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Seminar on Abelian Functions 2023

Organized by
Takanori Ayano

December 4, 2023 and January 9, 2024.

Abstract

We report on the seminar on Abelian functions held via Zoom on December 4, 2023 and January 9, 2024.

2020 *Mathematics Subject Classification.*
14H40, 14H42, 14H70, 14H81, 14K25, 30F10, 32G20, 33E05.

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theta function, sigma function, Riemann surface,
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Preface

The theory of elliptic functions was one of the main subjects of research in the 19th century and the concrete theory of elliptic functions was constructed. It is applied to many fields in mathematics and science. As the technology advances, there are movements to apply Abelian functions, which are generalizations of elliptic functions to many variables, to many fields in mathematics and science. The elliptic sigma function $\sigma(u)$ and the elliptic function $\wp(u)$, which are defined and studied by Weierstrass, are generalized to the multivariable sigma functions and the Abelian functions associated with hyperelliptic curves by Klein and Baker about 100 years ago. The hyperelliptic sigma functions and the hyperelliptic Abelian functions are generalized and extensively studied for the last three decades.

The purpose of this seminar is to learn the latest findings about Abelian functions and related fields from lectures by experts and discuss them. The 1st seminar on Abelian functions was held via Zoom on December 4, 2023. In this seminar, 62 people registered for participation. The 2nd seminar on Abelian functions was held via Zoom on January 9, 2024. In this seminar, 45 people registered for participation. There were lively discussions among participants. We are grateful to the participants of the seminar for their contribution. In this report, we record the slides of the lectures.

The website of this seminar is as follows:

`https://sites.google.com/view/abelianfunction/%E3%83%9B%E3%83%BC%E3%83%A0`

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Organizers

Takanori Ayano

Department of Mathematics, Graduate School of Science, Osaka Metropolitan University, 3-3-138, Sugimoto, Sumiyoshi-ku, Osaka, 558-8585, Japan

Email address: ayano@omu.ac.jp

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佐藤のグラスマン多様体とテータ関数入門

Atsushi Nakayashiki
Tsuda University

OCAMI アーベル関数論セミナー

December 4, 2023

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$$
\begin{tikzcd}
\text{代数曲線} \arrow[r] \arrow[d] & \begin{array}{l} \text{周期 } \omega_1, \omega_2,\ \tau = \omega_1^{-1}\omega_2 \\ \theta(z,\tau) : \text{リーマンテータ関数} \\ \sigma(u) : \text{シグマ関数(多変数)} \end{array} \arrow[d] \\
\begin{array}{c} \text{佐藤のグラスマン多様体} \\ (\text{UGM}) \end{array} \arrow[r, leftrightarrow] & \text{KP階層の解}
\end{tikzcd}
$$

・この図式は可換だが、右回りと左回りでは、解の記述の仕方が異なる。

→ これを用いて、テータ関数、シグマ関数の性質を調べることが出来る。

ここでは、楕円曲線を例にとり、これを説明する。

話の予定

1. 楕円曲線とテータ関数、シグマ関数

2. 佐藤のグラスマン多様体(UGM)

3. 代数曲線のデータのUGMへの埋め込み

4. 展開

5. 退化

6. まとめ

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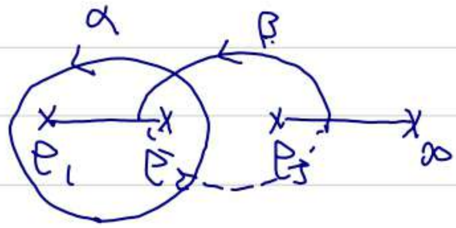
1. 楕円曲線とテータ関数，シグマ関数

$X: \ y^2 = 4x^3 - g_2 x - g_3, \quad g_2^3 - 27 g_3^2 \neq 0$

$= 4(x-e_1)(x-e_2)(x-e_3)$



$\longrightarrow du = \frac{dx}{y}, \quad dr = \frac{x\,dx}{y}$

$\{\alpha, \beta\}$ s.t. $\alpha \cdot \beta = 1$

$\int_\alpha du = 2\omega_1, \quad \int_\beta du = 2\omega_2$

$\int_\alpha dr = -2\eta_1, \quad \int_\beta dr = -2\eta_2$

$\longrightarrow \tau = \frac{\omega_2}{\omega_1} \qquad \mathrm{Im}\,\tau > 0$

$\longrightarrow$ テータ関数：$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z, \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i (n+a)^2 \tau + 2\pi i (n+a)(z+b)}$

$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z + m + n\tau) = e^{2\pi i a m - 2\pi i b n} e^{-\pi i n^2 \tau - 2\pi i n z} \theta\begin{bmatrix} a \\ b \end{bmatrix}(z, \tau)$

特に

$\theta_{11}(z, \tau) = \theta\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}(z, \tau)$ とおくと

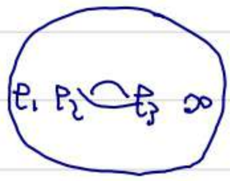
- $\theta_{11}(-z, \tau) = -\theta_{11}(z, \tau)$
- $\theta_{11}(z, \tau) = \theta_{11}'(0, \tau) z + \frac{\theta'''(0, \tau)}{3!} z^3 + \cdots, \qquad \theta_{11}'(0, \tau) \neq 0$

→ シグマ関数

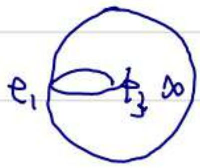
$$\sigma(u) = \frac{2\omega_1}{\theta_{11}'(0,\tau)} e^{\frac{1}{2}\omega_1^{-1}\eta_1 u^2}\theta_{11}((2\omega_1)^{-1}u,\tau)$$

展開: $\sigma(u) = u - \frac{g_2}{240}u^5 - \frac{g_3}{840}u^7 + \cdots$

- u^nの係数は g_2, g_3 の同次多項式 ($\deg g_j = 2j$)
- $\deg u = -1$ とおくと, $\deg \sigma(u) = -1$

退化: $X =$ 

$\xrightarrow{e_2 \to e_1}$ (g_2, g_3 を $g_2^3 - 27g_3^2 = 0$ に近づける)



$$\sigma(u) \longmapsto \frac{1}{2\sqrt{3e_1}} e^{-\frac{e_1}{2}u^2}\left(e^{\sqrt{3e_1}u} - e^{-\sqrt{3e_1}u}\right)$$

- これらは古典的によく知られており, 微分方程式, 楕円積分を用いて導くことができる(種数1)。
- 種数2以上の場合に同様にして調べることは難しい。

→ 1つの考え方: UGM を使う。

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2. 佐藤のグラスマン多様体 (UGM)

KP方程式

$u = u(x, y, t)$

$$3u_{yy} + (-4u_t + 6uu_x + u_{xxx})_x = 0$$

・未知関数の変換 $u = 2\partial_x^2 \log \tau$

・τの方が解の記述が簡単 (τ ~ 指数関数, テータ関数
u ~ 楕円関数)

KP方程式 $\hookleftarrow$ KP階層

・無限変数 $t = (t_1, t_2, \cdots)$, $t_1 = x$, $t_2 = y$, $t_3 = t$

・無限連立非線形微分方程式

佐藤理論

$$\{\text{KP階層の解}\} \overset{\text{“1対1”}}{\longleftrightarrow} \text{佐藤のグラスマン多様体 (UGM)}$$

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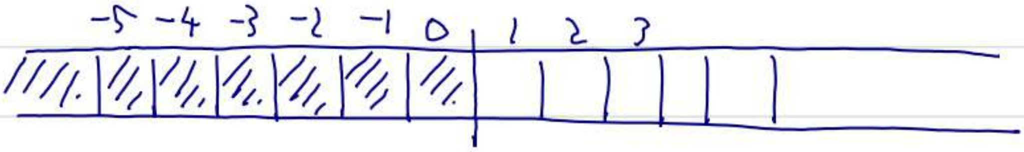
$$V = \mathbb{C}((z)) = \underset{\substack{\parallel \\ V_\phi}}{\mathbb{C}[z^{-1}]} \oplus \underset{\substack{\parallel \\ V_0}}{z\mathbb{C}[[z]]}$$

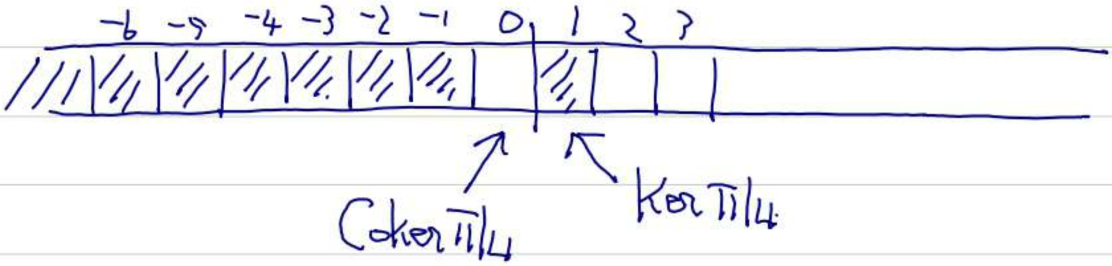
$$\pi : V \longrightarrow V_\phi$$
$$\sum_{n=-\infty}^{+\infty} a_n z^n \longmapsto \sum_{n\leq 0} a_n z^n$$

$$\mathrm{UGM} = \{ V_\phi\text{と同じ大きさを持つ}V\text{の部分空間} \}$$
$$= \{ U \subset V \mid \dim \operatorname{Ker} \pi|_U = \dim \operatorname{Coker} \pi|_U < \infty \}$$

例 $U = V_\phi = \langle z^0, z^{-1}, z^{-2}, \cdots \rangle \in \mathrm{UGM}$



例 $U = \langle z, z^{-1}, z^{-2}, \cdots \rangle \in \mathrm{UGM}$



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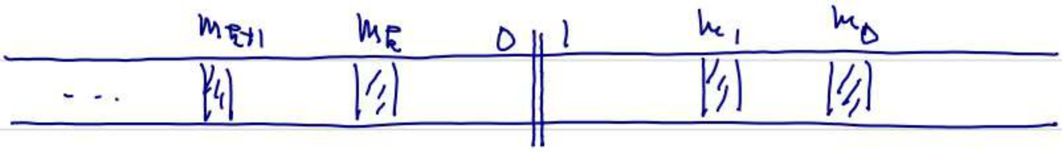
一般に

$\mathbb{U} = \langle f_0, f_1, f_2, \dots \rangle$: frame（基底）

$f_i = z^{m_i} + O(z^{m_i+1})$ $m_0 > m_1 > m_2 > \cdots$

▨ $\longleftrightarrow m_i$ としたとき



このとき

$\mathbb{U} \in UGM \iff$ ・十分左の箱はすべて▨

・$\#\{\|$ より左にある □ $\} = \#\{\|$ より右にある ▨ $\}$

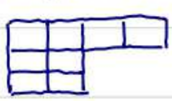
$\iff m_i = -i \quad i \gg 0$

この条件を満たす $M = (m_0, m_1, m_2, \dots)$ を
マヤ図形という。

・{マヤ図形} $\longleftrightarrow$ {分割}

$M = (m_0, m_1, \dots) \mapsto \lambda(M) = (m_0+0, m_1+1, m_2+2, \cdots)$

例 $M = (4, 1, 0, -3, -4, \dots) \to \lambda(M) = (4+0, 1+1, 0+2, -3+3, -4+4 \cdots)$

$= (4, 2, 2) =$ 

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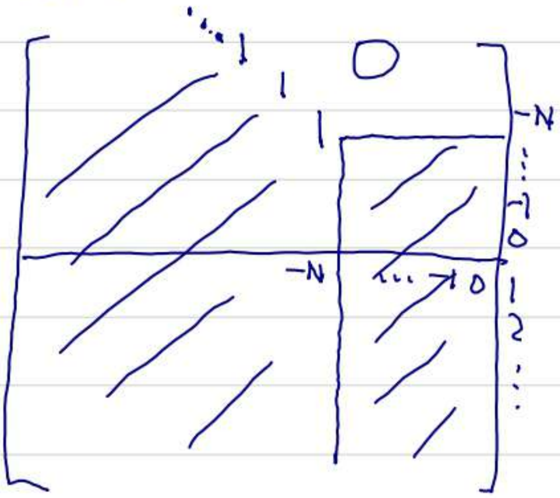
行列表示

$$\sum a_n z^n \longleftrightarrow \begin{bmatrix} \vdots \\ a_{-1} \\ a_0 \\ a_1 \\ \vdots \end{bmatrix}$$

とすると、$U \in M$ の frame は

$\xi =$



$= (\xi_{ij})_{i\in\mathbb{Z},\, j\leq 0}$

行 $\mathbb{Z}$

列 $\mathbb{Z}_{\leq 0}$ $(0, -1, -2, \cdots)$

マヤ図形 $M = (m_0, m_1, \cdots)$ に対し

$$\xi_M = \det(\xi_{m_i j})_{-i, j \leq 0} \quad : \text{プリュッカー座標}$$

例

$$
\xi = \begin{array}{c c}
\begin{bmatrix}
 & \ddots & & \\
 & & 1 & O \\
 & & 1 & \\
 & & b_{-1} & a_{-1} \\
 & & b_0 & a_0 \\
 & & b_1 & a_1 \\
 & & b_2 & a_2 \\
 & & \vdots & \vdots
\end{bmatrix}
&
\begin{array}{l}
\vdots \\
-3 \leftarrow m_3 \\
-2 \leftarrow m_2 \\
-1 \leftarrow m_1 \\
0 \\
1 \\
2 \leftarrow m_0 \\
\
\end{array}
\end{array}
$$

$$M = (2, -1, -2, -3, \cdots)$$

$$
\xi_M = \det \begin{array}{c c}
\begin{bmatrix}
\ddots & 1 & 0 & 0 & 0 \\
\cdots & * & 1 & 0 & 0 \\
\cdots & * & * & b_{-1} & a_{-1} \\
\cdots & * & * & b_2 & a_2
\end{bmatrix}
&
\begin{array}{l}
-3 \\
-2 \\
-1 \\
2
\end{array}
\end{array}
= \det \begin{bmatrix} b_{-1} & a_{-1} \\ b_2 & a_2 \end{bmatrix}
$$

有限次detに帰着

シューア関数

$$e^{\sum_{n=1}^{\infty} t_n u^n} = \sum_{n=0}^{\infty} p_n(t) u^n$$

$$p_0 = 1,\ p_1 = t_1,\ p_2 = t_2 + \frac{t_1^2}{2},\ \cdots$$

分割 $\lambda = (\lambda_1, \cdots, \lambda_\ell)$, $\lambda_1 \geqq \cdots \geqq \lambda_\ell \geqq 0$ に対し

$$
S_\lambda(t) = \det \begin{bmatrix}
p_{\lambda_1} & p_{\lambda_1+1} & \cdots & p_{\lambda_1+\ell-1} \\
p_{\lambda_2-1} & p_{\lambda_2} & & \vdots \\
\vdots & & \ddots & \vdots \\
p_{\lambda_\ell-(\ell-1)} & \cdots & & p_{\lambda_\ell}
\end{bmatrix}
$$

・$|\lambda|=\lambda_1+\cdots+\lambda_\ell$ $\deg t_i=i$ とおくと

$S_\lambda(t)$ は $|\lambda|$ 次同次多項式

定理 (Sato-Sato '82)

・$\xi\in UGM \Rightarrow \tau(t)=\sum \xi_M S_{\lambda(M)}(t)$ は KP階層の解。

・逆に KP階層の任意の形式的べき級数解は このようにして得られる。

逆構成：

$$\Psi^*(t,z)=\frac{\tau(t+[z])}{\tau(t)}e^{-\sum t_n z^n} \quad : \text{ adjoint wave function}$$

$$[z]=\left(z,\frac{z^2}{2},\frac{z^3}{3},\cdots\right)$$

として

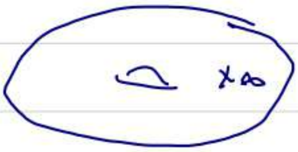
$$U=\mathrm{Span}_{\mathbb{C}}\{\tau(t)\Psi^*(t,z) \text{ の } t \text{ に関する展開係数}\}$$

※ UGMを使って解を作ると このように 解は級数の形で表示される。

3. 代数曲線のデータのUGMへの埋め込み

・代数曲線からUGMの点を構成できる。

$X : y^2 = 4x^3 - g_2 x - g_3$



・∞における局所座標 z

s.t. $x = z^{-2}$, $y = -2z^{-3}\underbrace{\left(1-\frac{g_2}{4}z^4-\frac{g_3}{4}z^6\right)}_{F(z)}^{1/2} = -2z^{-3}(1+O(z))$

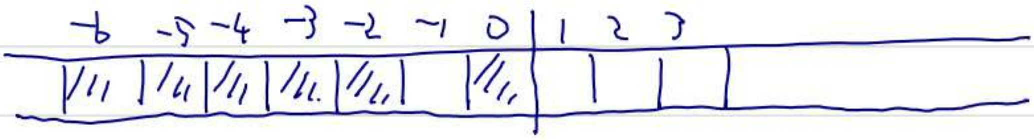
より

$W = H^0(X, \mathcal{O}(*\infty)) =$ X上の有理型関数で∞にのみ高々極を持つもの全体

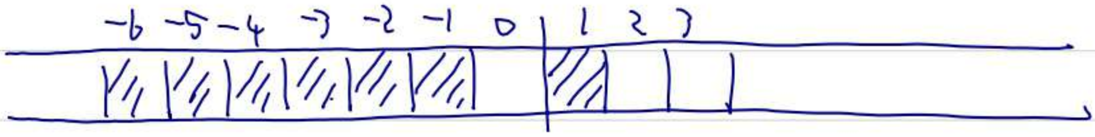
$= \mathbb{C}[x, y]$.

$W : \ x^i, x^i y, \ i \geq 0$

$\sim z^{-2i}, z^{-2i-3}(1+O(z)), \ i \geq 0$



$\therefore U = zW \in UGM \ : \ z^{-2i+1}, z^{-2i-2}(1+O(z)) \ \ i \geq 0$



$\sqcup$ の frame の行列表示：

$$F(z)=\left(1-\frac{g_2}{4}z^4-\frac{g_3}{4}z^6\right)^{1/2}=\sum_{n=0}^{\infty}a_n z^n$$

$a_0=1,\quad a_{2n+1}=0$

$a_2=0$

$a_4=-\frac{1}{8}g_2$

$a_6=-\frac{1}{8}g_3$

$$\xi=\left[\begin{array}{ccccc}
 & \ddots & & & \\
1 & & & & \\
0 & 1 & & & \\
0 & 0 & 1 & & \\
0 & 0 & 0 & 1 & \\
a_4 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 \\
a_6 & 0 & a_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
a_8 & 0 & a_6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
a_{10} & 0 & a_8 & 0 & 0 \\
\vdots & & \vdots & & \vdots
\end{array}\right]
\begin{array}{l}
\vdots \\
-4 \\
-3 \\
-2 \\
-1 \\
0 \\
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
\
\end{array}$$

$$\tau(t)=\sum \xi_M S_{\lambda(M)}(t)=S_{(1)}(t)+\frac{1}{8}g_2 S_{(15)}(t)+\cdots$$

ここで

- ξ_M は g_2, g_3 の $|\lambda(M)|-1$ 次同次式

が分かる。

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$\sqcup$に対応する解のシグマ関数による表示

Krichever構成 $\longrightarrow \Psi^*(t,P) = z\Phi(t,z)$, Φ s.t.

- $\Phi \sim z^{-1}(1+O(z))e^{-\sum t_n z^{-n}}$ at ∞ (essential singularity)
- $X\setminus\{\infty\}$で正則.

$$\Rightarrow \Phi \sim \frac{\sigma(u+\sum_{j=1}^{\infty} b_{1j}t_j)}{\sigma(u)\sigma(\sum_{j=1}^{\infty} b_{1j}t_j)} e^{-\sum_{n=1}^{\infty} t_n \int^P d\hat{r}_n}, \quad u = \int_\infty^P du = z + \cdots$$

$u \to u + 2\omega_j$ に対して、不変(周期的)になるように作る。

これに対し.

$$\Psi^* = z\Phi = \frac{\tau(t+[z])}{\tau(t)} e^{-\sum t_n z^{-n}}$$

となるような $\tau(t)$ を求める。

以下が知られている:

- $du = \dfrac{dx}{y} = \sum_{j=1}^{\infty} b_{1j} z^{j-1} dz \qquad b_{1,1} = 1$

- $\hat{\omega}(P_1,P_2) = \dfrac{2y_1y_2 + 4x_1x_2(x_1+x_2) - g_2(x_1+x_2) - 2g_3}{4y_1y_2} dx_1 dx_2, \qquad P_i = (x_i, y_i) \in X$

$$= \left(\frac{1}{(z_1-z_2)^2} + \sum_{i,j=1}^{\infty} \hat{q}_{i,j} z_1^{i-1} z_2^{j-1}\right) dz_1 dz_2, \qquad z_i = z(P_i)$$

- $\log\sqrt{\dfrac{du}{dz}} = \log\left(1+\sum_{j=2}^{\infty} b_{1j} z^{j-1}\right)^{1/2} = \sum_{n=1}^{\infty} c_n \dfrac{z^n}{n}$

として

$$\hat{q}(t) = \sum_{i,j=1}^{\infty} \hat{q}_{ij} t_i t_j$$

とすると

$$\tau(t) = e^{-\sum_{n=1}^{\infty} c_n t_n + \frac{1}{2}\hat{q}(t)} \sigma\left(\sum_{j=1}^{\infty} b_{1j} t_j\right)$$

$$\therefore\ e^{-\sum c_n t_n + \frac{1}{2}\hat{q}(t)} \sigma\left(t_1 + \sum_{j=2}^{\infty} b_{1j} t_j\right) = \sum_{M} \xi_M S_{\lambda(M)}(t)$$

$t_j = 0$ $j \geqq 2$ とおいて．これから $\sigma(u)$ の展開の u^n の係数は g_2, g_3 の $(n-1)$ 次同次多項式になることが分かる。

同様にして．指標付シグマ関数の展開も調べることが出来る：

$$\sigma_1(u) = e^{\frac{1}{2}\omega_1^{-1}\eta_1 u^2} \cdot \frac{\theta\left[\begin{smallmatrix}\frac{1}{2}\\ 0\end{smallmatrix}\right]((2\omega_1)^{-1}u)}{\theta\left[\begin{smallmatrix}\frac{1}{2}\\ 0\end{smallmatrix}\right](0)} = e^{-\eta_1 u} \frac{\sigma(u+\omega_1)}{\sigma(\omega_1)} = 1 + O(u^2)$$

の展開の u^n の係数は．e_1, g_2, g_3 の n 次同次多項式．$(\deg e_1 = 2)$

UGM ： $U = H^0(X, \mathcal{O}(P_1 + *\infty))$ ， $P_1 = (e_1, 0) \in X$.

5. 退化

$$X:\ y^2 = 4x^3 - g_2x - g_3 = 4(x-e_1)(x-e_2)(x-e_3) \xrightarrow{e_2 \to e_1} y^2 = 4(x-e_1)^2(x-e_3)$$

$e_1+e_2+e_3=0 \quad \therefore\ e_3=-2e_1$

これを調べるため:

$$x \longrightarrow x+e_3$$

とシフトし.

$$Y:\ y^2 = 4x(x-\tilde{e}_1)(x-\tilde{e}_2) \quad , \quad \tilde{e}_j = e_j - e_3$$

を考える。

局所座標 w:

s.t. $x = w^{-2}$, $y = -2w^{-3}F(w)$, $F(w) = \{(1-\tilde{e}_1w^2)(1-\tilde{e}_2w^2)\}^{1/2}$

とする。

$$W = H^0(Y, \mathcal{O}(*\infty)) = \mathbb{C}[x,y] \ :\ x^i,\ x^iy,\ i \geqq 0$$

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$$U = wW : w^{-2i+1},\ w^{-2i-2}\{(1-\tilde{e}_1 w^2)(1-\tilde{e}_2 w^2)\}^{1/2},\ i \geqq 0$$

$$\xrightarrow[(\tilde{e}_2 \to \tilde{e}_1)]{e_2 \to e_1} w^{-2i+1},\ w^{-2i-2}(1-\tilde{e}_1 w^2),\ i \geqq 0$$

$\underset{\sim\sim}{\text{基底のとりかえ}}$

$$w,\ w^{-1}(1-\tilde{e}_1 w^2),\ w^{-3}(1-\tilde{e}_1 w^2), \cdots,\ w^{-2i-2}(1-\tilde{e}_1 w^2),$$

$$= w,\ w^{-2i-1}(1-\tilde{e}_1 w^2),\ w^{-2i-2}(1-\tilde{e}_1 w^2),\ i \geqq 0$$

$$\xrightarrow[\text{ゲージ変換}]{\times(1-\tilde{e}_1 w^2)^{-1}} \frac{w}{1-\tilde{e}_1 w^2},\quad w^{-i}\ (i \geqq 1)$$

$\underset{\sim\sim}{\text{部分分数分解}}$

$$\frac{1}{1-\sqrt{\tilde{e}_1}\,w} - \frac{1}{1+\sqrt{\tilde{e}_1}\,w},\quad w^{-i}\ (i \geqq 1) \qquad ①$$

[KP階層のソリトン解のframe $(M \geqq N)$

$A = (a_{ij}) : M \times N,\quad p_1, \dots, p_M \in \mathbb{C}$

$$z^{-(N-1)} \sum_{i=1}^{M} \frac{a_{ij}}{1-p_i w}\ (1 \leqq j \leqq N),\quad z^{-i}\ (i \geqq N)$$

$\longrightarrow$ $\tau(t)$の解析的表示は知られている.]

$M=2$
$N=1$
$A = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

これから極限で：

$$\tau(t) = e^{\sum_{n=1}^{\infty} t_n \sqrt{e_1}^{\,n}} - e^{\sum_{n=1}^{\infty} t_n (-\sqrt{e_1})^n}$$

$$= e^{\sum_{n:\mathrm{odd}} t_n \sqrt{\tilde{e}_1}^{\,n}} - e^{-\sum_{n:\mathrm{odd}} t_n \sqrt{\tilde{e}_1}^{\,n}}$$

$$\tilde{e}_1 = e_1 - e_3 \underset{\substack{\uparrow \\ e_3 = -2e_1}}{=} 3e_1$$

$$= e^{\sum_{n:\mathrm{odd}} t_n \sqrt{3e_1}^{\,n}} - e^{-\sum_{n:\mathrm{odd}} t_n \sqrt{3e_1}^{\,n}}$$

一方

$$\tau(t) = e^{-\sum c_n' t_n + \frac{1}{2}\hat{q}'(t)} \sigma\left(t_1 + \sum_{j=2}^{\infty} b_{1j}' t_j\right)$$

($x \to x - e_3$ の影響で、c_n, $\hat{q}_{ij}$, b_{ij} が少し変わる。)
とゲージ変換

これから $(t_j = 0,\ j \geq 2,\ t_1 = u)$ として

$$\sigma(u) \longrightarrow \frac{1}{2\sqrt{3e_1}} e^{-\frac{1}{2}e_1 u^2}\left(e^{\sqrt{3e_1}u} - e^{-\sqrt{3e_1}u}\right)$$

が得られる。

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6. まとめ

佐藤のグラスマン多様体(UGM)を利用して

1. 楕円シグマ関数の代数的展開

2. 楕円曲線の有理曲線への退化にともなう
   シグマ関数の極限

を調べることが出来ることを説明した。

→ これは高種数にも拡張できる。

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reference.

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    - Sigma function as a tau function, IMRN 2010-3, 373-394
- J. Bernatska, V. Enolski and A. Nakayashiki
    - Sato Grassmannian and degenerate sigma function, CMP 374 (2020) 627-660
- A. Nakayashiki :
  Vertex operators of the KP hierarchy and singular algebraic curves, arXiv: 2309.08850

関連する最近の研究

・リーマンテータ関数, KP階層のテータ解の退化極限

Takashi Ichikawa:

- Tau functions and KP solutions on families of algebraic curves, arXiv: 2208.07013

- Periods of tropical curves and associated KP solutions, CMP 402 (2023), 1707–1723

D. Agostini, C. Fevola, Y. Mandelshtam and B. Sturmfels:

- KP solitons from tropical limits, arXiv: 2101.10392

Y. Kodama:

- KP solitons and the Riemann theta functions, arXiv: 2308.06902

Y. Fedorov, J. Komeda, S. Matsutani, E. Previato and K. Aomoto:

- The sigma function over a family of cyclic trigonal curves with a singular fiber, arXiv: 1909.03858

「第 2 回 アーベル関数論セミナー」 Abelian integrals and solitons

supported by 大阪公立大学数学研究所「数学・理論物理の協働・共創...

Takashi Ichikawa (ichikawn@cc.saga-u.ac.jp)

Saga University

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Contents

アブストラクト：Riemann 面の Schottky 一意化表示を用いて Abel 微分と積分の明示公式を導き，トロピカル曲線に付随したソリトンの構成と計算に応用する。

§1. Aim of this seminar

§2–4. We review of the theory of Abelian integrals on Riemann surfaces containing:

- Definition and basic facts of Abelian differentials and (period) integrals.
- Their computable variational formulas by the Schottky uniformization theory.

§3–8. Combining the above formulas with tropical geometry, we study the program:

$$\boxed{\text{Quasi-periodic solutions to soliton equations} \overset{(*)}{\Rightarrow} \text{All(?) soliton solutions.}}$$

Here

- quasi-periodic solution: given by Krichever,...using general Riemann theta functions.
- soliton equation: KP, Toda lattice (and Universal?) hierarchies containing
    - KP equation: $3\partial_y^2 u = \partial_x \left(4\partial_t u - \partial_x^3 u - 6u\,\partial_x u\right)$ $\leftrightarrow$ shallow water waves
      $\subset$ KP hierarchy $\sim$ Sato Grassmannian (Nakayashiki's talk), Characterizing Jacobians.
    - Toda lattice: $d_t^2 u_n = e^{u_{n-1}-u_n} - e^{u_n-u_{n+1}}$ $\leftrightarrow$ nonlinearly interacting chains
      $\subset$ Toda hierarchy $\sim$ 2D gravity, String theory, Enumerative geometry,...
- $(*)$: taking limit under rational degeneration of Riemann surfaces.
- soliton: expressed by compositions of rational, exponential, logarithm functions.
- We also discuss their calculation, regularity and nonrational degeneration.

§2. Abelian differentials and period integrals

Set up.

- $\boldsymbol{R}$: Riemann surface: $\left\{\begin{array}{l}\text{compact } \mathbf{1}\text{-dim. complex mfd.}\\ \text{proper smooth algebraic curve}/\mathbb{C}\end{array}\right\}$, genus $\boldsymbol{g} = \sharp(\text{holes})$.
- Abelian differential (of the 1st kind): meromorphic (holomorphic) 1-form on $\boldsymbol{R}$
  $\overset{\text{Riemann-Roch}}{\Rightarrow} \dim\{\text{hol. diff. on } \boldsymbol{R}\} = -\deg(\mathcal{O}_{\boldsymbol{R}}) - 1 + \boldsymbol{g} + \dim H^0(\mathcal{O}_{\boldsymbol{R}}) = \boldsymbol{g}$.
- $\{a_i, b_i\}_{1\le i\le g}$: symp. basis of $H_1(\boldsymbol{R})$; $(a_i, a_j) = (b_i, b_j) = 0$, $(a_i, b_j) = \delta_{ij}$
  $\Rightarrow {}^{\exists 1}\{w_i\}_{1\le i\le g}$: holomorphic abelian diff. on $\boldsymbol{R}$ normalized i.e., $\int_{a_i} w_j = \delta_{ij}$
  $\Rightarrow \left(P_{ij} = \int_{b_i} w_j\right)_{i,j}$: period matrix of $\boldsymbol{R}$ ($\Rightarrow P_{ij} = P_{ji}$, $\mathrm{Im}(P_{ij}) > 0$).

Abel's theorem and Riemann theta function.

- Jacobian of $\boldsymbol{R} \overset{\text{Abel}}{\cong} \mathbb{C}^g/(\mathbb{Z}^g + \mathbb{Z}^g(P_{ij}))$; $\sum_k (p_k - q_k) \mapsto (\sum_k \int_{q_k}^{p_k} w_i)_i / \sim$.
- $\Theta_{\boldsymbol{R}}(z) := \sum_{u\in\mathbb{Z}^g} \exp\left(\pi\sqrt{-1}\left(u(P_{ij})u^T + 2zu^T\right)\right) = \Theta_{\boldsymbol{R}}(z+v)$ $(v \in \mathbb{Z}^g)$.

Question. Can we calculate exact values of P_{ij}?

§3. Explicit variational formulas of differentials and periods

Schottky uniformization. Γ: hyperbolic subgroup of $PGL_2(\mathbb{C})$, free of rank g
$\Rightarrow R_\Gamma = \Omega_\Gamma/\Gamma$: genus g Riemann surface; $\Omega_\Gamma = \mathbb{P}^1 \backslash$(limit set of Γ) (Indra's Pearls).

Explicit formula of w_i, P_{ij} (Schottky, Manin-Drinfeld). $\Gamma = \langle \gamma_i \rangle$, $\gamma_i^{\pm n}(a) \overset{n\to\infty}{\longrightarrow} \alpha_{\pm i}$.
Under $b_i \leftrightarrow \gamma_i$ and convergence conditions, the differentials w_i and integrals P_{ij} satisfy

$$2\pi\sqrt{-1}w_i \overset{\text{Riemann}}{=} d_z\left(\int_a^{\gamma_i(a)} \sum_{\gamma\in\Gamma} \frac{d\gamma(\zeta)}{\gamma(\zeta)-z}\right) = \sum_{\gamma\in\Gamma/\langle\gamma_i\rangle} \frac{(\gamma(\alpha_i)-\gamma(\alpha_{-i}))dz}{(z-\gamma(\alpha_i))(z-\gamma(\alpha_{-i}))},$$

$$\exp(2\pi\sqrt{-1}P_{ij}) = \exp\left(2\pi\sqrt{-1}\int_a^{\gamma_i(a)} w_j\right) = \prod_{\gamma\in\langle\gamma_i\rangle\backslash\Gamma/\langle\gamma_j\rangle} \psi_{ij}(\gamma),$$

where $a \in \Omega_\Gamma$, $\psi_{ij}(\gamma) = \begin{cases} \text{multiplier of } \gamma_i & (i=j, \gamma \in \langle\gamma_i\rangle), \\ \frac{(\alpha_i-\gamma(\alpha_j))(\alpha_{-i}-\gamma(\alpha_{-j}))}{(\alpha_{-i}-\gamma(\alpha_j))(\alpha_i-\gamma(\alpha_{-j}))} & (\text{otherwise}). \end{cases}$

Computable variational formula. $\Delta = (V = \{\text{vertices}\}, E = \{\text{edges}\})$: graph
$\Rightarrow {}^\exists R_\Delta$: family of Riemann surfaces, genus $= \dim H_1(\Delta)$

s.t. $\begin{cases} R_\Delta\text{: deformation by } y_e\ (e \in E) \text{ of singular curve } \bigcup_{v\in V} P_v(=\mathbb{P}^1) \overset{\text{dual}}{\leftrightarrow} \Delta, \\ R_\Delta\text{: Schottky uniformized by } \Gamma_\Delta = \mathrm{Im}(\pi_1(\Delta) \overset{\exists}{\hookrightarrow} PGL_2(\mathbb{C})), \\ \exp(2\pi\sqrt{-1}P_{ij})\text{: computable Laurent series } U_{ij} \text{ in } y_e/\mathbb{Z}[x_{h(\in\pm E)},\ldots]. \end{cases}$

§4. Calculation of universal periods U_{ij}

Example 1. $\Delta = \{$single vertex, g loops$\}$.

$x_{\pm i}, y_i$: variables $(i = 1,\ldots, g)$, $\Gamma_\Delta = \langle \gamma_i \rangle$; $\frac{\gamma_i(z)-x_i}{z-x_i} = y_i \frac{\gamma_i(z)-x_{-i}}{z-x_{-i}}$

$\Rightarrow$ R_Δ: universal deformation of $\mathbb{P}^1/(x_i = x_{-i})_i$ by $y_1,\ldots, y_g$.

Furthermore,

$$\gamma_i(z) - \gamma_i(w) = \frac{(x_i-x_{-i})^2(z-w)y_i}{(z-x_{-i}-y_i(z-x_i))(w-x_{-i}-y_i(w-x_i))} = \frac{O(y_i)}{(z-x_{-i})(w-x_{-i})}$$

$$\Rightarrow \quad \psi_{ij}(\gamma) = 1 + \frac{(x_i-x_{-i})(\gamma(x_j)-\gamma(x_{-j}))}{(x_{-i}-\gamma(x_j))(x_i-\gamma(x_{-j}))} = 1 + O(y_k) \text{ if } \gamma \notin \langle \gamma_i \rangle \cdot \langle \gamma_j \rangle$$

$$\Rightarrow \quad U_{ij} = \begin{cases} y_i + O(y_k y_l) & (i = j), \\ \dfrac{(x_i - x_j)(x_{-i} - x_{-j})}{(x_{-i} - x_j)(x_i - x_{-j})} + O(y_k) & (i \neq j). \end{cases}$$

Summary. $^\exists$Computable theory of Abelian integrals/$\mathbb{Z}[x_h,\ldots]((y_e))$ containing

Abel's theorem: Jacobian of $R_\Delta \cong \mathbb{G}_m^g / \langle (U_{ij})_{1\le i\le g} \,|\, 1 \le j \le g \rangle$.

§5. Tropical curves and master functions of solitons

Tropical curve (of weights $\mathbf{0}$): graph $\Delta = (V, E)$ with length function $l : E \to \mathbb{R}_{>0}$

$$\Rightarrow \begin{cases} \text{genus} := \dim H_1(\Delta), \\ \text{period matrix: bilinear form on } H_1(\Delta)^{\otimes 2} \text{ s.t. } \langle e, e' \rangle = \begin{cases} \pm l(e) & (e = \pm e'), \\ 0 & (e \neq \pm e'). \end{cases} \end{cases}$$

Theorem (Proof of Agostini-Fevola-Mandelshtam-Sturmfels' conjecture (2021)).

C: tropical curve of genus g with (sym. pos.-def.) period matrix B_C

$\Rightarrow$ *$^{\exists}\{R_s\}_{s>0}$: family of Riemann surfaces of genus g with symp. H_1-basis s.t.*

(1) *(Period matrices of R_s) $\times (2\pi\sqrt{-1}) = \log s \cdot B_C + {}^{\exists}B_0$ as $s \downarrow 0$.*

(2) *$D_{C,\alpha} := \{u \in \mathbb{Z}^g; |u - \alpha|_{B_C}$: minimal$\}$: Delaunay finite set for $\alpha \in \mathbb{R}^g$*

$$\Rightarrow M_{C,\alpha}(z) \overset{z \in \mathbb{C}^g}{=} \sum_{u \in D_{C,\alpha}} \exp\left(\frac{uB_0u^T}{2} + zu^T\right): \text{ master } \tau\text{-function of solitons.}$$

Corollary. $u(x, y, t) = 2\partial_x^2 \log M_{C,\alpha}(xr_1 + yr_2 + tr_3 + c)$: *KP soliton*

$$; \begin{cases} B_0,\ r_i: \text{ explicitly represented by the above } x_{h(\in \pm E)}, \\ c: \text{ any constant vector.} \end{cases}$$

§6. Sketch of the proof

Proof of (1). $[\ ,\ ] : H_1(\Delta)^{\otimes 2} \to \bigoplus_{e\in E} \mathbb{Z} y_e$ s.t. $[e, e'] = \begin{cases} \pm y_e & (e = \pm e'), \\ 0 & (e \neq \pm e') \end{cases}$

$$\Rightarrow \quad U_{ij} = \prod_{e\in E} y_e^{[\gamma_i,\gamma_j]_e} (U_{ij}(0) + O(y_{e'}))$$

$\overset{y_e = s^{l(e)}}{\Rightarrow}$ (1), where $B_0 = (\log U_{ij}(0))_{i,j}$: explicit(?).

Proof of (2). ($z \in \mathbb{C}^g$, $\alpha \in \mathbb{R}^g$).

$\Theta_{R_s}(z) = \sum_{u\in\mathbb{Z}^g} \exp\left(\pi\sqrt{-1}\left(uP_s u^T + 2zu^T\right)\right)$; P_s: period matrix of R_s,

$\Theta_C(\alpha) := \max_{u\in\mathbb{Z}^g} \left\{\alpha B_C u^T - \frac{1}{2} u B_C u^T\right\}$: tropical theta function for C

$\Rightarrow M_{C,\alpha}(z) = \lim_{s\to 0} s^{\Theta_C(\alpha)} \cdot \Theta_{R_s}(z - (\log s)\alpha B_C)$ satisfies (2).

Example 2. $C = \{$single vertex, g loops with length $1\} \Rightarrow B_C = I_g$.

$\Gamma_\Delta = \langle \gamma_i \rangle_{1\le i\le g}$; $\frac{\gamma_i(z)-x_i}{z-x_i} = y_i \frac{\gamma_i(z)-x_{-i}}{z-x_{-i}}$: as in Example 1

$$\Rightarrow \quad M_{C,\alpha}(z) = \sum_{u\in\{0,1\}^g} \left(\prod_{i<j} \left(\frac{(x_i - x_j)(x_{-i} - x_{-j})}{(x_{-i} - x_j)(x_i - x_{-j})}\right)^{u_i u_j} \prod_{i=1}^g e^{z_i u_i}\right)$$

: gives the g soliton for $\alpha = (1/2, \ldots, 1/2)$.

§7. Supplementary comments

Regularity of $M_{C,\alpha}$ $\overset{\text{def}}{\Leftrightarrow}$ $M_{C,\alpha}(z) > 0$ $(z \in \mathbb{R}^g) \Rightarrow \partial_x^2 \log M_{C,\alpha}$: real, nonsingular.

M-curve $\overset{\text{def}}{=}$ real alg. curve with max. number $(=$ genus $+ 1)$ of components

$\Rightarrow$ ${}^{\exists}R_s$: M-curves as plane figures obtained by R_Δ taking $x_h \in \mathbb{R}$, $y_e = \pm s^{l(e)}$

$\Rightarrow$ $2\pi\sqrt{-1}\int_{b_i} w_j \in \mathbb{R}$ $\therefore$ $B_0 = (\log|U_{ij}(0)|)_{i,j}$: explicit and $M_{C,\alpha}$: regular.

Master function for general tropical curve.

$C = (V, E, l)$: tropical curve with weight function $w : V \to \mathbb{Z}_{\geq 0}$,

R_v: Riemann surface of genus $w(v)$, $w_{v,i}$: normalized Abelian diff. on R_v

$$\Rightarrow \sum_{u \in D_{C,\alpha}} \exp\left(\tfrac{1}{2} u B_0 u^T + z u^T\right) \prod_{v \in V} \Theta_{R_v}\left((z_{v,i})_i + \int_{u|_{R_v}} (w_{v,i})_i\right)$$

: master function of quasi-periodic solitons $\left(z \in \mathbb{C}^{h_1(\Delta)}, (z_{v,i})_i \in \mathbb{C}^{w(v)}\right)$.

Summary.

- Singular curve with tropical dual graph $\leftrightarrow$ Degenerating Riemann surfaces
  $\Rightarrow$ Master function of solitons reflecting the degeneration rates $(:l(e):)_{e \in E}$.
- Arithmetic of Abelian differentials and integrals $\Rightarrow$ Regularity of master functions.

§8. Related results (Are there direct connections with our results?)

Line KP soliton. Kodama and Williams studied the regularity,... of line KP solitons:

$$\sum_{I} \Delta_I(A) \prod_{m<l} (\kappa_{i_l} - \kappa_{i_m}) \prod_{i \in I} e^{\kappa_i x + \kappa_i^2 y + \kappa_i^3 t}; \; I \in \begin{pmatrix} [n] \\ k \end{pmatrix} \; (A \in \mathrm{Gr}_{k,n}, \kappa_i),$$

by the theory of positive Grassmannians, and Nakayashiki, Abenda-Grinevich and Kodama showed that line KP solitons are obtained as limits of quasi-periodic KP solutions.

Sato Grassmannian approach. By the analysis of the KP hierarchy using Sato Grassmannian, Nakayashiki and others constructed KP solutions of mixed type with solitons and quasi-periodic functions by degenerations of hyperelliptic curves:

$$y^2 = f(x); \; f(x): \text{ polynomial of } x \text{ without multiple root}$$

or trigonal curves. Further, Nakayashiki gave KP solutions on elliptic background using the action of vertex operators, and also there were works of Li-Zhang and Kakei.

Tropical theta and ultra-discretization. Tropical theta functions of tropical curves were introduced by Mikhalkin and Zharkov for studying tropical Jacobians, and used to construct solutions to the ultra-discrete Toda lattice and box-ball system by Inoue, Iwao and Takenawa.

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