Some class of cubic Pisot numbers with finiteness property

（有限性条件を満たす３次の Pisot 数のいくつかのクラスについて）

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Abstract

Let $\beta = \alpha^{-1} > 1$ and define $T : [0,1] \to [0,1)$ by $Tx = \beta x - \lfloor \beta x \rfloor$. Using the β-transformation T, we get β-expansion $\sum_{n=1}^{\infty} w_n \alpha^n$ where $w_n = \lfloor \beta T^{n-1} x \rfloor$. Akiyama classified all cubic Pisot units with finite β-expansion property. In this thesis, we show a generalization of Akiyama's theorem by using some reduction theorem and find a new class of cubic Pisot numbers with finite β-expansion property. Moreover, we also give another application of this reduction theorem.

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Contents

1 Introduction

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $[\![m, n]\!] = [m, n] \cap \mathbb{Z}$ where $m, n \in \mathbb{Z}$ with $m < n$. Denote by $\lfloor y \rfloor$ the integer part of a real number y and define $\{y\} = y - \lfloor y \rfloor$. Let $\beta > 1$ and we write

$$\alpha = \beta^{-1}.$$

Then by using the β-transformation $T : x \mapsto \beta x - \lfloor \beta x \rfloor$, we have the expansion

$$x = \sum_{n=1}^{\infty} w_n \alpha^n$$

and write

$$d_\beta(x) = w_1 w_2 \cdots$$

where $x \in [0, 1]$ and $w_n = \lfloor \beta T^{n-1} x \rfloor$. This expansion is called β-expansion and was introduced by Rényi ([8]).

We say that $w_1 w_2 \cdots \in \mathbb{N}_0^{\mathbb{N}}$ is finite if $\sharp\{n \in \mathbb{N} \mid w_n > 0\} < \infty$. In [6], Frougny and Solomyak studied the condition

$$\text{(F)} \quad \mathbb{Z}[\alpha] \cap [0, 1) = \{x \in [0, 1) \mid d_\beta(x) \text{ is finite}\}.$$

Moreover, it is shown that if β has property (F), then β is a Pisot number and $d_\beta(1)$ is finite. Here an algebraic integer β is a Pisot number if $\beta > 1$ and all conjugates other than β have modulus less than 1.

Several sufficient conditions for property (F) are also known. Let $x^d - a_{d-1}x^{d-1} - \cdots - a_0$ be the minimal polynomial of β. If $a_{d-1} \geq a_{d-2} \geq \cdots \geq a_0 \geq 1$, then β has property (F) and this β is called of Frougny-Solomyak type (see [6]). If $a_{d-1} > a_{d-2} + \cdots + a_0$, $a_j \geq 0$ and $a_0 \geq 1$, then β has property (F) and this β is called of Hollander type (see [7]). When a Pisot number β is given, Akiyama found an algorithm determines whether or not β has property (F) ([1]). This Akiyama's algorithm is effective for fixed β. On the other hand, in most cases, it does not work well for non-fixed β. So, in general, it is difficult to show that β has property (F). In [2], Akiyama classified all cubic Pisot units with property (F).

Theorem 1.1 (**[2]**) *Let β be a cubic Pisot unit. Then β has property (F) if and only if β is a root of the polynomial $x^3 - ax^2 - bx - 1$ with the following integer coefficients:*

(A1)$_{unit}$ $b = a + 1,\ a \geq 0$. *(H)$_{unit}$* $b = 0,\ a \geq 1$.

(FS)$_{unit}$ $1 \leq b \leq a$. *(A2)$_{unit}$* $b = -1,\ a \geq 2$.

For $\boldsymbol{r} \in \mathbb{R}^d$, define $\tau_{\boldsymbol{r}} : \mathbb{Z}^d \to \mathbb{Z}^d$ by $\tau_{\boldsymbol{r}}(\boldsymbol{a}) = (a_2, \cdots, a_d, -\lfloor \boldsymbol{r} \cdot \boldsymbol{a} \rfloor)$ where $\boldsymbol{a} = (a_1, \cdots, a_d)$ and $\boldsymbol{r} \cdot \boldsymbol{a} = \sum_{1 \leq n \leq d} r_n a_n$. $\tau_{\boldsymbol{r}}$ is a forgetful generalization of β-transformation T when β is an algebraic integer with degree $d+1$. Indeed, let the minimal polynomial of β be $x^{d+1} - a_d x^d - \cdots - a_1 x - a_0$ and define $r_k = \sum_{1 \leq i \leq k} a_{k-i} \alpha^i$ $(1 \leq k \leq d)$, $r_{d+1} = 1$ and $\varphi(\boldsymbol{a}) = \{\boldsymbol{r} \cdot \boldsymbol{a}\}$. Then we get the following commutative diagram. (Note φ is bijective and $r_k = \beta^{d+1-k} - \sum_{0 \leq i \leq d-k} a_{k+i} \beta^i$ for $1 \leq k \leq d+1$.)

$$\begin{array}{ccc} \mathbb{Z}^d & \xrightarrow{\tau_{\boldsymbol{r}}} & \mathbb{Z}^d \\ \varphi \downarrow & & \downarrow \varphi \\ \mathbb{Z}[\beta] \cap [0,1) & \xrightarrow[T]{} & \mathbb{Z}[\beta] \cap [0,1) \end{array} \tag{1}$$

We say that $\tau_{\boldsymbol{r}}$ is a *shift radix system property* (*SRS property* for short) if for each $\boldsymbol{a} \in \mathbb{Z}^d$, there is $k > 0$ such that $\tau_{\boldsymbol{r}}^k(\boldsymbol{a}) = 0$. In [3], it is shown that SRS property is a generalization of property (F). Furthermore, when a sufficiently small polyhedron $H \subset \mathbb{R}^d$ is given, they found an algorithm such that determines whether $\tau_{\boldsymbol{r}}$ has SRS property for each $\boldsymbol{r} \in H$. Through the above-mentioned algorithm with help of extensive computer calculations, they found some regions of $\boldsymbol{r} \in \mathbb{R}^2$ satisfying that $\tau_{\boldsymbol{r}}$ has SRS property ([4]).

In this thesis, we aim to prove the following theorem as a generalization of Theorem 1.1.

Theorem 1.2 (**[9]**) *Let β be a cubic Pisot number with minimal polynomial $x^3 - ax^2 - bx - c$ satisfying one of the following conditions:*

(A1) $0 \leq a < b,\ 2a - 2b + c \geq -1$ *and* $b - 2c \geq -1$.

(FS) $a \geq b \geq c \geq 1$.

(H$_m$) $a \geq b \geq 0,\ a - b - c \leq 0$ *and* $b - c = -1$.

(H) $a \geq b \geq 0,\ a - b - c > 0$ *and* $c \geq 1$.

(A2) $-a < -c \leq b < 0$ *and* $a + b - c \geq 0$.

Then β has property (F).

Here we make a few comments.

Except for (H_m), Theorem 1.2 can be shown by the framework in [4] (see section 3.4 for precise argument). On the other hand, (H_m) is one of complicated cases (see subsection 3.4.2 for details) and we can show property (F) for (H_m) by using the set which is smaller than *set of witnesses* (see subsection 3.4.3 for the definition and details) . In this thesis, we give another proof of Theorem 1.2 by using a "reduction" theorem (see section 2.1 for details) and hand computation.

Bassino characterized of the form $d_\beta(1)$ when β is a cubic Pisot number ([5]). As a result of Bassino, we have that if $-a < b < 0$ and $d_\beta(1)$ is finite, then $d_\beta(1)$ has the form

$$d_\beta(1) = (a-1)(a+b)(b+c)c.$$

So $a > c \geq 1$ and $b + c \geq 0$ are necessary conditions for property (F) in the case $-a < b < 0$.

Now (FS) and (H) are exactly the same as Frougny-Solomyak type and as cubic Hollander type, respectively. Notice that $(H)_{unit}$ with $a = 1$ (i.e., $(a, b, c) = (1, 0, 1)$) is a case in (H_m).

Moreover in the last section, we consider some conditions (on β) different from Theorem 1.2, and show such β has property (F) by reduction theorem.

2 Preliminary

For $\beta > 1$, let

$$F = F_\beta := \{w_1 w_2 \cdots \in d_\beta([0,1)) \mid w_1 w_2 \cdots \text{ is finite}\}.$$

2.1 Sufficient Condition for Property (F)

Let $\beta = \alpha^{-1} > 1$ be an algebraic integer with minimal polynomial

$$x^3 - ax^2 - bx - c.$$

So

$$1 = a\alpha + b\alpha^2 + c\alpha^3.$$

Define

$$\mu(l_1, l_2, l_3) = l_1 c\alpha + l_2(b\alpha + c\alpha^2) + l_3 = (l_1, l_2, l_3) \begin{pmatrix} c & & O \\ b & c & \\ a & b & c \end{pmatrix} \begin{pmatrix} \alpha \\ \alpha^2 \\ \alpha^3 \end{pmatrix}.$$

Let

$$Q = \{\boldsymbol{\ell} \in \mathbb{Z}^3 \mid |\mu(\boldsymbol{\ell})| < 1\}.$$

For $r \in \mathbb{N}$, define

$$Q_0(r) = Q \cap [\![-r, r]\!]^3$$

and

$$Q_1(r) = \{\boldsymbol{\ell} = (l_1, l_2, l_3) \in Q_0(r) \mid (l_0, l_1, l_2) \in Q_0(r) \text{ for some } l_0\}.$$

Let

$$\lambda(l_1, l_2) = l_1 c\alpha + l_2(b\alpha + c\alpha^2).$$

So $\mu(l_1, l_2, l_3) = \lambda(l_1, l_2) + l_3$. For $\boldsymbol{\ell} = (l_1, l_2, l_3) \in Q$, we set $\boldsymbol{\ell}^s = (l_2, l_3)$ and

$$R(\boldsymbol{\ell}) = \{x \in [0, 1) \mid x + \mu(\boldsymbol{\ell}) \in [0, 1)\}.$$

In order to prove Theorem 1.2, we use the following "reduction" theorem.

Theorem 2.1 (See [10] in section 6, Corollary 6.5) *Let $(n, r) \in \{0, 1\} \times \mathbb{N}$ satisfy one of the following conditions:*

(1) $-r \leq \lfloor \beta \rfloor - a \leq r - 1$ *and*

$$M_n(r) := \max\{\lambda(\boldsymbol{\ell}^s) \mid \boldsymbol{\ell} \in Q_n(r)\} \leq r.$$

(2) $-r \leq \lfloor \beta \rfloor - a \leq r - 1$ *and*

$$M_{*,n}(r) := \max\{|\lfloor \lambda(\boldsymbol{\ell}^s) + Tx \rfloor| \mid \boldsymbol{\ell} \in Q_n(r), x \in R(\boldsymbol{\ell})\} \leq r.$$

Then each $\boldsymbol{\ell} \in Q_n(r)$ with $\mu(\boldsymbol{\ell}) > 0$ satisfies $d_\beta(\mu(\boldsymbol{\ell})) \in F$ if and only if β has property (F).

2.2 Sufficient Condition That Establishes $M_n(r) \leq r$

First, we find some coefficient conditions of minimal polynomial $x^3 - ax^2 - bx - c$ which implies $M_n(r) \leq r$ for some $(n, r) \in \{0, 1\} \times \mathbb{N}$.

Theorem 2.2 *Let β be a cubic Pisot number with minimal polynomial $x^3 - ax^2 - bx - c$.*

(1) If there is an integer $r \geq 1$ such that

(I) $0 \leq a < b$, $ra - rb + (r-1)c \geq -1$, $b - rc \geq 0$ *and* $r \geq 2$,

then $M_1(r) \le r$.

(2) *Suppose that there is an integer* $r \ge 1$ *satisfying one of the following conditions:*

 (II) $-a < -c \le b < 0$, $ra + (r-s)b - rc \ge s$ *and* $rb + sc \le -s$ *for some* $s \in [\![0, r-1]\!]$.

 (III) $-a < -c \le b < 0$, $ra + (r-1)b - rc \ge 0$, $rb + c = -1$ *and* $a + b - c < 0$. *(In condition (III), note* $r \ge 2$.*)*

Then $M_0(r) \le r$.

We show Theorem 2.2 in next section.

Example 2.1 *Suppose* $r \ge 2$ *and* $n \in \mathbb{N}$. *Let* $\beta > 1$ *be a root of* $x^3 - ax^2 - bx - c$ *with* $a = nr(r-1) + n$, $b = nr^2$ *and* $c = nr$. *Then* β *is a cubic Pisot number and* (a, b, c) *satisfies the condition (I).*

Example 2.2 *Let* $\beta > 1$ *be a root of* $x^3 - ax^2 - (a+1)x - 1$ *with* $a \ge 1$. *This* β *is a cubic Pisot unit with property (F) (see [2]). In this case,* (a, b, c) *satisfies the condition (I) with* $r = 2$.

Example 2.3 *Let* $\beta > 1$ *be a root of* $x^3 - 5x^2 + 3x - 3$. *Then* β *is a cubic Pisot number and* (a, b, c) *satisfies condition (II) with* $(r, s) = (2, 1)$.

Example 2.4 *Let* $\beta > 1$ *be a root of* $x^3 - 6x^2 + 4x - 4$. *This* β *is a cubic Pisot number and satisfies condition (II) with* $(r, s) = (3, 2)$.

Example 2.5 *Suppose* $r \ge 2$ *and* $n \in \mathbb{N}$. *Let* $\beta > 1$ *be a root of* $x^3 - ax^2 - bx - c$ *with* $a = nr(r+1) - n - 1$, $b = -nr$ *and* $c = nr^2 - 1$. *Then* β *is a cubic Pisot number and* (a, b, c) *satisfies condition (III).*

2.3 An approach to guarantee property (F)

Now, $x \xrightarrow{w} y$ means $x \in [0, 1)$, $w = \lfloor \beta x \rfloor$ and $y = Tx$.

Proposition 2.3 *Let* $\boldsymbol{\ell} = (l_1, l_2, l_3) \in Q$ *satisfy* $\mu(\boldsymbol{\ell}) \ge 0$. *Then we have* $\mu(\boldsymbol{\ell}) \xrightarrow{w} \mu(l_2, l_3, l_4)$ *where* $l_4 = -\lfloor \lambda(l_2, l_3) \rfloor$ *and* $w = l_1 c + l_2 b + l_3 a - l_4$.

Proof. Since

$$\begin{aligned}
&\beta = a + b\alpha + c\alpha^2 \\
&\beta(b\alpha + c\alpha^2) = b + c\alpha \\
&\beta(c\alpha) = c,
\end{aligned}$$

we have

$$\begin{aligned}\beta\mu(\boldsymbol{\ell}) &= l_1c + l_2b + l_3a + l_2c\alpha + l_3(b\alpha + c\alpha^2)\\ &= l_1c + l_2b + l_3a + \lambda(l_2, l_3).\end{aligned}$$

Let $l_4 = -\lfloor\lambda(l_2, l_3)\rfloor$ and $w = l_1c + l_2b + l_3a - l_4$. Therefore

$$\lfloor\beta\mu(\boldsymbol{\ell})\rfloor = l_1c + l_2b + l_3a + \lfloor\lambda(l_2, l_3)\rfloor = l_1c + l_2b + l_3a - l_4 = w$$

and

$$T(\mu(\boldsymbol{\ell})) = \beta\mu(\boldsymbol{\ell}) - \lfloor\beta\mu(\boldsymbol{\ell})\rfloor = \mu(l_2, l_3, l_4).$$

Hence we get the assertion. □

Define

$$\begin{aligned}&Q_{\geq 0} = \{\boldsymbol{\ell} \in Q \mid \mu(\boldsymbol{\ell}) \geq 0\}\\ &\Psi(l_1, l_2, l_3) = (l_2, l_3, -\lfloor\lambda(l_2, l_3)\rfloor).\end{aligned}$$

Then we have the following commutative diagram. (Notice that μ is bijective and that $c\alpha = \beta^2 - a\beta - b$ and $b\alpha + c\alpha^2 = \beta - a$.)

$$\begin{array}{ccc} Q_{\geq 0} & \xrightarrow{\Psi} & Q_{\geq 0} \\ \mu\big\downarrow & & \big\downarrow\mu \\ \mathbb{Z}[\beta] \cap [0, 1) & \xrightarrow[T]{} & \mathbb{Z}[\beta] \cap [0, 1) \end{array} \tag{2}$$

In order to check the finiteness of $d_\beta(\mu(\boldsymbol{\ell}))$ for $\boldsymbol{\ell} \in Q_n(r)$ $(n = 0, 1)$ with $\mu(\boldsymbol{\ell}) > 0$, the following statement is useful.

Corollary 2.4 *Suppose that $x \in \mathbb{Z}[\beta] \cap [0, 1)$ of the form $x = \mu(l_1, l_2, l_3)$ $(l_i \in \mathbb{Z})$ has the following β-expansion*

$$d_\beta(x) = w_1w_2\cdots.$$

Then we have

$$\begin{aligned}&w_n = l_nc + l_{n+1}b + l_{n+2}a - l_{n+3}\\ &l_{n+3} = -\lfloor\lambda(l_{n+1}, l_{n+2})\rfloor.\end{aligned}$$

In particular,

$$\mu(l_1, l_2, l_3) \xrightarrow{w_1} \mu(l_2, l_3, l_4) \xrightarrow{w_2} \cdots \xrightarrow{w_{n-1}} \mu(l_n, l_{n+1}, l_{n+2}) \xrightarrow{w_n} \cdots \tag{†}$$

Proof. Let $x_0 = x$ and $x_n = Tx_{n-1}$ for $n \in \mathbb{N}$. Then by Proposition 2.3, we get the assertion. □

Remark 2.1 *The directed graph* (†) *can be interpreted in the terminology in [3] and [4] as follows: Define* $\boldsymbol{r} = (\lambda(1,0), \lambda(0,1)) \in \mathbb{R}^2$ *and* $\tau_{\boldsymbol{r}}(\eta,\theta) = (\theta, -\lfloor \lambda(\eta,\theta) \rfloor)$*. The map* $\mathbb{Z}^2 \to Q_{\geq 0}$ *defined by* $\boldsymbol{a} = (a_1, a_2) \mapsto (a_1, a_2, -\lfloor \boldsymbol{r} \cdot \boldsymbol{a} \rfloor) = (a_1, \tau_{\boldsymbol{r}}(\boldsymbol{a}))$ *is bijective. So we can conclude that* Ψ *is a counterpart of* $\tau_{\boldsymbol{r}}$ *via commutative diagrams (1) and (2). Since* $x = \mu(l_1, l_2, l_3) \in [0,1)$*, we have* $l_3 = -\lfloor \lambda(l_1, l_2) \rfloor$ *and so* $\tau_{\boldsymbol{r}}(l_1, l_2) = (l_2, l_3)$*. Similarly,* $\tau_{\boldsymbol{r}}(l_n, l_{n+1}) = (l_{n+1}, l_{n+2})$*. Thus the iteration of* $\tau_{\boldsymbol{r}}$ *generates the graph:*

$$(l_1, l_2) \to (l_2, l_3) \to \cdots \to (l_n, l_{n+1}) \to \cdots .$$

Our approach to prove Theorem 1.2 is the following. First, we find $(n, r) \in \{0,1\} \times \mathbb{N}$ satisfying one of the following conditions:

(1) $-r \leq \lfloor \beta \rfloor - a \leq r - 1$ and $M_n(r) \leq r$.

(2) $-r \leq \lfloor \beta \rfloor - a \leq r - 1$ and $M_{*,n}(r) \leq r$.

Then by Theorem 2.1 and Corollary 2.4, it suffices to show $d_\beta(\mu(l_2, l_3, -\lfloor \lambda(l_2, l_3) \rfloor)) \in F$ for any $\boldsymbol{\ell} = (l_1, l_2, l_3) \in Q_n(r)$ with $\mu(\boldsymbol{\ell}) > 0$ because $d_\beta(\mu(\boldsymbol{\ell})) \in F$ is equivalent to $d_\beta(\mu(l_2, l_3, -\lfloor \lambda(l_2, l_3) \rfloor)) \in F$. So we need to estimate each element of

$$\Lambda_n(a,b,c) := \{\lambda(l_2, l_3) \mid \boldsymbol{\ell} = (l_1, l_2, l_3) \in Q_n(r) \text{ with } \mu(\boldsymbol{\ell}) > 0 \text{ for some } l_1\}.$$

Now, since α is cubic over $\mathbb{Q}$, we have that $(\eta, \theta) \in \mathbb{Z}^2 - \{(0,0)\}$ implies $\lambda(\eta, \theta) \notin \mathbb{Z}$ by definition of λ. Hence if $(\eta, \theta) \in \mathbb{Z}^2 - \{(0,0)\}$, then

$$\lfloor \lambda(-\eta, -\theta) \rfloor = -\lfloor \lambda(\eta, \theta) \rfloor - 1$$

and so we shall seek the value $\lfloor \lambda \rfloor$ for each $\lambda \in \Lambda_n(a,b,c)$ with $\lambda > 0$ in next section.

3 Proof of Theorem 1.2

3.1 Proof of Theorem 2.2

Now, notice that

$$\lfloor \beta \rfloor = a + \lfloor b\alpha + c\alpha^2 \rfloor.$$

According to [2], we have

$$|b - 1| < a + c.$$

Thus we have

Remark 3.1 *Let β be a cubic Pisot number with minimal polynomial $x^3 - ax^2 - bx - c$. Then*

$$a + c \geq \begin{cases} b & \text{if } b > 0 \\ 2 - b & \text{if } b \leq 0. \end{cases}$$

In [5], Bassino determined $d_\beta(1)$ for each cubic Pisot number β. In particular,

$$\lfloor \beta \rfloor = \begin{cases} a+1 & \text{if } a < b \\ a & \text{if } 0 \leq b \leq a \\ a-1 & \text{if } -a < b < 0 \\ a-2 & \text{if } b \leq -a. \end{cases} \tag{$\sharp$}$$

Let

$$p = \begin{cases} 1 & \text{if } a < b \\ 0 & \text{if } 0 \leq b \leq a \\ -1 & \text{if } -a < b < 0 \\ -2 & \text{if } b \leq -a. \end{cases}$$

Then $\lfloor \beta \rfloor = a + p$ by ($\sharp$). Since $\lfloor b\alpha + c\alpha^2 \rfloor = p$, we have $p < b\alpha + c\alpha^2 < p + 1$. Thus

$$\lambda(0,1) = b\alpha + c\alpha^2 \in \begin{cases} (1,2) & \text{if } a < b \\ (0,1) & \text{if } 0 \leq b \leq a \\ (-1,0) & \text{if } -a < b < 0 \\ (-2,-1) & \text{if } b \leq -a. \end{cases} \tag{$\sharp\sharp$}$$

Remark 3.2 $|c|\alpha < 1$ *holds.*

Proof. Let $\alpha^{(j)}$ $(j = 1, 2)$ be other conjugates of α. Since β is a Pisot number, $|\alpha^{(j)}| > 1$. Thus we have $|c|\alpha = \left|\alpha^{(1)}\alpha^{(2)}\right|^{-1} < 1$. □

By ($\sharp$) and Remark 3.2, we see that

$$c \leq \begin{cases} a+1 & \text{if } a < b \\ a & \text{if } 0 \leq b \leq a \\ a-1 & \text{if } -a < b < 0 \\ a-2 & \text{if } b \leq -a. \end{cases}$$

Lemma 3.1

(1) Let $a < b$. (Then $c \geq 1$ by Remark 3.1.)

(i) $0 < \lambda(1,0) < 1 < \lambda(0,1) < 2$.

(ii) $\lambda(\eta,\theta)$ *is increasing in* η *and* θ.

(iii) $\lambda(\eta,\theta) < \lambda(\eta - 1, \theta + 1)$.

(2) Let $0 \leq b \leq a$ and $c \geq 1$.

(i) $0 < \lambda(1,0) < 1$ and $0 < \lambda(0,1) < 1$.

(ii) $\lambda(\eta, \theta)$ is increasing in η and θ.

(iii) If $b < c$, then $\lambda(\eta, \theta) < \lambda(\eta+1, \theta-1)$.

(3) Let $-a < -c \leq b < 0$.

(i) $-1 < \lambda(0,1) < 0 < \lambda(1,0) < 1$.

(ii) $\lambda(\eta, \theta)$ is increasing in η and decreasing in θ.

(iii) $\lambda(\eta, \theta) < \lambda(\eta+1, \theta+1)$.

Proof. (1),(2),(3)-(i) : By Remark 3.2 and (♯♯).

(1),(2),(3)-(ii) : Since $c \geq 1$, $\lambda(\eta, \theta)$ is increasing in η. If $b\alpha + c\alpha^2 > 0$, then $\lambda(\eta, \theta)$ is increasing in θ. If $b\alpha + c\alpha^2 < 0$, then $\lambda(\eta, \theta)$ is decreasing in θ.

(1)-(iii) : Since $b \geq a+1 \geq c \geq 1$,

$$\lambda(-1,1) = (b-c)\alpha + c\alpha^2 > 0$$

and so

$$\lambda(\eta, \theta) < \lambda(\eta, \theta) + \lambda(-1,1) = \lambda(\eta-1, \theta+1).$$

(2)-(iii) : Since $c > b \geq 0$,

$$\lambda(1,-1) = (c-b)\alpha - c\alpha^2 = (c-b-1)\alpha + (1-c\alpha)\alpha > 0$$

and so

$$\lambda(\eta, \theta) < \lambda(1,-1) + \lambda(\eta, \theta) = \lambda(\eta+1, \theta-1).$$

(3)-(iii) : Since $c \geq -b \geq 1$,

$$\lambda(1,1) = (c+b)\alpha + c\alpha^2 > 0$$

and so

$$\lambda(\eta, \theta) < \lambda(1,1) + \lambda(\eta, \theta) = \lambda(\eta+1, \theta+1).$$

□

3.1.1 Case $0 \le a < b$

Suppose that (a, b, c) satisfies the conditions:

$$0 \le a < b,\ ra - rb + (r-1)c \ge -1 \text{ and } b - rc \ge 0 \text{ for some } r \in \mathbb{Z}_{\ge 2}.$$

Lemma 3.2 *Define*

$$S_1(r) = \{(l_2, l_3) \mid (l_1, l_2, l_3) \in Q_1(r) \textit{ for some } l_1\}.$$

Then

$$S_1(r) \subset \{(\eta, \theta) \in [\![-r, r]\!]^2 \mid \eta\theta \le 0,\ |\eta| \le |\theta| + 1,\ |\theta| \le |\eta| + 1\}.$$

Proof. Let $\boldsymbol{l} = (l_1, l_2, l_3) \in Q_0(r)$ be given. Since $-Q_0(r) = Q_0(r)$ and $Q_1(r) \subset Q_0(r)$, it suffices to show that

(1)' $l_2 > 0$ implies $l_3 \le 0$.

(2)' $l_3 \ge 0$ implies $|l_2| \le l_3 + 1$.

(3)' Suppose that $\boldsymbol{l} \in Q_1(r)$. Then $l_2 \ge 0$ implies $|l_3| \le l_2 + 1$.

(1)' : Assume that $l_2 > 0$ and $l_3 > 0$. Then

$$\mu(\boldsymbol{l}) \ge \mu(-r, 1, 1) = (b - rc)\alpha + c\alpha^2 + 1 > 1,$$

contradicting $|\mu(\boldsymbol{l})| < 1$.

(2)' : First, we show that $l_3 \ge 0$ implies $l_2 \ge -(l_3 + 1)$. Assume that $l_3 \ge 0$ and $l_2 \le -(l_3 + 2)$. Then by Lemma 3.1-(1)-(i),

$$\begin{aligned} \mu(\boldsymbol{l}) &\le \mu(r, -(l_3 + 2), l_3) \\ &= (rc - b)\alpha - c\alpha^2 - \lambda(0, 1) + l_3(1 - \lambda(0, 1)) \\ &\le -\lambda(0, 1) < -1, \end{aligned}$$

contradicting $|\mu(\boldsymbol{l})| < 1$. Since $Q_0(r) = -Q_0(r)$, we have that $l_3 \le 0$ implies $l_2 \le -l_3 + 1$. In particular, (2)' holds in case $l_3 = 0$. Consider the case $l_3 > 0$. By (1), $l_2 \le 0$. Hence $|l_2| \le l_3 + 1$.

(3)' : Since $Q_1(r) = -Q_1(r)$, it suffices to show that $l_2 \in [\![0, r-1]\!]$ implies $|l_3| \le l_2 + 1$. First, we show that $l_2 \in [\![0, r-1]\!]$ implies $l_3 \ge -(l_2 + 1)$. Let $l_0 \in [\![-r, r]\!]$ satisfy $|\mu(l_0, l_1, l_2)| < 1$. Then by (2), $|l_1| \le l_2 + 1$.

Case $l_2 = 0$. Assume $l_3 \le -2$. Then by Remark 3.2, we have

$$\mu(\boldsymbol{l}) \le \mu(1, 0, -2) = c\alpha - 2 < -1,$$

contradicting $|\mu(\boldsymbol{\ell})| < 1$.

Case $l_2 > 0$. Then by (1), $l_1 \leq 0$. Assume $l_3 \leq -(l_2+2)$. Since $b \leq a+c$ by Remark 3.1, we have $\mu(-1,1,-1) < 0$. Hence, since $l_2 c \leq (r-1)c \leq b-c \leq a$ by Remark 3.1,

$$\mu(\boldsymbol{\ell}) \leq \mu(0, l_2, -l_2-2) = \mu(-l_2, l_2, -l_2) + l_2 c\alpha - 2 < a\alpha - 2 < -1,$$

contradicting $|\mu(\boldsymbol{\ell})| < 1$. □

Proof of Theorem 2.2-(1). Let $(\eta, \theta) \in [\![-r, r]\!]^2$ such that $(\zeta, \eta, \theta) \in Q_1(r)$ for some $\zeta \in [\![-r, r]\!]$. If $\theta \leq 0$, then

$$\lambda(\eta, \theta) \leq \lambda(r, 0) = rc\alpha < r$$

by Lemma 3.1-(1)-(ii) and Remark 3.2. Now, we show that $\theta \geq 1$ implies $\lambda(\eta, \theta) \leq \lambda(1-r, -r)$. Indeed, suppose that $1 \leq \theta \leq r-1$. Then by Lemma 3.2-(1) and (2),

$$-\theta - 1 \leq \eta \leq 0.$$

On the other hand, by Lemma 3.2-(3),

$$\theta \leq -\eta + 1.$$

Thus $\eta \leq -\theta + 1$. So by Lemma 3.1-(1)-(ii) and (iii),

$$\lambda(\eta, \theta) \leq \lambda(-\theta+1, \theta) < \lambda(-\theta + 1 - (r-\theta), \theta + (r-\theta)) = \lambda(1-r, r).$$

Suppose that $\theta = r$. By Lemma 3.2-(1), $\eta \leq 0$. Moreover by Lemma 3.2-(3), $r \leq -\eta + 1$. Hence $\eta \in \{-r, 1-r\}$ and so by Lemma 3.1-(1)-(ii),

$$\lambda(\eta, \theta) = \lambda(\eta, r) \leq \lambda(1-r, r).$$

Thus it suffices to show $\lambda(1-r, -r) < r$. Since $ra - rb + (r-1)c \geq -1$, $b - rc \geq 0$ and $r \geq 2$, we have

$$\begin{aligned} r - \lambda(1-r, r) &= (ra - rb + (r-1)c)\alpha + (rb - rc)\alpha^2 + rc\alpha^3 \\ &\geq -\alpha + (r-1)b\alpha^2 + rc\alpha^3 \\ &\geq -\alpha + \alpha\lambda(0,1) > 0. \end{aligned}$$

Hence in case (I), we have $M_1(r) < r$. □

We use the following remark later.

Remark 3.3

$$\sharp\{(\eta, \theta) \in [\![-r, r]\!]^2 \mid \eta\theta \leq 0,\ |\eta| \leq |\theta| + 1,\ |\theta| \leq |\eta| + 1\} \leq 6r + 1.$$

3.1.2 Case $-a < -c \le b < 0$

Suppose that (a, b, c) satisfies one of the following conditions:

(II) $-a < -c \le b < 0$, $ra + (r-s)b - rc \ge s$ and $rb + sc \le -s$ for some $r \in \mathbb{N}$ and $s \in [\![0, r-1]\!]$.

(III) $-a < -c \le b < 0$, $ra + (r-1)b - rc \ge 0$, $rb + c = -1$ and $a + b - c < 0$ for some $r \in \mathbb{N}$. (Note $r \ne 1$.)

Notice that if $a + b - c < 0$, then we have

$$2c\alpha \ge (c-b)\alpha > a\alpha > 1.$$

Lemma 3.3 *Define*

$$S_0(r) = \{(l_2, l_3) \mid (l_1, l_2, l_3) \in Q_0(r) \text{ for some } l_1\}.$$

(1) Under condition (II), we have

(i) $\lambda(r, s-r) < r$

(ii) $S_0(r) \subset \{(\eta, \theta) \in [\![-r, r]\!]^2 \mid s - 2r + \eta \le \theta \le 2r - s + \eta\}$.

(2) Under condition (III), we have

(i) $\lambda(r, 1-r) < r$

(ii) $S_0(r) \subset \{(\eta, \theta) \in [\![-r, r]\!]^2 \mid 1 - 2r + \eta \le \theta \le 2r - 1 + \eta\}$.

Proof. (1)-(i) : Notice that $1 > c\alpha \ge -b\alpha$, $ra + (r-s)b - rc \ge s$ and $b + c \ge 0$. So

$$\begin{aligned} r - \lambda(r, s-r) &= (ra + (r-s)b - rc)\alpha + (rb + (r-s)c)\alpha^2 + rc\alpha^3 \\ &\ge s\alpha + (sb + (r-s)(b+c))\alpha^2 + rc\alpha^3 \\ &= s\alpha(1 + b\alpha) + (r-s)(b+c)\alpha^2 + rc\alpha^3 > 0. \end{aligned}$$

(1)-(ii) : First, we show that $\mu(r, r, s-r-1) < -1$. Since $rb + sc \le -s$ and $c \le a - 1$, we have

$$\begin{aligned} \mu(r, r, s-r-1) &= ((s-r-1)a + rb + rc)\alpha + ((s-r-1)b + rc)\alpha^2 + (s-r-1)c\alpha^3 \\ &\le ((s-r-1)a + (r-s)c - s)\alpha + \alpha\lambda(r, s-r) - (b\alpha^2 + c\alpha^3) \\ &\le (-a-r)\alpha + \alpha\lambda(r, s-r) - (b\alpha^2 + c\alpha^3) \\ &= -1 + \alpha(-r + \lambda(r, s-r)) < -1 \end{aligned}$$

by (1)-(i). Next, we show that $l_3 \geq s - 2r + l_2$ if $(l_1, l_2, l_3) \in Q_0(r)$. Let $\boldsymbol{\ell} = (l_1, l_2, l_3) \in Q_0(r)$ be given. Assume that $l_3 \leq s - 2r + l_2 - 1$. Then by Lemma 3.1-(3)-(i),

$$\begin{aligned}\mu(\boldsymbol{\ell}) &\leq \mu(r, l_2, s - 2r + l_2 - 1) = \lambda(r, l_2) + s - 2r + l_2 - 1\\ &< \lambda(r, l_2 + (r - l_2)) + s - 2r + l_2 - 1 + (r - l_2)\\ &= \lambda(r, r) + s - r - 1 = \mu(r, r, s - r - 1) < -1,\end{aligned}$$

contradicting $|\mu(\boldsymbol{\ell})| < 1$. Hence $l_3 \geq s - 2r + l_2$. So, since $Q_0(r) = -Q_0(r)$, we have $l_3 \leq 2r - s + l_2$ if $(l_1, l_2, l_3) \in Q_0(r)$.

(2)-(i) : Since $ra + (r-1)b - rc \geq 0$ and $rb + c = -1$, we have

$$\begin{aligned}r - \lambda(r, 1 - r) &= (ra + (r-1)b - rc)\alpha + (rb + (r-1)c)\alpha^2 + rc\alpha^3\\ &\geq ((r-2)c - 1)\alpha^2 + rc\alpha^3 =: \delta.\end{aligned}$$

So if $r = 2$, then (recall $2c\alpha > 1$)

$$\delta = -\alpha^2 + 2c\alpha^3 > 0$$

and if $r \geq 3$, then

$$\delta \geq (c-1)\alpha^2 + rc\alpha^3 > 0.$$

(2)-(ii) : In the same way as (1)-(ii), we get the assertion. □

Proof of Theorem 2.2-(2). Let $(\eta, \theta) \in [\![-r, r]\!]^2$ such that $|\mu(\zeta, \eta, \theta)| < 1$ for some $\zeta \in [\![-r, r]\!]$.

Case $ra + (r-s)b - rc \geq s$ and $rb + sc \leq -s$. If $\eta < r - s$, then by Lemma 3.1-(3)-(ii), (iii) and Lemma 3.3-(1)-(i),

$$\begin{aligned}\lambda(\eta, \theta) &\leq \lambda(r - s - 1, -r) < \lambda(r - s - 1 + (s+1), -r + (s+1))\\ &= \lambda(r, s - r + 1) < \lambda(r, s - r) < r.\end{aligned}$$

If $r - s \leq \eta$, then by Lemma 3.1-(3)-(ii), (iii) and Lemma 3.3-(1),

$$\lambda(\eta, \theta) \leq \lambda(\eta, s - 2r + \eta) < \lambda(\eta + (r - \eta), s - 2r + \eta + (r - \eta)) = \lambda(r, s - r) < r.$$

Case $ra + (r-1)b - rc \geq 0$ and $rb + c = -1$. If $\eta < r - 1$, then by Lemma 3.1-(3)-(ii), (iii) and Lemma 3.3-(2)-(i),

$$\lambda(\eta, \theta) \leq \lambda(r - 2, -r) < \lambda(r, 2 - r) < \lambda(r, 1 - r) < r.$$

If $r - 1 \leq \eta$, then by Lemma 3.1-(3)-(ii), (iii) and Lemma 3.3-(2),

$$\lambda(\eta, \theta) \leq \lambda(\eta, 1 - 2r + \eta) < \lambda(\eta + (r - \eta), 1 - 2r + \eta + (r - \eta)) = \lambda(r, 1 - r) < r.$$

Hence in case (II) and (III), we have $M_0(r) < r$. □

We use the following remark later.

Remark 3.4 *Let $r \in \mathbb{N}$ and $s \in [\![0, r-1]\!]$. Then*

$$\sharp\{(\eta,\theta) \in [\![-r,r]\!]^2 \mid s-2r+\eta \le \theta \le 2r-s+\eta\} \le 4r(r+1)-s(s+1)+1.$$

In particular,

$$\sharp\{(\eta,\theta) \in [\![-r,r]\!]^2 \mid 1-2r+\eta \le \theta \le 2r-1+\eta\} \le 4r(r+1)-1.$$

3.2 Reduction

3.2.1 Cases: (FS) or (H)

Proposition 3.4 *Let (a,b,c) satisfy (FS) or (H). Then $M_0(1) < 1$. Moreover in case (FS), $(\eta,\theta) \ne \pm(1,1)$ if $(\zeta,\eta,\theta) \in Q_0(1)$.*

Proof. Let $(\eta,\theta) \in [\![-1,1]\!]^2$. Suppose that (a,b,c) satisfies (H). By Lemma 3.1-(2)-(ii), we see that

$$\lambda(\eta,\theta) \le \lambda(1,1) = (c+b)\alpha + c\alpha^2 \le (a-1)\alpha + c\alpha^2 < a\alpha < 1.$$

Next, suppose that (a,b,c) satisfies (FS). Let $(\zeta,\eta,\theta) \in Q_0(1)$. If $(\eta,\theta) = (1,1)$, then

$$\mu(\zeta,1,1) \ge \mu(-1,1,1) = (b-c)\alpha + c\alpha^2 + 1 > 1,$$

contradicting $|\mu(\zeta,\eta,\theta)| < 1$. Hence $(\eta,\theta) \ne (1,1)$ if $(\zeta,\eta,\theta) \in Q_0(1)$. Similarly, $(\eta,\theta) \ne (-1,-1)$ if $(\zeta,\eta,\theta) \in Q_0(1)$. So by Lemma 3.1-(2)-(i) and (ii),

$$\lambda(\eta,\theta) \le \max\{\lambda(0,1), \lambda(1,0)\} < 1.$$

Hence we get the assertion. □

3.2.2 Cases: (A2) or (A1) with $b-2c \ge 0$

Corollary 3.5 *Let β be a cubic Pisot number with minimal polynomial $x^3 - ax^2 - bx - c$.*

(1) If (a,b,c) satisfies (A1) with $b-2c \ge 0$, then $M_1(2) < 2$.

(2) If (a,b,c) satisfies (A2), then $M_0(1) < 1$.

Proof. In case (1), apply Theorem 2.2-(I) putting $r=2$. In case (2), apply Theorem 2.2-(II) putting $(r,s) = (1,0)$. □

3.2.3 Cases: (H_m) or (A1) with $b-2c=-1$

In the following, define

$$r=\begin{cases}1 & \text{if } (a,b,c) \text{ satisfies } (\mathrm{H}_m)\\ 2 & \text{if } (a,b,c) \text{ satisfies (A1) with } b-2c=-1\end{cases}$$

and we show

Proposition 3.6

(1) If (a,b,c) satisfies (H_m), then $M_{*,0}(r)\le r$.

(2) If (a,b,c) satisfies (A1) with $b-2c=-1$, then $M_{*,1}(r)\le r$.

In order to prove Proposition 3.6, we use the following statements.

Lemma 3.7 *Let $\boldsymbol{\ell}=(l_1,l_2,l_3)\in Q_0(r)$. Then*

(1) If (a,b,c) satisfies (H_m), then $l_2l_3>0$ implies $\boldsymbol{\ell}\in\{(-1,1,1),(1,-1,-1)\}$.

(2) Let (a,b,c) satisfy (A1) with $b-2c=-1$. Then

(i) $l_2l_3>0$ implies $\boldsymbol{\ell}\in\{(-2,1,1),(2,-1,-1)\}$.

(ii) $l_3=0$ implies $|l_2|\le 1$.

(iii) Suppose $\boldsymbol{\ell}\in Q_1(2)$. Then $l_2=0$ implies $|l_3|\le 1$.

Proof of Lemma 3.7. Since $-Q_1(2)=Q_1(2)$ and $-Q_0(r)=Q_0(r)$, it suffices to show that

(i) If (a,b,c) satisfies (H_m) or (A1) with $b-2c=-1$, then $l_2>0$ and $l_3>0$ imply $(l_1,l_2,l_3)=(-r,1,1)$.

(ii) If (a,b,c) satisfies (A1) with $b-2c=-1$, then $l_3=0$ implies $l_2\le 1$.

(iii) Suppose $\boldsymbol{\ell}\in Q_1(2)$. If (a,b,c) satisfies (A1) with $b-2c=-1$, then $l_2=0$ implies $l_3\le 1$.

(i) : Assume that $l_2>0$, $l_3>0$ and $l_1\ge -(r-1)$. Since $b-rc=-1$, we have

$$\begin{aligned}\mu(l_1,l_2,l_3)&\ge \mu(-(r-1),1,1)\\&=-(r-1)c\alpha+b\alpha+c\alpha^2+1\\&=(c-1)\alpha+c\alpha^2+1>1,\end{aligned}$$

contradicting $|\mu(l_1, l_2, l_3)| < 1$. Therefore $l_2 > 0$ and $l_3 > 0$ imply $l_1 = -r$. When (a, b, c) satisfies (H_m), the proof is completed. So we consider the case that (a, b, c) satisfies (A1) with $b-2c = -1$. By Lemma 3.1-(1)-(ii) and (iii),

$$1 < \mu(-2, 1, 2) < \mu(-2, 2, 1) < \mu(-2, 2, 2).$$

Hence if $l_2 > 0$ and $l_3 > 0$, then $(l_1, l_2, l_3) = (-2, 1, 1)$.

(ii) : Assume that (a, b, c) satisfies (A1) with $b - 2c = -1$ and $(l_2, l_3) = (2, 0)$. Since $b - 2c = -1$,

$$\begin{aligned}\mu(l_1, 2, 0) - 1 \geq \mu(-2, 2, 0) - 1 &= -2c\alpha + 2(b\alpha + c\alpha^2) - 1 \\ &= (2b - a - 2c)\alpha + (2c - b)\alpha^2 - c\alpha^3 \\ &= (b - a - 1)\alpha + \alpha^2 - c\alpha^3 \\ &= (b - a - 1)\alpha + (1 - c\alpha)\alpha^2 > 0.\end{aligned}$$

Thus $\mu(l_1, 2, 0) > 1$, contradicting $|\mu(l_1, l_2, l_3)| < 1$.

(iii) : Assume that (a, b, c) satisfies (A1) with $b - 2c = -1$ and $(l_2, l_3) = (0, 2)$. Since $(l_1, 0, 2) \in Q_1(r)$, $|\mu(l_0, l_1, 0)| < 1$ for some $l_0 \in [\![-2, 2]\!]$. So by (ii), we have $|l_1| \leq 1$. Moreover,

$$\mu(l_1, 0, 2) \geq \mu(-1, 0, 2) = -c\alpha + 2 > 1,$$

contradicting $|\mu(l_1, 0, 2)| < 1$. □

Remark 3.5 *Let (a, b, c) satisfy (A1) with $b - 2c = -1$ and define*

$$S_1(2) = \{(l_2, l_3) \mid (l_1, l_2, l_3) \in Q_1(2)\}.$$

Then $\sharp S_1(2) \leq 15$ (by Lemma 3.7).

Remark 3.6

(1) Let (a, b, c) satisfy (A1) with $b - 2c = -1$.

(i) $0 < \lambda(2, -1) < 1$.

(ii) $0 < \lambda(-1, 1) < \lambda(1, 0) < 1 < \lambda(0, 1) < 2$.

(iii) $1 < \lambda(-2, 2) < \lambda(-1, 2) < 2$.

(iv) If $2a - b - c = -2$, then $2 < \lambda(1, 1) < 3$ and if $2a - b - c \geq -1$, then $1 < \lambda(1, 1) < 2$.

(2) Let (a, b, c) satisfy (H_m). Then

(i) $0 < \lambda(1,0) < 1$ *and* $0 < \lambda(0,1) < 1$.

(ii) $0 < \lambda(1,-1) < 1$.

(iii) $1 < \lambda(1,1) < 2$.

Proof of Remark 3.6. See Appendix. □

Remark 3.7 *If* (a,b,c) *satisfies (A1), then* $2a-b-c \geq -2$.

Proof. By the assumption, we see that $2b+c \leq 2a+2c+1 \leq 2a+b+2$. Therefore $2a-b-c \geq -2$. □

Now, we show Proposition 3.6. Here, let

$$n = \begin{cases} 0 & \text{if } (a,b,c) \text{ satisfies } (\mathrm{H}_m) \\ 1 & \text{if } (a,b,c) \text{ satisfies (A1) with } b-2c=-1. \end{cases}$$

Proof of Proposition 3.6. Let $(l_1,l_2,l_3) \in Q_n(r)$ and $x \in R(\boldsymbol{\ell})$. Moreover (l_1,l_2,l_3) belongs to one of the following cases (note Remark 3.7):

Case 1. $(l_1,l_2,l_3) \neq \pm(-r,1,1)$.

Case 2. $(l_1,l_2,l_3) = \pm(-r,1,1)$ and (a,b,c) satisfies (A1) with $b-2c=-1$ and $2a-b-c \geq -1$.

Case 3. $(l_1,l_2,l_3) = \pm(-r,1,1)$ and (a,b,c) satisfies (H_m) or (A1) with $b-2c=-1$ and $2a-b-c=-2$.

Case 1. Suppose that (a,b,c) satisfies (H_m). By Lemma 3.7-(1) and Remark 3.6-(2)-(i),

$$-1 < \min\{\lambda(-1,0), \lambda(0,-1)\} \leq \lambda(l_2,l_3) \leq \max\{\lambda(1,0), \lambda(0,1)\} < 1.$$

Suppose that (a,b,c) satisfies (A1) with $b-2c=-1$. If $l_3 \leq 0$, then by Lemma 3.1-(1)-(i) and (ii),

$$\lambda(l_2,l_3) \leq \lambda(2,0) = 2\lambda(1,0) < 2.$$

If $l_3 = 1$, then by Lemma 3.7-(2) and Remark 3.6-(1)-(ii), we have

$$\lambda(l_2,l_3) \leq \lambda(0,1) < 2.$$

If $l_3 = 2$, then by Lemma 3.7-(2), Lemma 3.1-(1)-(ii) and Remark 3.6-(iii), we have

$$\lambda(l_2,l_3) = \lambda(l_2,2) \leq \lambda(-1,2) < 2.$$

Similarly, we can show $\lambda(l_2, l_3) > -2$.

Case 2. By Remark 3.6-(iv), we have $-2 < \lambda(l_2, l_3) < 2$.

So in Case 1 and Case 2, we have

$$-r < \lambda(l_2, l_3) + Tx < r + 1$$

and so

$$|\lfloor \lambda(l_2, l_3) + Tx \rfloor| \leq r.$$

Case 3. Since $x + \mu(l_1, l_2, l_3) \in [0, 1)$ and $b - rc = -1$, we have

$$x \begin{cases} < \alpha - c\alpha^2 & \text{if } (l_1, l_2, l_3) = (-r, 1, 1) \\ \geq 1 - \alpha + c\alpha^2 & \text{if } (l_1, l_2, l_3) = (r, -1, -1). \end{cases}$$

Now, note that $\lfloor \beta \rfloor = a + r - 1$. So, since $\lfloor \lambda(1, 1) \rfloor = r$ by Remark 3.6-(1)-(iv) and (2)-(iii), we see that

$$\lfloor \beta(1 - \alpha + c\alpha^2) \rfloor = a - 1 + \lfloor \lambda(1, 1) \rfloor = \lfloor \beta \rfloor = a + r - 1.$$

Hence

$$T(1 - \alpha + c\alpha^2) = \beta(1 - \alpha + c\alpha^2) - (a + r - 1) = -r + \lambda(1, 1).$$

Therefore

$$Tx \begin{cases} < 1 - c\alpha & \text{if } (l_1, l_2, l_3) = (-r, 1, 1) \\ \geq -r + (b + c)\alpha + c\alpha^2 & \text{if } (l_1, l_2, l_3) = (r, -1, -1). \end{cases}$$

Since $r - 1 < b\alpha + c\alpha^2 < r$, we see that

$$0 < \lambda(1, 1) + Tx < b\alpha + c\alpha^2 + 1 < r + 1,$$
$$1 > \lambda(-1, -1) + Tx \geq -r.$$

So

$$|\lfloor \lambda(l_2, l_3) + Tx \rfloor| \leq r.$$

Hence $M_{*,n}(r) \leq r$. □

3.3 Proof of Theorem 1.2

Let (a, b, c) satisfy the condition of Theorem 1.2. In summary of Corollary 3.5, Proposition 3.4 and Proposition 3.6, we set

$$(r, n) = \begin{cases} (2, 1) & \text{if } (a, b, c) \text{ satisfies (A1)}, \\ (1, 0) & \text{otherwise}. \end{cases}$$

Following the approach in subsection 2.3, we seek the value $\lfloor \lambda(l_2, l_3) \rfloor$ with $\lambda(l_2, l_3) > 0$ and $(l_1, l_2, l_3) \in Q_n(r)$ for some l_1

Remark 3.8

(1) Let (a, b, c) *satisfy (A1) with* $b - 2c \geq 0$.

(i) $0 < \lambda(1, 0) < 1 < \lambda(0, 1)$.

(ii) $0 < \lambda(-2, 1) < \lambda(-1, 1) < 1$.

(iii) $1 < \lambda(0, 1) < \lambda(-2, 2) < \lambda(-1, 2) < 2$.

(2) Let (a, b, c) *satisfy (FS). Then* $0 < \lambda(1, 0) < \lambda(0, 1) < 1$ *and* $0 < \lambda(-1, 1) < 1$.

(3) Let (a, b, c) *satisfy (H) with* $b - c < 0$*. Then*

(i) $0 < \lambda(0, 1) < 1$ *and* $0 < \lambda(1, 0) < 1$.

(ii) $0 < \lambda(1, -1) < 1$.

(iii) $0 < \lambda(1, 1) < 1$.

(4) Let (a, b, c) *satisfy (A2). Then*

(i) $0 < \lambda(0, -1) < 1$ *and* $0 < \lambda(1, 0) < 1$.

(ii) $0 < \lambda(1, 1) < \lambda(1, -1) < 1$.

Proof of Remark 3.8. See Appendix. □

Proof of Theorem 1.2. First, we show that by Remark 3.6 and Remark 3.8, we have determined each integer part of an element in $\Lambda_n(a, b, c)$. Indeed, it is clear in case (H) and (A2) by Remark 3.8-(3) and (4) (see Table 6 and 7). So consider the case (A1), (FS) and (H_m). Let $\boldsymbol{\ell} = (l_1, l_2, l_3) \in Q_n(r)$ satisfy $\mu(\boldsymbol{\ell}) > 0$.

Case (FS). Since $(l_1, l_2, l_3) \in Q_0(r)$, $(l_2, l_3) \neq (1, 1)$ by Proposition 3.4. So we have determined each integer part of an element in $\Lambda_0(a, b, c)$ by Remark 3.8-(2) (see Table 4).

Case (A1) *with* $b - 2c \geq 0$ *or with* $b - 2c = -1$ *and* $2a - b - c \geq -1$. By Lemma 3.2, Lemma 3.7, Remark 3.8-(1) and Remark 3.6-(1), we have determined each integer part of an element in $\Lambda_1(a, b, c)$ (see Table 2 and 3).

Case (H_m) *or* (A1) *with* $b - 2c = -1$ *and* $2a - b - c = -2$. Notice that by Lemma 3.7-(1) and (2)-(i), $(l_2, l_3) = \pm(1, 1)$ implies $(l_1, l_2, l_3) = \pm(-r, 1, 1)$. If $(l_2, l_3) = (1, 1)$, then

$$\mu(-r, 1, 1) = (-rc + b)\alpha + c\alpha^2 + 1 > 1 - \alpha > 0$$

and so $\lambda(1,1) \in \Lambda_n(a,b,c)$. Consider the case $(l_2,l_3) = (-1,-1)$. Then $\mu(r,-1,-1) < 0$ and so $\lambda(-1,-1) \notin \Lambda_n(a,b,c)$. Hence we have determined each integer part of an element in $\Lambda_n(a,b,c)$ by Lemma 3.7, Remark 3.6 (see Table 1 and 5).

Recall that it suffices to show $d_\beta(\mu(l_2,l_3,-\lfloor\lambda(l_2,l_3)\rfloor)) \in F$ for each $\boldsymbol{\ell} = (l_1,l_2,l_3) \in Q_n(r)$ with $\mu(\boldsymbol{\ell}) > 0$. Now, by Corollary 2.4 and Figure k $(1 \le k \le 7)$ which describes each β-expansion, we can see $d_\beta(\mu(\boldsymbol{\ell})) \in F$ for such $\boldsymbol{\ell}$. □

3.4 Comparison with results in [4]

In this subsection, we compare Theorem 1.2 with results in [4].

For $\boldsymbol{r} = (r_1,r_2) \in \mathbb{R}^2$, define $\tau_{\boldsymbol{r}}(l_1,l_2) = (l_2, -\lfloor r_1l_1 + r_2l_2\rfloor)$. The set $\mathcal{D}_2^0$ defined by

$$\mathcal{D}_2^0 = \{\boldsymbol{r} \in \mathbb{R}^2 \mid \forall(l_1,l_2) \in \mathbb{Z}^2,\ \exists k \ge 0 : \tau_{\boldsymbol{r}}^k(l_1,l_2) = (0,0)\}.$$

By Corollary 2.4 and Remark 2.1, we see that $(\lambda(1,0),\lambda(0,1)) \in \mathcal{D}_2^0$ if and only if β has property (F).

In [4], some regions in $\mathcal{D}_2^0$ are introduced as follows.

3.4.1 Cases except for ($\mathbf{H}_m$)

Fact 3.1 (Theorem 3.3 in [4]) *If* $|\lambda(1,0)| + |\lambda(0,1)| \le 1$, $\lambda(1,0) > 0$ *and* $\lambda(0,1) > 0$, *then* $(\lambda(1,0),\lambda(0,1)) \in \mathcal{D}_2^0$.

In the case (H), $(\lambda(1,0),\lambda(0,1))$ satisfies the condition of Fact 3.1.

Fact 3.2 (Theorem 3.4 in [4]) *Let* $p_1 = \lambda(1,0)$ *and* $p_2 = \lambda(0,1)$. *Suppose that* $|p_1| + |p_2| < 1$ *and there is unique* $k \in \{1,2\}$ *such that* $p_k < 0$. *Then* $(\lambda(1,0),\lambda(0,1)) \in \mathcal{D}_2^0$ *if and only if* $\sum_{1\le j\le 2/k} p_{3-kj} \ge 0$.

In the case (A2), $(\lambda(1,0),\lambda(0,1))$ satisfies the condition of Fact 3.2.

Fact 3.3 (Theorem 3.5 in [4]) *If* $0 \le \lambda(1,0) \le \lambda(0,1) \le 1$ *then* $(\lambda(1,0),\lambda(0,1)) \in \mathcal{D}_2^0$.

In the case (FS), $(\lambda(1,0),\lambda(0,1))$ satisfies the condition of Fact 3.3.

In the following, denote by $\Delta(A,B,C)$ the plane closed triangle with vertices $A,B,C \in \mathbb{R}^2$ and let γ_q be the positive root of the polynomial $qx^3 + qx^2 - qx - q + 1$. Notice that $\gamma_1 = \frac{\sqrt{5}-1}{2}$ and $\frac{3}{4} < \gamma_2 < 1$.

Fact 3.4 (Theorem 4.6 in [4]) *If* $\lambda(1,0) < \gamma_1^2$ *or* $\lambda(0,1) > \frac{1}{\gamma_1}\lambda(1,0) + \gamma_1$, *then* $(\lambda(1,0),\lambda(0,1)) \in \mathcal{D}_2^0$.

Fact 3.5 (Lemma 5.3 in [4]) *Let H be the convex hull of the four points $(\gamma_1^3, 1)$, $(\gamma_1\gamma_2, \gamma_1 + \gamma_2)$, $(\frac{2}{3}, 1)$ and $(\frac{2}{3}, \frac{2}{3\gamma_2} + \gamma_2)$. If $(\lambda(1,0), \lambda(0,1)) \in H$, then $(\lambda(1,0), \lambda(0,1)) \in \mathcal{D}_2^0$.*

Consider the case (A1) with $b - 2c \geq 0$. By Remark 3.8-(1), we see that $(\lambda(1,0), \lambda(0,1)) \in \Delta((0,1), (\frac{1}{2}, 1), (\frac{2}{3}, \frac{4}{3}))$ (use $\lambda(-2,1) > 0$, $\lambda(-1,2) < 2$ and $\lambda(0,1) > 1$). So, since $\gamma_1 + \gamma_2 > \frac{4}{3}$ and $\frac{2}{3\gamma_2} + \gamma_2 > \frac{4}{3}$, $(\lambda(1,0), \lambda(0,1))$ satisfies the condition of Fact 3.4 or Fact 3.5. For example, if $(a, b, c) = (1, 2, 1)$, then $\lambda(1,0) > \frac{1}{3} > \gamma_1^3$ and so $(\lambda(1,0), \lambda(0,1))$ satisfies the condition of Fact 3.5.

Fact 3.6 (Lemma 5.7 in [4]) *Let H be the convex hull of the four points $(\frac{2}{3}, 1)$, $(\frac{2}{3}, \frac{4}{3})$, $(\frac{29}{30}, 1)$ and $(\frac{29}{30}, \frac{31}{30})$. If $(\lambda(1,0), \lambda(0,1)) \in H$, then $(\lambda(1,0), \lambda(0,1)) \in \mathcal{D}_2^0$.*

Consider the case (A1) with $b - 2c = -1$ and $2a - b - c \geq -1$. By Remark 3.6-(1), we see that $(\lambda(1,0), \lambda(0,1)) \in \Delta((\frac{1}{2}, 1), (\frac{2}{3}, \frac{4}{3}), (\frac{3}{4}, \frac{5}{4}))$ (use $\lambda(2,-1) > 0$, $\lambda(1,1) < 2$ and $\lambda(-2,2) > 1$). So $(\lambda(1,0), \lambda(0,1))$ satisfies the condition of Fact 3.5 or Fact 3.6.

Fact 3.7 (Lemma 5.10 in [4]) *If $\frac{2}{3} \leq \lambda(1,0) \leq \frac{5}{6}$ and $2 - \lambda(1,0) \leq \lambda(0,1) \leq \frac{1}{2}\lambda(1,0) + 1$, then $(\lambda(1,0), \lambda(0,1)) \in \mathcal{D}_2^0$.*

Consider the case (A1) with $b - 2c = -1$ and $2a - b - c = -2$. Since $\lambda(0,1) < \frac{1}{2}\lambda(1,0) + 1$ by Remark 3.6-(1)-(iii) and $\lambda(0,1) > 2 - \lambda(1,0)$ by Remark 3.6-(1)-(iv), we have $2 - \lambda(1,0) < \lambda(0,1) < \frac{1}{2}\lambda(1,0) + 1$ and so $\lambda(1,0) > \frac{2}{3}$. Moreover, since $b - 2c = -1$ and $2a - b - c = -2$, we see that

$$(a, b, c) = (3n, 4n + 1, 2n + 1), \; n \geq 0.$$

Then

$$5 - 6\lambda(1,0) = (5a - 6c)\alpha + 5\alpha\lambda(0,1) > (5a - 6c + 5)\alpha = (3n - 1)\alpha$$

and so $n \geq 1$ implies $\lambda(1,0) < \frac{5}{6}$. If $n = 0$, then $1 = \alpha^2 + \alpha^3$ and $\lfloor \beta \rfloor = 1$. Therefore

$$\begin{aligned} 5 - 6\lambda(1,0) &= -6\alpha + 5\alpha^2 + 5\alpha^3 = 5\alpha^2 - \alpha^3 - 6\alpha^4 \\ &= -\alpha^3 - \alpha^4 + 5\alpha^5 = -\alpha^4 + 4\alpha^5 - \alpha^6 \\ &= \alpha^4(3\alpha - 1) + \alpha^5(1 - \alpha) > 0. \end{aligned}$$

Hence $(\lambda(1,0), \lambda(0,1))$ satisfies the condition of Fact 3.7.

3.4.2 Case ($\mathbf{H}_m$)

Fact 3.8 (Lemma 5.1 and Lemma 5.11 in [4]) *If* $\lambda(1,0) \le \frac{5}{6}$ *and* $1-\lambda(1,0) \le \lambda(0,1) \le \lambda(1,0)$, *then* $(\lambda(1,0), \lambda(0,1)) \in \mathcal{D}_2^0$.

Notice that if (a,b,c) satisfies (H_m), then $1-\lambda(1,0) < \lambda(0,1) < \lambda(1,0)$ by Remark 3.6-(2)-(ii) and (iii).

Remark 3.9 *Define*

$$\mathcal{D}_2 = \{\boldsymbol{r} \in \mathbb{R}^2 \mid \forall (l_1,l_2) \in \mathbb{Z}^2 \text{ the sequence } \{\tau_{\boldsymbol{r}}^k(l_1,l_2)\}_{k=0}^{\infty} \text{ is ultimately periodic}\}.$$

Clearly, $\mathcal{D}_2^0 \subset \mathcal{D}_2$. *Moreover in [4], it is shown that* $\mathcal{E}_2 \subset \mathcal{D}_2 \subset \overline{\mathcal{E}_2}$ *and* $\mathcal{D}_2^0 \subset \mathcal{E}_2$ *(Corollary 2.5 in [4]) where*

$$\mathcal{E}_2 = \{(x,y) \in \mathbb{R}^2 \mid x < 1,\ -x-1 < y < x+1\}.$$

The shape of $\mathcal{D}_2^0$ *near* $x=1$ *is very complicated (for example, see Figure 1 in [3]) and mostly unknown. Here we can find* (a,b,c) *satisfying condition* (H_m) *such that* $(\lambda(1,0), \lambda(0,1)) \in \mathcal{D}_2^0$ *and* $c\alpha = \lambda(1,0)$ *is arbitrarily close to* 1. *Indeed, let*

$$(a_n, b_n, c_n) = (n, n-1, n),\ n \in \mathbb{N}$$

and β_n *be a cubic Pisot number which is a root of* $x^3 - a_n x^2 - b_n x - c_n$. *Then* (a_n,b_n,c_n) *satisfies* (H_m). *Moreover, since* $\lfloor \beta_n \rfloor = a_n = n$, *we have*

$$1 < \frac{\beta_n}{n} < 1 + \frac{1}{n}$$

and so, letting $\alpha_n = \beta_n^{-1}$, *we have*

$$\lim_{n\to\infty} c_n \alpha_n = \lim_{n\to\infty} \left(\frac{\beta_n}{n}\right)^{-1} = 1.$$

3.4.3 Set of witnesses

Let β be a cubic Pisot number with minimal polynomial $x^3 - ax^2 - bx - c$. Recall

$$\mu(l_1,l_2,l_3) = \lambda(1,0)l_1 + \lambda(0,1)l_2 + l_3 \text{ and } Q = \{(l_1,l_2,l_3) \in \mathbb{Z}^3 \mid |\mu(l_1,l_2,l_3)| < 1\}.$$

According to the terminology in [4], we say that $\mathcal{V} \subset \mathbb{Z}^2$ is a set of witnesses (for β) if $\{\pm(1,0), \pm(0,1)\} \subset \mathcal{V}$ and if for each $(z_1,z_2) \in \mathcal{V}$ and (z_2,z_3) with $(z_1,z_2,z_3) \in Q$, $(z_2,z_3) \in \mathcal{V}$. Define

$$S_n(r) = \{(\eta,\theta) \mid (\zeta,\eta,\theta) \in Q_n(r) \text{ for some } \zeta\} \subset [\![-r,r]\!]^2.$$

Remark 3.10 *$S_n(r)$ is a set of witnesses if and only if $M_n(r) \le r$.*

Proof. It suffices to show that

(i) If $M_n(r) \le r$, then $(\eta, \theta) \in S_n(r)$ implies $\{(\theta, -\lfloor \lambda(\eta, \theta) \rfloor), (\theta, -\lfloor \lambda(\eta, \theta) \rfloor - 1)\} \subset S_n(r)$.

(ii) If $M_n(r) > r$, then there is $(\eta, \theta) \in S_n(r)$ such that $(\theta, -\lfloor \lambda(\eta, \theta) \rfloor - 1) \notin S_n(r)$.

(iii) There are ζ_1 and ζ_2 such that $\{(\zeta_1, 1, 0), (\zeta_2, 0, 1)\} \subset Q_1(1)$. (Note $Q_n(r) = -Q_n(r)$.)

(i) : Let $(\eta, \theta) \in S_n(r)$ be given. Since $\lambda(\eta, \theta) \neq \pm r$ and $M_n(r) \le r$, we have $-r < \lambda(\eta, \theta) < r$. Thus

$$-r \le \lfloor \lambda(\eta, \theta) \rfloor \le \lfloor \lambda(\eta, \theta) \rfloor + 1 \le r.$$

Hence $\{(\eta, \theta, -\lfloor \lambda(\eta, \theta) \rfloor), (\eta, \theta, -\lfloor \lambda(\eta, \theta) \rfloor - 1)\} \subset Q_n(r)$ and so

$$\{(\theta, -\lfloor \lambda(\eta, \theta) \rfloor), (\theta, -\lfloor \lambda(\eta, \theta) \rfloor - 1)\} \subset S_n(r).$$

(ii) : Since $M_n(r) > r$, there is $(\zeta, \eta, \theta) \in Q_n(r)$ such that $\lambda(\eta, \theta) > r$. Then $(\eta, \theta) \in S_n(r)$ and $-\lfloor \lambda(\eta, \theta) \rfloor - 1 \le -r - 1$. Hence $(\theta, -\lfloor \lambda(\eta, \theta) \rfloor - 1) \notin [\![-r, r]\!]^2$ and so $(\theta, -\lfloor \lambda(\eta, \theta) \rfloor - 1) \notin S_n(r)$.

(iii) : Define $\zeta_2 = -\mathrm{sgn}(c)$ and

$$\zeta_1 = \begin{cases} 1 & \text{if } b \le -a \\ 0 & \text{if } -a < b \le a \\ -1 & \text{if } a < b. \end{cases}$$

By Remark 3.2, $\mu(\zeta_2, 0, 1) = 1 - |c|\alpha \in (0, 1)$. Thus $(\zeta_2, 0, 1) \in Q_0(1)$. If $(\zeta_1, 1, 0) \in Q_0(1)$, then $(\zeta_1\zeta_2, \zeta_2, 0) \in Q_0(1)$ and so $(\zeta_2, 0, 1) \in Q_1(1)$. Therefore it suffices to show $(\zeta_1, 1, 0) \in Q_1(1)$. Let

$$\zeta_1' = \begin{cases} 1 & \text{if } \zeta_1 \neq 0 \\ \zeta_2 & \text{if } \zeta_1 = 0 \end{cases}$$

Recall ($\sharp$) and ($\sharp\sharp$) in section 3.1.

Case 1: $\zeta_1 = 0$. By ($\sharp\sharp$), we have

$$\mu(\zeta_1, 1, 0) = b\alpha + c\alpha^2 \in (-1, 1).$$

Hence $(\zeta_1', \zeta_1, 1) = (\zeta_2, 0, 1) \in Q_0(1)$ and so $(\zeta_1, 1, 0) \in Q_1(1)$.

Case 2: $\zeta_1 = 1$. Since $b\alpha + c\alpha^2 \in (-2, -1)$ by ($\sharp\sharp$), we have

$$\mu(\zeta_1, 1, 0) = c\alpha + b\alpha + c\alpha^2 < c\alpha - 1 < 0.$$

Notice that $c \geq 2$ by Remark 3.1 (i.e., $c + a \geq 2 - b$ in the case 2) and that $\lfloor \beta \rfloor = a - 2$ by ($\sharp$). So we have

$$\mu(\zeta_1, 1, 0) = (c + b)\alpha + c\alpha^2 \geq -(a - 2)\alpha + c\alpha^2 > -(a - 2)\alpha > -1.$$

Therefore $\mu(\zeta_1, 1, 0) \in (-1, 0)$ and so

$$\mu(\zeta_1', \zeta_1, 1) = \mu(\zeta_1, 1, 0) + 1 \in (0, 1).$$

Hence $(\zeta_1, 1, 0) \in Q_1(1)$.

Case 3: $\zeta_1 = -1$. Since $b\alpha + c\alpha^2 \in (1, 2)$ by ($\sharp\sharp$), we have

$$\mu(\zeta_1, 1, 0) = -c\alpha + b\alpha + c\alpha^2 > 1 - c\alpha > 0.$$

Notice that $\lfloor \beta \rfloor = a + 1$ by ($\sharp$). Since $a + c \geq b$ by Remark 3.1, we have

$$\mu(\zeta_1, 1, 0) = (b - c)\alpha + c\alpha^2 \leq a\alpha + c\alpha^2 < (a + 1)\alpha < 1.$$

So $\mu(\zeta_1, 1, 0) \in (0, 1)$ and so

$$\mu(\zeta_1', \zeta_1, 1) = -\mu(\zeta_1, 1, 0) + 1 \in (0, 1).$$

Hence $(\zeta_1, 1, 0) \in Q_1(1)$. □

Denote the set of (η, θ) in Table k by L_k ($k = 1, 2, 3, 4, 5, 6, 7$). Let

$$(r, n) = \begin{cases} (1, 0) & \text{if } k = 4, 5, 6, 7 \\ (2, 1) & \text{if } k = 1, 2, 3. \end{cases}$$

First, consider cases $k = 2, 3, 4, 6, 7$. In these cases, $L_k \cup \{(0, 0)\}$ is a set of witnesses. Indeed, note $M_1(r) \leq \min\{M_n(r), r\}$. So it suffices to show $L_k \cup \{(0, 0)\} = S_1(r)$ by Remark 3.10. Define

$$P_k = \{(\eta, \theta, \iota) \mid (\eta, \theta) \in L_k,\ \iota = -\lfloor \lambda(\eta, \theta) \rfloor\}.$$

Clearly, $P_k \subset Q_0(r)$ and so $-P_k \subset Q_0(r)$ because $Q_0(r) = -Q_0(r)$. On Table k, we can verify (tediously)

$$L_k \subset \{(\theta, \iota) \mid (\eta, \theta, \iota) \in P_k \cup (-P_k) \text{ for some } \eta\},$$

that is, $(\eta,\theta)\in L_k$ implies $(\zeta,\eta,\theta)\in P_k\cup(-P_k)$ for some ζ. In particular, $(\eta,\theta)\in L_k$ implies $(\zeta,\eta)\in L_k$ for some ζ (note $L_k=-L_k$). So for any $(\eta,\theta)\in L_k$, there are ζ and ζ' such that $\{(\zeta,\eta,\theta),(\zeta',\zeta,\eta)\}\subset Q_0(r)$. Thus $L_k\cup\{(0,0)\}\subset S_1(r)$. On the other hand, we see

$$\sharp L_k=\begin{cases}14 & \text{if } k=2\\ 12 & \text{if } k=3\\ 6 & \text{if } k=4\\ 8 & \text{if } k=6,7.\end{cases}$$

Therefore by Remark 3.5, Remark 3.3 and Proposition 3.4, we have $\sharp S_1(r)\le \sharp S_n(r)\le \sharp L_k+1$. So $S_1(r)=L_k\cup\{(0,0)\}$.

Next, for $k=1,5$, any subset of $L_k\cup\{(0,0)\}$ is not a set of witnesses. Indeed, note $L_k\cup\{(0,0)\}\subset [\![-r,r]\!]^2$ and $\lfloor\lambda(1,1)\rfloor=r$. Let

$$C_k=\begin{cases}\{(0,1,-2),(1,-2,1),(-2,1,1),(1,1,-3)\} & \text{if } k=1\\ \{(0,-1,1),(-1,1,1),(1,1,-2)\} & \text{if } k=5.\end{cases}$$

Then $C_k\subset Q$. So, since each set of witnesses contains $\pm(0,1)$, it contains $(1,-r-1)$, but $(1,-r-1)\notin [\![-r,r]\!]^2$.

4 Concluding Remarks

Let β be a cubic Pisot number with minimal polynomial x^3-ax^2-bx-c and $c\ge 1$.

4.1 β which does not have property (F)

Recall that if $b\le -a$, then $\lfloor\beta\rfloor=a-2$ and so $c\le a-2$.

Proposition 4.1 (Compare Example 4.7 in [3]) *If $b\le -a$, then β does not have property (F).*

Proof. Notice that

$$\mu(1,1,1)=(a+b+c-1)\alpha+\alpha\mu(1,1,1).$$

Since $c\le a-2$ and $a+b+c\ge 2$ by Remark 3.1,

$$1\le a+b+c-1\le a-3$$

and so

$$(1-\alpha)\mu(1,1,1)=(a+b+c-1)\alpha\in(0,1-\alpha).$$

Hence $\mu(1,1,1) \in (0,1)$ and $\lambda(1,1) = \mu(1,1,1) - 1 \in (-1,0)$. Let $(l_1, l_2, l_3) = (1,1,1)$ and $l_{n+3} = -\lfloor \lambda(l_{n+1}, l_{n+2}) \rfloor$ for all $n \in \mathbb{N}$. Then $l_n = 1$ for all $n \in \mathbb{N}$ and by Corollary 2.4,

$$d_\beta(\mu(1,1,1)) = (a+b+c-1)^\infty.$$

Hence β does not have property (F). $\square$

Proposition 4.2 (Compare Example 4.8 in [3]) *If (a,b,c) satisfies $-a < b < 0$, $2b + c \leq -2$ and $a - b - 2c \leq 1$, then β does not have property (F).*

Proof. By the assumption, $a + b - c < 0$. Moreover $b + c \geq 0$ because $a - b - 2c \leq 1$ and $c \leq a - 1$. First, we show

$$-\lfloor \lambda(k,l) \rfloor = \begin{cases} -1 & \text{if } (k,l) \in \{(2,1),(1,-1)\} \\ 1 & \text{if } (k,l) \in \{(-1,-1),(1,2)\} \\ 2 & \text{if } (k,l) = (-1,1). \end{cases}$$

Case $(k,l) = (-1,-1)$. Notice that $-a < -c \leq b < 0$. So $\lambda(-1,-1) > -c\alpha > -1$. Hence by Lemma 3.1-(3)-(iii),

$$-1 < \lambda(-1,-1) = \lambda(0,0) - \lambda(1,1) < \lambda(0,0) = 0.$$

Hence $-\lfloor \lambda(-1,-1) \rfloor = 1$.

Case $(k,l) = (1,-1)$. Notice that $\lambda(1,-1) < 2(a-1)\alpha < 2$. Since $a + b - c \leq -1$, we have

$$\lambda(1,-1) \geq (a+1)\alpha - c\alpha^2 > a\alpha > 1.$$

Thus $1 < \lambda(1,-1) < 2$ and so $-\lfloor \lambda(1,-1) \rfloor = -1$.

Case $(k,l) = (2,1)$. Notice that

$$\lambda(2,1) = c\alpha + \lambda(1,1) > 0.$$

Since $a - b - 2c \leq 1$ and $\lambda(1,-1) > 1$, we have

$$\lambda(2,1) - 1 \geq (a-1)\alpha + c\alpha^2 - 1 = \alpha(-1 + \lambda(1,-1)) > 0.$$

So $1 < \lambda(2,1) < 2$ and $-\lfloor \lambda(2,1) \rfloor = -1$.

Case $(k,l) = (1,2)$. Notice that

$$\lambda(1,2) = \lambda(2,1) - \lambda(1,-1) > -1$$

because $1 < \lambda(2,1)$ and $\lambda(1,-1) < 2$. Since $2b + c \leq -2$, we have

$$\lambda(1,2) \leq -2\alpha(1 - c\alpha) < 0.$$

So $-\lfloor \lambda(1,2) \rfloor = 1$.

Let $(l_1, l_2, l_3) = (-1, -1, 1)$ and $l_{n+3} = -\lfloor \lambda(l_{n+1}, l_{n+2}) \rfloor$ for $n \in \mathbb{N}$. Then

$$(l_{6n+1}, l_{6n+2}, l_{6n+3}, l_{6n+4}, l_{6n+5}) = (-1, -1, 1, 2, 1) \text{ for } n \in \mathbb{N}_0.$$

Hence by Corollary 2.4, we see that

$$\begin{aligned} &d_\beta(\mu(-1,-1,1)) \\ &= ((a-b-c-2)(2a+b-c-1)(a+2b+c+1)(2c+b-a+1)(c-b-a-1))^\infty \end{aligned}$$

and so $d_\beta(\mu(-1,-1,1))$ is not finite. □

4.2 Some cubic Pisot numbers with property (F) and $r \geq 2$

Proposition 4.3 *Let β be a cubic Pisot number with minimal polynomial $x^3 - ax^2 - bx - c$.*

(1) Let (a,b,c) satisfy $-a < -c \leq b < 0$, $2a + b - 2c \geq 1$, $2b + c \leq -2$ and $a + b - c < 0$. Then β has property (F) if and only if $a - b - 2c \geq 2$.

(2) Let (a,b,c) satisfy one of the following conditions:

(i) $-a < -c \leq b < 0$, $2a + b - 2c \geq 0$ and $2b + c = -1$.

(ii) $-a < -c \leq b < 0$, $2a + 2b - c \geq 0$, $a + 3b + c \leq -2$ and $a - 2b - 3c \leq 2$.

Then β has property (F).

Proposition 4.4 *Let β be a cubic Pisot number with minimal polynomial $x^3 - ax^2 - bx - c$ satisfying $0 \leq a < b$, $3a - 3b + 2c \geq -1$ and $b - 3c \geq 0$. Then β has property (F).*

4.3 Proof of Proposition 4.3 and Proposition 4.4

Remark 4.1 *Let (a,b,c) satisfy the condition of Proposition 4.3-(2)-(ii). Then we have*

(1) $3b + 2c \leq -3$. *(4)* $3a + b - 3c \geq 2$.

(2) $a + b - c < 0$.

(3) $a - b - 2c \geq 2$. *(5)* $2a + b - 2c \leq 0$.

Proof of Remark 4.1. See Appendix. □

We set

$$(r, n) = \begin{cases} (2, 0) & \text{if } (a, b, c) \text{ satisfies Proposition 4.3-(1) or (2)-(i)} \\ (3, 0) & \text{if } (a, b, c) \text{ satisfies Proposition 4.3-(2)-(ii).} \\ (3, 1) & \text{if } (a, b, c) \text{ satisfies Proposition 4.4.} \end{cases}$$

Observation 4.1 *Let (a, b, c) satisfy one of conditions in Proposition 4.3 or Proposition 4.4. Then $M_n(r) < r$.*

Proof. Put

$$s = \begin{cases} 1 & \text{if } (a, b, c) \text{ satisfies Proposition 4.3-(1)} \\ 2 & \text{if } (a, b, c) \text{ satisfies Proposition 4.3-(2)-(ii).} \end{cases}$$

By Remark 4.1-(1) and (4), we have

$$-a < -c \leq b < 0, \ 3a + b - 3c \geq 2 \text{ and } 3b + 2c \leq -3$$

if (a, b, c) satisfies the Proposition 4.3-(2)-(ii). So by Theorem 2.2, we have desired result. □

Following the approach in subsection 2.3, we seek the value $\lfloor \lambda(l_2, l_3) \rfloor$ with $\lambda(l_2, l_3) > 0$ and $\boldsymbol{\ell} = (l_1, l_2, l_3) \in Q_n(r)$ with $\mu(\boldsymbol{\ell}) > 0$ for some l_1.

Remark 4.2

(1) Let (a, b, c) satisfy the condition of Proposition 4.3-(1) with $a - b - 2c \geq 2$. Then

(i) $0 < \lambda(-1, -2) < \lambda(0, -1) < \lambda(1, 0) < \lambda(2, 1) < 1$.

(ii) $1 < \lambda(1, -1) < \lambda(2, 0) < \lambda(1, -2) < \lambda(2, -1) < 2$.

(iii) $0 < \lambda(1, 1) < \lambda(2, 2) < 1$.

(iv) If $a + 2b \geq 0$, then $0 < \lambda(0, -2) < 1$ and if $a + 2b \leq -1$, then $1 < \lambda(0, -2) < 2$.

(2) Let (a, b, c) satisfy the condition of Proposition 4.3-(2)-(i) with $a + b - c < 0$. Then

(i) $0 < \lambda(0,-1) < \lambda(0,-2) < \lambda(1,0) < 1$ *and* $\lambda(1,2) > 0$.

(ii) $1 < \lambda(1,-1) < \lambda(1,-2) < \lambda(2,-1) < 2$.

(iii) $0 < \lambda(1,2) < \lambda(1,1) < \lambda(2,2) < 1$.

(iv) $1 < \lambda(2,1) < \lambda(2,0) < 2$.

(3) Let (a,b,c) *satisfy the condition of Proposition 4.3-(2)-(ii). Then we have*

(i) $0 < \lambda(-2,-3) < \lambda(-1,-2) < \lambda(0,-1) < \lambda(1,0) < \lambda(2,1) < 1$.

(ii) $0 < \lambda(1,1) < \lambda(2,2) < \lambda(3,3) < 1$.

(iii) $2 < \lambda(2,-1) < \lambda(3,0) < 3$.

(iv) $1 < \lambda(3,2) < \lambda(2,0) < \lambda(3,1) < 2$.

(v) $1 < \lambda(-1,-3) < \lambda(0,-2) < \lambda(1,-1) < \lambda(0,-3) < \lambda(1,-2) < 2$.

(vi) $2 < \lambda(1,-3) < \lambda(2,-2) < \lambda(3,-1) < 3$.

(4) Let (a,b,c) *satisfy the condition of Proposition 4.4 with* $2a - 2b + c \leq -2$. *Then*

(i) $0 < \lambda(1,0) < 1$ *and* $1 < \lambda(0,1) < 2$.

(ii) $0 < \lambda(-2,1) < \lambda(-1,1) < 1$.

(iii) $1 < \lambda(-3,2) < \lambda(-2,2) < 2$.

(iv) $2 < \lambda(-1,2) < \lambda(-3,3) < \lambda(-2,3) < 3$.

Proof of Remark 4.2. See Appendix. □

Proof of Proposition 4.3. (1) : Suppose that $a-b-2c \leq 1$. Then by Proposition 4.2, β does not have property (F).

Suppose that $a - b - 2c \geq 2$. By Lemma 3.3-(1) and Remark 4.2-(1), we have determined each integer part of an element in $\Lambda_0(a,b,c)$ (see Table k ($k = 8, 9$)). So we see that β has property (F) by Figure k ($k = 8, 9$) and Corollary 2.4.

(2)-(i) : If $a+b-c \geq 0$, then the proof has already completed by Theorem 1.2. So consider the case $a + b - c < 0$. By Lemma 3.3-(2) and Remark 4.2-(2), we have determined each integer part of an element in $\Lambda_0(a,b,c)$ (see Table 10). Hence we see that β has property (F) by Figure 10 and Corollary 2.4.

(2)-(ii) : By Lemma 3.3-(1) and Remark 4.2-(3), we have determined each integer part of an element in $\Lambda_0(a,b,c)$ (see Table 11). Hence we see that β has property (F) by Figure 11 and Corollary 2.4. □

Proof of Proposition 4.4. Notice that if $2a - 2b + c \geq -1$, then we have already shown that β has property (F) by Theorem 1.2-(A1). So consider the case $2a - 2b + c \leq -2$. By Lemma 3.2 and Remark 4.2-(4), we have determined each integer part of an element in $\Lambda_1(a, b, c)$ (see Table 12). Hence we see that β has property (F) by Figure 12 and Corollary 2.4. $\square$

4.4 Comparison with results in [4]

Each case of Proposition 4.3 and Proposition 4.4 can be shown by the framework in [4].

4.4.1 Cases in Proposition 4.3

Fact 4.1 (Lemma 5.2 in [4]) *If* $(\lambda(1,0), \lambda(0,1)) \in \Delta((\frac{1}{2}, -\frac{1}{2}), (\frac{2}{3}, -\frac{1}{3}), (\frac{2}{3}, -\frac{2}{3}))$, *then* $(\lambda(1,0), \lambda(0,1)) \in \mathcal{D}_2^0$.

Fact 4.2 (Lemma 5.5 in [4]) *Let* H *be the convex hull of the four points* $(\frac{2}{3}, -\frac{1}{3})$, $(\frac{2}{3}, -\frac{2}{3})$, $(\frac{29}{30}, -\frac{14}{15})$ *and* $(\frac{29}{30}, -\frac{29}{30})$. *If* $(\lambda(1,0), \lambda(0,1))$ *is in* H *and not on the line connecting* $(\frac{2}{3}, -\frac{1}{3})$ *and* $(\frac{29}{30}, -\frac{14}{15})$, *then* $(\lambda(1,0), \lambda(0,1)) \in \mathcal{D}_2^0$.

First, suppose that (a, b, c) satisfies the condition of Proposition 4.3-(1) with $a - b - 2c \geq 2$. Then by Remark 4.2-(1), $(\lambda(1,0), \lambda(0,1))$ belongs to the convex hull of the four points $(\frac{1}{2}, -\frac{1}{2})$, $(\frac{2}{3}, -\frac{2}{3})$, $(\frac{3}{4}, -\frac{1}{2})$ and $(\frac{2}{3}, -\frac{1}{3})$ (use $1 < \lambda(1,-1)$, $\lambda(2,1) < 1$, $\lambda(2,-1) < 2$ and $0 < \lambda(1,1)$). So $(\lambda(1,0), \lambda(0,1))$ satisfies the condition of Fact 4.1 or Fact 4.2.

Next, suppose that (a, b, c) satisfies the condition of Proposition 4.3-(2)-(ii). Then by Remark 4.2-(3), $(\lambda(1,0), \lambda(0,1)) \in \Delta((\frac{2}{3}, -\frac{2}{3}), (\frac{4}{5}, -\frac{3}{5}), (\frac{5}{7}, -\frac{4}{7}))$ (use $2 < \lambda(2,-1)$, ($1 < \lambda(-1,-3)$ and $\lambda(1,-2) < 2$). So $(\lambda(1,0), \lambda(0,1))$ satisfies the condition of Fact 4.2.

Fact 4.3 (Lemma 5.12 in [4]) *If* $\frac{2}{3} \leq \lambda(1,0) \leq \frac{5}{6}$ *and* $-\frac{1}{2}\lambda(1,0) \leq \lambda(0,1) \leq \lambda(1,0) - 1$, *then* $(\lambda(1,0), \lambda(0,1)) \in \mathcal{D}_2^0$.

Suppose that (a, b, c) satisfies the condition of Proposition 4.3-(2)-(i) with $a + b - c < 0$. Since $\lambda(1,-1) > 1$ by Remark 4.2-(2)-(ii) and $\lambda(1,2) > 0$ by Remark 4.2-(2)-(iii), we have $-\frac{1}{2}\lambda(1,0) < \lambda(0,1) < \lambda(1,0) - 1$ and $\lambda(1,0) > \frac{2}{3}$. On the other hand, since $\lambda(2,-1) < 2$ by Remark 4.2-(2)-(ii) and $\lambda(2,2) < 1$ by Remark 4.2-(2)-(iii), we have $\lambda(0,1) < \frac{5}{6}$. Hence $(\lambda(1,0), \lambda(0,1))$ satisfies the condition of Fact 4.3.

4.4.2 Case: Proposition 4.4

Suppose that (a,b,c) satisfies the condition of Proposition 4.4 with $2a-2b+c \leq -2$. Notice that $\lambda(-3,1) > 0$ because $b-3c \geq 0$. Then by Remark 4.2-(4), $(\lambda(1,0),\lambda(0,1)) \in \Delta((0,1),(\frac{2}{5},\frac{6}{5}),(\frac{3}{7},\frac{9}{7}))$ (use $\lambda(-3,1) > 0$, $2 < \lambda(-1,2)$ and $\lambda(-2,3) < 3$). So $(\lambda(1,0),\lambda(0,1))$ satisfies the condition of Fact 3.4 or Fact 3.5 For example, if $(a,b,c) = (9,12,4)$, then $(\lambda(1,0),\lambda(0,1))$ satisfies the condition of Fact 3.5.

4.4.3 Set of witnesses

Denote the set of (η,θ) in Table k by L_k ($k = 8,9,10,11,12$). We show $L_k \cup \{(0,0)\}$ is a set of witnesses. Recall

$$S_n(r) = \{(\eta,\theta) \mid (\zeta,\eta,\theta) \in Q_n(r) \text{ for some } \zeta\}$$

in section 3.4.3 and

$$(r,n) = \begin{cases} (2,0) & \text{if } k = 8,9,10 \\ (3,0) & \text{if } k = 11 \\ (3,1) & \text{if } k = 12 \end{cases}$$

in section 4.3. Note $M_1(r) \leq \min\{M_n(r), r\}$. In the case $k = 10, 12$, we can show $S_1(r) = L_k \cup \{(0,0)\}$ in a similar way as section 3.4.3 (by using Remark 3.3 and Remark 3.4) and so by Remark 3.10, $L_k \cup \{(0,0)\}$ is a set of witnesses. Consider the case $k = 8,9,11$. In these cases, we show

$$L_k \cup \{(0,0)\} = S_0(r) \cup \{\pm(-1,r)\}.$$

Indeed, on Table k, we can verify

$$L_k \cup \{(0,0)\} \subset \{(\theta,\iota) \mid (\eta,\theta,\iota) \in P_k \cup (-P_k) \text{ for some } \eta\} \cup \{\pm(-1,r)\}$$

where

$$P_k = \{(\eta,\theta,\iota) \mid (\eta,\theta) \in L_k,\ \iota = -\lfloor \lambda(\eta,\theta) \rfloor\}.$$

Thus $L_k \cup \{(0,0)\} \subset S_0(r) \cup \{\pm(-1,r)\}$. On the other hand, note

$$S_0(r) \cup \{\pm(-1,r)\} \subset \{(\eta,\theta) \in [\![-r,r]\!]^2 \mid s-2r+\eta \leq \theta \leq 2r-s+\eta\}.$$

So by Remark 3.4, we have $\sharp(S_0(r) \cup \{\pm(-1,r)\}) \leq \sharp L_k + 1$. Hence $L_k \cup \{(0,0)\} = S_0(r) \cup \{\pm(-1,r)\}$.

Now $S_0(r)$ is a set of witnesses by Remark 3.10. Note

$$\{\pm(r,-\lfloor\lambda(-1,r)\rfloor), \pm(r,-\lfloor\lambda(-1,r)\rfloor - 1)\} \subset S_0(r)$$

because $\lfloor\lambda(-1,r)\rfloor = -r$. So, since $L_k \cup \{(0,0)\} = S_0(r) \cup \{\pm(-1,r)\}$, we have $L_k \cup \{(0,0)\}$ is a set of witnesses.

Appendix: Proof of remarks

In the following proof, we use $0 < c\alpha < 1$ without mentioning Remark 3.2.

Proof of Remark 3.6. (1)-(i) : Since $b - 2c = -1$, we have

$$\lambda(2, -1) = \alpha - c\alpha^2 = \alpha(1 - c\alpha) \in (0, 1).$$

(1)-(ii) : By Lemma 3.1-(1)-(i), it suffices to show $0 < \lambda(-1, 1) < \lambda(1, 0)$. By (2)-(i),

$$\lambda(-1, 1) = \lambda(-2, 1) + \lambda(1, 0) < \lambda(1, 0).$$

Since $b - c = c - 1 \geq 0$, we have

$$\lambda(-1, 1) = (c - 1)\alpha + c\alpha^2 > 0.$$

(1)-(iii) : First, we show $\lambda(-1, 2) < 2$. Since $2a - 2b + c \geq -1$, we have

$$\begin{aligned} -c\alpha + 2(b\alpha + c\alpha^2) - 2 &\leq (2a + 1)\alpha + 2c\alpha^2 - 2(a\alpha + b\alpha^2 + c\alpha^3) \\ &= \alpha + (2c - 2b)\alpha^2 - 2c\alpha^3 \\ &= \alpha - (b - 1)\alpha^2 - (b + 1)\alpha^3 \\ &= -(b - a - 1)\alpha^2 - \alpha^3 + c\alpha^4 < 0. \end{aligned}$$

because $a < b$. Next, we show $1 < \lambda(-2, 2)$. Since $b - 2c = -1$, we have

$$\begin{aligned} \lambda(-2, 2) - 1 &= (b - 1)\alpha + 2c\alpha^2 - 1 \\ &= (b - a - 1)\alpha + (2c - b)\alpha^2 - c\alpha^3 \\ &= (b - a - 1)\alpha + \alpha^2(1 - c\alpha) > 0. \end{aligned}$$

Hence we have

$$1 < \lambda(-2, 2) = \lambda(-1, 2) - \lambda(1, 0) < \lambda(-1, 2) < 2$$

by Lemma 3.1-(1)-(i).

(1)-(iv) : Notice that by Lemma 3.1-(1)-(i),

$$1 < \lambda(1, 1) = \lambda(1, 0) + \lambda(0, 1) < 3.$$

If $2a - b - c = -2$, then by (2)-(iii),

$$\begin{aligned} \lambda(1, 1) - 2 &= (2a + 2)\alpha + c\alpha^2 - 2 \\ &= 2\alpha + (-2b + c)\alpha^2 - 2c\alpha^3 \\ &= \alpha(2 - \lambda(-1, 2)) > 0. \end{aligned}$$

If $2a - b - c \geq -1$, then by (2)-(iii),

$$\lambda(1,1) - 2 \leq \alpha + (-2b + c)\alpha^2 - 2c\alpha^3 = \alpha(1 - \lambda(-1,2)) < 0.$$

(2)-(i) : By Lemma 3.1-(2)-(i).
(2)-(ii) : By (2)-(i),

$$\lambda(1,-1) = \lambda(1,0) - \lambda(0,1) < \lambda(1,0) < 1.$$

Since $b - c \leq -1$, we have

$$\lambda(1,-1) = (c - b)\alpha - c\alpha^2 \geq \alpha(1 - c\alpha) > 0.$$

(2)-(iii) : By (2)-(i),

$$\lambda(1,1) = \lambda(1,0) + \lambda(0,1) < 2.$$

Since $a - b - c \leq 0$, we have

$$\lambda(1,1) - 1 \geq a\alpha + c\alpha^2 - 1 = (c - b)\alpha^2 - c\alpha^3 = \alpha\lambda(1,-1) > 0$$

by (2)-(ii). □

Proof of Remark 3.8. (1)-(i) : By Lemma 3.1-(1)-(i).
(1)-(ii) : Since $b - 2c \geq 0$, we have

$$\lambda(-2,1) \geq c\alpha^2 > 0.$$

Since $a + c \geq b$ by Remark 3.1, we have

$$\lambda(-1,1) \leq a\alpha + c\alpha^2 < (a+1)\alpha < 1.$$

Hence by (1)-(i),

$$0 < \lambda(-2,1) = \lambda(-1,0) + \lambda(-1,1) < \lambda(-1,1).$$

(1)-(iii) : By (1)-(i) and (ii),

$$1 < \lambda(0,1) < \lambda(0,1) + \lambda(-2,1) = \lambda(-2,2) < \lambda(-2,2) + \lambda(1,0) = \lambda(-1,2).$$

Hence, since $2a - 2b + c \geq -1$ and $\lambda(-2,2) > 1$,

$$\begin{aligned}
\lambda(-1,2) - 2 &= (-2a + 2b - c)\alpha + (-2b + 2c)\alpha^2 - 2c\alpha^3 \\
&\leq \alpha + (-2b + 2c)\alpha^2 - 2c\alpha^3 \\
&= \alpha(1 - \lambda(-2,2)) < 0.
\end{aligned}$$

(2) : Since $b \geq c$, we have

$$\lambda(-1,1) = \lambda(0,1) - \lambda(1,0) = (b-c)\alpha + c\alpha^2 > 0.$$

Hence by Lemma 3.1-(2)-(i),

$$0 < \lambda(1,0) < \lambda(0,1) < 1.$$

(3)-(i) : By Lemma 3.1-(2)-(i).
(3)-(ii) : By (3)-(i),

$$\lambda(1,-1) = \lambda(1,0) - \lambda(0,1) < \lambda(1,0) < 1.$$

Since $b - c \leq -1$, we have

$$\lambda(1,-1) = (c-b)\alpha - c\alpha^2 \geq \alpha(1-c\alpha) > 0.$$

(3)-(iii) : By (3)-(i), $\lambda(1,1) = \lambda(1,0) + \lambda(0,1) > 0$. Since $a - b - c \geq 1$, we have

$$\lambda(1,1) - 1 \leq (a-1)\alpha + c\alpha^2 - 1 = -\alpha + (c-b)\alpha^2 - c\alpha^3 = \alpha(-1 + \lambda(1,-1)) < 0$$

by (3)-(ii).
(4)-(i) : By Lemma 3.1-(3)-(i).
(4)-(ii) : Since $-c \leq b$, we have

$$\lambda(1,1) \geq c\alpha^2 > 0.$$

So by (4)-(i),

$$0 < \lambda(1,1) < \lambda(1,1) + \lambda(0,-2) = \lambda(1,-1).$$

Hence, since $a + b - c \geq 0$ and $\lambda(1,1) > 0$, we have

$$\lambda(1,-1) - 1 \leq a\alpha - c\alpha^2 - 1 = (-b-c)\alpha^2 - c\alpha^3 = -\alpha\lambda(1,1) < 0.$$

□

Proof of Remark 4.1. (1) : Since $c < a$ and $a + 3b + c \leq -2$, we have

$$3a + 2c < a + 3b + c \leq -2.$$

Hence $3a + 2c \leq -3$.

(2) : Since $b+c \geq 0$ and $a+3b+c \leq -2$, we have

$$a+b-c = a+3b+c-2(b+c) \leq a+3b+c \leq -2.$$

(3) : Since $2a+2b-c \geq 0$ and $a+3b+c \leq -2$,

$$a-b-2c = 2a+2b-c-(a+3b+c) \geq 2.$$

(4) : Since $2a+2b-c \geq 0$, we have by (3),

$$3a+b-2c = a-b-2c+2a+2b-c \geq 2.$$

(5) : Since $a+3b+c \leq -2$ and $a-2b-3c \leq 2$,

$$2a+b-2c = a+3b+c+a-2b-3c \leq 0.$$

□

Proof of Remark 4.2. (1)-(i) : Since $2b+c \leq -2$, we have

$$\lambda(-1,-2) \geq 2\alpha(1-c\alpha) > 0.$$

Since $a-b-2c \geq 2$, we have

$$\lambda(2,1) \leq (a-2)\alpha + c\alpha^2 < (a-1)\alpha < 1.$$

Hence by Lemma 3.1-(3)-(iii),

$$0 < \lambda(-1,-2) < \lambda(0,-1) < \lambda(1,0) < \lambda(2,1) < 1.$$

(1)-(ii) : Since $a+b-c \leq -1$, we have

$$\lambda(1,-1) \geq (a+1)\alpha - c\alpha^2 > a\alpha > 1.$$

Since $2a+b-2c \geq 1$, we have

$$\lambda(2,-1) - 2 \leq (2a-1)\alpha - c\alpha^2 - 2 = \alpha(-1+\lambda(-1,-2)) < 0$$

by (1)-(i). Hence by (1)-(i) and Lemma 3.1-(3)-(iii),

$$1 < \lambda(1,-1) < \lambda(2,0) < \lambda(2,0) + \lambda(-1,-2) = \lambda(1,-2) < \lambda(2,-1) < 2.$$

(1)-(iii) : Since $2b+c \leq -2$, we have

$$\lambda(2,2) \leq (c-2)\alpha + 2c\alpha^2 = \alpha(c-2+\lambda(2,0)) < c\alpha < 1$$

by (1)-(ii). Hence by Lemma 3.1-(3)-(iii),

$$0 = \lambda(0,0) < \lambda(1,1) < \lambda(2,2) < 1.$$

(1)-(iv) : By (1)-(i), $\lambda(0,-2) = 2\lambda(0,-1) \in (0,2)$. If $a + 2b \geq 0$, then by (1)-(i),

$$\lambda(0,-2) - 1 = -(a+2b)\alpha - \alpha\lambda(2,1) \leq -\alpha\lambda(2,1) < 0.$$

If $a + 2b \leq -1$, then by (1)-(i), we have

$$\lambda(0,-2) - 1 = -(a+2b)\alpha - \alpha\lambda(2,1) \geq \alpha(1 - \lambda(2,1)) > 0.$$

(2)-(i) : Recall that

$$2c\alpha \geq (c-b)\alpha > a\alpha > 1$$

because $a + b - c < 0$ and $\lfloor \beta \rfloor = a - 1$. Since $2b + c = -1$, we have

$$\lambda(1,2) = \lambda(1,0) - \lambda(0,-2) = \alpha(-1 + 2c\alpha) > 0.$$

So by Lemma 3.1-(3)-(i),

$$0 < \lambda(0,-1) < 2\lambda(0,-1) = \lambda(0,-2) < \lambda(1,0) < 1.$$

(2)-(ii) : Since $a + b - c \leq -1$, we have

$$\lambda(1,-1) \geq (a+1)\alpha - c\alpha^2 > a\alpha > 1.$$

Since $2a + b - 2c \geq 0$, we have

$$\lambda(2,-1) - 2 \leq 2a\alpha - c\alpha^2 - 2 = -\alpha\lambda(1,2) < 0$$

by (2)-(i). Hence by (2)-(i) and Lemma 3.1-(3)-(iii),

$$1 < \lambda(1,-1) < \lambda(1,-1) + \lambda(0,-1) = \lambda(1,-2) < \lambda(2,-1) < 2.$$

(2)-(iii) : By (2)-(i), $\lambda(1,2) < \lambda(1,2) + \lambda(0,-1) = \lambda(1,1)$. Since $2b + c = -1$, we have

$$\lambda(2,2) - 1 = (c-1)\alpha + 2c\alpha^2 - 1 = \alpha(c - a - 1 + \lambda(2,-1)) < (c - a + 1)\alpha \leq 0$$

by (2)-(ii). Hence

$$\lambda(1,2) < \lambda(1,1) < 2\lambda(1,1) = \lambda(2,2) < 1.$$

(2)-(iv) : By (2)-(i) and (iii), $\lambda(2,1) = \lambda(1,-1) + \lambda(1,2) > 1$. So by (2)-(i),

$$1 < \lambda(2,1) < \lambda(2,1) + \lambda(0,-1) = \lambda(2,0) = 2c\alpha < 2.$$

(3)-(i) : By Remark 4.1-(1),

$$\lambda(-2,-3) \geq 3\alpha - 3c\alpha^2 = 3\alpha(1 - c\alpha) > 0.$$

By Remark 4.1-(3),

$$\lambda(2,1) \leq (a-2)\alpha + c\alpha^2 < (a-1)\alpha < 1.$$

So by Lemma 3.1-(3)-(iii), we have

$$0 < \lambda(-2,-3) < \lambda(-1,-2) < \lambda(0,-1) < \lambda(1,0) < \lambda(2,1) < 1.$$

(3)-(ii) : Since $-a < -c \leq b < 0$, we have

$$\lambda(1,1) \geq c\alpha^2 > 0.$$

By Remark 4.1-(1),

$$\lambda(3,3) \leq (c-3)\alpha + 3c\alpha^2 < c\alpha < 1.$$

So by Lemma 3.1-(3)-(iii), we have $0 < \lambda(1,1) < \lambda(2,2) < \lambda(3,3) < 1$.

(3)-(iii) : By (3)-(i) and Remark 4.1-(5),

$$\lambda(2,-1) - 2 \geq 2a\alpha - c\alpha^2 - 2 = \alpha\lambda(-1,-2) > 0.$$

So by Lemma 3.1-(3)-(iii), we have

$$2 < \lambda(2,-1) < \lambda(3,0) = 3c\alpha < 3.$$

(3)-(iv) : Since $a - 2b - 3c \leq 2$, we have by (3)-(iii),

$$\lambda(3,2) - 1 \geq (a-2)\alpha + 2c\alpha^2 - 1 = \alpha(-2 + \lambda(2,-1)) > 0.$$

Since $2a + 2b - c \geq 0$, we have by Remark 4.1-(1),

$$\lambda(3,1) \leq (c - 2b - 3)\alpha + c\alpha^2 < (2a-3)\alpha + c\alpha^2 < 2(a-1)\alpha < 2.$$

So by (3)-(i) and Lemma 3.1-(3)-(iii),

$$1 < \lambda(3,2) = \lambda(2,0) - \lambda(-1,-2) < \lambda(2,0) < \lambda(3,1) < 2.$$

(3)-(v) : Since $a+3b+c \leq -2$, we have by (3)-(iv),

$$\lambda(-1,-3)-1 \geq (a+2)\alpha - 3c\alpha^2 - 1 = \alpha(2-\lambda(3,1)) > 0.$$

Since $2a+2b-c \geq 0$, we have by (3)-(ii),

$$\lambda(1,-2)-2 \leq 2a\alpha - 2c\alpha^2 - 2 = -\alpha\lambda(2,2) < 0.$$

Since $\lambda(0,-3) = \lambda(-1,-2) + \lambda(1,-1) > \lambda(1,-1)$ by (3)-(i), we have

$$1 < \lambda(-1,-3) < \lambda(0,-2) < \lambda(1,-1) < \lambda(0,-3) < \lambda(1,-2) < 2$$

by Lemma 3.1-(3)-(iii).

(3)-(vi) : By (3)-(v),

$$\lambda(1,-3) = \lambda(1,-1) + \lambda(0,-2) > 2.$$

By (3)-(v) and Remark 4.1-(4),

$$\lambda(3,-1)-3 \leq (3a-2)\alpha - c\alpha^2 - 3 = \alpha(-2+\lambda(-1,-3)) < 0.$$

So by Lemma 3.1-(3)-(iii), we have $2 < \lambda(1,-3) < \lambda(2,-2) < \lambda(3,-1) < 3$.

(4)-(i) : By Lemma 3.1-(1)-(i).

(4)-(ii) : By (4)-(i), $\lambda(-2,1) = \lambda(-1,1) - \lambda(1,0) < \lambda(-1,1)$. Hence, since $b-3c \geq 0$ and $b-c \leq a$ by Remark 3.1, we have

$$0 < (b-2c)\alpha + c\alpha^2 = \lambda(-2,1) < \lambda(-1,1) = (b-c)\alpha + c\alpha^2 < (a+1)\alpha < 1.$$

(4)-(iii) : By (4)-(i),

$$\lambda(-3,2) \geq b\alpha + 2c\alpha^2 > \lambda(0,1) > 1.$$

Hence by (4)-(ii), we have

$$1 < \lambda(0,1) + \lambda(-2,1) = \lambda(-3,2) < \lambda(-2,2) = 2\lambda(-1,1) < 2.$$

(4)-(iv) : Since $2a-2b+c \leq -2$, we have

$$\lambda(-1,2)-2 \geq (2a+2)\alpha + 2c\alpha^2 - 2 = \alpha(2-\lambda(-2,2)) > 0$$

by (4)-(iii). Since $3a-3b+2c \geq -1$ and $b-3c \geq 0$, we have

$$\lambda(-2,3)-3 \leq (3a+1)\alpha + 3c\alpha^2 - 3 = \alpha + (3c-3b)\alpha^2 - 3c\alpha^3 < \alpha(1-\lambda(0,2)) < 0.$$

Hence by (4)-(i) and (ii),

$$2 < \lambda(-1,2) < \lambda(-1,2) + \lambda(-2,1) = \lambda(-3,3) < \lambda(-3,3) + \lambda(1,0) < \lambda(-2,3) < 3.$$

□

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Table 1: (A1) with $b-2c=-1$ and $2a-b-c=-2$

η	θ	$\lfloor\lambda(\eta,\theta)\rfloor$
-2	2	1
-2	1	-1
-1	2	1
-1	1	0
-1	0	-1
0	1	1
0	-1	-2
1	1	2
1	0	0
1	-1	-1
1	-2	-2
2	-1	0
2	-2	-2

Table 2: (A1) with $b-2c=-1$ and $2a-b-c\geq-1$

η	θ	$\lfloor\lambda(\eta,\theta)\rfloor$
-2	2	1
-2	1	-1
-1	2	1
-1	1	0
-1	0	-1
-1	-1	-2
0	1	1
0	-1	-2
1	1	1
1	0	0
1	-1	-1
1	-2	-2
2	-1	0
2	-2	-2

Table 3: (A1) with $b-2c\geq 0$

η	θ	$\lfloor\lambda(\eta,\theta)\rfloor$
-2	2	1
-2	1	0
-1	2	1
-1	1	0
-1	0	-1
0	1	1
0	-1	-2
1	0	0
1	-1	-1
1	-2	-2
2	-1	-1
2	-2	-2

Table 4: (FS)

η	θ	$\lfloor\lambda(\eta,\theta)\rfloor$
-1	1	0
-1	0	-1
0	1	0
0	-1	-1
1	0	0
1	-1	-1

Table 5: (H_m)

η	θ	$\lfloor\lambda(\eta,\theta)\rfloor$
-1	1	-1
-1	0	-1
0	1	0
0	-1	-1
1	1	1
1	0	0
1	-1	0

Table 6: (H) with $b - c < 0$

η	θ	$\lfloor\lambda(\eta,\theta)\rfloor$
-1	1	-1
-1	0	-1
-1	-1	-1
0	1	0
0	-1	-1
1	1	0
1	0	0
1	-1	0

Table 7: (A2)

η	θ	$\lfloor\lambda(\eta,\theta)\rfloor$
-1	1	-1
-1	0	-1
-1	-1	-1
0	1	-1
0	-1	0
1	1	0
1	0	0
1	-1	0

Table 8: Proposition 4.3-(1) with $a + 2b \leq -1$

η	θ	$\lfloor \lambda(\eta, \theta) \rfloor$
-2	1	-2
-2	0	-2
-2	-1	-1
-2	-2	-1
-1	2	-2
-1	1	-2
-1	0	-1
-1	-1	-1
-1	-2	0
0	2	-2
0	1	-1
0	-1	0
0	-2	1
1	2	-1
1	1	0
1	0	0
1	-1	1
1	-2	1
2	2	0
2	1	0
2	0	1
2	-1	1

Table 9: Proposition 4.3-(1) with $a + 2b \geq 0$

η	θ	$\lfloor \lambda(\eta, \theta) \rfloor$
-2	1	-2
-2	0	-2
-2	-1	-1
-2	-2	-1
-1	2	-2
-1	1	-2
-1	0	-1
-1	-1	-1
-1	-2	0
0	2	-1
0	1	-1
0	-1	0
0	-2	0
1	2	-1
1	1	0
1	0	0
1	-1	1
1	-2	1
2	2	0
2	1	0
2	0	1
2	-1	1

Table 10: Proposition 4.3-(2)-(i) with $a+b-c<0$

η	θ	$\lfloor\lambda(\eta,\theta)\rfloor$
-2	1	-2
-2	0	-2
-2	-1	-2
-2	-2	-1
-1	2	-2
-1	1	-2
-1	0	-1
-1	-1	-1
-1	-2	-1
0	2	-1
0	1	-1
0	-1	0
0	-2	0
1	2	0
1	1	0
1	0	0
1	-1	1
1	-2	1
2	2	0
2	1	1
2	0	1
2	-1	1

Table 11: Proposition 4.3-(2)-(ii)

η	θ	$\lfloor\lambda(\eta,\theta)\rfloor$
-3	1	-3
-3	0	-3
-3	-1	-2
-3	-2	-2
-3	-3	-1
-2	2	-3
-2	1	-3
-2	0	-2
-2	-1	-1
-2	-2	-1
-2	-3	0
-1	3	-3
-1	2	-2
-1	1	-2
-1	0	-1
-1	-1	-1
-1	-2	0
-1	-3	1
0	3	-2
0	2	-2
0	1	-1
0	-1	0
0	-2	1
0	-3	1
1	3	-2
1	2	-1
1	1	0
1	0	0
1	-1	1
1	-2	1
1	-3	2
2	3	-1
2	2	0
2	1	0
2	0	1
2	-1	2
2	-2	2
3	3	0
3	2	1
3	1	1
3	0	2
3	-1	2

Table 12: Proposition 4.4

η	θ	$\lfloor \lambda(\eta, \theta) \rfloor$
-3	3	2
-3	2	1
-2	3	2
-2	2	1
-2	1	0
-1	2	2
-1	1	0
-1	0	-1
0	1	1
0	-1	-2
1	0	0
1	-1	-1
1	-2	-3
2	-1	-1
2	-2	-2
2	-3	-3
3	-2	-2
3	-3	-3

Figure 1: (A1) with $b-2c=-1$ and $2a-b-c=-2$

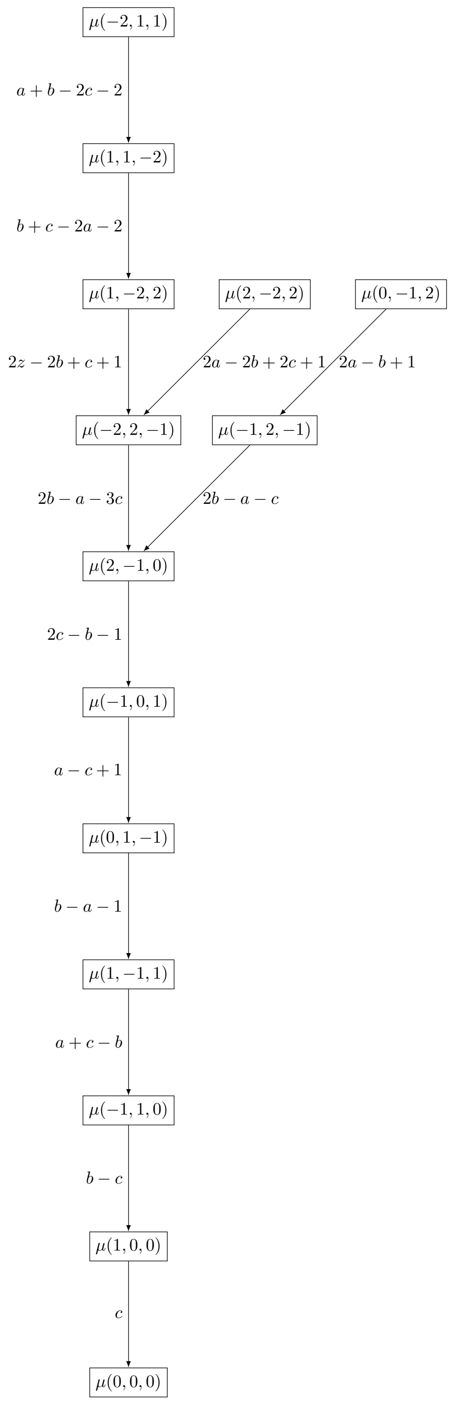

Figure 2: (A1) with $b - 2c = -1$ and $2a - b - c \geq -1$

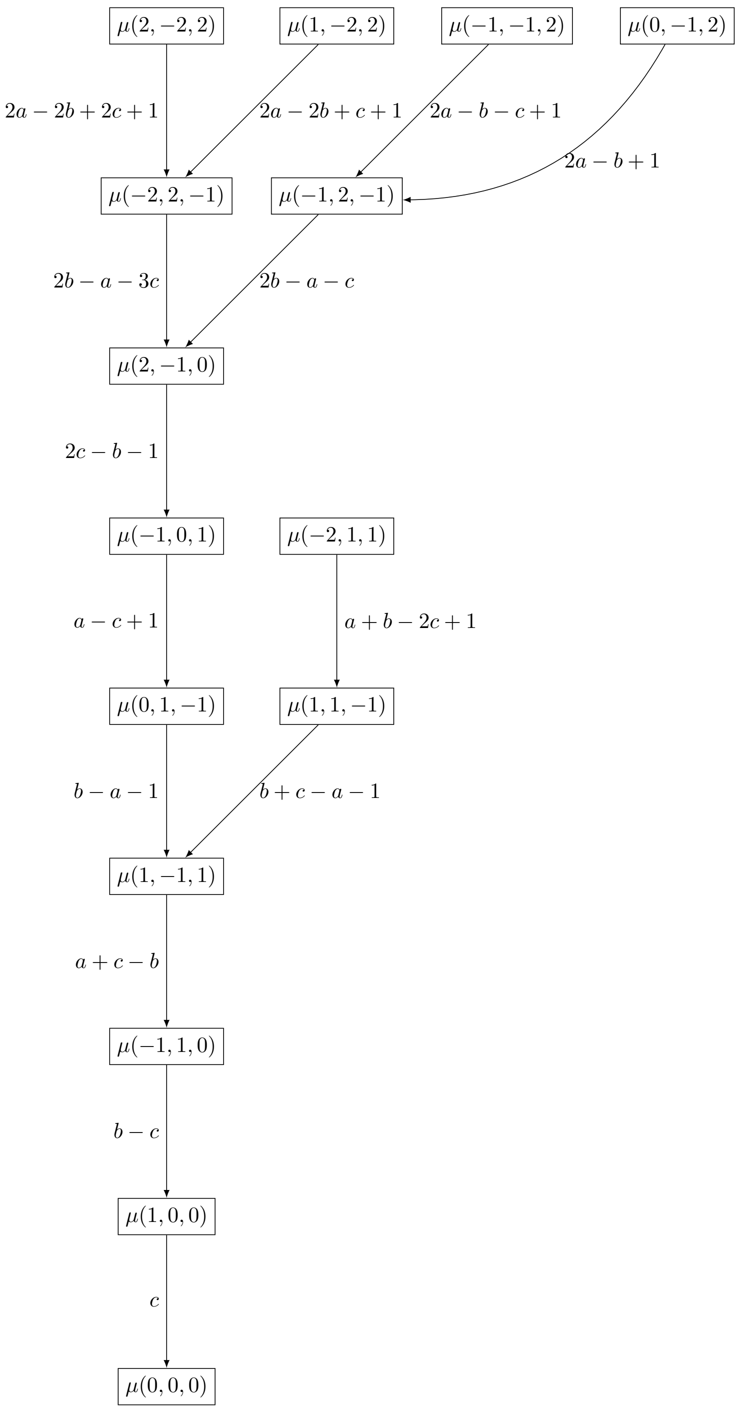

Figure 3: (A1) with $b - 2c \geq 0$

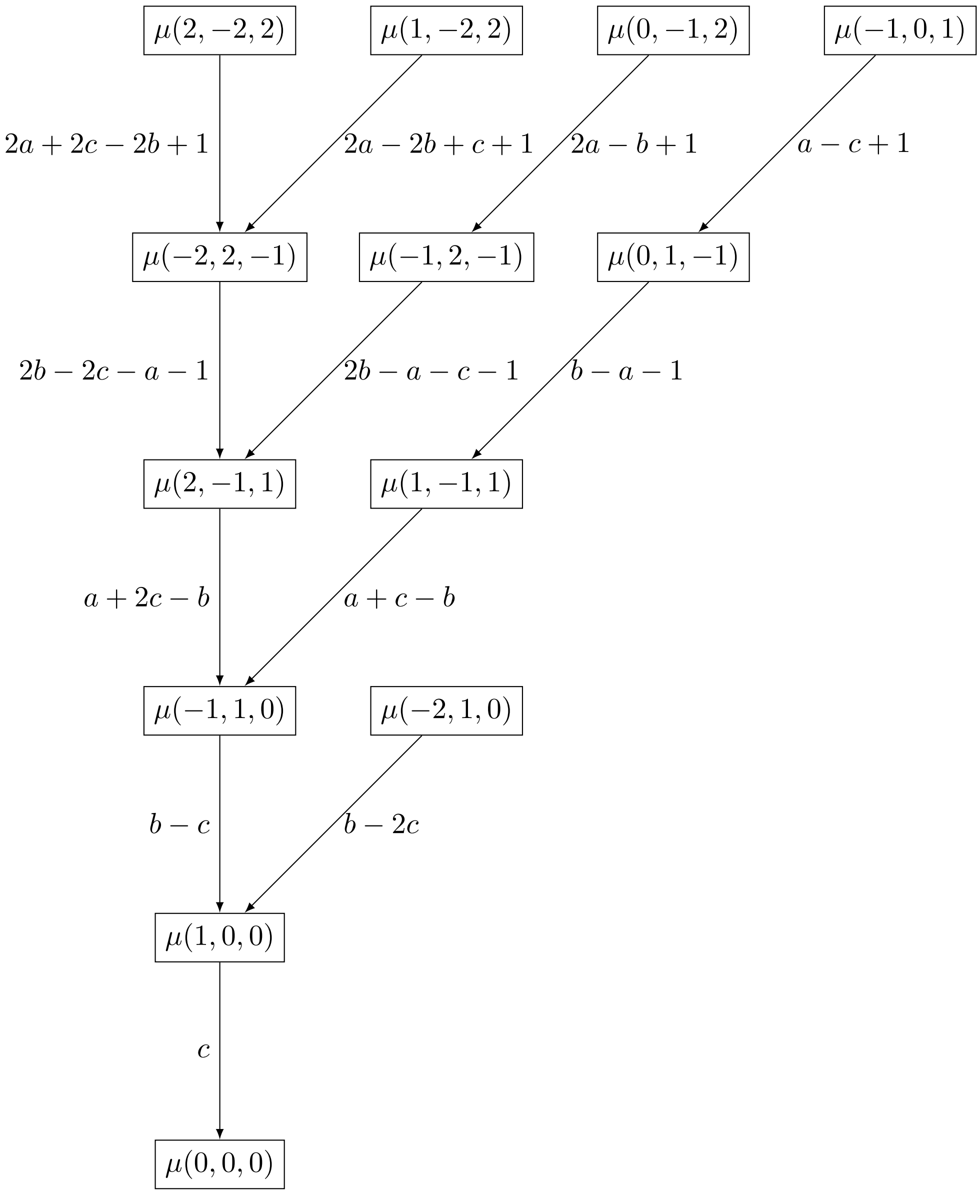

Figure 4: (FS)

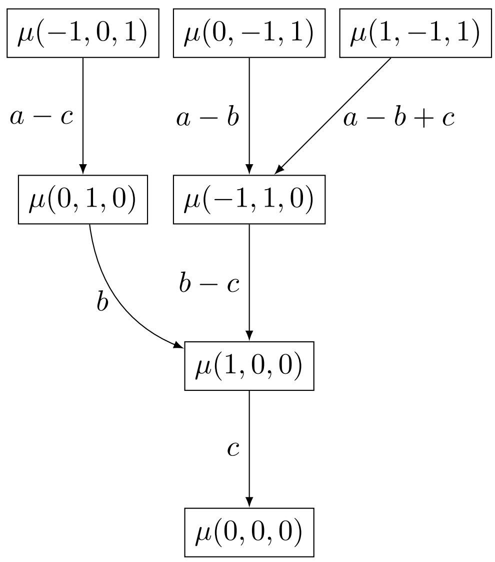

Figure 5: (H_m)

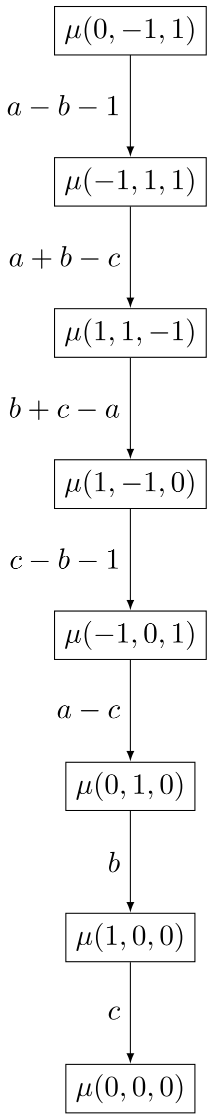

Figure 6: (H) with $b - c < 0$

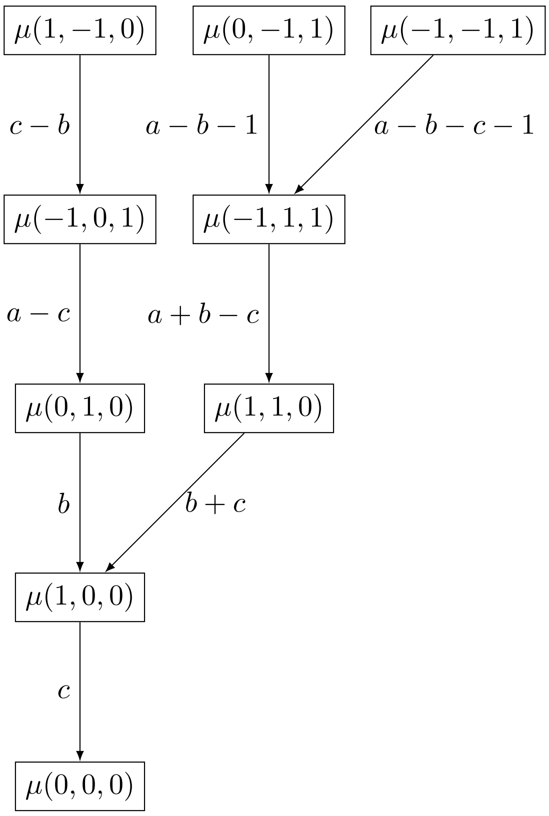

Figure 7: (A2)

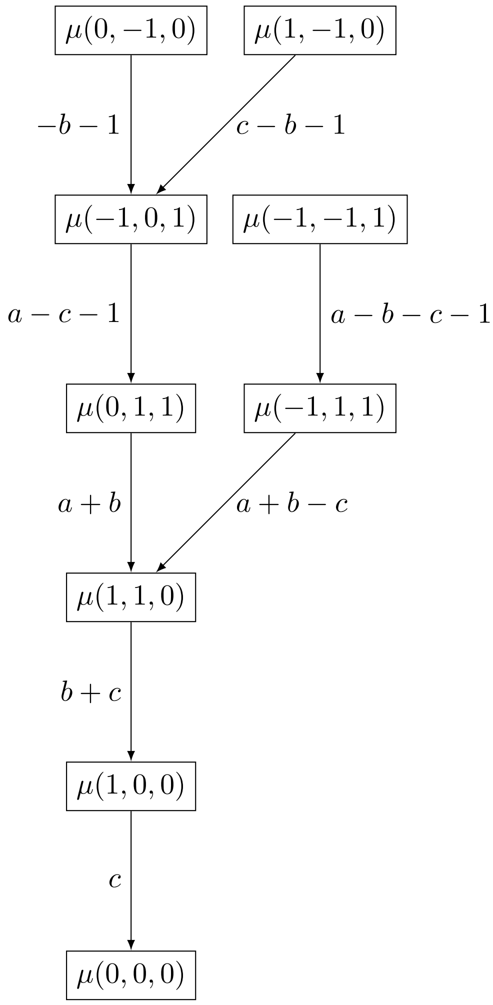

Figure 8: Proposition 4.3-(1) with $a + 2b \leq -1$

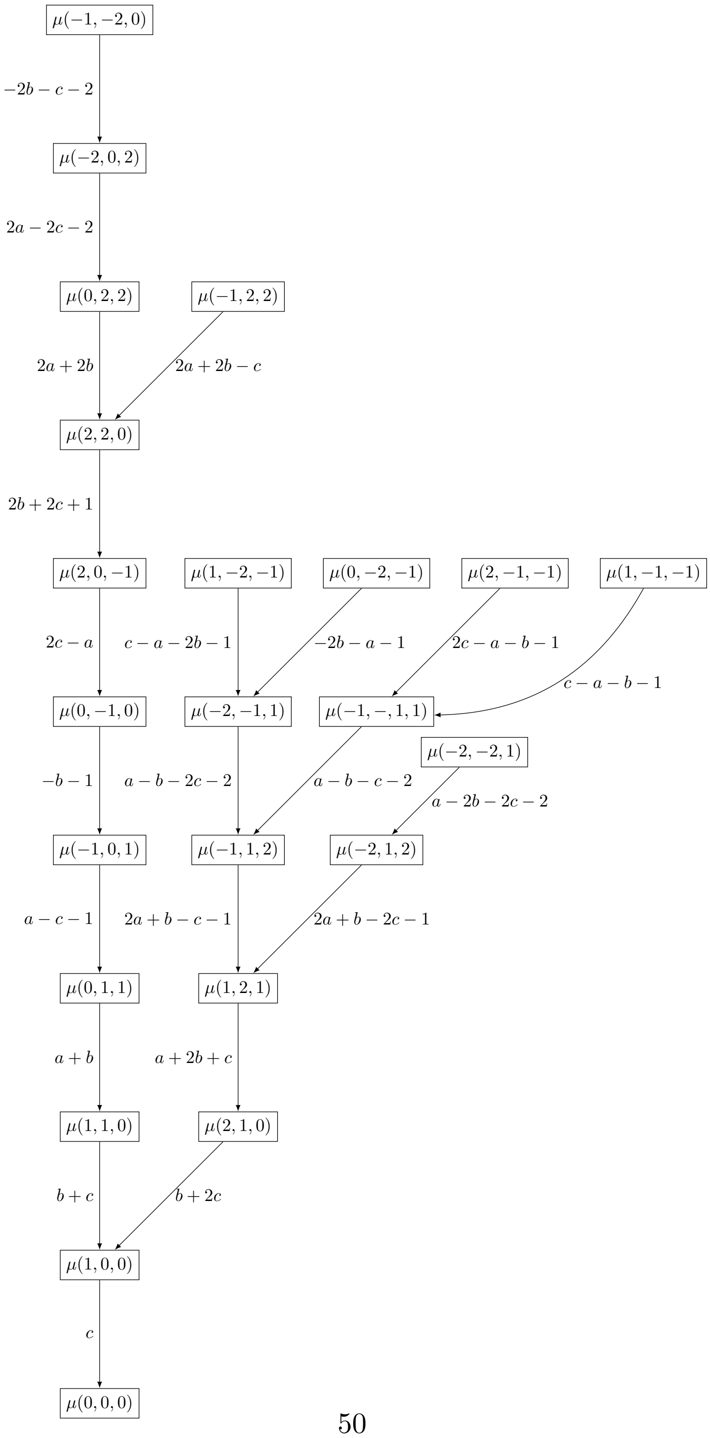

Figure 9: Proposition 4.3-(1) with $a + 2b \geq 0$

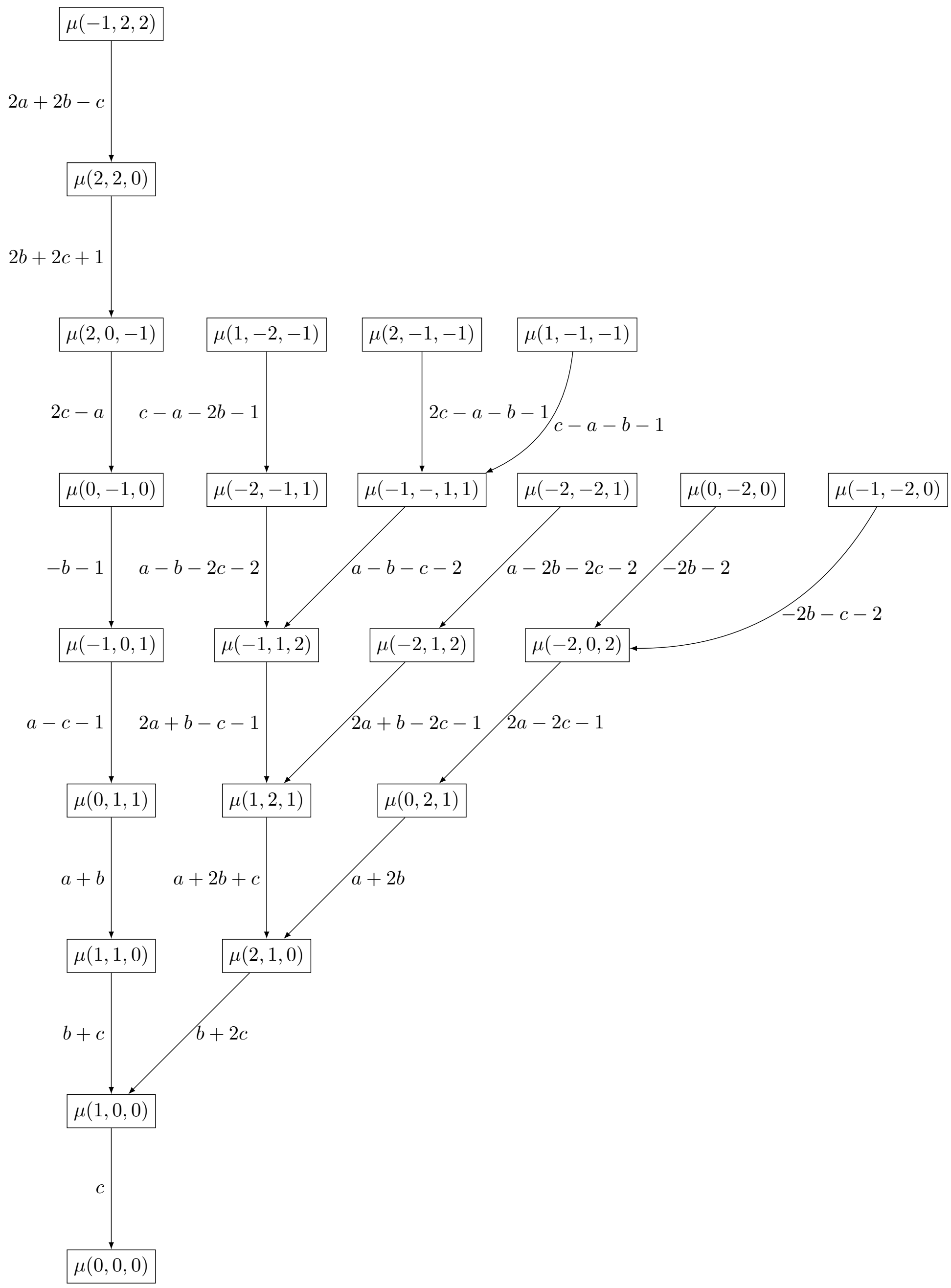

Figure 10: Proposition 4.3-(2)-(i) with $a + b - c < 0$

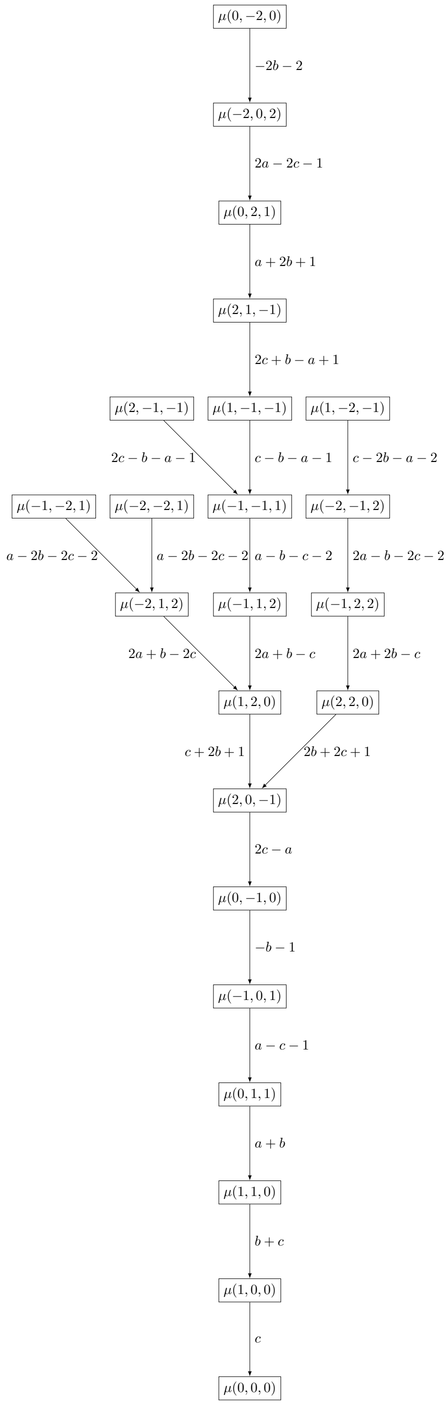

Figure 11: Proposition 4.3-(2)-(ii)

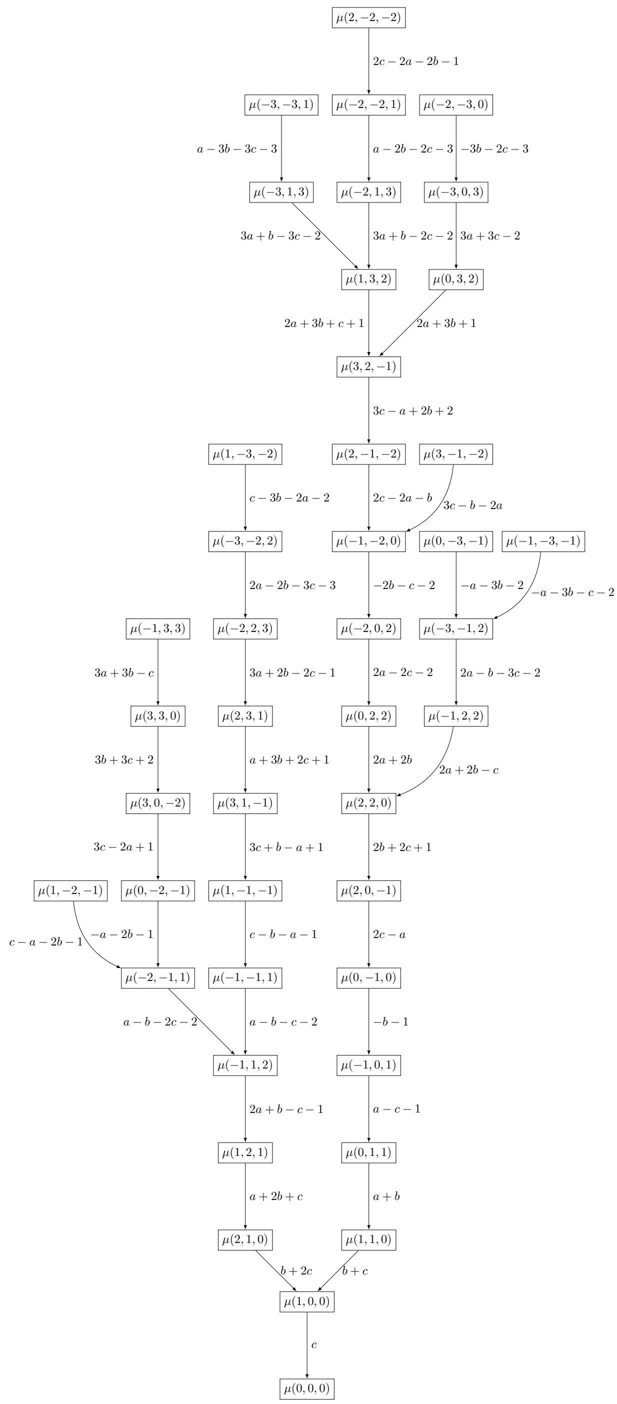

Figure 12: Proposition 4.4 with $2a - 2b + c \leq -2$