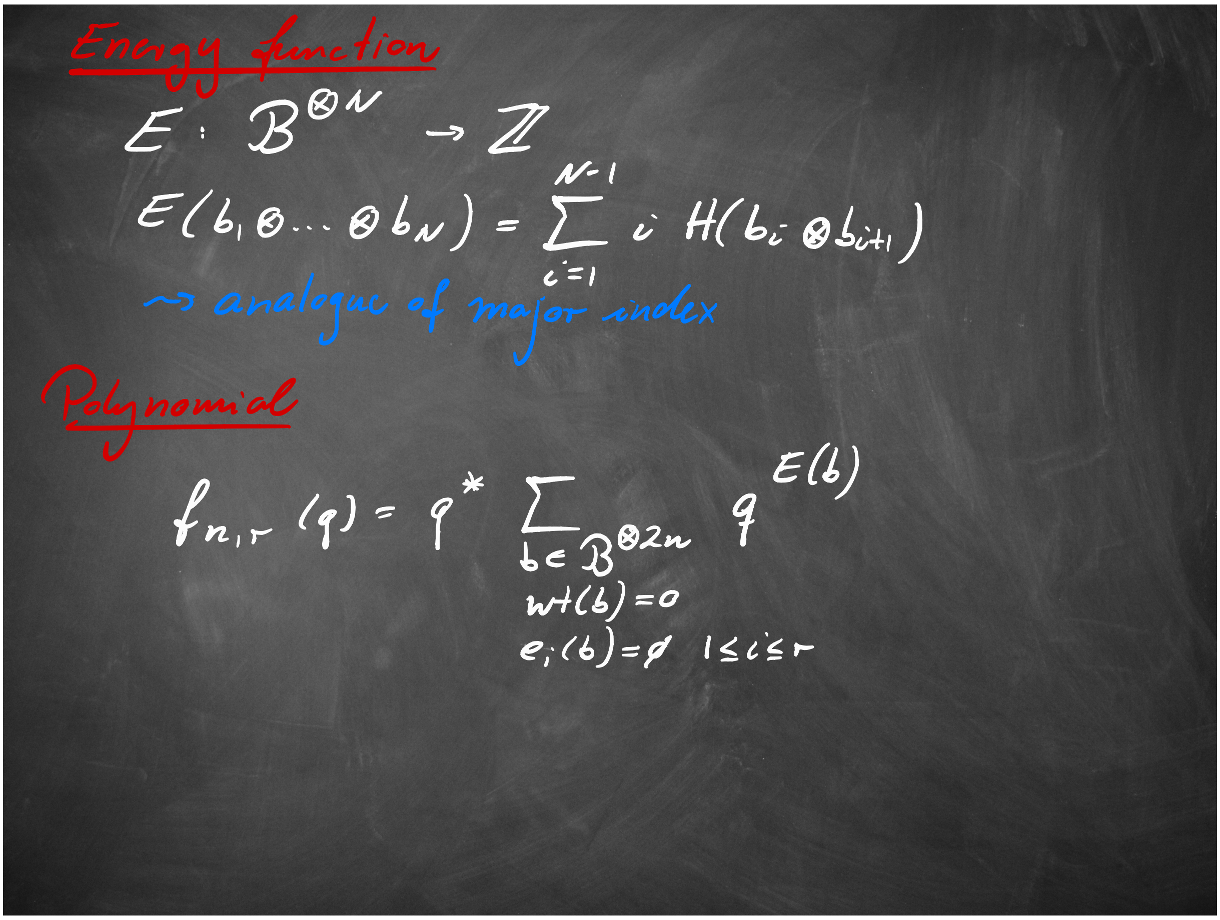
Def X finite set
$C = \langle c \rangle$ cyclic group acting on X
ζ is $|C|^{th}$ root of unity
$f(q) \in \mathbb{Z}[q]$

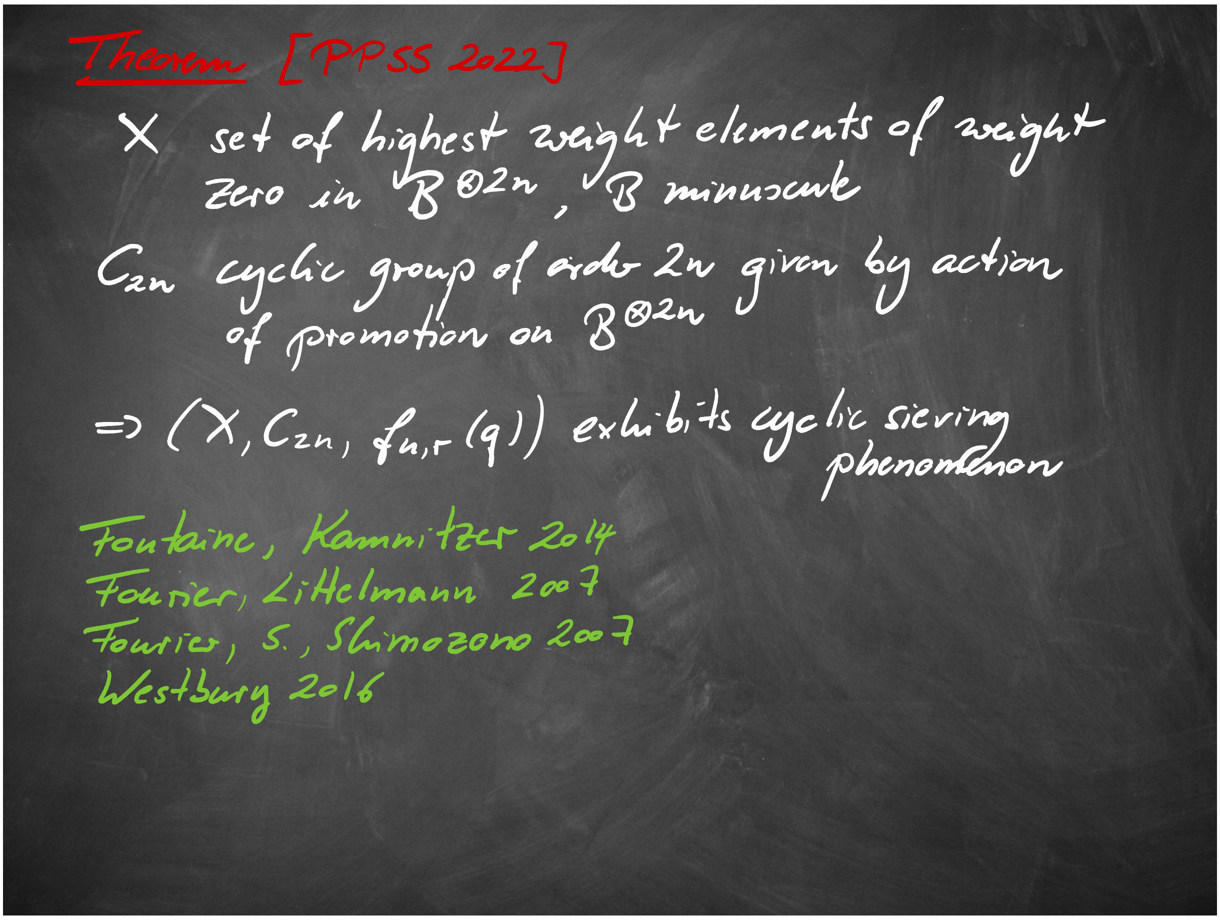
Then (X, C, f) exhibits the *cyclic sieving phenomenon*
if $|X^{c^d}| = f(\zeta^d)$ $\forall d \geq 0$
(X^{c^d}: fixed pt set under c^d)

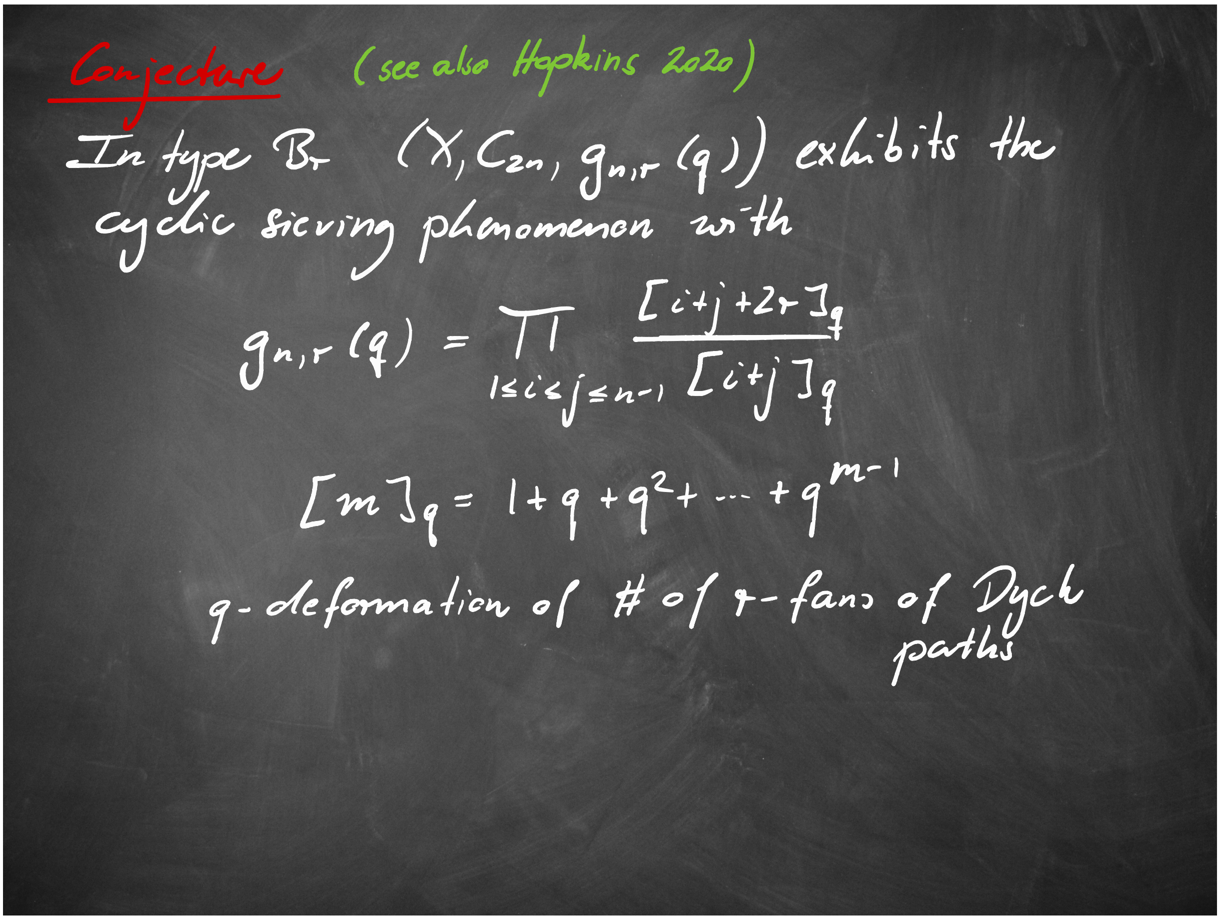

Cyclic sieving phenomenon for
• oscillating tableaux
• r-fans of Dyck paths
using the promotion action.
Polynomial f(q) requires the energy function.
Local energy: B affine crystal
H : B ⊗ B → Z
H(e_i (b_1 ⊗ b_2)) = H(b_1 ⊗ b_2) + { +1 i=0 ε_0(b_1) > φ_0(b_2); -1 i=0 ε_0(b_1) ≤ φ_0(b_2); 0 else

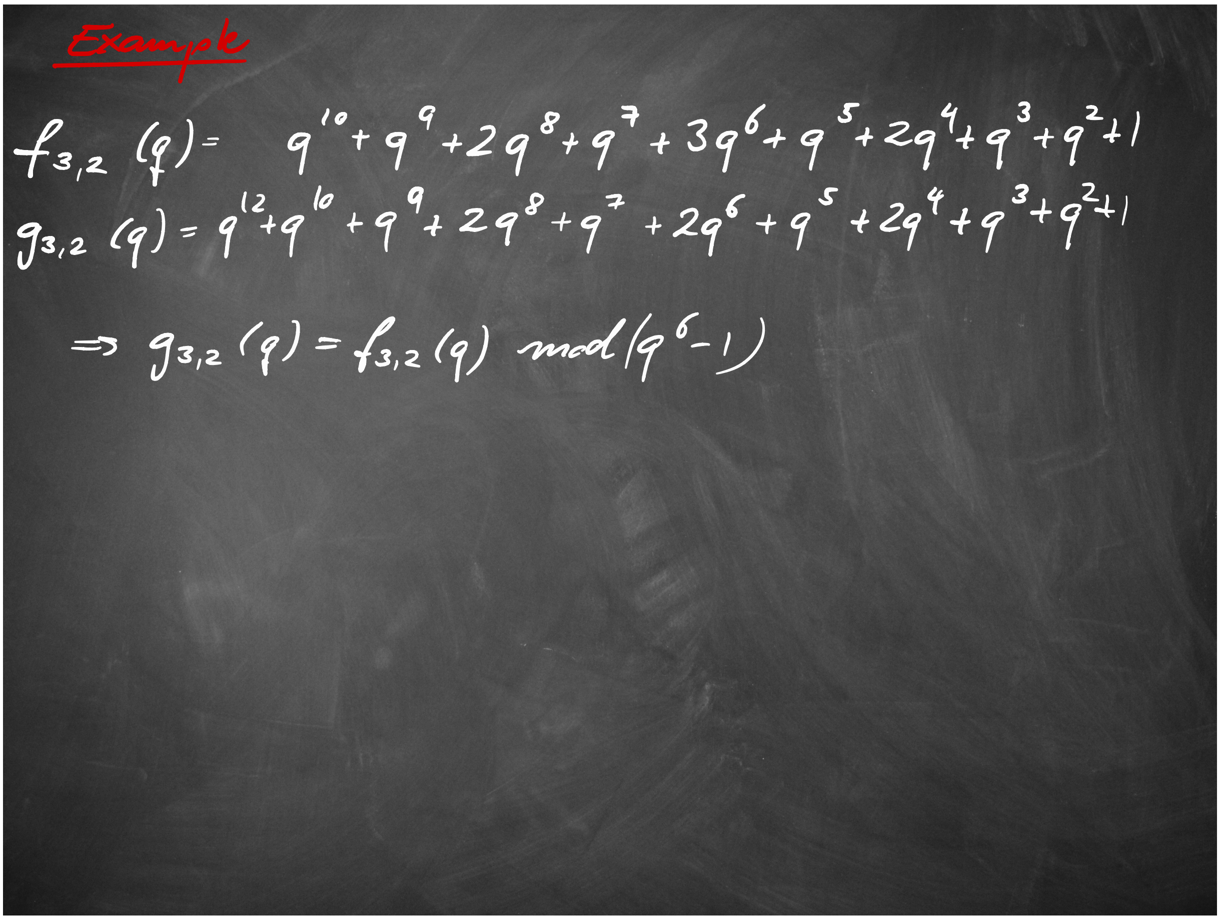

Example
C_□^af type C_r^(1)
order
1 < 2 < ... < r < r̄ < ... < 1̄
H(a⊗b) = { 0 if a ≤ b; 1 if a > b
C_2^(1)
1
2
2̄
1̄
0


Example
B^{af}_{spin} type $B_r^{(1)}$
$B_2^{(1)}$
of - signs in ε_i
$H\left(\begin{matrix}\varepsilon_r \\ \vdots \\ \varepsilon_2 \\ \varepsilon_1\end{matrix} \otimes \begin{matrix}+ \\ \vdots \\ + \\ +\end{matrix}\right) = \left\lfloor \frac{m\left(\begin{matrix}\varepsilon_r \\ \vdots \\ \varepsilon_1\end{matrix}\right)+1}{2} \right\rfloor$


Energy function
$E : B^{\otimes N} \to \mathbb{Z}$
$E(b_1 \otimes \cdots \otimes b_N) = \sum_{i=1}^{N-1} i\, H(b_i \otimes b_{i+1})$
⇝ analogue of major index
Polynomial
$f_{n,r}(q) = q^* \sum_{\substack{b \in B^{\otimes 2n} \\ wt(b)=0 \\ e_i(b)=\phi \ 1 \le i \le r}} q^{E(b)}$


Theorem [PPSS 2022]
X set of highest weight elements of weight zero in $B^{\otimes 2n}$, B minuscule
C_{2n} cyclic group of order 2n given by action of promotion on $B^{\otimes 2n}$
$\Rightarrow (X, C_{2n}, f_{n,r}(q))$ exhibits cyclic sieving phenomenon
Fontaine, Kamnitzer 2014
Fourier, Littelmann 2007
Fourier, S., Shimozono 2007
Westbury 2016


Conjecture (see also Hopkins 2020)
In type B_r $(X, C_{2n}, g_{n,r}(q))$ exhibits the cyclic sieving phenomenon with
$g_{n,r}(q) = \prod_{1 \le i \le j \le n-1} \frac{[i+j+2r]_q}{[i+j]_q}$
$[m]_q = 1 + q + q^2 + \cdots + q^{m-1}$
q-deformation of # of r-fans of Dyck paths


Example
$f_{3,2}(q) = q^{10}+q^9+2q^8+q^7+3q^6+q^5+2q^4+q^3+q^2+1$
$g_{3,2}(q) = q^{12}+q^{10}+q^9+2q^8+q^7+2q^6+q^5+2q^4+q^3+q^2+1$
$\Rightarrow g_{3,2}(q) = f_{3,2}(q) \mod (q^6-1)$


Thank you!

Lecture 3: Diagram algebras, insertion algorithms, and plethysm

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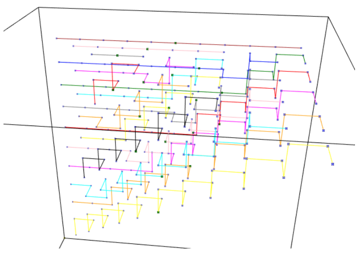
Department of Mathematics, UC Davis

based on joint work with Rosa Orellana (Dartmouth), Franco Saliola (UQAM), Mike Zabrocki (York), Algebraic Combinatorics (2022)
OSSZ, Laura Colmenarejo (NCSU) arXiv:2208.07258
COSSZ J. Algebra (2020)



Integrable systems and quantum groups
Osaka, Japan
March 8, 2023

RSK Application: Diagram algebras Uniform block permutation algebra The plethysm problem

Goal

- Exploration of variants of RSK
  - Insertion of multisets instead of integers
  - Enumerative manifestations of double centralizer theorem:

$$V = \bigoplus_{\lambda} V_\lambda = \bigoplus_{\lambda} U_\lambda \otimes W_\lambda \qquad \text{operators } \mathcal{A}, \mathcal{B} \text{ acting}$$

  $\mathcal{A}$ only acting on U_λ, $\mathcal{B}$ only acting on W_λ
- Applications to partition algebras
  - Insertion
    partition diagrams $\longrightarrow$ (standard tableau, multiset-valued tableau)
  - Well behaved with respect to subalgebras
  - dimensions of irreducibles = number of tableaux
- Uniform block permutation algebra $\to$ plethysm

Outline

The Robinson–Schensted–Knuth correspondence

- Robinson 1938: permutations in S_n
  $\longrightarrow \bigcup_{\lambda \vdash n} \mathsf{SYT}(\lambda) \times \mathsf{SYT}(\lambda)$
- Schensted 1961: words of length n in $[k] = \{1, 2, \ldots, k\}$
  $\longrightarrow \bigcup_{\lambda \vdash n} \mathsf{SSYT}_{[k]}(\lambda) \times \mathsf{SYT}(\lambda)$
- Knuth 1970: generalized permutations over $[n]$ and $[k]$ of length ℓ
  $\longrightarrow \bigcup_{\lambda \vdash \ell} \mathsf{SSYT}_{[k]}(\lambda) \times \mathsf{SSYT}_{[n]}(\lambda)$

Generalized permutations

A, B ordered alphabets (i.e. $A = [n]$, $B = [k]$)

Definition

A generalized permutation is a two-line array $w = \begin{pmatrix} a_1 & a_2 & \cdots & a_\ell \\ b_1 & b_2 & \cdots & b_\ell \end{pmatrix}$ such that

- $a_1, \ldots, a_\ell \in A$, $b_1, \ldots, b_\ell \in B$
- $a_i \leqslant_A a_{i+1}$ for $1 \leqslant i \leqslant \ell - 1$
- $b_i \leqslant_B b_{i+1}$ whenever $a_i = a_{i+1}$

Example

Generalized permutation from [6] to [5]:

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 6 & 6 & 6 \\ 1 & 5 & 5 & 2 & 3 & 1 & 3 & 5 & 5 & 1 & 1 & 2 & 3 \end{pmatrix}$$

Row insertion

$$\begin{array}{|c|c|c|c|c|c|} \hline 6 \\ \cline{1-1} 5 & 7 \\ \cline{1-2} 4 & 5 & 6 \\ \cline{1-3} 2 & 3 & 4 & 4 \\ \cline{1-4} 1 & 1 & 2 & 2 & 3 & 5 \\ \hline \end{array} \quad \longleftarrow \quad 2$$

$$= \begin{array}{|c|c|c|c|c|c|} \hline 6 & 7 \\ \cline{1-2} 5 & 5 \\ \cline{1-2} 4 & 4 & 6 \\ \cline{1-3} 2 & 3 & 3 & 4 \\ \cline{1-4} 1 & 1 & 2 & 2 & 2 & 5 \\ \hline \end{array}$$

RSK correspondence

generalized permutation $w = \begin{pmatrix} a_1 & a_2 & \cdots & a_\ell \\ b_1 & b_2 & \cdots & b_\ell \end{pmatrix}$
Row insert $b_1, b_2, \ldots, b_\ell$ one by one
Record new box when inserting b_i by a_i

Theorem (Knuth 1970)

$\exists$ *bijection*

$$\text{generalized permutation from } A \text{ to } B \mapsto (P, Q)$$

- $\mathrm{shape}(P) = \mathrm{shape}(Q)$
- *P is semistandard tableau with entries in B*
- *Q is semistandard tableau with entries in A*

RSK and representation theory

Schensted 1961

- Combinatorial bijection

$$\{\text{words of length } n \text{ in } [k]\} \longrightarrow \bigcup_{\lambda \vdash n} \mathrm{SSYT}_{[k]}(\lambda) \times \mathrm{SYT}(\lambda)$$

- Enumerative result

$$k^n = \sum_{\lambda \vdash n} \#\mathrm{SSYT}_{[k]}(\lambda) \cdot \#\mathrm{SYT}(\lambda)$$

- Representation theory interpretation
  $GL_k \times S_n$-module $V^{\otimes n}$ where $V = \mathbb{C}^k$ (commuting actions)

$$V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} W_k^\lambda \otimes S^\lambda$$

  W_k^λ is a simple left GL_k-module
  S^λ is a simple right S_n-module

Outline

Variant

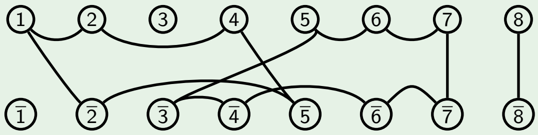
- Encoding of partition diagrams as generalized permutations with multisets
- RSK algorithm gives pairs of standard multiset tableaux
- Well behaved with respect to subalgebras
- Matches the representation theory and dimensions of Halverson and Jacobson (2018)
- New map from standard multiset tabelaux to Bratteli diagrams (different from Benkart and Halverson (2017))

Partition diagrams

Partition of two alphabets $[k]$ and $[\overline{k}]$

Example

$\pi = \{\{1,2,4,\overline{2},\overline{5}\},\{3\},\{5,6,7,\overline{3},\overline{4},\overline{6},\overline{7}\},\{8,\overline{8}\},\{\overline{1}\}\}$ represented by:

$\pi =$ 

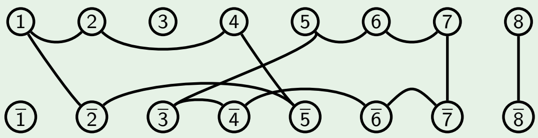
Partition algebra

$$P_k(n) = \text{span}_{\mathbb{C}}\{\pi \mid \pi \vdash [k] \cup [\overline{k}]\}$$

(Non)propagating blocks

Example

$\pi = \{\{1,2,4,\overline{2},\overline{5}\},\{3\},\{5,6,7,\overline{3},\overline{4},\overline{6},\overline{7}\},\{8,\overline{8}\},\{\overline{1}\}\}$ represented by:

$\pi =$ 

A block is propagating if it contains vertices from both $[k]$ and $[\overline{k}]$.

Example

$\{1,2,4,\overline{2},\overline{5}\}$ is propagating.

Otherwise, the block is non-propagating.

Example

$\{3\}$ and $\{\overline{1}\}$ are non-propagating.

The correspondence

$\pi = \{\pi_1, \pi_2, \ldots, \pi_r\}$ set partition of $[k] \cup [\overline{k}]$

Order: last letter order

- $\pi_{j_1}, \pi_{j_2}, \ldots, \pi_{j_p}$ propagating blocks of π ordered as $\pi_{j_1}^+ < \cdots < \pi_{j_p}^+$, where $\pi_j^+ = \pi_j \cap [k]$ and $\pi_j^- = \pi_j \cap [\overline{k}]$
- $\sigma_{i_1}, \ldots, \sigma_{i_a} \subseteq [k]$ non-propagating blocks in $[k]$ ordered as $\sigma_{i_1} < \cdots < \sigma_{i_a}$
- $\tau_{i_1}, \ldots, \tau_{i_b} \subseteq [\overline{k}]$ non-propagating blocks in $[\overline{k}]$ ordered as $\tau_{i_1} < \cdots < \tau_{i_b}$

$$(P, Q) = RSK \begin{pmatrix} \pi_{j_1}^+ & \pi_{j_2}^+ & \cdots & \pi_{j_p}^+ \\ \pi_{j_1}^- & \pi_{j_2}^- & \cdots & \pi_{j_p}^- \end{pmatrix}$$

$T = P$ by adjoining row containing $n - p - b$ empty cells followed by $\tau_{i_1}, \ldots, \tau_{i_b}$

$S = Q$ by adjoining row containing $n - p - a$ empty cells followed by $\sigma_{i_1}, \ldots, \sigma_{i_a}$

The correspondence – example

Example

$\pi = \{\{2,3,4,\overline{4},\overline{5}\}, \{5,\overline{2},\overline{3}\}, \{1,6,\overline{7},\overline{8}\}, \{7,8\}, \{9,\overline{6}\}, \{\overline{1}\}, \{\overline{9}\}\} \in P_9(18)$

$$\begin{pmatrix} \pi_{j_1}^+ & \pi_{j_2}^+ & \cdots & \pi_{j_p}^+ \\ \pi_{j_1}^- & \pi_{j_2}^- & \cdots & \pi_{j_p}^- \end{pmatrix} = \begin{pmatrix} \{2,3,4\} & \{5\} & \{1,6\} & \{9\} \\ \{\overline{4},\overline{5}\} & \{\overline{2},\overline{3}\} & \{\overline{7},\overline{8}\} & \{\overline{6}\} \end{pmatrix}$$

Apply RSK:

$$P = \begin{array}{|c|c|} \hline \overline{45} & \overline{78} \\ \hline \overline{23} & \overline{6} \\ \hline \end{array} \qquad Q = \begin{array}{|c|c|} \hline 5 & 9 \\ \hline 234 & 16 \\ \hline \end{array}$$

Adjoin new rows:

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline \overline{45} & \overline{78} \\ \cline{1-2} \overline{23} & \overline{6} \\ \hline & & & & & & & & & & & & \overline{1} & \overline{9} \\ \hline \end{array} \qquad S = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 5 & 9 \\ \cline{1-2} 234 & 16 \\ \hline & & & & & & & & & & & & & 78 \\ \hline \end{array}$$

RSK | Application: Diagram algebras | Uniform block permutation algebra | The plethysm problem

The correspondence – Theorem

$\mathrm{SMT}_{[k]}(\lambda)$ = set of standard multiset tableaux over alphabet $[k]$

Theorem (COSSZ'20)

Let $n \geqslant 2k$. $\exists$ bijection

$$\Psi : \{\textit{set partitions of } [k] \cup [\overline{k}]\} \longrightarrow \bigcup_{\lambda \vdash n} \mathrm{SMT}_{[\overline{k}]}(\lambda) \times \mathrm{SMT}_{[k]}(\lambda)$$

Enumerative result

$$B(2k) = \sum_{\lambda \vdash n} \#\mathrm{SMT}_{[k]}(\lambda)^2$$

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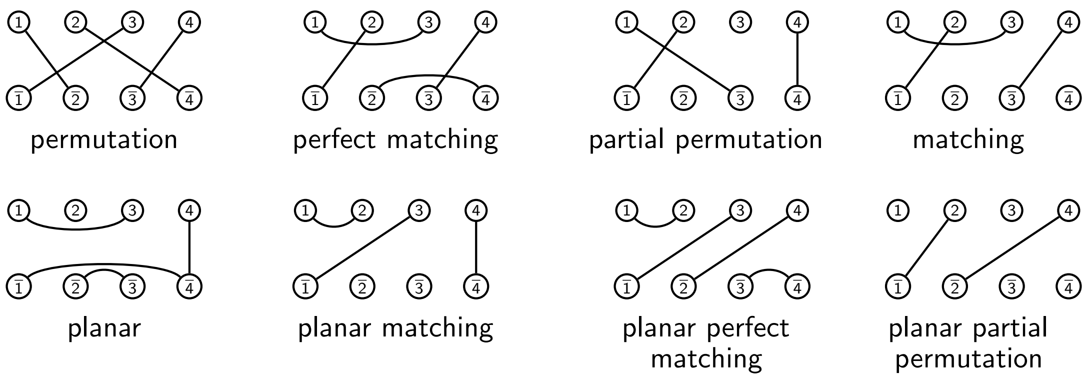
Restriction to subalgebras

Subclasses of set partitions



Subalgebras of the partition algebra $P_k(n)$

Subalgebra A_k	Diagrams spanning the subalgebra	Dimension
Partition algebra $P_k(n)$	all diagrams	$B(2k)$
Group algebra of symmetric group $\mathbb{C}S_k$	permutations	$k!$
Brauer algebra $B_k(n)$	perfect matchings	$(2k-1)!!$
Rook algebra $R_k(n)$	partial permutations	$\sum_{i=0}^{k} \binom{k}{i}^2 i!$
Rook-Brauer algebra $RB_k(n)$	matchings	$\sum_{i=0}^{k} \binom{2k}{2i} (2i-1)!!$
Temperley–Lieb algebra $TL_k(n)$	planar perfect matchings	$\frac{1}{k+1} \binom{2k}{k}$
Motzkin algebra $M_k(n)$	planar matchings	$\sum_{i=0}^{k} \frac{1}{i+1} \binom{2i}{i} \binom{2k}{2i}$
Planar rook algebra $PR_k(n)$	planar partial permutations	$\binom{2k}{k}$
Planar algebra $PP_k(n)$	planar diagrams	$\frac{1}{2k+1} \binom{4k}{2k}$

Properties under Ψ

A_k subalgebra of partition algebra
$\mathrm{SMT}_{A_k}(\lambda)$ set of standard multiset-valued tableaux under Ψ for A_k

Definition

$T \in \mathrm{SMT}_{A_k}(\lambda)$

- T is matching if the first row contains sets of size less than or equal to 2 and all other rows contain only sets of size 1.
- Two sets S and S' are non-crossing if there do not exist elements $a, b \in S$ and $c, d \in S'$ such that $a < c < b < d$ or $c < a < d < b$.
- We say that $c \in [k]$ is between a set S if there exist $a, b \in S$ such that $a < c < b$.
- T is planar if
  - it has two rows
  - the sets in the first row are pairwise non-crossing
  - no element belonging to one of the sets in the second row is between any set in the tableau

Tableaux for subalgebras

Under the bijection Ψ, the tableaux are characterized as follows:

A_k	diagrams spanning A_k	properties characterizing SMT_{A_k}	
		sizes of entries in first row	other properties
$P_k(n)$	all diagrams	—	—
$PP_k(n)$	planar diagrams	—	planar
$\mathbb{C}S_k$	permutations	0	matching
$B_k(n)$	perfect matchings	0, 2	matching
$R_k(n)$	partial permutations	0, 1	matching
$RB_k(n)$	matchings	0, 1, 2	matching
$TL_k(n)$	planar perfect matchings	0, 2	matching & planar
$M_k(n)$	planar matchings	0, 1, 2	matching & planar
$PR_k(n)$	planar partial permutations	0, 1	matching & planar

Tableaux for subalgebras

Corollary

Let $n \geqslant 2k$ and $\lambda \vdash n$. For each of the algebras A_k let $V_{A_k}^{\overline{\lambda}}$ be the irreducible A_k-representation indexed by $\overline{\lambda}$. Then

$$\dim\left(V_{A_k}^{\overline{\lambda}}\right) = \#\mathsf{SMT}_{A_k}(\lambda).$$

Corollary

If $n \geqslant 2k$, then for each subalgebra A_k of the partition algebra $P_k(n)$, we have

$$\dim(A_k) = \sum_{\lambda \vdash n} \left(\#\mathsf{SMT}_{A_k}(\lambda)\right)^2.$$

Diagram algebras

- Restrict diagonal action of GL_n on $V^{\otimes k}$ to $S_n \subseteq GL_n$: for $\sigma \in S_n$

$$\sigma(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = \sigma v_{i_1} \otimes \cdots \otimes \sigma v_{i_k}$$

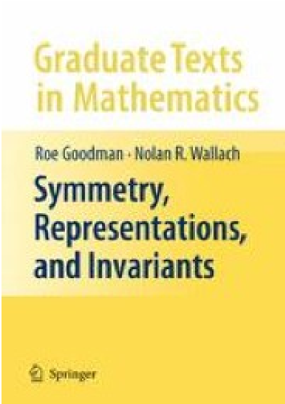
- What commutes with this action?
  Answer: Partition algebra $P_k(n)$ Martin, Jones 1990s
- Basis: set partitions of $\{1, 2, \ldots, k\} \cup \{\overline{1}, \overline{2}, \ldots, \overline{k}\}$

Remark

- S_k and GL_n form a centralizer pair
- $P_k(n)$ and S_n form a centralizer pair

Martin and Jones

See-Saw pairs



(See book by Goodman, Wallach)

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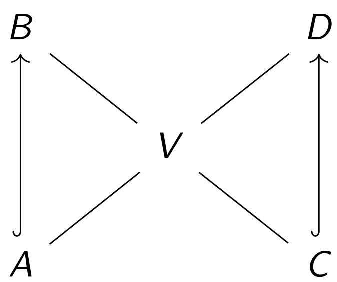
See-Saw pairs

$A \hookrightarrow B$ algebra embedding $\qquad \mathrm{Res}^B_A \; V^\lambda_B = \bigoplus_\mu \left(V^\mu_A\right)^{\oplus c_{\lambda\mu}}$

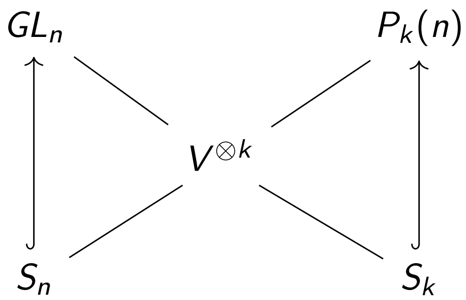


- B and C centralizer pair
- A and D centralizer pair

1. Indices for the simple modules for B and C are the same.
2. Indices for the simple modules for A and D are the same.

$$\mathrm{Res}^D_C \; V^\mu_D = \bigoplus_\lambda \left(V^\lambda_C\right)^{\oplus c_{\lambda\mu}}$$

Our See-Saw pair



$$\mathrm{Res}^{GL_n}_{S_n} V^{\lambda}_{GL_n} = \bigoplus_{\mu} \left(V^{\mu}_{S_n}\right)^{\oplus r_{\lambda\mu}}$$

$$\mathrm{Res}^{P_k(n)}_{S_k} V^{\mu}_{P_k(n)} = \bigoplus_{\lambda} \left(V^{\lambda}_{S_k}\right)^{\oplus r_{\lambda\mu}}$$

Idea: Restrict representations of $P_k(n)$ to S_k

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The approach

$\mathcal{U}_k$ uniform block permutation algebra

$$\underbrace{S_k \hookrightarrow}_{\text{special cases of plethysm}} \quad \mathcal{U}_k \quad \underbrace{\hookrightarrow P_k(n)}_{\text{generalized LR coefficients}}$$

Goal: Combinatorial model for the representation theory of $\mathcal{U}_k$

Outline

1. RSK algorithm and representation theory (review)
2. Application: Diagram algebras
3. Uniform block permutation algebra
4. The plethysm problem

Uniform block permutations

Tanabe 1997, Kosuda 2006

Party algebra, centralizer algebra for complex reflection groups

Definition

The set partition $d = \{d_1, d_2, \ldots, d_\ell\}$ of $[k] \cup [\bar{k}]$ is uniform if $|d_i \cap [k]| = |d_i \cap [\bar{k}]|$ for all $1 \leqslant i \leqslant \ell$. Let

$$\mathcal{U}_k = \big\{d \vdash [k] \cup [\bar{k}] : d \text{ uniform}\big\}.$$

Example

$$d = \{\{2, \bar{4}\}, \{5, \bar{7}\}, \{1, 3, \bar{1}, \bar{2}\}, \{4, 6, \bar{3}, \bar{6}\}, \{7, 8, 9, \bar{5}, \bar{8}, \bar{9}\}\}$$

Think of d as a size-preserving bijection

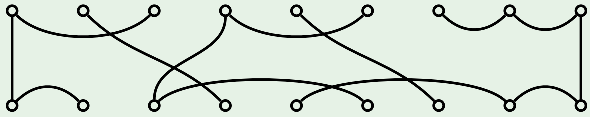
$$\begin{pmatrix} \{2\} & \{5\} & \{1,3\} & \{4,6\} & \{7,8,9\} \\ \{4\} & \{7\} & \{1,2\} & \{3,6\} & \{5,8,9\} \end{pmatrix}$$

$\Rightarrow$ Elements of $\mathcal{U}_k$ are called uniform block permutations

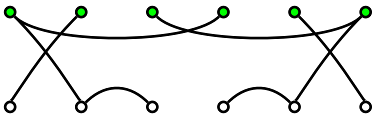
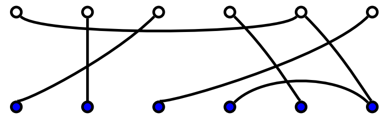
Uniform block permutations – continued

Example

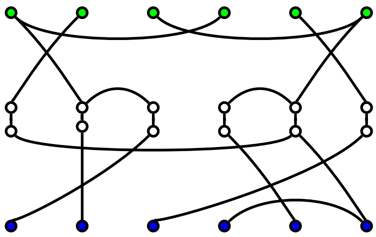
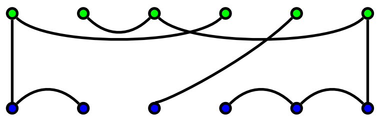
Diagram for $\{\{1,3,\bar{1},\bar{2}\},\{2,\bar{4}\},\{4,6,\bar{3},\bar{6}\},\{5,\bar{7}\},\{7,8,9,\bar{5},\bar{8},\bar{9}\}\}$



The product of

$d =$  and $d' =$ 

is obtained by stacking the diagrams of d and d':

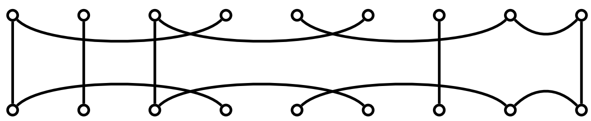
$dd' =$  $=$ 

Idempotents

For every set partition π of $[k]$ we define:

$$e_\pi = \{A \cup \bar{A} : A \in \pi\} \in \mathcal{U}_k$$

where $\bar{A} = \{\bar{i} : i \in A\}$. For example,

$e_{2|7|14|36|589} =$ 

Lemma

The set $E(\mathcal{U}_k) = \{e_\pi : \pi \vdash [k]\}$ is a complete set of idempotents in $\mathcal{U}_k$.

Maximal subgroups

Definition

M finite monoid, e idempotent

Maximal subgroup: G_e = unique largest subgroup of M containing e

Lemma

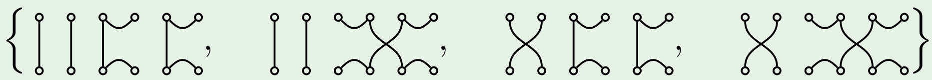
The maximal subgroup of $\mathcal{U}_k$ at the idempotent e_π is

$$G_{e_\pi} = \{d \in \mathcal{U}_k : \mathrm{top}(d) = \mathrm{bot}(d) = \pi\}$$

Example

For $\pi = \{\{1\}, \{2\}, \{3,4\}, \{5,6\}\}$

$G_{e_\pi} =$

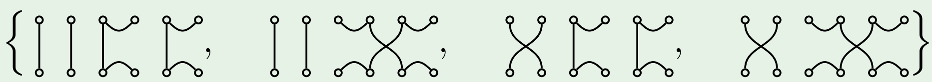


Maximal subgroups – continued

Example

For $\pi = \{\{1\}, \{2\}, \{3,4\}, \{5,6\}\}$ with $\mathrm{type}(\pi) = (1^2 2^2)$

$G_{e_\pi} =$



Theorem

For $\pi \vdash [k]$ with $\mathrm{type}(\pi) = (1^{a_1} 2^{a_2} \ldots k^{a_k})$

$$G_{e_\pi} \simeq S_{a_1} \times S_{a_2} \times \cdots \times S_{a_k}$$

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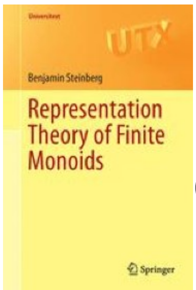
Representation theory of $\mathcal{U}_k$



(See book by Steinberg 2016)

Indexing set of simple modules

$$I_k = \left\{ \left(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)} \right) : \lambda^{(i)} \text{ are partitions such that } \sum_{i=1}^{k} i|\lambda^{(i)}| = k \right\}$$

Example

$I_3 = \{((3), \emptyset, \emptyset), ((2,1), \emptyset, \emptyset), ((1,1,1), \emptyset, \emptyset), ((1), (1), \emptyset), (\emptyset, \emptyset, (1))\}$

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Characters, symmetric functions, and plethysm

Theorem (OSSZ 2022)

Multiplicity of $V^{\mu}_{S_k}$ *in* $\mathrm{Res}^{\mathcal{U}_k}_{S_k} V^{\vec{\lambda}}_{\mathcal{U}_k}$ *is* $\langle s_{\lambda^{(1)}}[s_1] s_{\lambda^{(2)}}[s_2] \cdots s_{\lambda^{(k)}}[s_k],\ s_{\mu} \rangle$

Outline

Plethysm via representations of GL_n

Definition

$GL_n(\mathbb{C})$ = invertible $n \times n$ matrices

- GL_n-representation $\rho\colon GL_n \to GL_m$
- GL_m-representation $\tau\colon GL_m \to GL_r$
- Composition is GL_n-representation

$$\tau \circ \rho\colon GL_n \to GL_r$$

Definition

Character of composition is plethysm:

$$\mathrm{char}(\tau \circ \rho) = \mathrm{char}(\tau)[\mathrm{char}(\rho)]$$

Frobenius map

R^n space of class functions of GL_n
Λ^n ring of symmetric functions of degree n

Power sum symmetric function p_λ

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_\ell}$$
$$p_r = x_1^r + x_2^r + \cdots$$

Schur function s_λ

$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} x^{\mathrm{wt}(T)}$$

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Frobenius map – continued

Definition

The Frobenius characteristic map is $\mathrm{ch}^n \colon R^n \to \Lambda^n$

$$\mathrm{ch}^n(\chi) = \sum_{\mu \vdash n} \frac{1}{z_\mu} \chi_\mu p_\mu$$

where $z_\mu = 1^{a_1} a_1! 2^{a_2} a_2! \cdots$ for $\mu = 1^{a_1} 2^{a_2} \cdots$

Remark

The irreducible character χ^λ indexed by λ under the Frobenius map is

$$\mathrm{ch}^n(\chi^\lambda) = s_\lambda$$

by the identity

$$s_\lambda = \sum_{\mu} \frac{1}{z_\mu} \chi_\mu^\lambda p_\mu$$

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Plethysm problem

Problem

Find a combinatorial interpretation for the coefficients $a^{\nu}_{\lambda\mu} \in \mathbb{N}$ in the expansion

$$s_\lambda[s_\mu] = \sum_\nu a^{\nu}_{\lambda\mu} s_\nu$$

Problem

Find a crystal on tableaux of tableaux which explains $a^{\nu}_{\lambda\mu}$.

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Thank you !

Remark (Take away)

Plethysm is hard!

Remark (Take away)

Integrable systems, representation theory and combinatorics all play hand in hand!

QUIVER HALL-LITTLEWOOD FUNCTIONS AND KOSTKA-SHOJI POLYNOMIALS

MARK SHIMOZONO

These lectures are dedicated to Prof. Masato Okado on the occasion of his 60-th birthday conference "Integrable Systems and Quantum Groups", March 4-8, 2023 at Osaka City/Metropolitan University.

They are based on joint work with Dan Orr [OS22]. They are inspired by Shoji's work [Sho04] on Green's polynomials for complex reflection groups and the paper of Finkelberg and Ionov [FI18] which according to Finkelberg was intended to be a coherent sheaf version of Shoji's construction.

There are two main constructions for Kostka-Shoji polynomials.

(I) Quiver Hall-Littlewood (QHL) series: these are multigraded characters of modules given by the Euler characteristic of global sections of a family of vector bundles on Lusztig's convolution diagram.

(II) QHL symmetric functions: these are elements of the tensor product of symmetric functions that are obtained by vertex operators.

In each case the Kostka-Shoji polynomials arise as coefficients of the irreducible character basis.

The QKS polynomials also appear as structure constants of Schur functions in a K-theoretic Hall algebra [OS22, §5].

1. Part I: Geometry

1.1. **Lusztig's convolution diagram $\mathcal{W}$.** Let $Q = (Q_0, Q_1)$ be a quiver (directed graph); Q_0 is the set of nodes and Q_1 is the set of arrows. For our purposes (see [OS22, Subsection 2O]) there is no loss of generality in assuming that for every $(i, j) \in Q_0^2$ there is at most one arrow from i to j. If $b \in Q_1$ is an arrow from i to j we say $i = ta$ and $j = ha$ (tail and head of b).

Lusztig's convolution diagram $\mathcal{W}$ [Lu90] is specified by Q and the following data:

- A sequence $\underline{i} = (i_1, i_2, \ldots, i_m)$ of quiver nodes i_k for $1 \le k \le m$.
- A sequence $\underline{a} = (a_1, a_2, \ldots, a_m)$ of positive integers $a_k \in \mathbb{Z}_{>0}$.

In our notation a superscript as in $V^{(i)}$ refers to data at node $i \in Q_0$. An index k as in a_k or $\mu(k)$ refers to data at the k-th position in a filtration.

Given $(\underline{i}, \underline{a})$, define a Q_0-graded $\mathbb{C}$- vector space $V^\bullet = \bigoplus_{i \in Q_0} V^{(i)}$ and a decreasing partial flag of Q_0-graded subspaces

$$V^\bullet = V(0)^\bullet \supset V(0)^\bullet \supset \cdots \supset V(m)^\bullet = 0$$

as follows. Let $V(m)^\bullet = 0$ be the zero Q_0-graded vector space. Then for k from m down to 1, let $V(k-1)$ be obtained from $V(k)$ by adding dimension a_k at vertex i_k. Let $V^\bullet = V(0)^\bullet$ be the final result.

For $i \in Q_0$ let $a^{(i)} = (a_k \mid i_k = i)$ be the sequence of dimension jumps at vertex i. Let $B^{(i)} \subset P^{(i)} \subset GL(V^{(i)})$ be the standard lower triangular Borel, standard lower triangular parabolic with diagonal block sizes given by $a^{(i)}$, and the general linear group on $V^{(i)}$.

Example 1. Let $Q_0 = \{0, 1\}$. Letting $\underline{i}$ and $\underline{a}$ be as below, we give the tuples $a^{(i)}$ and the dimension vectors of the spaces $V(k)$.

k	1	2	3	4	5
i_k	0	1	0	1	1
a_k	1	3	2	2	1
$a^{(0)}$	1		2		
$a^{(1)}$		3		2	1

$a^{(0)} = (1, 2) \quad a^{(1)} = (3, 2, 1)$

k	0	1	2	3	4	5
$\dim V(k)^{(0)}$	3	2	2	0	0	0
$\dim V(k)^{(1)}$	6	6	3	3	1	0

A flag of type $(\underline{i}, \underline{a})$ is a sequence $F(\cdot)$ of Q_0-graded vector spaces

$$V^\bullet = F(0)^\bullet \supset F(1)^\bullet \supset \cdots \supset F(m)^\bullet = 0$$

such that for all $1 \le k \le m$:

$$\dim(F(k-1)^{(i)}/F(k)^{(i)}) = \begin{cases} a_k & \text{if } i_k = i \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathrm{Fl}_{\underline{i},\underline{a}}$ be the space of flags of type $(\underline{i}, \underline{a})$.

Let $G = \prod_{i \in Q_0} GL(V^{(i)})$ and $\mathrm{Fl} = \prod_{i \in Q_0} GL(V^{(i)})/P^{(i)}$ the product of partial flag varieties. There is an isomorphism

$$\mathrm{Fl}_{\underline{i},\underline{a}} \cong \mathrm{Fl} := \prod_{i \in Q_0} \mathrm{Fl}^{(i)}$$

Let $E = \bigoplus_{b \in Q_1} \mathrm{Hom}_{\mathbb{C}}(V^{(tb)}, V^{(hb)})$ be the space of representations of Q, the space of linear maps associated with $V^\bullet$.

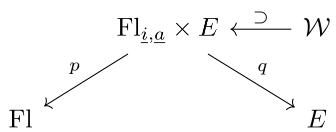
Let $T^{Q_1} = (\mathbb{C}^*)^{Q_1}$. It acts on E such that the copy of $\mathbb{C}^*$ for $a \in Q_1$ acts on $\mathrm{Hom}(V^{(ta)}, V^{(ha)})$ by scaling. The group $\mathcal{G} = G \times T^{Q_1}$ acts on E.

Say that $F(\cdot) \in \mathrm{Fl}_{\underline{i},\underline{a}}$ is strictly ϕ-stable for $\phi \in E$ if

(1) $$\phi_b(F(k-1)^{(tb)}) \subset F(k)^{(hb)} \qquad \text{for all } b \in Q_1,\ 1 \le k \le m.$$

Define the convolution diagram [Lu90]

$$\mathcal{W} := \{(F(\cdot), \phi) \in \mathrm{Fl}_{\underline{i},\underline{a}} \times E \mid F(\cdot) \text{ is strictly } \phi\text{-stable}\}.$$



The map q is $\mathcal{G}$-equivariant.

Example 2. Let Q be the one loop quiver and $n = \sum_{k=1}^{m} a_k$. Then $\mathcal{W} = T^*(GL_n/P_{\underline{a}})$ is the cotangent bundle on the partial flag variety where $P_{\underline{a}}$ is the lower triangular parabolic with block sizes $\underline{a}$. The space $E = \mathfrak{gl}_n$ affords the adjoint action of $G = GL(n)$. Let $\mathfrak{n}_{\underline{a}}$ be the nilradical of $\mathrm{Lie}(P_{\underline{a}})$. The map q is the parabolic Springer resolution. Its image is the nilpotent adjoint orbit closure $X_{\underline{a}} = \overline{\mathrm{Ad}(G) \cdot \mathfrak{n}_{\underline{a}}} \subset E$.

1.2. **$\mathcal{O}_{\mathcal{W}}$ modules $\mathcal{W}^{\mu(\cdot)}$ and quiver HL series.** In [OS22] we consider a family of $\mathcal{G}$-equivariant $\mathcal{O}_{\mathcal{W}}$-modules $\mathcal{W}^{\mu(\cdot)}$.

Given $(\underline{i}, \underline{a})$ we require one more input, namely, a sequence of dominant weights

$$\mu(\cdot) = (\mu(1), \mu(2), \dots, \mu(m)) \qquad \mu(k) \in X_+(GL_{a_k}).$$

At each vertex $i \in Q_0$ let $\mu^{(i)} \in X(GL(V^{(i)}))$ be the concatenation of the $\mu(k)$ for $i_k = i$.

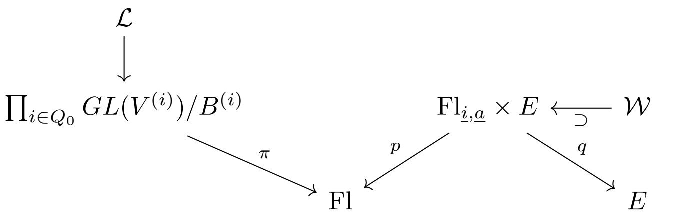
Example 3. Let $Q_0 = \{0, 1\}$ with $\underline{i}, \underline{a}$ as in the previous example. We choose a sequence of weights $\mu(\cdot)$ below.

k	1	2	3	4	5
i_k	0	1	0	1	1
a_k	1	3	2	2	1
$\mu(k)$	(2)	$(3,2,2)$	$(1,1)$	$(2,1)$	(1)
$\mu^{(0)}$	2		11		
$\mu^{(1)}$		322		21	1

We have $\mu^{(0)} = (2, 1, 1)$ and $\mu^{(1)} = (3, 2, 2, 2, 1, 1)$.

Say that $(\underline{i}, \underline{a}, \mu(\cdot))$ is *dominant* if each $\mu^{(i)}$ is dominant.

In [OS22] a vector bundle $\mathcal{W}_{\mu(\cdot)}$ on $\mathcal{W}$ is defined as follows. Let $\mathcal{L}_{\mu^{(i)}}$ be the standard line bundle of weight $\mu^{(i)}$ on $GL(V^{(i)})/B^{(i)}$ and $\mathcal{L} = \boxtimes_{i \in Q_0} \mathcal{L}_{\mu^{(i)}}$ the outer tensor product, which is a line bundle on the product of complete flag varieties $\prod_{i \in Q_0} GL(V^{(i)})/B^{(i)}$, and let π be the projection to Fl.



Define $\mathcal{W}_{\mu(\cdot)} = p^* \pi_*(\mathcal{L})$; it is a vector bundle on $\mathcal{W}$. Define the Quiver Hall-Littlewood (QHL) series to be the T^{Q_1}-equivariant Euler characteristic of global sections of $\mathcal{W}_{\mu(\cdot)}$.

$$\chi_{\mu(\cdot)}^{(\underline{i},\underline{a})} = \sum_{p \geq 0} (-1)^p \mathrm{ch}_{\mathcal{G}} H^p(\mathcal{W}, \mathcal{W}_{\mu(\cdot)})$$

For $\lambda^\bullet \in \prod_{i \in Q_0} X_+(GL(V^{(i)}))$ let $\chi^{\lambda^\bullet}$ be the irreducible character of G. We define the quiver Kostka-Shoji (QKS) polynomial $\mathcal{K}_{\lambda^\bullet, \mu(\cdot)}^{(\underline{i},\underline{a})}(t_{Q_1}) \in R(T^{Q_1}) \cong \mathbb{Z}[t_b^{\pm 1} \mid b \in$

$Q_1]$ as the coefficient of $\chi^{\lambda^\bullet}$ in $\chi^{(\underline{i},\underline{a})}_{\mu(\cdot)}$.

$$\chi^{(\underline{i},\underline{a})}_{\mu(\cdot)} = \sum_{\lambda^\bullet} \mathcal{K}^{(\underline{i},\underline{a})}_{\lambda^\bullet,\mu(\cdot)}(t_{Q_1})\chi^{\lambda^\bullet}$$

Since $\mathcal{W}_{\mu(\cdot)}$ may be viewed as a bundle over the product Fl of partial flag varieties, the QKS polynomials can be computed using Bott's formula for the Euler characteristic of a standard line bundle on the flag variety. We refer the reader to [OS22, Subsection 2L] for an explicit alternating sum formula for the QKS polynomial.

Example 4. Let Q be the single loop quiver and $n = \sum_k a_k$. Let $\rho = (n-1, \dots, 1, 0) \in \mathbb{Z}^n = X(GL(n))$ and let $J = \sum_{w \in S_n} (-1)^w w$ be the antisymmetrizer over the symmetric group S_n. Let $\mu \in X(GL_n)$ be the concatenation of all the $\mu(k)$. We have

$$\chi^{(\underline{i},\underline{a})}_{\mu(\cdot)} = J(z^\rho)^{-1} J\left(z^{\rho+\mu} \prod_{\alpha \in \Phi^+(\mathfrak{n}_{\underline{a}})} \frac{1}{1 - tz^\alpha}\right)$$

Here the QKS polynomials are parabolic (also called generalized) Kostka polynomials [SW00].

If $n = 2$, $\underline{a} = (1,1)$ and $\mu(\cdot) = ((0),(0))$

$$\begin{aligned}\chi^{(\underline{i},\underline{a})}_{\mu(\cdot)} &= J(x^\rho)^{-1} J\left(\frac{x^\rho}{1 - tx_1/x_2}\right)\\ &= \sum_{k \geq 0} t^k (x_1x_2)^{-k} s_{(2k,0)}(x_1, x_2).\end{aligned}$$

There are always two main problems to solve. The first is geometric.

Conjecture 1. [OS22, Conjecture 2.14] *Vanishing: If* $(\underline{i}, \underline{a}, \mu(\cdot))$ *is dominant then*

$$H^p(\mathcal{W}, \mathcal{W}_{\mu(\cdot)}) = 0 \qquad \text{for } p > 0. \tag{2}$$

Corollary 2. $\mathcal{K}^{(\underline{i},\underline{a})}_{\lambda^\bullet,\mu(\cdot)}(t_{Q_1}) \in \mathbb{Z}_{\geq 0}[t_b^{\pm} \mid b \in Q_1]$.

The second is to obtain an explicit combinatorial formula for the positive polynomials.

In all the following examples we assume dominance holds.

We say the data $(\underline{i}, \underline{a})$ is *Borel* if $a_k = 1$ for all k. In the Borel case, for any $\mu(\cdot)$, each $\mu(k)$ is a single row weight. We call the data *parabolic* in the general case.

In discussing the combinatorics of the known cases below it is important to know the following.

Remark 1. Fix $(\underline{i}, \underline{a})$ and consider $\mu(\cdot)$ and $\lambda^\bullet \in \prod_{i \in Q_0} X_+(GL(V^{(i)}))$. Let $N^\bullet \in \mathbb{Z}^{Q_0}$ be a tuple of integers, one for each vertex. Denote by $\mu(\cdot) + N^\bullet$ be the result of adding $N^{(i)}$ to each of the parts of the weight $\mu(k)$ if $i_k = i$. Similarly let $\lambda^\bullet + N^\bullet$ be defined by adding $N^{(i)}$ to every part of every weight $\lambda^{(i)} \in X_+(GL(V^{(i)}))$. It is not hard to see that

$$\mathcal{K}^{(\underline{i},\underline{a})}_{\lambda^\bullet+N^\bullet,\mu(\cdot)+N^\bullet}(t_{Q_1}) = \mathcal{K}^{(\underline{i},\underline{a})}_{\lambda^\bullet,\mu(\cdot)}(t_{Q_1}). \tag{3}$$

In particular every coefficient polynomial $\mathcal{K}^{(\underline{i},\underline{a})}_{\lambda^\bullet,\mu(\cdot)}(t_{Q_1})$ is equal to another such in which all of the weights $\mu^{(i)}$ are partitions (have all nonnegative parts) with at most $\dim(V^{(i)})$ parts for all $i \in Q_0$.

In the Borel case this doesn't matter much since all weights are single rows. However in the parabolic case, this adding $N^\bullet$ causes a number of "full-sized" columns to be added to a partition.

Example 5. Let Q be the single loop quiver.

- In the Borel case the QKS polynomials are the Kostka Foulkes polynomials [Mac79, §III.6].
  - Vanishing was proved in [Bro93].
  - The Kostka-Foulkes polynomials have a Young tableau formula [LS78].
  - They also have a fermionic formula (rigged configurations) [KR86].
  - They give the dimensions of the quotients for the filtration of the action of a principal nilpotent on a weight space [Bry89].
  - They are the isotypic components of the one-dimensional sum for the tensor product $\otimes_k B^{1,\mu_k}$ of "single row" type A KR crystals, graded by the energy function [NY97].
- In the parabolic case the QKS polynomials are known as parabolic or generalized Kostka polynomials [SW00].
  - Suppose all $\mu(k)$ are single columns. The QKS polynomials are the Kostka-Foulkes with grading reversed. They have the following descriptions:
    - The intersection cohomology of $X_{\underline{a}}$ [Lu83].
    - A tricky (catabolizable) tableau formula [Las91].
    - Via the Tanisaki ideal of $\mathbb{C}[X_{\underline{a}} \cap \mathfrak{h}]$ [GP82].
    - One dimensional sum for tensor products of single column type A KR crystals [NY97].
  - Suppose all $\mu(k)$ are rectangles all of which have the same number of columns. The QKS polynomials are isotypic components of the coordinate ring of the nilpotent adjoint orbit closure $X_{\underline{a}}$.
    - Vanishing and the analogue of Lusztig's formula for weight multiplicity was proved in [W89].
    - The parabolic Kostka polynomials (via [Sh01] and [Sh02]) equals the sl_n-invariant Demazure characters in the highest weight module $V(s\Lambda_0)$ of $\hat{sl}_n$ [KMOTU00].
  - Suppose all $\mu(k)$ are rectangles. Let $\mu(k)$ be an $a_k \times b_k$ rectangle R_k for all k.
    - Vanishing was proved in [Bro93].
    - Geometric character has a tableau formula [Sh01].
    - The tableau formula equals the one-dimensional sum for any type A affine KR crystal $\otimes_k B^{a_k,b_k}$ [ScWa99] [Sh02].
    - The equality of the above one-dimensional sums with the type A fermionic formula is proved in [KSS02]. This is the untwisted type A case of the remarkable $X = M$ conjecture of M. Okado and collaborators given in [HKOTY02] for the untwisted affine root systems and in [HKOTT02] for the twisted affine root systems. Here X means the one-dimensional sums which are the energy-graded characters of arbitrary tensor products of KR crystals of any affine Lie algebra and M is their fermionic formula.

    * In [SW00] it is conjectured that the rectangular parabolic Kostka polynomials agree with the rectangle product case of Lascoux-Leclerc-Thibon functions. It was proved in [GH07].
- General case
    * The vanishing conjecture in this case is due to [Bro93]. A proof was announced in [Ka23].
    * A catabolizable tableau conjecture was given in [SW00]. It was proved in [BMP] which considered more general characters by allowing more general ideals of roots as opposed to just the roots of the nilradical of a parabolic. They studied (affine Borel) modules built from tensoring with affine highest weight vectors and applying Demazure operators.

Example 6. Let Q be the cyclic quiver, where $Q_0 = \mathbb{Z}/r\mathbb{Z}$ and $Q_1 = \{(i, i+1) \mid i \in \mathbb{Z}/r\mathbb{Z}\}$. Borel case:

- The Borel cyclic quiver QKS polynomials were defined in [FI18].
- For 2 nodes they were conjectured in [FI18] to be equal to those defined in [Sho04]. This was proved by Shoji in [Sho18].
- The cyclic quiver QKS were given in intersection cohomology interpretation in [AH08].

Parabolic case: These were defined at the same time as the general case in [OS22] and specifically studied in [OS22a].

- A tableau formula is given in[OS22a] for the case of rectangles all at vertex $r-1$, and zero weights at other vertices.
- If all are single columns at vertex $r-1$ the QHL symmetric function for the cyclic quiver was shown to be equal to certain wreath Hall-Littlewood polynomials [Ha03, §7.2.4] which are obtained from Haiman's wreath H-Macdonald polynomial by taking the coefficient of the lowest occurring power of q. This single-columns-at-one-vertex case is not directly related to the single-rows-at-one-vertex case, unlike the situation for the single node cyclic quiver, where the two are related by degree reversal (after transposing).

Example 7. For any quiver whose connected components are directed cycles and directed paths, a catabolizable tableau conjecture is given in [OS22a]. For the case of the A_2-quiver, the answer is a truncated Littelwood-Richardson coefficient [**?**].

2. Part II: Creation operators for Quiver HL functions

The second method of construction of QKS polynomials in [OS22] is by creating symmetric functions by vertex operators. This was inspired by Jing [**?**], Garsia and Procesi [GP82], and a joint work with Zabrocki [SZ01].

Let Λ be the Hopf algebra of symmetric functions over $\mathbb{F} = \mathrm{Frac}(R(T^{Q_1})) = \mathbb{Q}(t_a \mid a \in Q_1)$.

For a triple (i, a, μ) with $i \in Q_0$, $a \in \mathbb{Z}_{>0}$ and $\mu \in X_+(GL_a)$ we define an operator $H_\mu^{(i,a)} \in \mathrm{End}(\Lambda^{\otimes Q_0})$. Then for $(\underline{i}, \underline{a}, \mu(\cdot))$ we define the Quiver Hall-Littlewood symmetric function by the sequence of operators acting on $1 \in \Lambda^{\otimes Q_0}$:

$$H_{\mu(\cdot)}^{(\underline{i},\underline{a})} = H_{\mu(1)}^{(i_1,a_1)} \circ \cdots \circ H_{\mu(m)}^{(i_m,a_m)}(1) \in \Lambda^{\otimes Q_0}.$$

2.1. **Symmetric function notation.** Let Λ be the Hopf algebra of symmetric functions over $\mathbb{Z}$. Let X represent a sequence of indeterminates $(x_1, x_2, \dots)$. Let $\mathbb{Z}[[x_i \mid i \in \mathbb{Z}_{>0}]]$ be the formal power series ring. Let $S_{\mathbb{Z}_{>0}}$ be the group of permutations of $\mathbb{Z}_{>0}$ that move finitely many elements. Then Λ is isomorphic to the subring $\Lambda[X]$ of $\mathbb{Z}[[x_i \mid i \in \mathbb{Z}_{>0}]]$ consisting of the series which are symmetric, that is, fixed by $S_{\mathbb{Z}_{>0}}$, and have bounded degree. Let $\hat{\Lambda}[X]$ consist of symmetric series with no condition on degree bound.

For indeterminates z, w define

$$\begin{aligned} \Omega[z] &= \frac{1}{1-z} \\ \Omega[-w] &= 1/\Omega[w] = (1-w) \\ \Omega[z+w] &= \Omega[z]\Omega[w] \end{aligned}$$

Ω behaves like an exponential.

The negative sign has a special meaning. It does not give the same result as using a variable and then specializing the variable to -1. For example $\Omega[uw] = (1-uw)^{-1}$ and setting $u = -1$ yields $(1+w)^{-1}$.

We use the suggestive notation $X = x_1 + x_2 + \cdots$. For an indeterminate u and extending the above notation using infinite sums and products we have

$$\begin{aligned} \Omega[uX] &= \prod_{i\ge 1} \Omega[ux_i] = \prod_{i \ge 1} \frac{1}{1-ux_i} = \sum_{k \ge 0} u^k h_k[X] \\ \Omega[-uX] &= \prod_{i \ge 1} (1-ux_i) = \sum_{k\ge 0} (-1)^k u^k e_k[X]. \end{aligned}$$

This is the definition of the homogeneous (h_k) and elementary (e_k) symmetric functions.

$\Lambda_{\mathbb{Z}} = \mathbb{Z}[h_1, h_2, \cdots]$ is a polynomial algebra over the integers. To connect with the usual presentation of the boson-fermion correspondence we define the power sums and their connection with Ω:

$$p_r[X] = \sum_{i \ge 1} x_i^r \qquad \text{for } r \ge 1$$

$$\Omega[uX] = \exp\left(\sum_{r \ge 1} \frac{1}{r} p_r[X] u^r \right).$$

Of course we must work over $\mathbb{Q}$ if using power sums. We have $\Lambda_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda_{\mathbb{Z}} = \mathbb{Q}[p_1, p_2, \dots]$.

We now give the Hopf structure. Let $S : \Lambda \to \Lambda$ denote the antipode. It is an involutive algebra automorphism denoted $f[X] \mapsto f[-X]$ for $f \in \Lambda$. It is enough to define it on the generating function $\Omega[uX]$ of algebra generators h_k and then take coefficients of powers of u.

$$\begin{aligned} \Omega[uX] &\mapsto \Omega[-uX] && \text{that is,} \\ h_k &\mapsto (-1)^k e_k && \text{for all } k \ge 0. \end{aligned}$$

Over $\mathbb{Q}$ it can be defined by $p_k[X] \mapsto -p_k[X]$ for all $k \ge 1$.

The tensor product $\Lambda \otimes_{\mathbb{Z}} \Lambda$ can be realized by series in two sets of variables X and Y which are separately symmetric in X and in Y and are of bounded degree. If $f, g \in \Lambda$ then we write $f[X]g[Y]$ for the element $f \otimes g$.

The coproduct $\Delta : \Lambda \to \Lambda \otimes \Lambda$ is an algebra homomorphism and is denoted $f \mapsto f[X+Y]$. Heuristically, $f[X+Y]$ means to plug both sets of variables X and Y into f. We do this on the generating function $\Omega[uX]$ of the $h_k[X]$ and then take the coefficient of powers of u:

$$\begin{aligned}\Delta(\Omega[uX]) &= \Omega[u(X+Y)] = \Omega[uX]\Omega[uY] \\ \Delta(h_k) &= \sum_{\substack{i,j\ge 0\\ i+j=k}} h_i[X]h_j[Y] = \sum_{\substack{i,j\ge 0\\ i+j=k}} h_i \otimes h_j.\end{aligned}$$

For power sums we get

$$p_k[X+Y] = \sum_{i\ge 1}(x_i^k + y_i^k) = p_k[X] + p_k[Y] = p_k \otimes 1 + 1 \otimes p_k,$$

that is, the p_k are primitive algebra generators of $\Lambda_{\mathbb{Q}}$.

Define the Hall pairing $\Lambda \otimes \Lambda \to \mathbb{Z}$ to be the one with respect to which the Schur functions s_λ are orthonormal:

$$\langle s_\lambda\,,\, s_\mu\rangle = \delta_{\lambda\mu} \qquad \text{for } \lambda,\mu \in \mathbb{Y} \text{ (Young's lattice of partitions)}$$

By the Cauchy identity, its reproducing kernel is:

$$\sum_\lambda s_\lambda[X]s_\lambda[Y] = \prod_{i,j\ge 1}\frac{1}{1-x_iy_j} = \Omega[XY].$$

The counit $\epsilon : \Lambda \to \mathbb{Z}$ is taking the coefficient of 1: $\epsilon(f) = \langle 1\,,\, f\rangle$ for all $f \in \Lambda$.

For a symmetric function $f \in \Lambda$, define the operator $f^\perp \in \mathrm{End}(\Lambda)$ (called f "perp" or "skewing by f") to be the adjoint operator to multiplication by f. It is defined by (for all $g, h \in \Lambda$)

$$\begin{aligned}\langle f^\perp(g)\,,\, h\rangle &= \langle g\,,\, fh\rangle = \langle \Delta(g)\,,\, f\otimes h\rangle \\ &= \sum_{(g)}\langle g_{(1)}\,,\, f\rangle\langle g_{(2)}\,,\, h\rangle.\end{aligned}$$

If f is homogeneous of degree d then $f^\perp$ has degree $-d$. In particular for any Z we have

$$\Omega[ZX]^\perp(1) = \sum_{\lambda\in\mathbb{Y}} s_\lambda[Z]s_\lambda[X]^\perp(1) = 1 \tag{4}$$

since $s_\lambda[X]^\perp$ has strictly negative degree for λ a nonempty partition.

Notation: $^\perp$ is taken with respect to the X variables.

Exercise 1. *Show that for all $f \in \Lambda$*

$$f[X]^\perp(\Omega[XY]) = f[Y]\Omega[XY]. \tag{5}$$

For $f[X] = \Omega[ZX]$ we have

$$\Omega[XZ]^\perp(\Omega[XY]) = \Omega[ZY]\Omega[XY] = \Omega[(X+Z)Y] \tag{6}$$

Exercise 2. *Show that*

$$\Omega[XZ]^\perp(f[X]) = f[X+Z] \qquad \text{for } f \in \Lambda.$$

For any $f, g \in \Lambda$ we have

$$\begin{aligned}(\Omega[XZ]^\perp \circ f[X])(g[X]) &= \Omega[XZ]^\perp(f[X]g[X]) \\ &= f[X+Z]g[X+Z] \\ &= f[X+Z]\Omega[XZ]^\perp(g[X]).\end{aligned}$$

Therefore

$$\Omega[XZ]^\perp \circ f[X] = f[X+Z] \circ \Omega[XZ]^\perp \qquad \text{in } \mathrm{End}(\Lambda). \tag{7}$$

If $f[X] = \Omega[XY]$ then in $\mathrm{End}(\Lambda)$

$$\Omega[XZ]^\perp \circ \Omega[XY] = \Omega[(X+Z)Y]\Omega[XZ]^\perp \tag{8}$$

$$= \Omega[ZY]\Omega[XY] \circ \Omega[XZ]^\perp. \tag{9}$$

2.2. **Bernstein operators.** We require the Bernstein operators that are used to create Schur functions. Define $\{S_m \mid m \in \mathbb{Z}\} \subset \mathrm{End}(\Lambda)$ as follows.

Let $\rho = (n-1, n-2, \dots, 1, 0) \in \mathbb{Z}^n$ and let $Z = (z_1, \dots, z_n)$ be a finite set of auxiliary variables. Define

$$R(Z) = \prod_{1 \le i < j \le n} \left(1 - \frac{z_j}{z_i}\right) = z^{-\rho} \prod_{1 \le i < j \le n} (z_i - z_j) = z^{-\rho} J(z^\rho)$$

where recall that J is the antisymmetrizer. For $\lambda \in \mathbb{Z}^n = X(GL(n))$ define

$$s_\lambda(Z) = J(z^\rho)^{-1} J(z^{\lambda+\rho}).$$

This is the Schur polynomial $s_\lambda(Z)$ if $\lambda \in X_+(GL(n)$.

Define the Bernstein operators $S_k \in \mathrm{End}(\Lambda)$ by the generating function

$$\begin{aligned}\sum_{k \in \mathbb{Z}} z^k S_k = S(z) &= \Omega[zX]\Omega[-z^{-1}X]^\perp \\ &= \sum_{i \ge 0} z^i h_i[X] \sum_{j \ge 0} (-z^{-1})^j e_j[X]^\perp \\ &= \sum_{k \in \mathbb{Z}} z^k \sum_{\substack{i,j \ge 0 \\ i-j=k}} (-1)^j h_i[X] e_j[X]^\perp.\end{aligned}$$

We compute the commutation relations using (8).

$$\begin{aligned}S(z)S(w) &= \Omega[zX]\Omega[-z^{-1}X]^\perp \Omega[wX]\Omega[-w^{-1}X]^\perp \\ &= \Omega[zX]\Omega[-z^{-1}w]\Omega[wX]\Omega[-z^{-1}X]^\perp\Omega[-w^{-1}X]^\perp \\ &= (1 - w/z)\Omega[(z+w)X]\Omega[-(z^{-1}+w^{-1})X]^\perp.\end{aligned}$$

Multiplying by z we have

$$zS(z)S(w) = (z-w)\Omega[(z+w)X]\Omega[-(z^{-1}+w^{-1})X]^\perp$$

Exchanging z and w gives

$$wS(w)S(z) = (w-z)\Omega[(z+w)X]\Omega[-(z^{-1}+w^{-1})X]^\perp$$

Note that the term $\Omega[(z+w)X]\Omega[-(z^{-1}+w^{-1})X]^\perp$ (the normal ordering of $S(z)S(w)$) is symmetric in z and w. We deduce that

$$zS(z)S(w) = -wS(w)S(z).$$

Taking the coefficient of $z^{m+1}w^n$ we obtain

$$S_m S_n = -S_{n-1}S_{m+1} \qquad \text{for all } m,n\in\mathbb{Z}.$$

This is the relation seen by switching rows in the Jacobi-Trudi determinantal formula for Schur functions.

Let $Z^* = z_1^{-1}+\cdots+z_n^{-1}$. We consider the composition of S operators.

$$\begin{aligned}
S(Z) &= S(z_1)S(z_2)\cdots S(z_n)\\
&= \Omega[z_1X]\Omega[-z_1^{-1}X]^\perp\cdots\Omega[z_nX]\Omega[-z_n^{-1}X]^\perp\\
&= \left(\prod_{1\le i<j\le n}(1-z_j/z_i)\right)\Omega[z_1X]\cdots\Omega[z_nX]\Omega[-z^{-1}X]^\perp\cdots\Omega[-z_n^{-1}X]^\perp\\
&= R(Z)\Omega[ZX]\Omega[-Z^*X]^\perp\\
&= z^{-\rho}J(z^\rho)\Omega[ZX]\Omega[-Z^*X]^\perp.
\end{aligned}$$

Let $\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_n\ge 0$. Letting $[z^\lambda]$ denote taking the coefficient of z^λ we have

$$\begin{aligned}
S_{\lambda_1}\circ\cdots\circ S_{\lambda_n}(1) &= [z^\lambda]S(z_1)\cdots S(z_n)(1)\\
&= [z^\lambda]z^{-\rho}J(z^\rho)\Omega[ZX]\Omega[-Z^*X]^\perp(1)\\
&= [z^{\lambda+\rho}]J(z^\rho)\sum_{\mu\in\mathbb{Y}}s_\mu[Z]s_\mu[X]\\
&= [z^{\lambda+\rho}]\sum_\mu J(z^{\mu+\rho})s_\mu[X]\\
&= s_\lambda[X].
\end{aligned}$$

2.3. **Modified Jing operators.** We use the modification of Jing's creation operators [J91] that was popularized by Garsia [GP82]. We define the operators $\{H_k \mid k\in\mathbb{Z}\}\subset \mathrm{End}(\Lambda)$ as follows.

$$\begin{aligned}
\sum_{k\in\mathbb{Z}}H_kz^k &= H(z)\\
&= S(z)\Omega[tz^{-1}X]^\perp\\
&= \Omega[zX]\Omega[-z^{-1}X]^\perp\Omega[tz^{-1}X]^\perp\\
&= \Omega[zX]\Omega[(t-1)z^{-1}X]^\perp.
\end{aligned}$$

We have

$$\begin{aligned}
H(z)H(w) &= \Omega[zX]\Omega[(t-1)z^{-1}X]^\perp\Omega[wX]\Omega[(t-1)w^{-1}X]^\perp\\
&= \Omega[(t-1)z^{-1}w]\Omega[(z+w)X]\Omega[(t-1)(z^{-1}+w^{-1})X]^\perp.
\end{aligned}$$

Note that

$$\Omega[(t-1)z^{-1}w] = \frac{1-w/z}{1-tw/z}.$$

We obtain

$$\begin{aligned} H(z_1)H(z_2)\cdots H(z_n) &= \left(\prod_{1\le i<j\le n}\Omega[(t-1)z_j/z_i]\right)\Omega[ZX]\Omega[(t-1)Z^*X]^\perp \\ &= R(Z)B(Z,t)\Omega[ZX]\Omega[(t-1)Z^*X]^\perp \qquad \text{where} \\ B(Z,t) &= \prod_{1\le i\le j\le n}(1-tz_j/z_i)^{-1}. \end{aligned}$$

Let $\mu = (\mu_1 \ge \mu_2 \ge \cdots \ge \mu_n)$ with $\mu_n \ge 0$. We have

$$\begin{aligned} H_{\mu_1}\circ\cdots H_{\mu_n}(1) &= [z^\mu]H(z_1)\cdots H(z_n)(1) \\ &= [z^\mu]R(Z)B(Z,t)\sum_\lambda s_\lambda[Z]s_\lambda[X] \\ &= [z^{\mu+\rho}]B(Z,t)\sum_\lambda J(z^{\lambda+\rho})s_\lambda[X] \\ &= \sum_\lambda s_\lambda[X][z^{\mu+\rho}]\sum_{w\in S_n}(-1)^w z^{w(\lambda+\rho)}B(Z,t). \end{aligned}$$

Let $H_\mu := H_{\mu_1}\circ\cdots\circ H_{\mu_n}(1)$. Taking the coefficient of $s_\lambda[Z]$ we have

$$\begin{aligned} K_{\lambda\mu}(t) &= \langle H_\mu\,,\, s_\lambda\rangle \\ &= \sum_{w\in S_n}(-1)^w[z^{\mu+\rho-w(\lambda+\rho)}]B(Z,t) \\ &= \sum_{w\in S_n}(-1)^w[z^{w(\lambda+\rho)-(\mu+\rho)}]\prod_{1\le i<j\le n}(1-tz_i/z_j)^{-1} \end{aligned}$$

where in the last step we replaced z_i by z_i^{-1} everywhere. This last formula is Lusztig's t-analogue of Kostant's weight multiplicity formula.

2.4. **Parabolic analogue.** Fix $a \in \mathbb{Z}_{>0}$ and let $Z = z_1 + z_2 + \cdots + z_a$. We define operators $\{H^a_\beta \mid \beta \in \mathbb{Z}^a\} \subset \mathrm{End}(\Lambda)$ by [SZ01]

$$\begin{aligned} \sum_{\beta\in\mathbb{Z}^a} z^\beta H^a_\beta = H^a(Z) &= S^{(a)}(Z)\Omega[tZ^*X]^\perp \\ &= R(Z)\Omega[ZX]\Omega[-Z^*X]^\perp\Omega[tZ^*X]^\perp \\ &= R(Z)\Omega[ZX]\Omega[(t-1)Z^*X]^\perp. \end{aligned}$$

Compare this with the composition of "single row" operators:

$$H(z_1)\circ\cdots\circ H(z_a) = B(Z,t)H^a(Z)$$

Given $\underline{a}$ and $\mu(\cdot)$, define the parabolic HL symmetric function

$$H^{\underline{a}}_{\mu(\cdot)} = H^{a_1}_{\mu(1)}\circ\cdots\circ H^{a_m}_{\mu(m)}(1).$$

Let $a_1 + a_2 + \cdots + a_m = n$, $P^+_{\underline{a}}$ the standard upper triangular parabolic with block sizes $\underline{a}$, and $\mathfrak{n}_{\underline{a}}$ the nilradical of $\mathrm{Lie}(P^+_{\underline{a}})$. Let $\Phi(\mathfrak{n}_{\underline{a}})$ be the set of roots of $\mathfrak{n}_{\underline{a}}$. Let $\mu \in \mathbb{Z}^n$ be the concatenation of the weights $\mu(1)$ through $\mu(m)$.

In [SZ01] it was shown that

$$\langle H^{\underline{a}}_{\mu(\cdot)}\,,\, s_\lambda\rangle = \sum_{w\in S_n} (-1)^w z^{w(\lambda+\rho)-(\mu+\rho)} \prod_{\alpha\in\Phi(\mathfrak{n}_{\underline{a}})} \frac{1}{1-tz^\alpha}.$$

Exercise 3. *Verify the above for* $\underline{a}=(1,2)$.

3. General quiver

Let $\mathbb{F} = \mathrm{Frac}(R(T^{Q_1})) = \mathbb{Q}(t_a \mid a \in Q_1)$. Let Λ be symmetric functions over $\mathbb{F}$ and $\Lambda^{\otimes Q_0}$ be the $|Q_0|$-th tensor power of Λ. For $f\in\Lambda$ and $i\in Q_0$ write $f[X^{(i)}]$ for $1\otimes\cdots\otimes 1\otimes f\otimes 1\otimes\cdots\otimes 1$ in which f occurs in the i-th tensor factor.

For $\lambda^\bullet \in \mathbb{Y}^{Q_0}$ define the tensor Schur basis of $\Lambda^{\otimes Q_0}$ by $s_{\lambda^\bullet} = \prod_{i\in Q_0} s_{\lambda^{(i)}}[X^{(i)}]$.

Let $\langle\cdot\,,\,\cdot\rangle : \Lambda^{\otimes Q_0}\otimes\Lambda^{\otimes Q_0}\to\mathbb{F}$ be the pairing for which the tensor Schur basis is orthonormal.

Let $f[X^{(i)}]^\perp \in \mathrm{End}(\Lambda^{\otimes Q_0})$ be the operator that is adjoint with respect to multiplication by $f[X^{(i)}]$. Note that here the $\perp$ is with respect to the variables $X^{(i)}$.

3.1. **General quiver parabolic creation operator.** Consider a triple (i,a,μ) with $i\in Q_0$, $a\in\mathbb{Z}_{>0}$ and $\mu\in X_+(GL_a)$. $Z = z_1+\cdots+z_a$. We define an operator $H^{(i,a)}_\mu \in \mathrm{End}(\Lambda^{\otimes Q_0})$. For $i\in Q_0$ let $S^{(i)}(Z)$ be the generating function for the composition of Bernstein operators acting on the i-th tensor factor:

$$S^{(i)}(Z) = R(Z)\Omega[ZX^{(i)}]\Omega[-Z^*X^{(i)}]^\perp$$

Let $\mathrm{Out}(i) = \{b\in Q_1 \mid tb = i\}$ be the set of arrows going out of node i. Define the quiver creation operator

$$\begin{aligned}\sum_{\beta\in\mathbb{Z}^a} H^{(i,a)}_\beta &= H^{(i,a)}(Z)\\ &= S^{(i)}(Z)\prod_{b\in\mathrm{Out}(i)}\Omega[t_b Z^* X^{(hb)}]^\perp.\end{aligned}$$

For $\underline{i} = (i_1,i_2,\dots,i_m)$, $\underline{a}=(a_1,a_2,\dots,a_m)$, and $\mu(\cdot) = (\mu(1),\mu(2),\dots,\mu(m))$ define the quiver Hall-Littlewood symmetric function

$$H^{\underline{i},\underline{a}}_{\mu(\cdot)} = H^{(i_1,a_1)}_{\mu(1)}\circ\cdots\circ H^{(i_m,a_m)}_{\mu(m)}\cdot 1 \in \Lambda^Q.$$

Theorem 3. [OS22] *For any* $\underline{i},\underline{a}$, *and* $\mu(\cdot)$ *such that all the* $\mu(k)$ *are polynomial weights (that is, all parts are nonnegative) we have*

$$H^{\underline{i},\underline{a}}_{\mu(\cdot)} = \sum_{\lambda^\bullet\in\mathbb{Y}^{Q_0}} \mathcal{K}^{\underline{i},\underline{a}}_{\lambda^\bullet,\mu(\bullet)}(t_{Q_1})s_{\lambda^\bullet}. \tag{10}$$

Remark 2. Due to Remark 1 every coefficient of a QHL series appears as a coefficient of a QHL symmetric function after shifting the arguments. Thus the two constructions give the same information.

3.2. **True number of torus parameters.** In our definition there is a parameter for every arrow. It was pointed out by Finkelberg that the dimension of dilation symmetry is not the number of edges in Q_1 but rather the dimension of $H_1(Q)$, that is, the dimension of the cycle space of the graph Q.

The edge space of Q is the free $\mathbb{Z}$-module $E(Q)$ with basis e_a where $a\in Q_1$. For every cycle C in the underlying undirected graph of Q, pick an orientation $\vec{C}$. For every edge $a\in Q_1$ whose undirected edge $|a|$ is in C, define $\mathrm{sgn}_{\vec{C}}(a)$ to be 1 or -1

according as the direction of $a \in Q_1$ agrees or disagrees with the direction in $\vec{C}$. Define the cycle vector $z_{\vec{C}} \in E(Q)$ by

$$z_{\vec{C}} = \sum_{\substack{a \in Q_1 \\ |a| \in C}} \operatorname{sgn}_{\vec{C}}(a)a.$$

Define the associated cycle monomial $t_{\vec{C}} \in R(T^{Q_1})$ by

$$t_{\vec{C}} = \prod_{\substack{a \in Q_1 \\ |a| \in C}} t_a^{\operatorname{sgn}_{\vec{C}}(a)}.$$

Example 8. (1) Let Q be the directed cyclic quiver. Taking the directed cycle $\vec{C} = 0 \to 1 \to 2 \to 0$ we get the cycle monomial $t_{\vec{C}} = t_{01}t_{12}t_{20}$.

(2) Let $Q_0 = \{0, 1, 2\}$ with $Q_1 = \{(0, 1), (0, 2), (1, 2)\}$. Taking $\vec{C} = 0 \to 1 \to 2 \to 0$ we see that the orientations of the edges $(0, 1)$ and $(1, 2)$ agree in Q_1 and on $\vec{C}$ while $(0, 2) \in Q_1$ disagrees with the direction in $\vec{C}$. Therefore $t_{\vec{C}} = t_{01}t_{12}t_{02}^{-1}$.

The cycle space $Z(Q_1) \subset E(Q_1)$ of Q_1 is by definition the subspace of $E(Q_1)$ given by the span of $z_{\vec{C}}$ as C runs over the cycles of the underlying undirected graph of Q. Since taking the opposite orientation of C just results in negating the corresponding cycle vector, the cycle space is independent of the choice or orientation for the cycles.

Say that a monomial in $R(T^{Q_1})$ is acyclic if it is not divisible by any cycle monomial of Q.

Proposition 4. [OS22] *Pick a basis* $\{z_{\vec{C}_1}, \dots, z_{\vec{C}_p}\}$ *of* $Z(Q_1)$. *Then for every* $\mathcal{K}_{\lambda^\bullet,\mu(\cdot)}(t_{Q_1})$ *there is a unique acyclic Laurent monomial* $m(t_{Q_1})$ *and a unique polynomial* $\mathcal{K}^{\mathrm{red}}_{\lambda^\bullet,\mu(\cdot)}(z_1, \dots, z_p)$ *with integer coefficients such that* $\mathcal{K}_{\lambda^\bullet,\mu(\cdot)}(t_{Q_1}) = m(t_{Q_1})\mathcal{K}^{\mathrm{red}}_{\lambda^\bullet,\mu(\cdot)}(t_{\vec{C}_1}, \dots, t_{\vec{C}_p})$.

Call the polynomials $\mathcal{K}^{\mathrm{red}}_{\lambda^\bullet,\mu(\cdot)}(z_1, \dots, z_p)$ the reduced QKS polynomial.

The Shoji-Finkelberg-Ionov polynomials, which have one parameter, are the reduced versions of our cyclic quiver Borel quiver Kostka-Shoji polynomials.

In particular for acyclic quivers the reduced QKS polynomial is just an integer.

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Tetrahedron equations associated with quantized six-vertex models

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Based on: A. Kuniba, S. Matsuike, AY, arXiv:2208.10258

Outline

Tetrahedron equation

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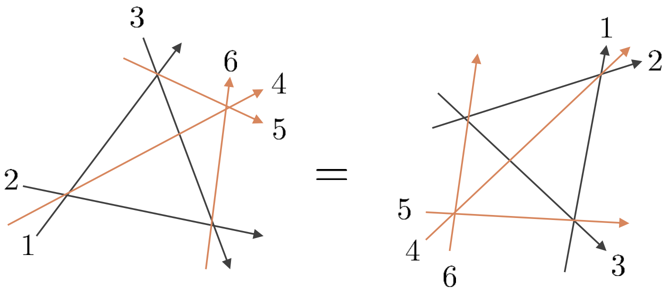

- Matrix equation on $V_1 \otimes \cdots \otimes V_6$ (V_i: linear space) [Zamolodchikov'81]
  - X_{ijk} ($X = A, B, C, D$) acts non-trivially only on $V_i \otimes V_j \otimes V_k$.

$$A_{124}B_{135}C_{236}D_{456} = D_{456}C_{236}B_{135}A_{124}$$

- 3D analog of Yang-Baxter equation (YBE)
  - We can construct a 3D version of transfer matrices similarly to YBE.
- Several solutions are known although less systematic than YBE.

  Zamolodchikov, Baxter, Bazhanov, Korepanov, Mangazeev, Sergeev, Stroganov, Kapranov, Voevodsky, Kazhdan, Soibelman, Carter, Saito, Kuniba, Okado, …

RLLL relation

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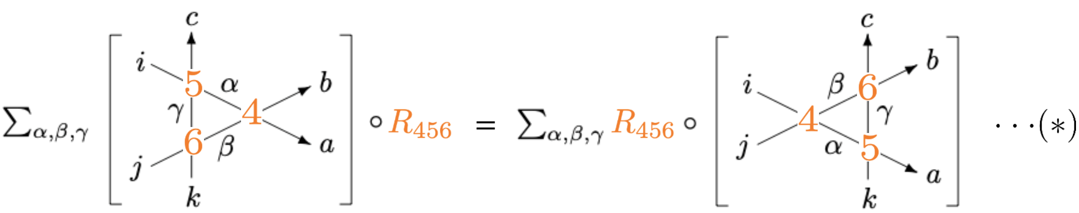
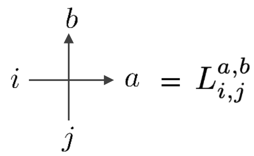
- Today, we focus on the *RLLL* type tetrahedron equation:

$$L_{124}L_{135}L_{236}R_{456} = R_{456}L_{236}L_{135}L_{124}$$

- If we specify the outer lines for 1,2,3-th spaces, this reads as



$$L(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{a,b} |a\rangle \otimes |b\rangle \otimes L_{i,j}^{a,b}\,|k\rangle$$



- For each (i, j, k, a, b, c), $(*)$ gives linear equations for R.
- If we can ansatz ``good" Ls, we can then obtain a solution to the *RLLL* type tetrahedron equation by solving these equations.
- In fact, it can be done by considering a quantized six vertex model for Ls.

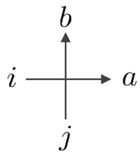
q-Oscillator algebra valued six vertex model 5/25

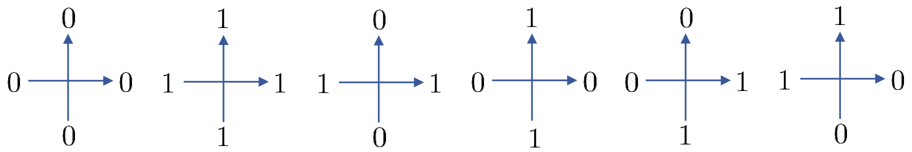
- q-Oscillator algebra O_q
  - Genetators: $\mathbf{k}, \mathbf{a}^{\pm}$
  - Relations $\mathbf{k}\mathbf{a}^{\pm} = q^{\pm}\mathbf{a}^{\pm}\mathbf{k}, \quad \mathbf{a}^{+}\mathbf{a}^{-} = 1 - \mathbf{k}^2, \quad \mathbf{a}^{-}\mathbf{a}^{+} = 1 - q^2\mathbf{k}^2$
  - Representation π_O on $F_+ = \oplus_{m \in \mathbb{Z}_{\geq 0}} \mathbb{C}\,|m\rangle$:

$$\pi_O : \mathbf{k}\,|m\rangle = q^m\,|m\rangle\,, \quad \mathbf{a}^{+}\,|m\rangle = |m+1\rangle\,, \quad \mathbf{a}^{-}\,|m\rangle = (1-q^2)\,|m-1\rangle$$

- L-operator $L^O \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes F_+)$ [Bazhanov-Sergeev'06]

$$L^O(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{a,b \in \{0,1\}} |a\rangle \otimes |b\rangle \otimes \pi_O((L^O)^{a,b}_{i,j})\,|k\rangle \quad (L^O)^{a,b}_{i,j} =$$





$1 \qquad 1 \qquad \mu\mathbf{k} \qquad -q\mu^{-1}\mathbf{k} \qquad \mathbf{a}^{+} \qquad \mathbf{a}^{-}$

μ: parameter

$RLLL$ relation for OOO 6/25

- Thm: [Bazhanov-Sergeev'06]
  - Consider the following $RLLL$ relation for L^O:

$$\underset{\mu_4}{\underline{L^O_{124}}}\,\underset{\mu_5}{\underline{L^O_{135}}}\,\underset{\mu_6}{\underline{L^O_{236}}}\,R^{OOO}_{456} = R^{OOO}_{456}\,L^O_{236}\,L^O_{135}\,L^O_{124}$$

  - $R^{OOO} \in \mathrm{End}(F_+^{\otimes 3})$ is uniquely determined and given by

$$(R^{OOO})^{a,b,c}_{i,j,k} = \delta^{a+b}_{i+j}\delta^{b+c}_{j+k}\left(\frac{\mu_3}{\mu_2}\right)^i\left(-\frac{\mu_1}{\mu_3}\right)^b\left(\frac{\mu_2}{\mu_1}\right)^k q^{ik+b(k-i+1)}\binom{a+b}{a}_{q^2}\,{}_2\phi_1\left(\begin{matrix}q^{-2b}, q^{-2i}\\ q^{-2a-2b}\end{matrix}; q^2, q^{-2c}\right)$$

$$(z;q)_\infty = \prod_{n\geq 0}(1-zq^n) \qquad (z;q)_m = \frac{(z;q)_\infty}{(zq^m;q)_\infty} \qquad {}_2\phi_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}; q, z\right) = \sum_{n\geq 0}\frac{(\alpha;q)_n(\beta;q)_n}{(\gamma;q)_n(q;q)_n}z^n$$

  - R^{OOO} also satisfies the $RRRR$ type tetrahedron equation:

$$R^{OOO}_{124}R^{OOO}_{135}R^{OOO}_{236}R^{OOO}_{456} = R^{OOO}_{456}R^{OOO}_{236}R^{OOO}_{135}R^{OOO}_{124}$$

- Thm: [Kapranov-Voevodsky'94]
  - R^{OOO} = intertwiner of irreps of quantum coordinate ring $A_q(A_2)$

$$R^{OOO} \circ (\pi_1 \otimes \pi_2 \otimes \pi_1(\Delta^{\mathrm{op}}(g))) = (\pi_2 \otimes \pi_1 \otimes \pi_2(\Delta(g))) \circ R^{OOO} \quad {}^{\forall}g \in A_q(A_2)$$

Outline

q-Weyl algebra

- Aim: Generalize the $RLLL$ approach by Bazhanov-Sergeev
- Recall: q-Oscillator algebra O_q
  - Genetators: $\mathbf{k}, \mathbf{a}^{\pm}$
  - Relations $\mathbf{k}\mathbf{a}^{\pm} = q^{\pm}\mathbf{a}^{\pm}\mathbf{k}, \quad \mathbf{a}^{+}\mathbf{a}^{-} = 1 - \mathbf{k}^2, \quad \mathbf{a}^{-}\mathbf{a}^{+} = 1 - q^2\mathbf{k}^2$
  - Representation π_O on $F_+ = \oplus_{m\in\mathbb{Z}_{\geq 0}}\mathbb{C}\,|m\rangle$:

$$\pi_O : \mathbf{k}\,|m\rangle = q^m\,|m\rangle\,, \quad \mathbf{a}^+\,|m\rangle = |m+1\rangle\,, \quad \mathbf{a}^-\,|m\rangle = (1-q^2)\,|m-1\rangle$$

- q-Weyl algebra W_q
  - Generators: $X^{\pm 1}, Z^{\pm 1}$
  - Relations: $XZ = qZX$
  - Representations π_X, π_Z on $F = \oplus_{m\in\mathbb{Z}}\mathbb{C}\,|m\rangle$:

$$\pi_X : X\,|m\rangle = q^m\,|m\rangle\,, \quad Z\,|m\rangle = |m+1\rangle \quad \text{(coordinate rep)}$$
$$\pi_Z : X\,|m\rangle = |m-1\rangle\,, \quad Z\,|m\rangle = q^m\,|m\rangle \quad \text{(momentum rep)}$$

- An embedding $O_q \hookrightarrow W_q$: $\mathbf{k} \mapsto X, \quad \mathbf{a}^+ \mapsto Z, \quad \mathbf{a}^- \mapsto Z^{-1}(1-X^2)$

q-Weyl algebra valued six vertex model

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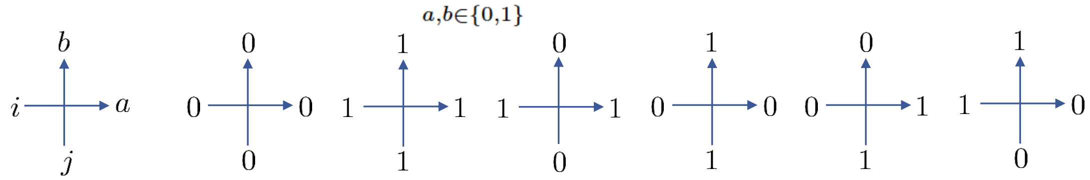
- L-operators L^A $(A = X, Z, O)$ [Kuniba-Matsuike-Y'22]
  - $L^A \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes F)$ $(A = X, Z)$ and $L^O \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes F_+)$

$$L^A(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{a,b\in\{0,1\}} |a\rangle \otimes |b\rangle \otimes \pi_A((L^A)^{a,b}_{i,j})\,|k\rangle \quad (A = X, Z, O)$$



$(L^X)^{a,b}_{i,j} = (L^Z)^{a,b}_{i,j}$	r	s	twX	$-qtX$	Z	$Z^{-1}(rs - t^2wX^2)$
$(L^O)^{a,b}_{i,j}$	1	1	$\mu\mathbf{k}$	$-q\mu^{-1}\mathbf{k}$	$\mathbf{a}^+$	$\mathbf{a}^-$

r, s, t, w, μ: parameters

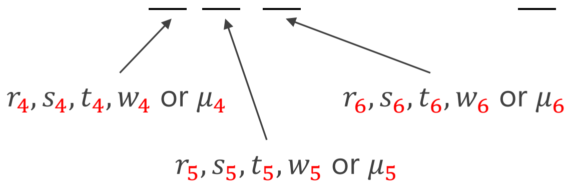
- Remark:
  - L^X for $(r, s, t, w) = (1, 1, \mu^{-1}, \mu^2)$ corresponds to L^O via the pullback.
  - L^Z doesn't have such a correspondence and behaves differently from L^O.
  - Slightly different but similar L^X was introduced in [Bazhanov-Mangazeev-Sergeev'10] but L^Z is new.

Family of $RLLL$ relations

10/25

- Our Problem:
  - Solve the following equation for R^{ABC} (A, B, C ∈ $\{X, Z, O\}$):

$$L^A_{124} L^B_{135} L^C_{236} R^{ABC}_{456} = R^{ABC}_{456} L^C_{236} L^B_{135} L^A_{124}$$



  - Each L has different parameters depending on its tensor compoment.

Main result

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- [Kuniba-Matsuike-Y'22]:
  - We solved $RLLL$ relations for the following ABCs.

ABC	feature	locally finiteness	$\sharp$(sector)
ZZZ	factorized	no	4
OZZ	${}_2\phi_1$	no	1
ZZO	${}_2\phi_1$	no	
ZOZ	${}_3\phi_2$-like	no	
OOZ	factorized	yes	1
ZOO	factorized	yes	
OZO	factorized	no	
OOO	${}_2\phi_1$	yes	1
XXZ	factorized	no	2
ZXX	factorized	no	
XZX	factorized	no	

  - For all cases, R^{ABC} are uniquely determined in each sector specified by appropriate parity conditions.
  - We obtained the explicit formulae for them, where their matrix elements are either factorized or expressed as q-hypergeometric series.

$RLLL$ relation for ZZZ

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- Examples of $RLLL$ relation for ZZZ:

$$R(1\otimes X\otimes X)=(1\otimes X\otimes X)R,\quad R(X\otimes X\otimes 1)=(X\otimes X\otimes 1)R,$$
$$-r_1r_3R(1\otimes Z\otimes 1)=(qt_1t_3w_1X\otimes Z\otimes X-r_2Z\otimes 1\otimes Z)R,$$
$$R(-qt_1t_3w_3X\otimes Z\otimes X+s_2Z\otimes 1\otimes Z)=s_1s_3(1\otimes Z\otimes 1)R,$$
$$t_1R(X\otimes Z\otimes Z^{-1}(r_3s_3-t_3^2w_3X^2)+s_2t_3Z\otimes 1\otimes X)=s_3t_2(Z\otimes X\otimes 1)R,$$
$$R(t_3w_3Z^{-1}(r_1s_1-t_1^2w_1X^2)\otimes Z\otimes X+s_2t_1w_1X\otimes 1\otimes Z)=s_1t_2w_2(1\otimes X\otimes Z)R.$$
$$\pi_Z : X\,|m\rangle=|m-1\rangle\,,\quad Z\,|m\rangle=q^m\,|m\rangle$$

- Writing down actions of π_Z, we obtain recursion relations for R^{ZZZ}:

$$R^{a,b,c}_{i,j-1,k-1}=R^{a,b+1,c+1}_{i,j,k},\quad R^{a,b,c}_{i-1,j-1,k}=R^{a+1,b+1,c}_{i,j,k},$$
$$(q^{a+c}r_2-q^jr_1r_3)R^{a,b,c}_{i,j,k}=q^{1+b}t_1t_3w_1R^{a+1,b,c+1}_{i,j,k},$$
$$(q^{i+k}s_2-q^bs_1s_3)R^{a,b,c}_{i,j,k}=q^{1+j}t_1t_3w_3R^{a,b,c}_{i-1,j,k-1},$$
$$q^jr_3s_3t_1R^{a,b,c}_{i-1,j,k}-q^{j+2}t_1t_3^2w_3R^{a,b,c}_{i-1,j,k-2}+q^{i+k}s_2t_3R^{a,b,c}_{i,j,k-1}=q^{a+k}s_3t_2R^{a,b+1,c}_{i,j,k},$$
$$q^jr_1s_1t_3w_3R^{a,b,c}_{i,j,k-1}-q^{j+2}t_1^2t_3w_1w_3R^{a,b,c}_{i-2,j,k-1}+q^{i+k}s_2t_1w_1R^{a,b,c}_{i-1,j,k}=q^{c+i}s_1t_2w_2R^{a,b+1,c}_{i,j,k}$$

- Fact: Recursion relations for ZZZ consists of 4 disjoint sets, which are specified with the parity pair $(d_1,d_2)=(a+c-j,b-i-k)$.

RLLL relation for *ZZZ*

- Thm: [Kuniba-Matsuike-Y'22]
  - $R^{ZZZ} \in \mathrm{End}(F^{\otimes 3})$ is uniquely determined in each sector and given by

$$R^{a,b,c}_{i,j,k} = \left(\frac{r_2}{t_1 t_3 w_1}\right)^{\frac{d_1}{2}} \left(\frac{s_2}{t_1 t_3 w_3}\right)^{\frac{d_2}{2}} \left(\frac{t_2}{s_1 t_3}\right)^{\frac{d_3}{2}} \left(\frac{t_2 w_2}{s_3 t_1 w_1}\right)^{\frac{d_4}{2}}$$

$$\times q^{\varphi} \frac{\Phi_{d_2}\left(\frac{s_1 s_3}{s_2}\right) \Phi_{d_3}\left(\frac{r_3 w_2}{s_3 w_1}\right) \Phi_{d_4}\left(\frac{r_1 w_3}{s_1 w_2}\right)}{\Phi_{-d_1}\left(\frac{q^2 r_1 r_3}{r_2}\right) \Phi_{d_3+d_4}\left(\frac{r_1 r_3 w_3}{s_1 s_3 w_1}\right)}, \qquad a,b,c,i,j,k \in \mathbb{Z}$$

$$\varphi = \frac{1}{4}\left((d_1 - d_2)(d_1 + d_2 + d_3 + d_4) + d_3 d_4\right) - d_1,$$

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} a + c - j \\ b - i - k \end{pmatrix}, \quad \begin{pmatrix} d_3 \\ d_4 \end{pmatrix} = \begin{pmatrix} -a - b + c + i + j - k \\ a - b - c - i + j + k \end{pmatrix}$$

$$\Phi_m(z) = \frac{1}{(zq^m; q^2)_\infty} \quad (m \in \mathbb{Z}),$$

- Features:
  - The matrix elements of R^{ZZZ} are factorized.
  - R^{ZZZ} is *not* locally finite.
  - There are 4 sectors specified with the parity pair (d_1, d_2).

RLLL relation for *OZZ*

- Thm: [Kuniba-Matsuike-Y'22]
  - $R^{OZZ} \in \mathrm{End}(F_+ \otimes F \otimes F)$ is uniquely determined and given by

$$R^{a,b,c}_{i,j,k} = \theta(i \geq 0) \left(\frac{r_2}{r_3}\right)^a \left(\frac{s_3}{s_2}\right)^i \left(\frac{t_2 w_2}{\mu s_2}\right)^{-b+j} \left(-\frac{\mu t_3}{r_3}\right)^{-c+k} \frac{(z;q^2)_a}{(q^2;q^2)_a} q^{(a-b+j-1)c-(i-b+j-1)k-aj+bi}$$

$$\times\, {}_2\phi_1\left(\begin{matrix} q^{-2i}, z^{-1}q^2 \\ z^{-1}q^{-2a+2} \end{matrix}; q^2, yq^{2i+2j-2a-2b}\right). \qquad a,i \in \mathbb{Z}_{\geq 0},\ b,c,j,k \in \mathbb{Z}$$

$$\mu = \mu_4, \quad y = \frac{r_3 w_3}{\mu^2 s_3}, \quad z = q^{2k-2c+2}\frac{\mu^2 s_2}{r_2 w_2}$$

- Features:
  - The matrix elements of R^{OZZ} are expressed as q−hypergeometric series.
  - R^{OZZ} is *not* locally finite.
  - There is only 1 sector.

RLLL relation for *OOZ*

- Thm: [Kuniba-Matsuike-Y'22]
  - $R^{OOZ} \in \mathrm{End}(F_+ \otimes F_+ \otimes F)$ is uniquely determined and non-trivial iff $\mu_1/\mu_2 = q^d$ for d ∈ ℤ. In that case, it is given by

$$R(d)^{a,b,c}_{i,j,k} = \theta(e \in \mathbb{Z})\theta(\min(i,j) \geq 0)\delta^{a+b}_{i+j} \qquad a,b,i,j \in \mathbb{Z}_{\geq 0},\ c,k \in \mathbb{Z}$$

$$\times\ s_3^i(\mu_2 t_3)^{-a}\left(\frac{\mu_2 s_3}{t_3 w_3}\right)^j\left(\frac{t_3^2 w_3}{r_3 s_3}\right)^e q^{cj-bk}\frac{(q^{2+2e-2j};q^2)_j(q^{2a+2};q^2)_{i-a}}{(q^2;q^2)_f(q^{2a-2e};q^2)_{e-a}}$$

$$e = \frac{1}{2}(a-c+j+k+d), \quad f = \frac{1}{2}(b+c+i-k-d)$$

- Features:
  - The matrix elements of R^{OOZ} are factorized.
  - R^{OOZ} is locally finite.
  - There is only 1 sector but R^{OOZ} is non-trivial if the parity of $2e$ is even.

RLLL relation for *OOO*

- Thm: [Bazhanov-Sergeev'06]
  - $R^{OOO} \in \mathrm{End}(F_+^{\otimes 3})$ is uniquely determined and given by

$$(R^{OOO})^{a,b,c}_{i,j,k} = \delta^{a+b}_{i+j}\delta^{b+c}_{j+k}\left(\frac{\mu_3}{\mu_2}\right)^i\left(-\frac{\mu_1}{\mu_3}\right)^b\left(\frac{\mu_2}{\mu_1}\right)^k q^{ik+b(k-i+1)}\binom{a+b}{a}_{q^2}{}_2\phi_1\left(\begin{matrix}q^{-2b}, q^{-2i} \\ q^{-2a-2b}\end{matrix};q^2,q^{-2c}\right)$$

$$a,b,c,i,j,k \in \mathbb{Z}_{\geq 0}$$

- Features:
  - The matrix elements of R^{OOO} are expressed as q−hypergeometric series.
  - R^{OOO} is locally finite.
  - There is only 1 sector.
  - R^{OOO} also satisfies the following tetrahedron equation:

$$R^{OOO}_{124}R^{OOO}_{135}R^{OOO}_{236}R^{OOO}_{456} = R^{OOO}_{456}R^{OOO}_{236}R^{OOO}_{135}R^{OOO}_{124}$$

  - R^{OOO} = intertwiner of irreps of quantum coordinate ring $A_q(A_2)$

$$R^{OOO} \circ (\pi_1 \otimes \pi_2 \otimes \pi_1(\Delta^{\mathrm{op}}(g))) = (\pi_2 \otimes \pi_1 \otimes \pi_2(\Delta(g))) \circ R^{OOO} \quad {}^\forall g \in A_q(A_2)$$

$$\pi_i : A_q(A_2) \to \mathrm{End}(F_+)$$

Outline

$RRRR$ equation as associaticity

- If we have $L_{124}L_{135}L_{236}R_{456} = R_{456}L_{236}L_{135}L_{124}$, we have

$$\begin{aligned}
&R_{124}R_{135}R_{236}R_{456}\underline{L_{\alpha\beta6}L_{\alpha\gamma5}L_{\beta\gamma4}}L_{\alpha\delta3}L_{\beta\delta2}L_{\gamma\delta1}\\
&= R_{124}R_{135}R_{236}L_{\beta\gamma4}L_{\alpha\gamma5}\underline{L_{\alpha\beta6}L_{\alpha\delta3}L_{\beta\delta2}}L_{\gamma\delta1}R_{456}\\
&= R_{124}R_{135}L_{\beta\gamma4}\underline{L_{\alpha\gamma5}L_{\beta\delta2}}L_{\alpha\delta3}\underline{L_{\alpha\beta6}L_{\gamma\delta1}}R_{236}R_{456}\\
&= R_{124}R_{135}L_{\beta\gamma4}L_{\beta\delta2}\underline{L_{\alpha\gamma5}L_{\alpha\delta3}L_{\gamma\delta1}}L_{\alpha\beta6}R_{236}R_{456}\\
&= R_{124}\underline{L_{\beta\gamma4}L_{\beta\delta2}L_{\gamma\delta1}}L_{\alpha\delta3}L_{\alpha\gamma5}L_{\alpha\beta6}R_{135}R_{236}R_{456}\\
&= L_{\gamma\delta1}L_{\beta\delta2}\underline{L_{\beta\gamma4}L_{\alpha\delta3}}L_{\alpha\gamma5}L_{\alpha\beta6}R_{124}R_{135}R_{236}R_{456}\\
&= L_{\gamma\delta1}L_{\beta\delta2}L_{\alpha\delta3}L_{\beta\gamma4}L_{\alpha\gamma5}L_{\alpha\beta6}R_{124}R_{135}R_{236}R_{456}
\end{aligned}$$

- $R_{456}R_{236}R_{135}R_{124}$ also gives an intertwiner for $\begin{cases} L_{\alpha\beta6}L_{\alpha\gamma5}L_{\beta\gamma4}L_{\alpha\delta3}L_{\beta\delta2}L_{\gamma\delta1} \\ L_{\gamma\delta1}L_{\beta\delta2}L_{\alpha\delta3}L_{\beta\gamma4}L_{\alpha\gamma5}L_{\alpha\beta6} \end{cases}$
- If they are irreducible and equivalent, we have

$$R_{124}R_{135}R_{236}R_{456} = R_{456}R_{236}R_{135}R_{124} \quad \text{(up to normalization)}$$

$RRRR$ equations for R^{ABC}

- For our $RLLL$ relations, we expect the following $RRRR$ equation holds:

$$R_{124}^{ABD}R_{135}^{ACE}R_{236}^{BCF}R_{456}^{DEF} = R_{456}^{DEF}R_{236}^{BCF}R_{135}^{ACE}R_{124}^{ABD}$$

$$A, B, C, D, E, F \in \{X, Z, O\}$$

- Remark:
  - Each tensor component is assigned with different parameters.
    - e.g. If $A = B = C = D = E = F = Z$, this depends on r_i, s_i, t_i, w_i $(i = 1, \dots, 6)$.
  - R^{ABC}s except for $ABC = OOZ, ZOO, OOO$ are not locally finite, so the convergence of $RRRR$ equation is non-trivial for such cases.
  - L^Z is not irreducible because $(L^Z)_{i,j}^{a,b}$ does not include X^{-1}.

i	j	a	b	$(L^Z)_{i,j}^{a,b}$
0	0	0	0	r
1	1	1	1	s
1	0	1	0	twX
0	1	0	1	$-qtX$
0	1	1	0	Z
1	0	0	1	$Z^{-1}(rs - t^2wX^2)$

$$\pi_Z : X\,|m\rangle = |m-1\rangle\,, \quad Z\,|m\rangle = q^m\,|m\rangle$$

$RRRR$ equations for R^{ABC}

- Conjecture: [Kuniba-Matsuike-Y'22]
  - The following $RRRR$ equations are valid:

$$R_{456}^{OOO}R_{236}^{OOO}R_{135}^{ZOO}R_{124}^{ZOO} = R_{124}^{ZOO}R_{135}^{ZOO}R_{236}^{OOO}R_{456}^{OOO}$$
$$R_{456}^{ZOO}R_{236}^{OOO}R_{135}^{OOO}R_{124}^{OOZ} = R_{124}^{OOZ}R_{135}^{OOO}R_{236}^{OOO}R_{456}^{ZOO}$$
$$R_{456}^{OOZ}R_{236}^{OOZ}R_{135}^{OOO}R_{124}^{OOO} = R_{124}^{OOO}R_{135}^{OOO}R_{236}^{OOZ}R_{456}^{OOZ}$$
$$R_{456}^{OOZ}R_{236}^{OOZ}R_{135}^{ZOO}R_{124}^{ZOO} = R_{124}^{ZOO}R_{135}^{ZOO}R_{236}^{OOZ}R_{456}^{OOZ}.$$

$$R_{456}^{OOO}R_{236}^{ZOO}R_{135}^{OOO}R_{124}^{OZO} = R_{124}^{OZO}R_{135}^{OOO}R_{236}^{ZOO}R_{456}^{OOO}$$
$$R_{456}^{OZO}R_{236}^{OOO}R_{135}^{OOZ}R_{124}^{OOO} = R_{124}^{OOO}R_{135}^{OOZ}R_{236}^{OOO}R_{456}^{OZO}$$
$$R_{456}^{OOO}R_{236}^{ZOO}R_{135}^{ZOO}R_{124}^{ZZO} = R_{124}^{ZZO}R_{135}^{ZOO}R_{236}^{ZOO}R_{456}^{OOO}$$
$$R_{456}^{ZOO}R_{236}^{OOO}R_{135}^{ZOO}R_{124}^{ZOZ} = R_{124}^{ZOZ}R_{135}^{ZOO}R_{236}^{OOO}R_{456}^{ZOO}$$
$$R_{456}^{ZOO}R_{236}^{ZOO}R_{135}^{OOO}R_{124}^{OZZ} = R_{124}^{OZZ}R_{135}^{OOO}R_{236}^{ZOO}R_{456}^{ZOO}$$
$$R_{456}^{ZZO}R_{236}^{OOO}R_{135}^{OOZ}R_{124}^{OOZ} = R_{124}^{OOZ}R_{135}^{OOZ}R_{236}^{OOO}R_{456}^{ZZO}$$
$$R_{456}^{ZOZ}R_{236}^{OOZ}R_{135}^{OOO}R_{124}^{OOZ} = R_{124}^{OOZ}R_{135}^{OOO}R_{236}^{OOZ}R_{456}^{ZOZ}$$
$$R_{456}^{OZZ}R_{236}^{OOZ}R_{135}^{OOZ}R_{124}^{OOO} = R_{124}^{OOO}R_{135}^{OOZ}R_{236}^{OOZ}R_{456}^{OZZ}$$
$$R_{456}^{ZOO}R_{236}^{ZOO}R_{135}^{ZOO}R_{124}^{ZZZ} = R_{124}^{ZZZ}R_{135}^{ZOO}R_{236}^{ZOO}R_{456}^{ZOO},$$
$$R_{456}^{ZZZ}R_{236}^{OOZ}R_{135}^{OOZ}R_{124}^{OOZ} = R_{124}^{OOZ}R_{135}^{OOZ}R_{236}^{OOZ}R_{456}^{ZZZ}.$$

$$R_{456}^{OOO}R_{236}^{OZO}R_{135}^{OZO}R_{124}^{OOO} = R_{124}^{OOO}R_{135}^{OZO}R_{236}^{OZO}R_{456}^{OOO}$$
$$R_{456}^{OOO}R_{236}^{OZO}R_{135}^{ZZO}R_{124}^{ZOO} = R_{124}^{ZOO}R_{135}^{ZZO}R_{236}^{OZO}R_{456}^{OOO}$$
$$R_{456}^{OZO}R_{236}^{OOO}R_{135}^{ZOZ}R_{124}^{ZOO} = R_{124}^{ZOO}R_{135}^{ZOZ}R_{236}^{OOO}R_{456}^{OZO}$$
$$R_{456}^{OOO}R_{236}^{ZZO}R_{135}^{OZO}R_{124}^{OZO} = R_{124}^{OZO}R_{135}^{OZO}R_{236}^{ZZO}R_{456}^{OOO}$$
$$R_{456}^{OOZ}R_{236}^{ZOZ}R_{135}^{OOO}R_{124}^{OZO} = R_{124}^{OZO}R_{135}^{OOO}R_{236}^{ZOZ}R_{456}^{OOZ}$$
$$R_{456}^{ZOO}R_{236}^{OZO}R_{135}^{OZO}R_{124}^{OOZ} = R_{124}^{OOZ}R_{135}^{OZO}R_{236}^{OZO}R_{456}^{ZOO}$$
$$R_{456}^{OZO}R_{236}^{OZO}R_{135}^{OZZ}R_{124}^{OOO} = R_{124}^{OOO}R_{135}^{OZZ}R_{236}^{OZO}R_{456}^{OZO}$$
$$R_{456}^{OOZ}R_{236}^{OZZ}R_{135}^{OZO}R_{124}^{OOO} = R_{124}^{OOO}R_{135}^{OZO}R_{236}^{OZZ}R_{456}^{OOZ}$$
$$R_{456}^{OZO}R_{236}^{OZO}R_{135}^{ZZZ}R_{124}^{ZOO} = R_{124}^{ZOO}R_{135}^{ZZZ}R_{236}^{OZO}R_{456}^{OZO},$$
$$R_{456}^{OOZ}R_{236}^{ZZZ}R_{135}^{OZO}R_{124}^{OZO} = R_{124}^{OZO}R_{135}^{OZO}R_{236}^{ZZZ}R_{456}^{OOZ}.$$

Rermark: Each equation is checked for over 10000 outer lines by computer.

R^{ZZZ} as intertwiner of $A_q(A_2)$ 21/25

- Proposition: [Kuniba-Matsuike-Y'22]
  - $R^{ZZZ} \in \mathrm{End}(F^{\otimes 3})$ satisfies the following intertwining relation of the quantum coordinate ring $A_q(A_2)$:

$$R^{ZZZ} \circ (\pi_1 \otimes \pi_2 \otimes \pi_1(\Delta^{\mathrm{op}}(g))) = (\pi_2 \otimes \pi_1 \otimes \pi_2(\Delta(g))) \circ R^{ZZZ} \quad {}^\forall g \in A_q(A_2)$$

  - $\pi_i = \pi_Z \circ \varrho_i$, where ϱ_1 and ϱ_2 are respectively given by t_{ij}: generators of $A_q(A_2)$

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \mapsto \begin{pmatrix} Z^{-1}(u_1 - g_1 h_1 X^2) & g_1 X & 0 \\ -q h_1 X & Z & 0 \\ 0 & 0 & u_1^{-1} \end{pmatrix}, \begin{pmatrix} u_2^{-1} & 0 & 0 \\ 0 & Z^{-1}(u_2 - g_2 h_2 X^2) & g_2 X \\ 0 & -q h_2 X & Z \end{pmatrix}$$

  - π_is are not irreducible. $\quad \pi_Z : X|m\rangle = |m-1\rangle, \quad Z|m\rangle = q^m |m\rangle$
  - Identification of parameters is done as follows:

$$u_1 = u_2 (=: u) \qquad g_1 h_1 = g_2 h_2 (=: p)$$

$$\frac{r_1}{t_1} = \frac{r_2}{t_2}, \quad \frac{s_2}{t_2} = \frac{s_3}{t_3}, \quad \frac{r_2}{r_1 r_3} = u, \quad \frac{s_1 s_3}{s_2} = u^2, \quad \frac{t_1^2 w_1}{r_1 s_1} = \frac{t_2^2 w_2}{r_2 s_2} = \frac{t_3^2 w_3}{r_3 s_3} = \frac{p}{u}.$$

Root of unity 22/25

- If we specialize q to a root of unity, the Fock spaces F, F_+ become finite dimensional. If we can formulate R^{ABC} in such cases…
- Extension of family of $RRRR$ equations:
  - Getting over its non locally finiteness, we obtain more family of $RRRR$ equations.
- Connection with physical models:
  - Finite dimensional solutions to tetrahedron equations are quite important because they can be used to construct tractable 3D transfer matrices.
  - [Bazhanov-Mangazeev-Sergeev'10] introduced $(L^X)'$ which is slightly different from L^X and solved $(R^{XXX})'$ at N-th root of unity. They found
    $(R^{XXX})' \cong$ Bazhanov-Baxter model
    (*spectral parameter dependent* solution to tetrahedron equation)

⬇ reduction [Bazhanov-Baxter'92]

generalized chiral Potts model
$\cong$ 2D R matrices associated with $U_q(A_{n-1}^{(1)})$ at root of unity

Other comments

- Boundary integrability in 3D:

$R(LLL) = (LLL)R$ (Yang-Baxter equation up to conjugation)	➡	RRRR=RRRR (Tetrahedron equation)
$K(LGLG) = (GLGL)K$ (reflection equation up to conjugation)	➡	RKRRKKR=RKKRRKR (3D reflection equation)

  - a q-Weyl algebra version of [Kuniba-Pasquier'18], [Kuniba-Okado-Y'19]?

- Reduction to 2D:
  - Generally, infinitely many solutions to the Yang-Baxter equation are obtained from one solution to the tetrahedron equation.
  - For R^{OOO}, they are identified with R matrices associated with

reduction	R matrices [Kuniba-Okado'14]
by trace	$U_q(A_{n-1}^{(1)})$, symmetric tensor rep.
by boundary vector	$U_q(D_{n+1}^{(2)})$, $U_q(A_{2n}^{(2)})$, $U_q(C_n^{(1)})$, Fock rep.

Other comments

- Characterization in terms of PBW bases:
  - Let us consider the transition matrix γ for PBW bases of quantum enveloping algebra $U_q(A_2)$:

$$e_2^{(a)} e_{12}^{(b)} e_1^{(c)} = \sum_{i,j,k} \gamma_{i,j,k}^{a,b,c} e_1^{(k)} e_{21}^{(j)} e_2^{(i)} \quad \cdots (*) \qquad i,j,k,a,b,c \in \mathbb{Z}_{\geq 0}$$

  - $e_i^{(a)}$: divided power given by $e_i^{(a)} = e_i^a/[a]!$
  - Theorem: [Sergeev'07], [Kuniba-Okado-Yamada'13]

$$\gamma_{i,j,k}^{a,b,c} = (R^{OOO})_{i,j,k}^{a,b,c}$$

  - Can we formulate R^{ABC} in this context?

Summary

25/25

- We considered three kinds of L-operators L^X, L^{Z}, L^{O} and $RLLL$ relations which they satisfy. They can be regarded as q-Oscillator or q-Weyl algebra valued six vertex models.
- We solved these $RLLL$ relations and obtained explicit formulae for R^{ABC}. For all cases, R^{ABC} are uniquely determined in each sector specified by appropriate parity conditions and their matrix elements are either factorized or expressed as q-hypergeometric series.
- By computer experiments, we conjectured $RRRR$ equations for R^{ABC}. This is motivated by earlier results about representation theoretic origin of R^{OOO}.
- We found R^{ZZZ} satisfies an intertwining relation for *reducible* representations of $A_q(A_2)$.