Cosmological Collider Signals of Non-Gaussianity from Higgs boson in GUT

(大統一理論におけるヒッグスボソンによる非ガウス性の宇宙論的兆候)

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Abstract

The Standard Model (SM) of elementary particles can explain various observed phenomena, and its correctness is no longer in doubt. On the other hand, it is also true that there are some observed facts that cannot be explained by the Standard Model. This indicates that there is a more fundamental theory beyond the SM. One of the candidates is the Grand Unified Theory (GUT), which unifies the three fundamental interactions: strong interaction, electromagnetic interaction, and weak interaction. According to the prediction of the renormalization group, the energy scale of GUT is about 10^{15-16} GeV. Since the energy scale of the LHC, which is currently the largest accelerator, is about 10^4 GeV, it is difficult to test the theory directly at such high energies with a terrestrial accelerator.

On the other hand, there is a natural phenomenon with an energy scale of about 10^{14} GeV: the inflation of the universe. Inflation is the exponential expansion of the universe in the early universe. Recent successes in precise observations of the universe have confirmed the existence of inflation. Therefore, we focus on the fact that the energy scales of inflation and GUT are very close, and search for evidence of GUT by observing their interaction.

In this thesis, we focus on the Higgs boson in GUT which is responsible for the gauge symmetry breaking from GUT gauge symmetry to the SM one, and the inflaton which is a scalar particle that causes inflation. As a model, we consider the action of gravity, the effective action of inflaton with slow-roll potential, the action of Higgs boson, the action of gauge bosons and fermions in GUT group. The most important feature of this model is that, considering a graph mediated by the Higgs bosons without vacuum expectation value (VEV) among three inflatons, there exists only loop graphs, but the Higgs boson has a non-vanishing VEV, we find the leading effect from tree graphs. By computing such three-point functions, we evaluate non-Gaussianity, which is an important observable quantity in inflation. The obtained non-Gaussianity is compared with the observed values, and the testability of GUT is discussed.

Contents

Chapter 1

Introduction

The Standard Model of elementary particles can explain a wide variety of elementary particle phenomena and has acquired reliability. On the other hand, there are certainly some phenomena that cannot be explained by the Standard Model, for example, the mechanism of generating tiny neutrino masses, the origin of dark matter and dark energy, and the hierarchy problem. To solve these problems, various theories beyond the Standard Model have been considered, such as extra dimensional scenario, supersymmetry, and Grand Unified Theory (GUT). GUT is the theory that unifies three of the four fundamental interactions that exist in nature: strong interaction, electromagnetic interaction, and weak interaction. The renormalization group method suggests that these three interactions are unified at a certain energy scale. Since the Standard Model is described by the Weinberg-Salam model unifying the electromagnetic and weak interactions and strong interaction separately, it is desirable that these interactions are unified. However, the energy scale of the GUT is expected to be about 10^{15} GeV, and it is difficult to verify directly such a high energy theory with terrestrial accelerators. For this reason, Cosmological Collider Physics has attracted much interests [1–94]. Cosmological Collider Physics is an approach that obtains information on high energy elementary particles by observing quantum fluctuations in space-time stretched by inflation through the cosmic microwave background radiation. That is, precise observation of the universe can provide information on elementary particles in high energy which cannot be reached by terrestrial accelerators.

Non-Gaussianity is the more than three point function of some quantum fluctuation in the curvature perturbations. Three point functions in models with only inflatons and gravitons were computed by Maldacena [95]. The effective field theory of inflation was proposed by C. Cheung, *et al.* [96], and its formalism has been used to calculate three-point functions in models with various particles. We focus on the case where the GUT scale is close to the energy scale of inflation and calculate three point function of inflaton by exchanging Higgs boson in GUT. The characteristic feature of this model is that the interaction between Higgs boson and inflaton generated by the (non-)vanishing Higgs boson vacuum expectation value (VEV) contributes to three point function of the inflaton at the (tree) 1-loop level as shown in Fig.1.1 and Fig.1.2. The results are found to be consistent with the current observational constraints on non-Gaussianity without drastic fine tuning of parameters, and it might be possible to detect the signature of the Higgs boson in GUT by 21cm spectrum, future LSS and future CMB depending on our model parameters.

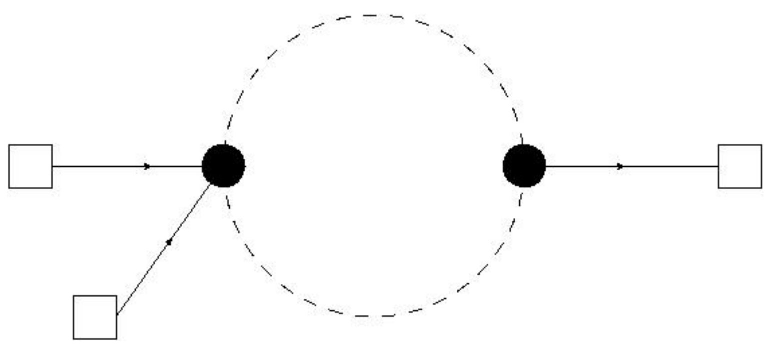


Figure 1.1: Leading graph of the inflaton three point function where the Higgs boson has no VEV inevitably becomes at one-loop level. The rigid line represents the inflaton, and the dotted line represents the Higgs boson in GUT. See Chapter 5 for notation.

This thesis is organized as follows. Chapters 1 to 5 are review parts, and chapters 6 to 8 are the main results of this thesis based on the work by N. Maru and myself [97]. First, we review the physics of inflation and the slow-roll condition. In Chapter 3, by solving the equation of the fluctuation generated by the inflation, we understand the behavior of the fluctuation and review its two-point function, the power spectrum. Subsequently, we introduce non-Gaussianity, an important observable quantity for inflation. In Chapter 4, we review the effective field theory of inflation. By viewing the inflaton as a Nambu-Goldstone boson of time translation, we see that it is possible to construct the effective action of inflation and review several advantages. In Chapter 5, we review the

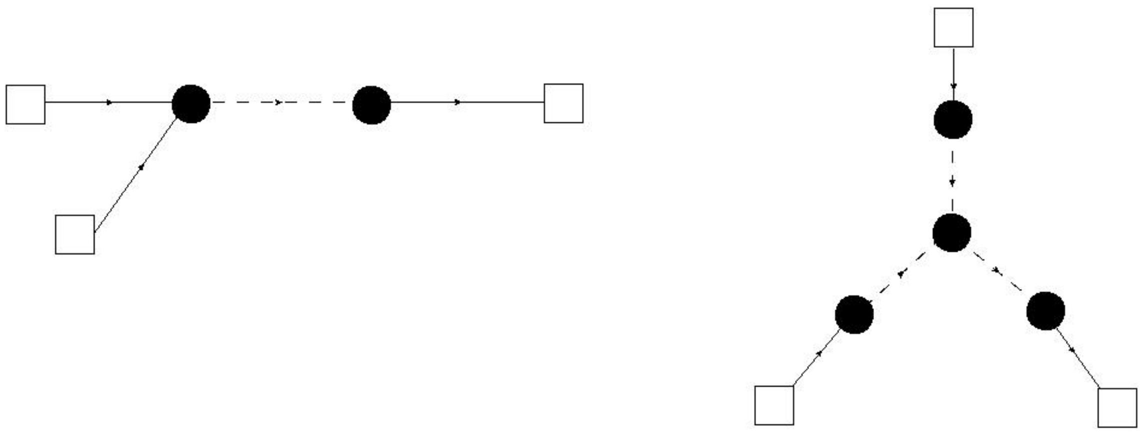


Figure 1.2: Leading graph of the inflaton three point function in the presence of spontaneous symmetry breaking. In the absence of spontaneous symmetry breaking, no such tree graph exists.

in-in formalism (Schwinger-Keldysh formalism) and Feynman rule, which are methods for computing physical quantities in field theories on curved spacetime. In Chapter 6, we actually set up our model. Furthermore, we introduce the Higgs potential in GUT and confirm that the Higgs boson interacts with the inflaton linearly after developing the vacuum expectation value due to the spontaneous symmetry breaking. Concretely, we compute three point function of the inflatons via the Higgs boson exchange at tree level by using the approximation in horizon exit. Non-Gaussianity is evaluated from the obtained three-point functions and the results are compared with the data observed by the Planck satellite. Conclusions are given in Chapter 8.

In Appendix A, we see what form of the perturbed metric in uniform isotropic spacetime is taken. In Appendix B, we consider the coordinate transformations as gauge transformations and investigate how the perturbed quantities in the metric are transformed. In Appendix C, we examine the transformation of matter appearing on the right-hand side of Einstein equations. In Appendix D, we combine these quantities to construct gauge invariants. Gauge fixing is also discussed. In Appendix E, we derive the Einstein equations for the perturbations to be satisfied. In Appendix F, we calculate how the determinant of the metric behaves when the coordinate transformations with respect to time are performed in the effective field theory.

Chapter 2

Inflation mechanism

In order to see what kind of physics the inflationary mechanism is, consider the following action

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm pl}^2}{2}R + P(\phi, X)\right], \tag{2.1}$$

where

$$X = -\frac{1}{2}g^{\mu\nu}\left(\partial_\mu\phi\right)\partial_\nu\phi \tag{2.2}$$

is the kinetic energy of a scalar field, and this scalar field is called the inflaton. The first term on the right-hand side is the Einstein-Hilbelt action, where $M_{\rm pl} = m_{\rm pl}/\sqrt{8\pi}$ is the reduced Planck mass and R is the Ricci scalar. The second term on the right-hand side, P, is Lagrangian for inflaton ϕ and

$$P = X - V(\phi) \tag{2.3}$$

for inflaton with canonical kinetic energy X. $V(\phi)$ is the potential energy of the inflaton ϕ. The energy density of the inflaton is given by the equation (C.53),

$$\rho = 2XP_{,X} - P \tag{2.4}$$

and the pressure is given by P. Since the curvature term can be regarded as zero once inflation starts, the Friedmann equation and the continuity equation in the Friedmann-Lemaître-Robertson-Walker (FLRW) metric are

$$3M_{\rm pl}^2H^2 = 2XP_{,X} - P, \tag{2.5}$$

$$M_{\rm pl}^2\dot{H} = -XP_{,X}, \tag{2.6}$$

$$(P_{,X} + 2XP_{,XX})\ddot{\phi} + 3HP_{,X}\dot{\phi} + 2XP_{,X\phi} - P_{,\phi} = 0. \tag{2.7}$$

The pressure P, energy density ρ, and inflaton ϕ are all functions of time t only, and the dot represents the derivative with respect to t.

Since the time derivative of the Hubble parameter H is

$$\dot{H} = \frac{d}{dt}\frac{\dot{a}}{a} = -\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}, \tag{2.8}$$

the quantity $\ddot{a}/a$ can be written as

$$\frac{\ddot{a}}{a} = \frac{\dot{a}^2}{a^2} + \dot{H} = H^2 \left\{ 1 - \left(-\frac{\dot{H}}{H^2} \right) \right\}. \tag{2.9}$$

We now define a quantity called the slow-roll parameter:

$$\epsilon := -\frac{\dot{H}}{H^2} = \frac{3XP_{,X}}{2XP_{,X} - P} = \frac{3XP_{,X}}{\rho}. \tag{2.10}$$

Using the slow-roll parameter ϵ, the quantity $\ddot{a}/a$ becomes

$$\frac{\ddot{a}}{a} = H^2(1-\epsilon). \tag{2.11}$$

Therefore, the slow-roll parameter ϵ must be smaller than 1 in order to satisfy the condition

$$\ddot{a} > 0 \tag{2.12}$$

that the universe accelerates and expands. Then, we define inflation as the condition

$$|\epsilon| \ll 1 \tag{2.13}$$

that the magnitude of ϵ is sufficiently smaller than 1. Using equation (2.10), the condition for inflation to occur is

$$|3XP_{,X}| \ll \rho. \tag{2.14}$$

For a canonical scalar field for which the Lagrangian is $P = X - V(\phi)$, the derivative with respect to X of the Lagrangian and energy density are

$$P_{,X} = 1, \quad \rho = 2X - (X - V) = X + V, \tag{2.15}$$

respectively, so that the condition follows:

$$X \ll V(\phi). \tag{2.16}$$

Since this implies that inflation will occur if the field moves slowly along the potential, such a situation is called slow-roll inflation. In the following, we will consider for a while the inflation induced by the potential energy of this canonical scalar field.

In the case of a canonical scalar field, the Friedmann equation (2.5) and the continuity equation (2.7) are

$$3M_{\rm pl}^2 H^2 = X + V(\phi), \tag{2.17}$$

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi}(\phi) = 0 \tag{2.18}$$

respectively. In order for the kinetic energy of the inflaton ϕ not to be large, the condition

$$\left|\ddot{\phi}\right| \ll \left|3H\dot{\phi}\right|,\ |V_{,\phi}(\phi)| \tag{2.19}$$

must be hold that the acceleration is small. In this case, the equation (2.18) becomes

$$3H\dot{\phi} \simeq -V_{,\phi}(\phi). \tag{2.20}$$

Approximation using this condition and the condition

$$X = \frac{1}{2}\dot{\phi}^2 \ll V(\phi) \tag{2.21}$$

that the kinetic energy is sufficiently smaller than the potential energy is called the slow-roll approximation.

It is convenient to define the following potential slow-roll parameter

$$\epsilon_V := \frac{M_{\rm pl}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2, \quad \eta_V := \frac{M_{\rm pl}^2 V_{,\phi\phi}}{V}, \quad \xi_V^2 := \frac{M_{\rm pl}^4 V_{,\phi} V_{,\phi\phi\phi}}{V^2}, \tag{2.22}$$

which is related to the flatness of the potential. Under the slow-roll approximation, they can be written as

$$\epsilon_V \simeq \epsilon, \quad \eta_V \simeq 2\epsilon - \frac{\dot{\epsilon}}{2H\epsilon}, \quad \xi_V^2 \simeq \left(2\epsilon_V - \frac{\dot{\eta}_V}{H\eta_V}\right)\eta_V \tag{2.23}$$

with ϵ defined by the equation (2.10). ξ_V^2 is a second order quantity.

We show that the potential slow-roll parameter can be written as in equation (2.23). Using the Friedmann equation and the continuous equation

$$3M_{\rm pl}^2H^2 \simeq V(\phi), \tag{2.24}$$
$$M_{\rm pl}^2\dot{H} = -X, \tag{2.25}$$
$$3H\dot{\phi} \simeq -V_{,\phi}(\phi), \tag{2.26}$$

ϵ is written as

$$\begin{aligned} \epsilon = \frac{3XP_{,X}}{\rho} &= \frac{3\dot{\phi}^2}{2V} \\ &\simeq \frac{3}{2V}\left(\frac{V_{,\phi}}{3H}\right)^2 \\ &= \frac{V_{,\phi}^2}{6VH^2} \\ &= \frac{3M_{\rm pl}^2V_{,\phi}^2}{6V^2} \\ &= \frac{M_{\rm pl}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \\ &= \epsilon_V \end{aligned} \tag{2.27}$$

from equation (2.10). Next, consider η_V. In order to find the second-order derivative of the potential, we perform the time derivative of ϵ, which is

$$\begin{aligned} \dot{\epsilon} \simeq \dot{\epsilon}_V &= M_{\rm pl}^2\frac{V_{,\phi}}{V}\frac{d}{dt}\frac{V_{,\phi}}{V} \\ &= M_{\rm pl}^2\frac{V_{,\phi}}{V}\left(-\frac{V_{,\phi}^2}{V^2}\dot{\phi} + \frac{V_{,\phi\phi}}{V}\dot{\phi}\right), \end{aligned} \tag{2.28}$$

then, we can write

$$V_{,\phi\phi} = \frac{V^2}{M_{\rm pl}^2V_{,\phi}\dot{\phi}}\dot{\epsilon} + \frac{V_{,\phi}^2}{V}. \tag{2.29}$$

Thus,

$$\begin{aligned}
\eta_V = \frac{M_{\rm pl}^2 V_{,\phi\phi}}{V} &= 2\frac{M_{\rm pl}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \frac{V V_{,\phi\phi}}{V_{,\phi}^2} \\
&\simeq 2\epsilon \frac{V V_{,\phi\phi}}{V_{,\phi}^2} \\
&= 2\epsilon \frac{V}{V_{,\phi}^2}\left(\frac{V^2}{M_{\rm pl}^2 V_{,\phi}\dot{\phi}}\dot{\epsilon} + \frac{V_{,\phi}^2}{V}\right) \\
&= 2\epsilon + M_{\rm pl}^2\left(\frac{V_{,\phi}}{V}\right)^2 \frac{V^3\dot{\epsilon}}{M_{\rm pl}^2 V_{,\phi}^3\dot{\phi}} \qquad \left(\epsilon \simeq \frac{M_{\rm pl}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2\right) \\
&= 2\epsilon + \frac{V\dot{\epsilon}}{V_{,\phi}\dot{\phi}} \\
&= 2\epsilon - \frac{V\dot{\epsilon}}{3H\dot{\phi}^2} \\
&= 2\epsilon - \frac{\dot{\epsilon}}{2H\epsilon} \qquad (2.30)
\end{aligned}$$

is obtained. In the last equality,

$$\epsilon = \frac{3\dot{\phi}^2}{2V} \qquad (2.31)$$

is used. Finally, consider ξ_V^2. In order to obtain the third-order derivative of the potential, we perform a time derivative on η_V, which yields

$$\begin{aligned}
\dot{\eta}_V &= -\frac{M_{\rm pl}^2 V_{,\phi} V_{,\phi\phi}}{V^2}\dot{\phi} + \frac{M_{\rm pl}^2 V_{,\phi\phi\phi}}{V}\dot{\phi} \\
&= -\frac{V_{,\phi}}{V}\dot{\phi}\eta_V + \frac{V\dot{\phi}}{M_{\rm pl}^2 V_{,\phi}}\frac{M_{\rm pl}^4 V_{,\phi}V_{,\phi\phi\phi}}{V^2} \\
&\simeq -\frac{V_{,\phi}}{V}\left(-\frac{V_{,\phi}}{3H}\right)\eta_V + \frac{V}{M_{\rm pl}^2 V_{,\phi}}\left(-\frac{V_{,\phi}}{3H}\right)\xi_V^2 \\
&= H\frac{V_{,\phi}^2}{3VH^2}\eta_V - H\frac{V}{3M_{\rm pl}^2 H^2}\xi_V^2 \\
&= HM_{\rm pl}^2\left(\frac{V_{,\phi}}{V}\right)^2\eta_V - H\xi_V^2 \\
&= 2H\epsilon_V\eta_V - H\xi_V^2. \qquad (2.32)
\end{aligned}$$

Therefore,

$$\xi_V^2 \simeq \left(2\epsilon_V - \frac{\dot{\eta}_V}{H\eta_V}\right)\eta_V \qquad (2.33)$$

is derived.□

With these parameters, the condition for inflation to occur is

$$|\epsilon_V|,\ |\eta_V| \ll 1, \tag{2.34}$$

and inflation ends when these quantities reach a magnitude of about 1.

Chapter 3

Two-point correlation function of inflaton

Consider linear density fluctuations in the action (2.1)

$$S=\int d^4x\sqrt{-g}\left[\frac{M_{\rm pl}^2}{2}R+P(\phi,X)\right]. \tag{3.1}$$

In this section, we will take the comoving gauge (uniform field gauge)

$$\delta\phi=0,\quad E=0 \tag{3.2}$$

and proceed with the discussion. The physical content is the same even if other gauges are adopted. See Appendices A to E for discussions and notations of gauges and perturbations. Under the comoving gauge, the gauge invariant, the comoving curvature fluctuation ζ, is equal to the scalar perturbation ψ. In the following, we consider the flat universe $K=0$. Vector perturbations are not considered because they do not increase. In other words, we consider

$$ds^2=a^2(\eta)\left[-(1+2A)d\eta^2+\{(1+2\psi)\gamma_{ij}+h_{ij}\}\,dx^idx^j\right] \tag{3.3}$$

as a metric. The four equations for scalar perturbations A,B,E, and ψ were (E.74), (E.75), (E.76), (E.77)

$$\begin{aligned}&3\mathcal{H}\psi'-[3\mathcal{H}^2-4\pi G\phi'^2\left(P_{,X}+2XP_{,XX}\right)]A-\left(\nabla^2+3K\right)\psi+\mathcal{H}\nabla^2\sigma\\&\quad=-4\pi G\left[a^2\left(P_{,\phi}-2XP_{,X\phi}\right)\delta\phi-\left(P_{,X}+2XP_{,XX}\right)\phi'\delta\phi'\right],\end{aligned} \tag{3.4}$$

$$\psi' - \mathcal{H}A - K\sigma = -4\pi G P_{,X}\,\phi'\delta\phi, \tag{3.5}$$

$$\sigma' + 2\mathcal{H}\sigma - A - \psi = 0, \tag{3.6}$$

$$\psi'' + 2\mathcal{H}\psi' - K\psi - \mathcal{H}A' - \left(2\mathcal{H}^2 + \mathcal{H}' + K\right)A = -4\pi G\left(P_{,X}\phi'\delta\phi' + a^2 P_{,\phi}\delta\phi\right) \tag{3.7}$$

respectively, where σ is

$$\sigma = E' - B = -B \tag{3.8}$$

because we adopted the comoving gauge. Let us eliminate A, σ from the above simultaneous equations and derive the equations of motion for the comoving curvature fluctuation $\psi = \zeta$. First, from the equation (3.5), we obtain

$$A = \frac{\zeta'}{\mathcal{H}}. \tag{3.9}$$

Substituting this into the left-hand side of the equation (3.7), we see that this equation is automatically satisfied. This is because the gauge fixing reduces the number of degrees of freedom, and thus the number of equations to be solved is also reduced. Let us define a quantity

$$Q_s := \frac{\phi'^2}{2\mathcal{H}^2}\left(P_{,X} + 2XP_{,XX}\right) \tag{3.10}$$

to represent the time evolution of the inflaton ϕ due to the expansion of the universe. Rewriting the left-hand side of the equation (3.4) by using Q_s, we obtain

$$\begin{aligned}
\text{LHS of (3.4)} &= 3\mathcal{H}\zeta' - 2\mathcal{H}^2\left(\frac{3}{2} - 4\pi G Q_s\right)\frac{\zeta'}{\mathcal{H}} - \nabla^2\zeta + \mathcal{H}\nabla^2\sigma \\
&= \mathcal{H}\left(8\pi G Q_s\zeta' - \frac{1}{\mathcal{H}}\nabla^2\zeta + \nabla^2\sigma\right) \\
&= \mathcal{H}\left(\frac{Q_s}{M_{\rm pl}^2}\zeta' - \frac{1}{\mathcal{H}}\nabla^2\zeta + \nabla^2\sigma\right).
\end{aligned} \tag{3.11}$$

Hence, we can obtain the equation

$$\nabla^2\sigma = -\frac{Q_s}{M_{\rm pl}^2}\zeta' + \frac{1}{\mathcal{H}}\nabla^2\zeta \tag{3.12}$$

for σ. Next, using equation (3.12), we eliminate σ from the equation (3.6). Rewriting σ, A, ψ using ζ by acting ∇^2 on both sides, we obtain

$$\left(-\frac{Q_s}{M_{\rm pl}^2}\zeta' + \frac{1}{\mathcal{H}}\nabla^2\zeta\right)' + 2\mathcal{H}\left(-\frac{Q_s}{M_{\rm pl}^2}\zeta' + \frac{1}{\mathcal{H}}\nabla^2\zeta\right) - \frac{1}{\mathcal{H}}\nabla^2\zeta' - \nabla^2\zeta = 0. \tag{3.13}$$

Performing the derivative of the second term, we can compute

$$\left(-\frac{Q_s}{M_{\rm pl}^2}\zeta'\right)' - \frac{\mathcal{H}'}{\mathcal{H}^2}\nabla^2\zeta - \frac{2\mathcal{H}Q_s}{M_{\rm pl}^2}\zeta' + \nabla^2\zeta = 0. \tag{3.14}$$

Multiplying both sides by $a^2 M_{\rm pl}^2$ and noting $\mathcal{H} = a'/a$ gives

$$a^2\left(Q_s\zeta'\right)' + 2aa'Q_s\zeta' - a^2 M_{\rm pl}^2\left(1 - \frac{\mathcal{H}'}{\mathcal{H}^2}\right)\nabla^2\zeta = 0. \tag{3.15}$$

Furthermore, the first and second terms can be combined into a single term using differentiation, which gives

$$\left(a^2 Q_s\zeta'\right)' - a^2 M_{\rm pl}^2\left(1 - \frac{\mathcal{H}'}{\mathcal{H}^2}\right)\nabla^2\zeta = 0. \tag{3.16}$$

Let us write

$$M_{\rm pl}^2\left(1 - \frac{\mathcal{H}'}{\mathcal{H}^2}\right) \equiv Q_s c_s^2 \tag{3.17}$$

to match the coefficients of the time and spatial derivative terms[1]. Then we arrive at the equation of motion

$$\left(a^2 Q_s\zeta'\right)' - a^2 Q_s c_s^2\nabla^2\zeta = 0 \tag{3.18}$$

for the comoving curvature fluctuation ζ. From the form of equation (3.18), we can see that c_s represents the propagation speed of the inflaton. From this fact, c_s is called the sound speed. The concrete form of sound speed c_s expressed by kinetic energy X and Lagrangian P is

$$\begin{aligned}
c_s^2 = \frac{M_{\rm pl}^2}{Q_s}\left(1 - \frac{\mathcal{H}'}{\mathcal{H}^2}\right) &= \frac{2\mathcal{H}^2 M_{\rm pl}^2}{\phi'^2\left(P_{,X} + 2XP_{,XX}\right)}\left(1 - \frac{\mathcal{H}'}{\mathcal{H}^2}\right) \\
&= \frac{2M_{\rm pl}^2}{\phi'^2\left(P_{,X} + 2XP_{,XX}\right)} \cdot \frac{a^2}{2M_{\rm pl}^2}(\rho + P) \quad \text{(using Friedmann eq.)} \\
&= \frac{a^2(\rho + P)}{\phi'^2\left(P_{,X} + 2XP_{,XX}\right)} \\
&= \frac{2a^2 X P_{,X}}{\phi'^2\left(P_{,X} + 2XP_{,XX}\right)} \quad \left(\text{energy density } \rho = 2XP_{,X} - P\right) \\
&= \frac{P_{,X}}{P_{,X} + 2XP_{,XX}} \qquad \left(\text{kinetic energy } X = \frac{1}{2a^2}\phi'^2\right)
\end{aligned} \tag{3.19}$$

[1] Since c_s is a dimensionless quantity, it is a quantity of $\mathcal{O}(1)$. In fact, $c_s = 1$ in the canonical model, as can be immediately seen from the expression (3.19) below.

from the definition.

The equation of motion (3.18) can be also derived from the following variation of the action

$$S_2 = \int d\eta d^3x\, a^2 Q_s \left[\zeta'^2 - c_s^2 (\partial\zeta)^2\right] \tag{3.20}$$

for ζ, where η is confomal time and $(\partial\zeta)^2$ means

$$(\partial\zeta)^2 := (\partial_i \zeta)\, \partial^i \zeta. \tag{3.21}$$

As we will show later, by expanding the action (2.1)

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{\rm pl}^2}{2} R + P(\phi, X)\right] \tag{3.22}$$

to the second order terms of the perturbation, we obtain the action (3.20).

The comoving curvature fluctuations ζ originate from quantum fluctuations at wavelengths smaller than the Hubble radius (k is the momentum, the wavelength is its inverse)

$$k \gg aH \qquad \left(\frac{1}{aH} \gg \frac{1}{k}\right) \tag{3.23}$$

at the early stage of inflation. Then, when the physical wavelength of the fluctuation becomes almost the same as the Hubble radius

$$k \sim aH, \tag{3.24}$$

the effect of gravity starts to become significant and the fluctuation behaves as a classical fluctuation. Finally, comoving curvature fluctuation ζ stops growing as

$$k \ll aH \tag{3.25}$$

and this is called frozen. To understand the process, we expand the comoving curvature fluctuation ζ into

$$\zeta(\eta, \boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3} \zeta(\eta, \boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{3.26}$$

$$\zeta(\eta, \boldsymbol{k}) = u(\eta, \boldsymbol{k}) a(\boldsymbol{k}) + u^*(\eta, -\boldsymbol{k}) a^\dagger(-\boldsymbol{k}) \tag{3.27}$$

in the Fourier space of wavenumber $\boldsymbol{k}$ where $a^\dagger(\boldsymbol{k})$ and $a(\boldsymbol{k})$ are the creation and annihilation operators and satisfy the commutation relation

$$\left[a(\boldsymbol{k}_1), a^\dagger(\boldsymbol{k}_2)\right] = (2\pi)^3 \delta^{(3)}(\boldsymbol{k}_1 - \boldsymbol{k}_2), \tag{3.28}$$

$$\left[a(\boldsymbol{k}_1), a(\boldsymbol{k}_2)\right] = \left[a^\dagger(\boldsymbol{k}_1), a^\dagger(\boldsymbol{k}_2)\right] = 0. \tag{3.29}$$

The conformal time η is

$$\eta = \int \frac{dt}{a} = -\frac{1}{aH} + \int \epsilon \frac{da}{a^2 H}. \tag{3.30}$$

We show that the conformal time η can be expressed as in equation (3.30):

$$\begin{aligned}
\eta = \int \frac{dt}{a} = \int \frac{da}{a\dot{a}} &= \int \frac{da}{a^2 H} \\
&= \int \left(-\frac{d}{da}\frac{1}{a}\right)\frac{1}{H} da \\
&= -\frac{1}{aH} + \int \frac{1}{a}\frac{d}{da}\frac{1}{H} da \\
&= -\frac{1}{aH} - \int \frac{1}{aH^2}\frac{dH}{da} da \\
&= -\frac{1}{aH} - \int \frac{1}{aH^2}\frac{dH}{dt}\frac{dt}{da} da \\
&= -\frac{1}{aH} - \int \frac{\dot{H}}{a\dot{a}H^2} da \\
&= -\frac{1}{aH} + \int \epsilon \frac{da}{a^2 H}. \quad \square
\end{aligned}$$

When the slow-roll parameter ϵ is sufficiently small and constant compared to 1, H is also constant and the conformal time becomes

$$\eta \simeq -\frac{1+\epsilon}{aH}. \tag{3.31}$$

The asymptotic past $a \to 0$ and asymptotic future $a \to \infty$ correspond to $\eta \to -\infty$ and $\eta \to -0$, respectively. From the equation of motion (3.18)

$$\left(a^2 Q_s \zeta'\right)' - a^2 Q_s c_s^2 \nabla^2 \zeta = 0 \tag{3.32}$$

and the Fourier expansion (3.27)

$$\begin{aligned}
\zeta(\eta, \boldsymbol{x}) &= \int \frac{d^3k}{(2\pi)^3} \zeta(\eta, \boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \\
\zeta(\eta, \boldsymbol{k}) &= u(\eta, \boldsymbol{k}) a(\boldsymbol{k}) + u^*(\eta, -\boldsymbol{k}) a^\dagger(-\boldsymbol{k}),
\end{aligned}$$

we obtain an equation satisfied by the Fourier mode u with wavenumber magnitude $k = |\boldsymbol{k}|$: Since it is

$$\left(a^2 Q_s u'\right)' + a^2 Q_s c_s^2 k^2 u = 0 \tag{3.33}$$

from the equation of motion (3.18), we obtain

$$a^2Q_su'' + (a^2Q_s)'u' + a^2Q_sc_s^2k^2u = 0 \tag{3.34}$$

by performing differentiation. Dividing both sides by a^2Q_s, we find that the equation satisfied for u is

$$u'' + \frac{(a^2Q_s)'}{a^2Q_s}u' + c_s^2k^2u = 0. \tag{3.35}$$

In the large scale limit $k \to 0$, the solution of this equation can be expressed as

$$u = c_1 + c_2\int d\eta\frac{1}{a^2Q_s} \quad (c_1, c_2 \text{ are integral constants})\,. \tag{3.36}$$

We show that solving the equation of motion (3.35) in the large scale limit $k \to 0$, the Fourier mode u can be expressed as in equation (3.36). Writing

$$u' \equiv U, \quad a^2Q_s \equiv A \tag{3.37}$$

for simplicity, the equation (3.35)

$$u'' + \frac{(a^2Q_s)'}{a^2Q_s}u' + c_s^2k^2u = 0 \tag{3.38}$$

in the large scale limit $k \to 0$ can be solved as follows:

$$\begin{aligned}
U' &= -\frac{A'}{A}U \\
\frac{U'}{U} &= -\frac{A'}{A} \\
\log U &= -\log A + c_3 \\
U &= \frac{c_2}{A} \quad (c_2 = e^{c_3}) \\
u' &= \frac{c_2}{A} \\
u &= c_1 + c_2\int d\eta\frac{1}{A} \\
u &= c_1 + c_2\int d\eta\frac{1}{a^2Q_s}. \quad \square
\end{aligned}$$

If Q_s changes slowly during inflation, the second term of the solution (3.36) decays and the mode function u approaches to a constant value c_1. On the other hand, in the small-scale limit $k \to \infty$, the third term on the left-hand side of the equation (3.35) becomes more dominant than the second term. In order to perform quantization for fluctuations sufficiently inside the Hubble radius at the beginning of inflation, it is necessary to consider which perturbation quantities correspond to the canonical fields. This canonical field is the field v (Mukhanov-Sasaki variables) defined by

$$v \equiv zu, \quad z := a\sqrt{2Q_s}. \tag{3.39}$$

This is because the kinetic energy term of the action (3.20)

$$S_2 = \int d\eta d^3x\, a^2 Q_s \left[\zeta'^2 - c_s^2(\partial\zeta)^2\right] \tag{3.40}$$

can be written as

$$\int d\eta d^3x \frac{v'^2}{2}. \tag{3.41}$$

The equation satisfied by the Mukhanov-Sasaki variable v is derived from the equation (3.35): The time derivative of the mode function u in terms of the Mukhanov-Sasaki variable v is

$$u = \frac{v}{z}, \quad u' = \frac{v'}{z} - \frac{z'v}{z^2}, \quad u'' = \frac{v''}{z} - \frac{2z'v'}{z^2} - \frac{z''}{z^2}v + \frac{2z'^2}{z^3}v. \tag{3.42}$$

Therefore, the left-hand side of equation (3.35)

$$u'' + \frac{(a^2Q_s)'}{a^2Q_s}u' + c_s^2k^2u = 0 \tag{3.43}$$

can be computed

$$\begin{aligned} \text{LHS of (3.35)} &= \frac{v''}{z} - \frac{2z'v'}{z^2} - \frac{z''}{z^2}v + \frac{2z'^2}{z^3}v + \frac{2z'}{z}\left(\frac{v'}{z} - \frac{z'v}{z^2}\right) + c_s^2k^2\frac{v}{z} \\ &= \frac{v''}{z} + \left(c_s^2k^2 - \frac{z''}{z}\right)\frac{v}{z}. \end{aligned} \tag{3.44}$$

Hence, the equation satisfied by the Mukhanov-Sasaki variable v is

$$v'' + \left(c_s^2k^2 - \frac{z''}{z}\right)v = 0. \tag{3.45}$$

From the definition of the sound speed c_s,

$$Q_s = \frac{\epsilon M_{\rm pl}^2}{c_s^2} \tag{3.46}$$

is satisfied. From this equation, if the background spacetime is close to the de Sitter universe in which the Hubble parameter H is constant, and if the sound speed c_s changes slowly, Q_s also changes slowly. In this case, by defining the change of Q_s as

$$\delta_{Q_s} := \frac{\dot{Q}_s}{HQ_s}, \tag{3.47}$$

the relation

$$\frac{z''}{z} = 2(aH)^2 \left(1 - \frac{1}{2}\epsilon + \frac{3}{4}\delta_{Q_s}\right) + \mathcal{O}\left(\epsilon^2\right) \tag{3.48}$$

hold.

We show that equation (3.48) holds. Since the relation between physical time t and conformal time η is $dt = ad\eta$, the relation

$$\frac{d}{d\eta} = \frac{dt}{d\eta}\frac{d}{dt} = a\frac{d}{dt} \tag{3.49}$$

for the derivative holds. Using this, we can calculate the conformal time derivative of $z = a\sqrt{2Q_s}$ to be

$$\begin{aligned} z' &= a\frac{d}{dt}a\sqrt{2Q_s} \\ &= a\dot{a}\sqrt{2Q_s} + a^2\frac{\dot{Q}_s}{\sqrt{2Q_s}}, \end{aligned} \tag{3.50}$$

$$z'' = a\dot{a}^2\sqrt{2Q_s} + a^2\ddot{a}\sqrt{2Q_s} + 3a^2\dot{a}\frac{\dot{Q}_s}{\sqrt{2Q_s}} + a^3\frac{\ddot{Q}_s}{\sqrt{2Q_s}} - a^3\frac{\dot{Q}_s^2}{2\sqrt{2}Q_s^{3/2}}. \tag{3.51}$$

Therefore,

$$\begin{aligned} \frac{z''}{z} &= \dot{a}^2 + a\ddot{a} + 3a\dot{a}\frac{\dot{Q}_s}{2Q_s} + a^2\frac{\ddot{Q}_s}{2Q_s} - a^2\frac{\dot{Q}_s^2}{4Q_s^2} \\ &= 2\dot{a}^2\left(\frac{1}{2} + \frac{a\ddot{a}}{2\dot{a}^2} + 3a\frac{\dot{Q}_s}{4\dot{a}Q_s} + a^2\frac{\ddot{Q}_s}{4\dot{a}^2Q_s} - a^2\frac{\dot{Q}_s^2}{8\dot{a}^2Q_s^2}\right) \\ &= 2\dot{a}^2\left(\frac{1}{2} + \frac{a^2}{2\dot{a}^2}\frac{\ddot{a}}{a} + \frac{3}{4}\frac{\dot{Q}_s}{HQ_s} + \frac{\ddot{Q}_s}{4H^2Q_s} - \frac{\dot{Q}_s^2}{8H^2Q_s^2}\right) \\ &= 2\dot{a}^2\left(\frac{1}{2} + \frac{1}{2H^2}H^2(1-\epsilon) + \frac{3}{4}\delta_{Q_s}\right) + \mathcal{O}(\epsilon^2) \\ &= 2(aH)^2\left(1 - \frac{1}{2}\epsilon + \frac{3}{4}\delta_{Q_s}\right) + \mathcal{O}\left(\epsilon^2\right) \end{aligned}$$

is derived. □

In the asymptotic past $k\eta \to -\infty$, the quantum fluctuation is sufficiently smaller than the Hubble radius $c_s k \gg aH$, and thus the relation

$$c_s^2 k^2 \gg \frac{z''}{z} \tag{3.52}$$

holds. In this case, the equation of motion (3.45)

$$v'' + \left(c_s^2 k^2 - \frac{z''}{z}\right) v = 0 \tag{3.53}$$

simplifies to

$$v'' + \omega_k^2 v \simeq 0, \quad \omega_k := c_s k. \tag{3.54}$$

If we choose the Bunch-Davis vacuum corresponding to a state with zero number of particles (a vacuum that is consistent with the solution of the flat spacetime field theory in the asymptotic past limit), we can write the solution as

$$v = \frac{e^{-i\omega_k \eta}}{\sqrt{2\omega_k}} = \frac{e^{-ic_s k\eta}}{\sqrt{2c_s k}} \qquad (k\eta \to -\infty). \tag{3.55}$$

With time evolution during inflation, z''/z in the equation of motion (3.45)

$$v'' + \left(c_s^2 k^2 - \frac{z''}{z}\right) v = 0 \tag{3.56}$$

becomes larger and larger, and the growth term of v due to gravity becomes important at around $c_s k = aH$. Since the sound speed is $c_s = 1$ in the canonical model $P = X - V(\phi)$, the condition $c_s k = aH$ coincides with the time

$$k = aH \tag{3.57}$$

when the wavelength of the fluctuation equals the Hubble radius. When Lagrangian P contains a nonlinear term of kinetic energy X, the condition $c_s \neq 1$ in general, thus the condition $c_s k = aH$ does not coincide with the time $k = aH$ when the fluctuation wavelength equals the Hubble radius. In the limit where the Hubble parameter H is constant, the equation (3.48)

$$\frac{z''}{z} = 2(aH)^2 \left(1 - \frac{1}{2}\epsilon + \frac{3}{4}\delta_{Q_s}\right) + \mathcal{O}\left(\epsilon^2\right) \tag{3.58}$$

remains only the first term, that is,

$$\frac{z''}{z} = 2(aH)^2 = \frac{2}{\eta^2}, \tag{3.59}$$

where we use the relation

$$\eta = -\frac{1}{aH} \tag{3.60}$$

which is valid when the slow-roll parameter ϵ is sufficiently small and H can be regarded as constant from the equation (3.31)

$$\eta \simeq -\frac{1+\epsilon}{aH}. \tag{3.61}$$

In this case, the solution of the equation of motion is given by

$$v = -\frac{\sqrt{\pi|\eta|}}{2}\left[c_1 H_{3/2}^{(1)}(x) + c_2 H_{3/2}^{(2)}(x)\right], \quad x = c_s k|\eta| \tag{3.62}$$

using the Hankel function of the first kind $H_{3/2}^{(1)}, H_{3/2}^{(2)}$ whose index is $\nu = 3/2$. Note that the explicit expression of the Hankel function is

$$H_{3/2}^{(1)}(x) = \left(H_{3/2}^{(2)}(x)\right)^* = -\sqrt{\frac{2}{\pi x}}\left(1+\frac{i}{x}\right)e^{ix} \tag{3.63}$$

and the coefficients are determined to be

$$c_1 = 1, \quad c_2 = 0 \tag{3.64}$$

by the condition that the limit of the asymptotic past $x \to \infty$ coincides with the solution (3.55)

$$v = \frac{e^{-i\omega_k \eta}}{\sqrt{2\omega_k}} = \frac{e^{-ic_s k\eta}}{\sqrt{2c_s k}} \qquad (k\eta \to -\infty). \tag{3.65}$$

Therefore, the solution of the mode function of the comoving curvature fluctuation is given by

$$\begin{aligned} u(\eta,k) = \frac{v}{z} = \frac{v}{a\sqrt{2Q_s}} &= \frac{1}{a\sqrt{2Q_s}}\frac{\sqrt{\pi|\eta|}}{2}\sqrt{\frac{2}{\pi c_s k|\eta|}}\left(1+\frac{i}{c_s k|\eta|}\right)e^{-ic_s k\eta} \\ &= \frac{H|\eta|}{2\sqrt{Q_s}}\sqrt{\frac{1}{c_s k}}\left(1+\frac{i}{c_s k|\eta|}\right)e^{-ic_s k\eta} \qquad \left(\eta = -\frac{1}{aH}\right) \\ &= \frac{iH}{2\left(c_s k\right)^{3/2}\sqrt{Q_s}}\left(1+ic_s k\eta\right)e^{-ic_s k\eta}. \end{aligned} \tag{3.66}$$

The mode function (3.66) is corrected by considering the deviation from de Sitter spacetime, but the correction to the spectrum of curvature fluctuation (3.70) obtained below is about $\mathcal{O}(\epsilon)$ and is not important.

The two-point correlation function when the curvature fluctuation ζ is frozen in the asymptotic future $c_s k \ll aH$ is given by the vacuum expectation value

$$\langle 0| \zeta(0,\boldsymbol{k}_1)\zeta(0,\boldsymbol{k}_2) |0\rangle \tag{3.67}$$

at $\eta \to 0$. The power spectrum P_ζ of the curvature fluctuation ζ is defined as follows:

$$\langle 0| \zeta\left(0,\boldsymbol{k}_1\right) \zeta\left(0,\boldsymbol{k}_2\right) |0\rangle \equiv P_\zeta\left(k_1\right)(2\pi)^3 \delta^{(3)}\left(\boldsymbol{k}_1+\boldsymbol{k}_2\right). \tag{3.68}$$

Using the Fourier expansion (3.27)

$$\begin{aligned} \zeta(\eta,\boldsymbol{x}) &= \frac{1}{(2\pi)^3}\int d^3\boldsymbol{k}\zeta(\eta,\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \\ \zeta(\eta,\boldsymbol{k}) &= u(\eta,\boldsymbol{k})a(\boldsymbol{k}) + u^*(\eta,-\boldsymbol{k})a^\dagger(-\boldsymbol{k}) \end{aligned}$$

and the commutation relation (3.28), (3.29)

$$\begin{aligned} \left[a\left(\boldsymbol{k}_1\right), a^\dagger\left(\boldsymbol{k}_2\right)\right] &= (2\pi)^3\delta^{(3)}\left(\boldsymbol{k}_1-\boldsymbol{k}_2\right), \\ \left[a\left(\boldsymbol{k}_1\right), a\left(\boldsymbol{k}_2\right)\right] &= \left[a^\dagger\left(\boldsymbol{k}_1\right), a^\dagger\left(\boldsymbol{k}_2\right)\right] = 0, \end{aligned}$$

the two-point correlation function is expressed as

$$\begin{aligned} \langle 0| \zeta\left(0,\boldsymbol{k}_1\right) \zeta\left(0,\boldsymbol{k}_2\right) |0\rangle &= \langle 0| \left\{u(0,\boldsymbol{k}_1)a(\boldsymbol{k}_1) + u^*(0,-\boldsymbol{k}_1)a^\dagger(-\boldsymbol{k}_1)\right\} \\ &\quad \times \left\{u(0,\boldsymbol{k}_2)a(\boldsymbol{k}_2) + u^*(0,-\boldsymbol{k}_2)a^\dagger(-\boldsymbol{k}_2)\right\} |0\rangle \\ &= u(0,\boldsymbol{k}_1)u^*(0,-\boldsymbol{k}_2)(2\pi)^3\delta^{(3)}\left(\boldsymbol{k}_1+\boldsymbol{k}_2\right). \end{aligned} \tag{3.69}$$

Consequently, the power spectrum is obtained by using the mode function (3.66) as follows:

$$P_\zeta(k) = |u(0,k)|^2 = \frac{H^2}{4Q_s c_s^3 k^3}. \tag{3.70}$$

Here, using the dimensionless quantity

$$\mathcal{P}_\zeta(k) := \frac{k^3}{2\pi^2}P_\zeta(k) = \frac{H^2}{8\pi^2 Q_s c_s^3} = \frac{H^2}{8\pi^2\epsilon M_{\rm pl}^2 c_s} \quad \left(Q_s = \frac{\epsilon M_{\rm pl}^2}{c_s^2}\right), \tag{3.71}$$

the vacuum expectation value of the curvature fluctuation when the power spectrum $P_\zeta(k)$ is added up by the mode k of all wavenumbers is denoted by

$$\begin{aligned} \langle\zeta^2\rangle &= \frac{1}{2\pi^2}\int dk\,k^2 P_\zeta(k) \\ &= \int \mathcal{P}_\zeta(k) d\ln k. \end{aligned} \tag{3.72}$$

This dimensionless quantity $\mathcal{P}_\zeta(k)$ is also called the power spectrum. This result was derived using a comoving gauge, but the same result can be obtained using a Newton gauge.

Since the curvature fluctuation ζ quickly approaches to a constant value for $c_s k < aH$, we only need to calculate the spectrum (3.70) when $c_s k = aH$ during inflation. The time when $c_s k = aH$ occurs later for larger wave number k, and therefore $\mathcal{P}_\zeta(k)$ has dependence on the k. To quantify this fact, we define the spectral index

$$n_s - 1 \equiv \left.\frac{d\ln\mathcal{P}_\zeta}{d\ln k}\right|_{c_s k = aH} \tag{3.73}$$

of the curvature fluctuation. When the change of sound speed c_s during inflation is slow, c_s, H is regarded as a constant and the logarithmic change of wavenumber $d\ln k$ at $c_s k = aH$ is approximately

$$d\ln k = d\ln a = \frac{da}{a} = \frac{\dot{a}}{a}dt = Hdt. \tag{3.74}$$

Using this, the spectral index (3.73) is expressed as

$$\begin{aligned} n_s - 1 = \left.\frac{d\ln\mathcal{P}_\zeta}{d\ln k}\right|_{c_s k = aH} &= \frac{1}{H}\frac{d}{dt}\ln\mathcal{P}_\zeta \\ &= \frac{1}{H}\frac{\dot{\mathcal{P}}_\zeta}{\mathcal{P}_\zeta} \\ &= \frac{1}{H}\frac{\epsilon c_s}{H^2}\frac{d}{dt}\frac{H^2}{\epsilon c_s} \\ &= \frac{\epsilon c_s}{H^3}\left(\frac{2H\dot{H}}{\epsilon c_s} - \frac{H^2\dot{\epsilon}}{\epsilon^2 c_s} - \frac{H^2\dot{c}_s}{\epsilon c_s^2}\right) \\ &= -2\left(-\frac{\dot{H}}{H^2}\right) - \frac{\dot{\epsilon}}{H\epsilon} - \frac{\dot{c}_s}{Hc_s} \\ &= -2\epsilon - \eta_s - s, \end{aligned} \tag{3.75}$$

where we define the rate of change of the slow-roll parameter ϵ and the sound speed c_s:

$$\eta_s := \frac{\dot{\epsilon}}{H\epsilon}, \quad s := \frac{\dot{c}_s}{Hc_s}. \tag{3.76}$$

Since

$$\epsilon,\ |\eta_s|,\ |s| \ll 1 \tag{3.77}$$

hold during inflation, the spectral index n_s is close to 1. In this case, from the definition of spectral exponent (3.73), we have

$$\mathcal{P}_\zeta \propto k^{n_s-1} = k^0, \tag{3.78}$$

which means that the amplitude is almost constant and independent of k. In other words, it is almost scale invariant. The initial spectrum of the observed temperature fluctuations of the background radiation (i.e., the spectrum before the fluctuations re-enter the Hubble radius) is close to scale-invariant, which is consistent with the theoretical prediction of inflation. Because ϵ, η_s, s are non-zero and their values are different for each model of inflation, it is possible to distinguish which model is consistent by observing its deviation from the scale invariance.

The spectrum $\mathcal{P}_\zeta(k)$ can be expressed as

$$\frac{\mathcal{P}_\zeta(k)}{\mathcal{P}_\zeta(k_0)} = \left(\frac{k}{k_0}\right)^{n_s-1} \tag{3.79}$$

i.e.

$$\mathcal{P}_\zeta(k) = \mathcal{P}_\zeta(k_0)\left(\frac{k}{k_0}\right)^{n_s-1} \tag{3.80}$$

by expanding around some wavenumber k_0, and is bounded by $\mathcal{P}_\zeta(k_0)$ and n_s from observations of background radiation. We also define a quantity

$$\alpha_s := \left.\frac{dn_s}{d\ln k}\right|_{c_s k = aH} \tag{3.81}$$

called running, which characterizes the scale dependence on n_s. Running α_s is a second-order quantity with respect to the slow-roll parameter. When running α_s is taken into account, a correction is added to the power spectrum (3.80).

Consider a canonical model whose Lagrangian is given by

$$P = X - V(\phi), \tag{3.82}$$

and find a concrete expression for the spectral index n_s and running α_s in terms of slow-roll parameters. In this case, since the sound speed is $c_s = 1$, we have

$$s = \frac{\dot{c}_s}{Hc_s} = 0. \tag{3.83}$$

According to equation (2.23)

$$\epsilon_V \simeq \epsilon, \quad \eta_V \simeq 2\epsilon - \frac{\dot{\epsilon}}{2H\epsilon} = 2\epsilon - \frac{1}{2}\eta_s, \tag{3.84}$$

since relations

$$\epsilon_V = \epsilon, \quad \eta_s = 4\epsilon_V - 2\eta_V \tag{3.85}$$

hold, the spectral index (3.75)

$$n_s - 1 = -2\epsilon - \eta_s - s \tag{3.86}$$

becomes

$$n_s - 1 = -6\epsilon_V + 2\eta_V. \tag{3.87}$$

Furthermore, from equation (2.23)

$$\epsilon_V \simeq \epsilon, \quad \eta_V \simeq 2\epsilon - \frac{\dot{\epsilon}}{2H\epsilon}, \quad \xi_V^2 \simeq \left(2\epsilon_V - \frac{\dot{\eta}_V}{H\eta_V}\right)\eta_V, \tag{3.88}$$

the relations

$$\frac{\dot{\epsilon}_V}{H} = -2\epsilon_V\eta_V + 4\epsilon_V^2, \quad \frac{\dot{\eta}_V}{H} = 2\epsilon_V\eta_V - \xi_V^2 \tag{3.89}$$

hold, thus the specific expression for the running α_s is

$$\begin{aligned}
\alpha_s &= \left.\frac{dn_s}{d\ln k}\right|_{c_s k = aH} \\
&= \frac{1}{H}\frac{d}{dt}n_s \\
&= \frac{1}{H}\left(-6\dot{\epsilon}_V + 2\dot{\eta}_V\right) \\
&= -6\left(-2\epsilon_V\eta_V + 4\epsilon_V^2\right) + 2\left(2\epsilon_V\eta_V - \xi_V^2\right) \\
&= 16\epsilon_V\eta_V - 24\epsilon_V^2 - 2\xi_V^2.
\end{aligned} \tag{3.90}$$

Next, let us consider the tensor perturbations generated by the inflation, i.e., the spectrum of gravitational waves. The gravitational waves h_{ij} has polarization states and can be written as

$$h_{ij} = h_+ e_{ij}^+ + h_\times e_{ij}^\times \tag{3.91}$$

using two polarization tensors e^{+}_{ij} and $e^{\times}_{ij}$. The polarization tensors e^{+}_{ij} and $e^{\times}_{ij}$ are symmetric tensors and satisfy the transverse wave condition and traceless condition

$$k^i e^{\lambda}_{ij} = k^j e^{\lambda}_{ij} = 0, \quad e^{\lambda}_{ii} = 0 \qquad (\lambda = +, \times). \tag{3.92}$$

In addition, the normalization condition and the orthogonality condition

$$e^{+}_{ij}(\boldsymbol{k}) e^{+*}_{ij}(-\boldsymbol{k}) = e^{\times}_{ij}(\boldsymbol{k}) e^{\times *}_{ij}(-\boldsymbol{k}) = 2, \quad e^{+}_{ij}(\boldsymbol{k}) e^{\times *}_{ij}(-\boldsymbol{k}) = 0 \tag{3.93}$$

are satisfied. Since the tensor perturbation equation is

$$\left(h^i{}_j\right)'' + 2\mathcal{H}\left(h^i{}_j\right)' + \left(2K - \nabla^2\right) h^i{}_j = 0, \tag{3.94}$$

the components h_+ and $h_\times$ satisfy equation

$$h''_\lambda + 2\mathcal{H} h'_\lambda + k^2 h_\lambda = 0 \tag{3.95}$$

in Fourier space respectively. For the tensor mode h_λ, expanding the action (2.1)

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{\rm pl}^2}{2} R + P(\phi, X) \right] \tag{3.96}$$

to the second order yields

$$S_t = \sum_{\lambda=+,x} \int d\eta d^3x \, a^2 Q_t \left[h'^2_\lambda - (\partial h_\lambda)^2 \right], \tag{3.97}$$

where we define

$$Q_t := \frac{M_{\rm pl}^2}{4}. \tag{3.98}$$

By variating this action with respect to h_λ, we can also obtain the equation of motion (3.95). In the model (2.1), the propagation velocity (sound speed) of the tensor mode is $c_t = 1$, as can be seen from the coefficient of the spatial derivative term in the equation (3.95).

Similar to the previous discussion on scalar perturbations, the canonical quantum field corresponding to the tensor mode h_λ is

$$v_\lambda := z_t h_\lambda, \quad z_t := a\sqrt{2Q_t}. \tag{3.99}$$

Using these, the equation of motion (3.95) becomes

$$v''_\lambda + \left(k^2 - \frac{z''_t}{z_t} \right) v_\lambda = 0. \tag{3.100}$$

The mode function with the Bunch-Davis vacuum as the initial state in the asymptotic past is

$$h_\lambda(\eta, k) = \frac{iHe^{-ik\eta}}{2k^{3/2}\sqrt{Q_t}}(1 + ik\eta) \tag{3.101}$$

at time η. This function approaches to

$$h_\lambda \to \frac{iH}{2k^{3/2}\sqrt{Q_t}} \tag{3.102}$$

after the crossing with the Hubble radius $k = aH$. Using the normalization condition

$$e^{+}_{ij}(\boldsymbol{k})e^{+*}_{ij}(-\boldsymbol{k}) = e^{\times}_{ij}(\boldsymbol{k})e^{\times *}_{ij}(-\boldsymbol{k}) = 2 \tag{3.103}$$

for the polarization tensor e^{λ}_{ij}, the power spectrum of gravitational waves becomes

$$\mathcal{P}_h(k) = \frac{4k^3}{2\pi^2}\left|h_\lambda(0, k)\right|^2 = \frac{H^2}{2\pi^2 Q_t} = \frac{2H^2}{\pi^2 M_{\rm pl}^2}. \tag{3.104}$$

Since the mode function H_λ freezes at $k < aH$, we only need to find the value at $k = aH$ in crossing. The spectral index of the gravitational wave is given by

$$\begin{aligned} n_t := \left.\frac{d\ln\mathcal{P}_h}{d\ln k}\right|_{k=aH} &= \frac{1}{H}\frac{d}{dt}\ln\mathcal{P}_h \\ &= \frac{1}{H^3}\frac{d}{dt}H^2 \\ &= -2\left(-\frac{\dot{H}}{H^2}\right) \\ &= -2\epsilon, \end{aligned} \tag{3.105}$$

and the running is

$$\begin{aligned} \alpha_t := \left.\frac{dn_t}{d\ln k}\right|_{k=aH} &= \frac{1}{H}\frac{d}{dt}n_t \\ &= -2\frac{\dot{\epsilon}}{H} \\ &= -2\epsilon\eta_s \qquad \left(\eta_s = \frac{\dot{\epsilon}}{H\epsilon}\right). \end{aligned} \tag{3.106}$$

In particular, when the canonical model $P = X - V(\phi)$, using relation $\eta_s = 4\epsilon_V - 2\eta_V$, the running is

$$\alpha_t = -8\epsilon_V^2 + 4\epsilon_V\eta_V. \tag{3.107}$$

We define the tensor-to-scalar ratio as an important physical quantity:

$$r := \frac{\mathcal{P}_h(k)}{\mathcal{P}_\zeta(k)} \tag{3.108}$$

From the power spectrum of the scalar perturbation (3.71)

$$\mathcal{P}_\zeta(k) = \frac{H^2}{8\pi^2 \epsilon M_{\rm pl}^2 c_s} \tag{3.109}$$

and the power spectrum of the tensor perturbation (3.104)

$$\mathcal{P}_h(k) = \frac{2H^2}{\pi^2 M_{\rm pl}^2}, \tag{3.110}$$

the expression for the tensor-to-scalar ratio becomes

$$r = 16 c_s \epsilon. \tag{3.111}$$

Since $c_s = 1$ in the canonical model $P = X - V(\phi)$, we can calculate (3.108) with $k = aH$. When $c_s \neq 1$, we only need to find the ratio r for the case where $\mathcal{P}_\zeta(k)$ and $\mathcal{P}_h(k)$ become almost constant. From the spectral index of the gravitational wave (3.105)

$$n_t = -2\epsilon \tag{3.112}$$

and the tensor-to-scalar ratio (3.108), we obtain

$$r = -8 c_s n_t \tag{3.113}$$

which is called consistency relation.

From the analysis of combined data of Planck, BICEP/Keck, WMAP polarization (WP), ACT and SPT power spectra on small scales when the sound speed is $c_s = 1$, we obtain the limits

$$\mathcal{P}_\zeta(k_0) = 2.198^{+0.056}_{-0.054} \times 10^{-9} \quad (68\%\text{CL}), \tag{3.114}$$

$$n_s(k_0) = 0.9585 \pm 0.0070 \quad (68\%\text{CL}), \tag{3.115}$$

$$r(k_0) < 0.036 \quad (95\%\text{CL}) \tag{3.116}$$

at $k_0 = 0.002\,\mathrm{Mpc}^{-1}$.

In the case of the canonical model $P = X - V(\phi)$, $H(k_0)$ is obtained from the scalar spectrum (3.71)

$$\mathcal{P}_\zeta(k) = \frac{k^3}{2\pi^2} P_\zeta(k) = \frac{H^2}{8\pi^2 Q_s c_s^3} = \frac{H^2}{8\pi^2 \epsilon M_{\mathrm{pl}}^2 c_s} \quad \left(Q_s = \frac{\epsilon M_{\mathrm{pl}}^2}{c_s^2}\right) \tag{3.117}$$

and the observed value (3.114):

$$H(k_0) = \sqrt{\epsilon(k_0)} \times 2\sqrt{2}\pi M_{\mathrm{pl}} \sqrt{\mathcal{P}_\zeta(k_0)} \simeq \sqrt{r(k_0)} \times 10^{15}\,\mathrm{GeV}. \tag{3.118}$$

Thus, the typical energy scale of inflation is estimated to be around 10^{14} GeV.

In addition to the spectral index n_s and tensor-to-scalar ratio r, the most important observables for identifying inflationary models is the parameter f_{NL}, which characterizes non-Gaussianity of curvature fluctuation. Non-Gaussianity is defined by

$$\langle\zeta\zeta\zeta\rangle = (2\pi)^7 \delta^{(3)}(\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3) \tilde{p}_\zeta^2 \left(\frac{9}{10} f_{\mathrm{NL}}\right) \frac{1}{(p_1 p_2 p_3)^2} \tag{3.119}$$

in the case where the configuration of the external momentum are equilateral

$$p := p_1 = p_2 = p_3, \tag{3.120}$$

where $\langle\zeta\zeta\zeta\rangle$ is the three-point correlation function of curvature fluctuation and $\tilde{p}_\zeta^2$ is a shorthand for the observed value (3.114). We sometimes call $\langle\zeta\zeta\zeta\rangle$ non-Gaussianity or f_{NL} non-Gaussianity. Since the constraints for non-Gaussianity f_{NL} are obtained from observations as shown in Fig.3.1, it is possible to select models by performing theoretical calculations and comparing their values. In Chapter 4, we will construct an effective field theory of inflation, and in Chapter 5, we will review the method of Feynman graphs in elementary particle theory for computing three-point correlation function.

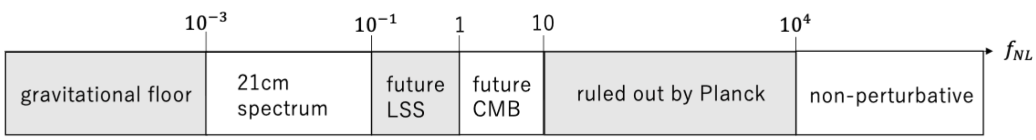


Figure 3.1: Schematic illustration of current and future constraints on non-Gaussianity (Figure taken from [104]).

Chapter 4

Effective field theory of inflation

Effective field theory is a method of writing down to Lagrangian all operators that are allowed by the symmetry of the system. This method allows us to investigate phenomenology at a given energy scale and plays the role of a guiding principle in particle theory and condensed matter physics because it is a systematic and consistent construction method. In discussing the effective field theory of inflation, let us consider inflation as Goldstone's theory of time translation. The discussion here is based on [96, 99].

4.1 Inflation as Goldstone boson theory

In order to construct an effective field theory of inflation, let us consider again what the physics of inflation is. What we know about inflation is that the universe is a quasi de Sitter spacetime, a period of accelerated expansion. It is important to note that it is not strictly de Sitter spacetime, but quasi de Sitter spacetime. This is because inflation will end at some point and move to the Big Bang cosmology. This means that the translational symmetry of time is spontaneously broken. Therefore, there exists a physical object that plays the role of a clock, which we choose as the simplest field to be a scalar field ϕ and call an inflaton. Using the coordinate invariance of general relativity, we can move to a coordinate system such that this physical clock is zero, i.e., a comoving gauge, by choosing a spatial slice where the time fluctuation is zero. In other words, the change of ϕ is adopted as a clock. Concretely, it is a differential homomorphic mapping of eigentime from an arbitrary coordinate system.

As an example, consider the case where we are in a spatially flat gauge with $\delta\phi(t, \boldsymbol{x}) \neq 0$. We consider a time diffeomorphism map

$$t \mapsto \tilde{t} = t + \delta t(t, \boldsymbol{x}) \tag{4.1}$$

such that we move from a scalar field and a spatial metric

$$\phi(t) = \phi_0(t) + \delta\phi(t, \boldsymbol{x}), \quad \gamma_{ij} = a^2(t)\left(\delta_{ij} + h_{ij}(t) + \frac{1}{2}h_{ik}h_{kj} + \cdots\right) \tag{4.2}$$

in a spatially flat gauge to a scalar field and a spatial metric

$$\phi = \phi_0(\tilde{t}), \quad \gamma_{ij} = a^2(\tilde{t})e^{2\zeta}\left(\delta_{ij} + h_{ij}(\tilde{t}) + \frac{1}{2}h_{ik}h_{kj} + \cdots\right) \tag{4.3}$$

in a comoving gauge. That is, consider what form $\delta t(t, \boldsymbol{x})$ should take. The inflaton undergoes the following transformation at the first order (which can be generalized to any order):

$$\begin{aligned}\phi(t) &= \phi_0(\tilde{t} - \delta t) + \delta\phi(\tilde{t} - \delta t, \boldsymbol{x}) \\ &= \phi_0(\tilde{t}) - \dot{\phi}_0(\tilde{t})\delta t + \delta\phi(\tilde{t}, \boldsymbol{x}).\end{aligned} \tag{4.4}$$

Since we want to move to a coordinate system such that the inflaton fluctuation is zero after the transformation, from

$$0 = \tilde{\delta\phi}(\tilde{t}, \boldsymbol{x}) = -\dot{\phi}_0(\tilde{t})\delta t(t, \boldsymbol{x}) + \delta\phi(t, \boldsymbol{x}) \tag{4.5}$$

we can take δt to be

$$\delta t = \frac{\delta\phi}{\dot{\phi}_0}. \tag{4.6}$$

Since the spatial metric is

$$\begin{aligned}\gamma_{ij} &= a^2(t)\left(\delta_{ij} + h_{ij}\right) \\ &= a^2(\tilde{t} - \delta t)\big(\delta_{ij} + h_{ij}(\tilde{t} - \delta t)\big) \\ &= \big(a^2(\tilde{t}) - 2a\dot{a}(\tilde{t})\delta t\big)\left(\delta_{ij} + h_{ij}\right) \\ &= a^2(\tilde{t})(1 - 2H\delta t)\left(\delta_{ij} + h_{ij}\right),\end{aligned} \tag{4.7}$$

the relation between the scalar fluctuation ζ and the inflaton fluctuation $\delta\phi$ can be read

$$\zeta = -H\delta t = -H\frac{\delta\phi}{\dot{\phi}_0}. \tag{4.8}$$

Now that we understand how to move to $\delta\phi(t, \boldsymbol{x}) = 0$, let us assume that we are now in a comoving gauge. Following the method of effective field theory, all possible degrees of freedom must be written in the action. Now, the possible degrees of freedom are the fluctuations of the metric. Lagrangian must be expanded in terms of fluctuations of the metric and all operators must be written down to match the symmetry of the problem. Now we are in a situation where the spatial coordinates on the spatial slice can be arbitrarily changed in a different way for each spatial slice, which means that the remaining gauge symmetry is a time-dependent diffeomorphism map of space:

$$x^i \mapsto \tilde{x}^i = x^i + \xi^i(t, \boldsymbol{x}). \tag{4.9}$$

Then, what is the most general Lagrangian in the comoving gauge? Lagrangian is a function of the metric $g_{\mu\nu}$ and must be written down as an invariant operator under a time-dependent diffeomorphism map of space. In addition to the terms of the ordinary Riemann tensor, which are invariant under the differential homomorphic mapping of space-time, many extra terms are now allowed because of the loosening of the symmetry of the system. The extra term describes the additional degrees of freedom eaten by graviton. For example, it is easy to verify that g^{00} is a scalar as in

$$\tilde{g}^{00} = \frac{\partial \tilde{t}}{\partial x^\mu} \frac{\partial \tilde{t}}{\partial x^\nu} g^{\mu\nu} = \delta^0_\mu \delta^0_\nu g^{\mu\nu} = g^{00} \tag{4.10}$$

under the diffeomorphism map of the space ($\tilde{t} = t$). Thus, g^{00} is free to enter the Lagrangian in the comoving gauge. The polynomial in g^{00} is the only quantity that can be allowed in Lagrangian as a term without derivatives. Given a slice of spacetime, it is also allowed to write a geometric object describing this slice. For example, the external curvature $K_{\mu\nu}$ on a time-constant surface is a tensor invariant under a diffeomorphism map of space, and can be written in Lagrangian. Let n^μ be a vector orthogonal to the time-constant surface, the external curvature is denoted by

$$K_{\mu\nu} = \gamma^\sigma_\mu \nabla_\sigma n_\nu, \tag{4.11}$$

where ∇ is the covariant derivative and $\gamma_{\mu\nu}$ is the induced metric on the spatial slice

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu. \tag{4.12}$$

In general, writing the Lagrangian as $\mathcal{L}(g)$ and considering the fluctuation $g+\delta g$ of the metric g, we can expand it as

$$\mathcal{L}(g+\delta g) = \mathcal{L}(g) + \left.\frac{\delta \mathcal{L}}{\delta g}\right|_g \delta g + \mathcal{O}\left((\delta g)^2.\right) \tag{4.13}$$

The first-order coefficient of the fluctuation δg is zero from the equation of motion. Thus, noting the fact that an arbitrary function of time can be multiplied by each term of the action, the most general Lagrangian can be written as

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}M_{\rm Pl}^2 R - c(t)g^{00} - \Lambda(t) + \frac{1}{2!}M_2(t)^4\left(\delta g^{00}\right)^2 + \frac{1}{3!}M_3(t)^4\left(\delta g^{00}\right)^3 \right. \\ \left. - \frac{\overline{M}_1(t)^3}{2}\left(\delta g^{00}\right)\delta K^\mu{}_\mu - \frac{\overline{M}_2(t)^2}{2}(\delta K^\mu{}_\mu)^2 - \frac{\overline{M}_3(t)^2}{2}\delta K^\mu{}_\nu \delta K^\nu{}_\mu + \dots\right], \tag{4.14}$$

where $M_2(t), M_3(t)$, etc. are arbitrary functions of time, and $\cdots$ denotes higher-order fluctuations and derivative terms. Note that the concrete expression of δg^{00} is

$$\delta g^{00} = g^{00} + 1, \tag{4.15}$$

and

$$\delta K_{\mu\nu} = K_{\mu\nu} - a^2 H \gamma_{\mu\nu} \tag{4.16}$$

is the fluctuation of the external curvature of the time constant surface with unperturbed background FLRW metric. Only the first three terms of the action (4.14) contain the first-order perturbations around the chosen FLRW solution, all other terms are explicitly the second-order or higher perturbations. The coefficients $c(t), \Lambda(t)$ are determined by the equations of motion of the background spacetime. Since the background spacetime is a flat FLRW universe

$$ds^2 = -dt^2 + a^2(t)d\boldsymbol{x}^2, \tag{4.17}$$

the left-hand side of the Einstein equation takes the form of the well-known Friedmann equation, and the right-hand side is contributed by the energy momentum tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm matter}}{\delta g^{\mu\nu}}, \tag{4.18}$$

we obtain

$$\begin{aligned} H^2 &= \frac{1}{3M_{\rm Pl}^2}[c(t) + \Lambda(t)], \\ \frac{\ddot{a}}{a} = \dot{H} + H^2 &= -\frac{1}{3M_{\rm Pl}^2}[2c(t) - \Lambda(t)] \end{aligned} \tag{4.19}$$

as the equation of motion. Solving these for c, Λ yields

$$c(t) = -M_{\rm Pl}^2 \dot{H}, \tag{4.20}$$

$$\Lambda(t) = M_{\rm Pl}^2 \left(3H^2 + \dot{H}\right), \tag{4.21}$$

then the action (4.14)

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} M_{\rm Pl}^2 R - c(t) g^{00} - \Lambda(t) + \frac{1}{2!} M_2(t)^4 \left(\delta g^{00}\right)^2 + \frac{1}{3!} M_3(t)^4 \left(\delta g^{00}\right)^3 \right. \\ \left. - \frac{\overline{M}_1(t)^3}{2} \left(\delta g^{00}\right) \delta K^\mu{}_\mu - \frac{\overline{M}_2(t)^2}{2} (\delta K^\mu{}_\mu)^2 - \frac{\overline{M}_3(t)^2}{2} \delta K^\mu{}_\nu \delta K^\nu{}_\mu + \dots \right] \tag{4.22}$$

can be written as

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} M_{\rm Pl}^2 R + M_{\rm Pl}^2 \dot{H} g^{00} - M_{\rm Pl}^2 \left(3H^2 + \dot{H}\right) \right. \\ + \frac{1}{2!} M_2(t)^4 \left(\delta g^{00}\right)^2 + \frac{1}{3!} M_3(t)^4 \left(\delta g^{00}\right)^3 \\ \left. - \frac{\overline{M}_1(t)^3}{2} \left(\delta g^{00}\right) \delta K^\mu{}_\mu - \frac{\overline{M}_2(t)^2}{2} (\delta K^\mu{}_\mu)^2 - \frac{\overline{M}_3(t)^2}{2} \delta K^\mu{}_\nu \delta K^\nu{}_\mu + \cdots \right]. \tag{4.23}$$

As mentioned above, the coefficients of all operators of actions are generally time dependent. However, we are interested in solutions where $H, \dot{H}$ does not change significantly in some Hubble time. Therefore, it is natural to assume that all operators do not change significantly in time as well. Under this assumption, Lagrangian is approximately time translation invariant. Hence, the time dependence generated by the loop effect is suppressed by a small breaking parameter.

It is important to note that this approach not only describes the most general Lagrnagian of scalar modes, but also the most general Lagrnagian of gravity. High energy effects are contained, for example, in the operator $\delta R_{\mu\nu\rho\sigma}$, which is a perturbation of the Riemann tensor. These contributions are higher order terms of the derivative.

Let us look at the relationship between the inflation model which we have dealt with in the previous chapters and effective field theory. A model with a canonical kinetic term and a slow-roll potential $V(\phi)$ can be written as

$$\begin{aligned} S_{\text{can}} &= \int d^4x\sqrt{-g}\left[-\frac{1}{2}(\partial\phi)^2 - V(\phi)\right] \\ &= \int d^4x\sqrt{-g}\left[-\frac{\dot{\phi}_0(t)^2}{2}g^{00} - V\left(\phi_0(t)\right)\right] \end{aligned} \tag{4.24}$$

in the comoving gauge. The Friedmann equation is

$$\dot{\phi}_0^2(t) = -2M_{\text{Pl}}^2\dot{H}, \tag{4.25}$$

$$V(\phi_0(t)) = M_{\text{Pl}}^2\left(3H^2 + \dot{H}\right) \tag{4.26}$$

with

$$P = X - V = \frac{1}{2}\dot{\phi}_0^2(t) - V(\phi_0(t)) \tag{4.27}$$

in equations (2.5) and (2.6)

$$\begin{aligned} 3M_{\text{pl}}^2H^2 &= 2XP_{,X} - P, \\ M_{\text{pl}}^2\dot{H} &= -XP_{,X} \end{aligned}$$

in the general Lagrangian P case. Hence, the action is

$$S_{\text{can}} = \int d^4x\sqrt{-g}\left[\frac{1}{2}M_{\text{Pl}}^2R + M_{\text{Pl}}^2\dot{H}g^{00} - M_{\text{Pl}}^2\left(3H^2 + \dot{H}\right)\right]. \tag{4.28}$$

Comparing this with the effective action (4.23)

$$\begin{aligned} S = \int d^4x\sqrt{-g}\Bigg[&\frac{1}{2}M_{\text{Pl}}^2R + M_{\text{Pl}}^2\dot{H}g^{00} - M_{\text{Pl}}^2\left(3H^2 + \dot{H}\right) \\ &+\frac{1}{2!}M_2(t)^4\left(\delta g^{00}\right)^2 + \frac{1}{3!}M_3(t)^4\left(\delta g^{00}\right)^3 \\ &-\frac{\overline{M}_1(t)^3}{2}\left(\delta g^{00}\right)\delta K^\mu{}_\mu - \frac{\overline{M}_2(t)^2}{2}(\delta K^\mu{}_\mu)^2 - \frac{\overline{M}_3(t)^2}{2}\delta K^\mu{}_\nu\delta K^\nu{}_\mu + \cdots\Bigg], \end{aligned}$$

we can see that except for the first three terms, all other terms are zero. That is, the action with a canonical kinetic term and slow-roll potential (4.28) is the simplest example of the effective action (4.23).

4.2 Stückelberg trick and Goldstone boson equivalence theorem

We have seen that the comoving gauge Lagrangian (4.23) is general expression and includes slow-roll inflation. However, the scalar degrees of freedom are not obvious, as in the scalar field ϕ of the slow-roll inflation (4.24). We will now rewrite the comoving gauge Lagrangian in a form where the scalar degrees of freedom are obvious, and further show that the action is of simpler form for the slow-roll inflation that we usually consider.

As a simple example, we consider the $U(1)$ gauge theory of electromagnetic interaction. As is well known, the invariant action for the gauge transformation

$$A_\mu \mapsto A'_\mu = A_\mu + \frac{1}{g}\partial_\mu \alpha \tag{4.29}$$

is

$$S = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right). \tag{4.30}$$

The action

$$S = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu\right), \tag{4.31}$$

which includes the mass term, is not gauge invariant:

$$S \mapsto S' = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 \left(A_\mu + \frac{1}{g}\partial_\mu \alpha\right)\left(A^\mu + \frac{1}{g}\partial^\mu \alpha\right)\right). \tag{4.32}$$

Introducing a scalar field that transforms as

$$\pi \mapsto \pi' = \pi - \alpha \tag{4.33}$$

into this action by replacement

$$\alpha \to \pi \tag{4.34}$$

yields a gauge-invariant action

$$S = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 \left(A_\mu + \frac{1}{g}\partial_\mu \pi\right)\left(A^\mu + \frac{1}{g}\partial^\mu \pi\right)\right). \tag{4.35}$$

In other words, the gauge symmetry is restored by introducing the field π into the action with the broken mass term. This method is called the Stückelberg trick.

Consider a field

$$\pi_c := \frac{m}{g}\pi. \tag{4.36}$$

which is a canonical normalization of π. The advantage of treating the Nambu-Goldstone boson in this way is that the mixing terms between the Nambu-Goldstone boson and the longitudinal wave component of the gauge field can be neglected at energy $E \gg m$. In fact, the mixing term takes the form

$$\frac{m^2}{g} A_\mu \partial^\mu \pi = m A_\mu \partial^\mu \pi_c, \tag{4.37}$$

which is negligible at energy $E \gg m$ because it is smaller than the canonical kinetic term $(\pi_c)^2$. This fact is called the Goldstone boson equivalence theorem.

Let us apply the same process to the case of a breaking of the differential homomorphic map of time. We consider the action

$$\int d^4x \sqrt{-g} \left[A(t) + B(t) g^{00}(x) \right] \tag{4.38}$$

as an example. Under the breaking of the differential homomorphic map of time

$$t \mapsto \tilde{t} = t + \xi^0(x), \quad \boldsymbol{x} \mapsto \tilde{\boldsymbol{x}} = \boldsymbol{x}, \tag{4.39}$$

the time-time component of the metric g^{00} transforms as in

$$g^{00}(x) \mapsto \widetilde{g}^{00}(\widetilde{x}(x)) = \frac{\partial \widetilde{x}^0(x)}{\partial x^\mu} \frac{\partial \widetilde{x}^0(x)}{\partial x^\nu} g^{\mu\nu}(x). \tag{4.40}$$

The action (4.38) becomes

$$\int d^4x \sqrt{-\widetilde{g}(\widetilde{x}(x))} \left| \frac{\partial \widetilde{x}}{\partial x} \right| \left[A(t) + B(t) \frac{\partial x^0}{\partial \widetilde{x}^\mu} \frac{\partial x^0}{\partial \widetilde{x}^\nu} \widetilde{g}^{\mu\nu}(\widetilde{x}(x)) \right] \tag{4.41}$$

under this transformation. Rewriting the integral variable as $\tilde{x}$ yields

$$\begin{aligned} \int d^4\widetilde{x} \sqrt{-\widetilde{g}(\widetilde{x})} \Big[& A\left(\widetilde{t} - \xi^0(x(\widetilde{x}))\right) \\ & + B\left(\widetilde{t} - \xi^0(x(\widetilde{x}))\right) \frac{\partial\left(\widetilde{t} - \xi^0(x(\widetilde{x}))\right)}{\partial \widetilde{x}^\mu} \frac{\partial\left(\widetilde{t} - \xi^0(x(\widetilde{x}))\right)}{\partial \widetilde{x}^\nu} \widetilde{g}^{\mu\nu}(\widetilde{x}) \Big]. \end{aligned} \tag{4.42}$$

We introduce the Nambu-Goldstone boson as in the $U(1)$ gauge theory case. Replace ξ^0 appearing in the action (4.42) by

$$\xi^0(x(\widetilde{x})) \to -\widetilde{\pi}(\widetilde{x}). \tag{4.43}$$

Dropping the tilde for readability gives us

$$\int d^4x\sqrt{-g(x)}\left[A(t+\pi(x))+B(t+\pi(x))\frac{\partial(t+\pi(x))}{\partial x^\mu}\frac{\partial(t+\pi(x))}{\partial x^\nu}g^{\mu\nu}(x)\right]. \tag{4.44}$$

If the transformation law of the Nambu-Goldstone boson π is

$$\pi(x)\mapsto\tilde{\pi}(\tilde{x}(x))=\pi(x)-\xi^0(x), \tag{4.45}$$

we can confirm that the action (4.44) is gauge invariant at all orders (not only for infinitesimal transformations). In other words, the differential homomorphic map of time is recovered.

Applying this process to the comoving gauge action (4.23)

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\left[\frac{1}{2}M_{\rm Pl}^2R+M_{\rm Pl}^2\dot{H}g^{00}-M_{\rm Pl}^2\left(3H^2+\dot{H}\right)\right.\\ &\quad +\frac{1}{2!}M_2(t)^4\left(\delta g^{00}\right)^2+\frac{1}{3!}M_3(t)^4\left(\delta g^{00}\right)^3\\ &\quad \left.-\frac{\overline{M}_1(t)^3}{2}\left(\delta g^{00}\right)\delta K^\mu{}_\mu-\frac{\overline{M}_2(t)^2}{2}(\delta K^\mu{}_\mu)^2-\frac{\overline{M}_3(t)^2}{2}\delta K^\mu{}_\nu\delta K^\nu{}_\mu+\cdots\right], \end{aligned}$$

we obtain the action with restored gauge symmetry:

$$\begin{aligned} S=\int d^4x\sqrt{-g}&\left[\frac{1}{2}M_{\rm Pl}^2R-M_{\rm Pl}^2\left(3H^2(t+\pi)+\dot{H}(t+\pi)\right)\right.\\ &+M_{\rm Pl}^2\dot{H}(t+\pi)\left(\partial_\mu(t+\pi)\partial_\nu(t+\pi)g^{\mu\nu}\right)\\ &+\frac{M_2(t+\pi)^4}{2!}\left(\partial_\mu(t+\pi)\partial_\nu(t+\pi)g^{\mu\nu}+1\right)^2\\ &\left.+\frac{M_3(t+\pi)^4}{3!}\left(\partial_\mu(t+\pi)\partial_\nu(t+\pi)g^{\mu\nu}+1\right)^3+\ldots\right]. \end{aligned} \tag{4.46}$$

As in the case of $U(1)$ gauge theory, we see that the Goldstone boson equivalence theorem simplifies the action. Let us consider the case $M_2=M_3=\cdots=0$, which involves ordinary slow-roll inflation. The leading term in the mixing of gravity and the Nambu-Goldstone boson π is of the form

$$M_{\rm Pl}^2\dot{H}\left(\partial_\mu\pi\right)\delta g^{0\mu}. \tag{4.47}$$

Canonical normalization of gravity and the Nambu-Goldstone boson π are

$$\pi_c := \sqrt{2} M_{\rm Pl}(-\dot{H})^{1/2}\pi, \quad \delta g_c^{\mu\nu} := M_{\rm Pl}\delta g^{\mu\nu}. \tag{4.48}$$

Recalling the definition of the slow-roll parameter (2.10)

$$\epsilon = -\frac{\dot{H}}{H^2}, \tag{4.49}$$

the mixing term (4.47) is evaluated as

$$M_{\rm Pl}^2 \dot{H} \left(\partial_\mu \pi\right) \delta g^{0\mu} \sim (-\dot{H})^{1/2} \left(\partial_\mu \pi_c\right) \delta g_c^{0\mu} \sim \epsilon^{1/2} H \left(\partial_\mu \pi_c\right) \delta g_c^{0\mu}. \tag{4.50}$$

Hence, when the energy scale is

$$E \gg \epsilon^{1/2} H, \tag{4.51}$$

the mixing term is smaller than the kinetic term

$$M_{\rm Pl}^2 \dot{H} \left(\partial_\mu \pi\right) \left(\partial^\mu \pi\right) \sim \left(\partial_\mu \pi_c\right) \left(\partial^\mu \pi_c\right) \tag{4.52}$$

and can be neglected. That is, when the slow-roll parameter ϵ is small, mixing term is negligible inside the horizon (UV). In such a case, the action (4.46)

$$\begin{aligned} S = \int d^4x \sqrt{-g} \Bigg[& \frac{1}{2} M_{\rm Pl}^2 R - M_{\rm Pl}^2 \left(3H^2(t+\pi) + \dot{H}(t+\pi)\right) \\ & + M_{\rm PI}^2 \dot{H}(t+\pi) \left(\partial_\mu (t+\pi) \partial_\nu (t+\pi) g^{\mu\nu}\right) \\ & + \frac{M_2(t+\pi)^4}{2!} \left(\partial_\mu (t+\pi) \partial_\nu (t+\pi) g^{\mu\nu} + 1\right)^2 \\ & + \frac{M_3(t+\pi)^4}{3!} \left(\partial_\mu (t+\pi) \partial_\nu (t+\pi) g^{\mu\nu} + 1\right)^3 + \ldots \Bigg], \end{aligned}$$

becomes surprisingly easy:

$$\begin{aligned} S_\pi = \int d^4x \sqrt{-g} \Bigg[& \frac{1}{2} M_{\rm Pl}^2 R - M_{\rm Pl}^2 \dot{H} \left(\dot{\pi}^2 - \frac{(\partial_i \pi)^2}{a^2} \right) \\ & + 2M_2^4 \left(\dot{\pi}^2 + \dot{\pi}^3 - \dot{\pi} \frac{(\partial_i \pi)^2}{a^2} \right) - \frac{4}{3} M_3^4 \dot{\pi}^3 + \ldots \Bigg]. \end{aligned} \tag{4.53}$$

Chapter 5

in-in formalism and Feynman rule

5.1 Path integral in in-in formalism

Now that we have constructed the effective action of inflation, let us construct the feynman rule in curved space-time as a method to compute the correlation function. In this section, we consider theories with Lagrangian

$$\mathscr{L}_{\rm cl} = \frac{1}{2}\mathcal{U}_{AB}\varphi'^A\varphi'^B + \mathcal{V}_A(\varphi)\varphi'^A + \mathcal{W}(\varphi) \tag{5.1}$$

on the FLRW background according to [105], and derive the Feynman rule in in-in formalism (Schwinger-Keldysh formalism) by constructing path integrals. We denote the scalar fields by φ^A, where A, B are subscripts to distinguish different fields, a prime denotes the derivative with respect to the conformal time and $\mathcal{U}_{AB}$ is a "measurement" for the fields. $\mathcal{V}_A(\varphi), \mathcal{W}(\varphi)$ are arbitrary functions of φ and its spatial derivative, but the dependence of the spatial derivative is not explicitly mentioned to avoid complications. The canonical conjugate momentum π of φ and Hamiltonian are defined by

$$\begin{aligned} \pi_A &= \frac{\partial \mathscr{L}_{\rm cl}[\varphi]}{\partial \varphi'^A}, \\ \mathscr{H}[\pi,\varphi] &= \pi_A\varphi'^A - \mathscr{L}_{\rm cl}[\varphi], \end{aligned} \tag{5.2}$$

which are explicitly given by

$$\begin{aligned} \pi_A &= \mathcal{U}_{AB}\varphi'^B + \mathcal{V}_A(\varphi), \\ \mathscr{H}[\pi,\varphi] &= \frac{1}{2}\pi_A\pi^A - \mathcal{V}_A\pi^A + \frac{1}{2}\mathcal{V}_A\mathcal{V}^A - \mathcal{W}. \end{aligned} \tag{5.3}$$

Now, let us compute the expectation value $\langle Q(\eta)\rangle$ for operator

$$Q(\eta) := \varphi^{A_1}(\eta, \boldsymbol{x}_1)\cdots\varphi^{A_N}(\eta, \boldsymbol{x}_N). \tag{5.4}$$

The expectation value $\langle \cdots \rangle$ is taken for the state $|\Omega\rangle$. The state $|\Omega\rangle$ is determined at some initial time slice $\eta = \eta_0$, and usually the vacuum state at $\eta = \eta_0$ is adopted.

Starting from the Heisenberg picture, consider the path integral expression for the expectation value of $\langle Q(\eta)\rangle$. In the Heisenberg picture, the initial state is time-independent and the expectation value $\langle Q(\eta)\rangle$ is given by

$$\langle Q\rangle = \langle \Omega|Q(\eta)|\Omega\rangle. \tag{5.5}$$

In order to derive the path integral expression for the expectation value of $\langle Q(\eta)\rangle$, let us first rewrite $\langle Q(\eta)\rangle$ in terms of the amplitudes between the in and out states. We can naturally adopt $|\Omega\rangle$ as the in state, but we do not know what the out state is because the inflationary spacetime is a non-equilibrium system. To solve this problem, we perform averaging. That is, we choose a time slice Σ_f at arbitrary time $\eta_f \geq \eta$ and insert the complete set of basis of states

$$1 = \int \prod_{\boldsymbol{x}} dO_\alpha(\eta_f, \boldsymbol{x}) \left|O_\alpha(\eta_f, \boldsymbol{x})\right\rangle \left\langle O_\alpha(\eta_f, \boldsymbol{x})\right|. \tag{5.6}$$

into the expectation value $\langle Q(\eta)\rangle$ on Σ_f:

$$\langle Q\rangle = \int \prod_{\boldsymbol{x}} dO_\alpha(\eta_f, \boldsymbol{x}) \left\langle \Omega \left|O_\alpha(\eta_f, \boldsymbol{x})\right\rangle \left\langle O_\alpha(\eta_f, \boldsymbol{x})\right| \varphi^{A_1}(\eta, \boldsymbol{x}_1) \cdots \varphi^{A_N}(\eta, \boldsymbol{x}_N) \right| \Omega\right\rangle \tag{5.7}$$

where $O_\alpha(\eta, \boldsymbol{x}_i)$ is a local operator consisting of field operators in Lagrangian. All of these operators are on the same time slice. There is also the ambiguity that the complete set of basis can be inserted in many different places, but the result is the same no matter where it is inserted.

The two factors $\langle \Omega|O_\alpha\rangle$ and $\langle O_\alpha|Q|\Omega\rangle$ on the right-hand side of Eq. (5.7) are in the form of S-matrix, which is suitable for the path integral expression. In particular, $\langle \Omega|O_\alpha\rangle$ resembles a conjugate quantity of the vacuum amplitude with the time order of the in and out states flipped. Therefore, we call it the anti-time-ordered factor and similarly $\langle O_\alpha|Q|\Omega\rangle$ time-ordered factor.

In order to write down the path integral expression for the two factors, we foliate between the initial slice Σ_0 and the final slice Σ_f in infinitely many time slices, as usual. Then, the complete set of basis

$$1 = \int \prod_{\boldsymbol{x}} d\varphi(\eta_i, \boldsymbol{x}) \left|\varphi(\eta_i, \boldsymbol{x})\right\rangle \left\langle \varphi(\eta_i, \boldsymbol{x})\right| \tag{5.8}$$

of the field operator φ^A and the complete set of basis

$$1 = \int \prod_{\boldsymbol{x}} d\pi(\eta_i, \boldsymbol{x}) \left|\pi(\eta_i, \boldsymbol{x})\right\rangle \left\langle \pi(\eta_i, \boldsymbol{x})\right| \tag{5.9}$$

of its conjugate operator π_A are inserted into each slices Σ_i. In particular, the time-ordered factor can be written as the path integral

$$\begin{aligned}\left\langle O_\alpha(\eta_f, \boldsymbol{x}) \left|Q(\eta)\right| \Omega\right\rangle = \int \mathcal{D}\varphi_+ \mathcal{D}\pi_+ \exp\left[i \int_{\eta_0}^{\eta_f} d\eta d^3x \left(\pi_{+A}\varphi_+'^A - \mathscr{H}\left[\pi_+, \varphi_+\right]\right)\right] \\ \times \varphi_+^{A_1}(\eta, \mathrm{x}_1) \cdots \varphi_+^{A_N}(\eta, \mathrm{x}_N) \left\langle O_\alpha(\eta_0, \boldsymbol{x}) | \varphi_+(\eta_f)\right\rangle \left\langle \varphi_+(\eta_0) | \Omega\right\rangle\end{aligned} \tag{5.10}$$

over the field configuration $\varphi_+^A(\eta, \boldsymbol{x})$ and its conjugate momentum $\pi_+^A(\eta, \boldsymbol{x})$, and the anti-time-ordered factor can be written as the path integral

$$\begin{aligned}\left\langle \Omega | O_\alpha(\eta_f, \boldsymbol{x})\right\rangle = \int \mathcal{D}\varphi_- \mathcal{D}\pi_- \exp\left[-i \int_{\eta_0}^{\eta_f} d\eta d^3x \left(\pi_{-A}\varphi_-'^A - \mathscr{H}\left[\pi_-, \varphi_-\right]\right)\right] \\ \times \left\langle \varphi_-(\eta_f) | O_\alpha(\eta_0, \boldsymbol{x})\right\rangle \left\langle \Omega | \varphi_-(\eta_0)\right\rangle\end{aligned} \tag{5.11}$$

over the field configuration $\varphi_-^A(\eta, \boldsymbol{x})$ and its conjugate momentum $\pi_-^A(\eta, \boldsymbol{x})$. In the expressions above, we introduced variables with $+$ and $-$ subscripts for time-ordered factors and anti-time-ordered factors.

We show that the path integral expressions (5.10) and (5.11) can be obtained (the subscript A is omitted here). First, we insert the complete set at each time into the time-ordered factors $\langle O_\alpha(\eta_f, \boldsymbol{x})|Q(\eta)|\Omega\rangle$:

$$\begin{aligned}\langle O_\alpha(\eta_f, \boldsymbol{x})|Q(\eta)|\Omega\rangle &= \left\langle O_\alpha(\eta_0, \boldsymbol{x}) \left| e^{-iH(\eta_f-\eta_0)} Q(\eta) \right| \Omega\right\rangle \\ &= \int \prod_{\boldsymbol{x}} d\varphi(\eta_f, \boldsymbol{x}) \left\langle O_\alpha(\eta_0, \boldsymbol{x}) | \varphi(\eta_f, \boldsymbol{x})\right\rangle \left\langle \varphi(\eta_f, \boldsymbol{x})\right| \\ &\quad \times e^{-iH(\eta_f-\eta_0)} Q(\eta)|\Omega\rangle \\ &= \int \prod_{\boldsymbol{x}} d\varphi(\eta_f, \boldsymbol{x})\, d\varphi(\eta_{f-1}, \boldsymbol{x}) \left\langle O_\alpha(\eta_0, \boldsymbol{x}) | \varphi(\eta_f, \boldsymbol{x})\right\rangle \\ &\quad \times \left\langle \varphi(\eta_f, \boldsymbol{x})\right| e^{-iH(\eta_f-\eta_{f-1})} \left|\varphi(\eta_{f-1}, \boldsymbol{x})\right\rangle \left\langle \varphi(\eta_{f-1}, \boldsymbol{x})\right| \\ &\quad \times e^{-iH(\eta_{f-1}-\eta_0)} Q(\eta)|\Omega\rangle\end{aligned} \tag{5.12}$$

Repeat this process, and divide the time into N equal parts

$$\varepsilon := \frac{\eta_f - \eta_0}{N} \tag{5.13}$$

and set $N\to\infty$, we can obtain

$$\begin{aligned}\langle O_\alpha(\eta_f)|Q(\eta)|\Omega(\eta_0)\rangle &= \lim_{N\to\infty}\int\prod_{n=0}^{N-1}\prod_{\boldsymbol{x}} d\varphi(\eta_n,\boldsymbol{x})\,\langle O_\alpha(\eta_0,\boldsymbol{x})|\varphi(\eta_f)\rangle\\ &\quad\times Q(\eta)G[\varphi_{n+1},\varphi_n;\varepsilon]\,\langle\varphi(\eta_0,\boldsymbol{x})|\Omega\rangle\,,\end{aligned}\tag{5.14}$$

where $\varphi_n=\varphi(\eta_n,\boldsymbol{x})$ and $\eta_f=\eta_N$. We denote $G[\varphi_{n+1},\varphi_n;\varepsilon]$ by Green's function, which is given by

$$\begin{aligned}G[\varphi_{n+1},\varphi_n;\varepsilon] &:= \left\langle\varphi_{n+1}\left|e^{-iH\varepsilon}\right|\varphi_n\right\rangle\\ &= \langle\varphi_{n+1}|(1-iH\varepsilon)|\varphi_n\rangle\\ &= \langle\varphi_{n+1}|\varphi_n\rangle - iH(\varphi_n)\,\varepsilon\,\langle\varphi_{n+1}|\varphi_n\rangle\\ &= \langle\varphi_{n+1}|\varphi_n\rangle\,e^{-iH(\varphi_n)\varepsilon}\\ &= \int\prod_{\boldsymbol{x}}\frac{d\pi_n}{2\pi}\exp\left[i\int d^3x\,\pi_n\{\varphi_{n+1}-\varphi_n\}\right]\exp\left[-iH(\varphi_n)\,\varepsilon\right]\\ &= \int\prod_{\boldsymbol{x}}\frac{d\pi_n}{2\pi}\exp\left[i\varepsilon\int d^3x\left(\pi_n\frac{\varphi_{n+1}-\varphi_n}{\varepsilon}-\mathscr{H}\right)\right].\end{aligned}\tag{5.15}$$

In the transformation of the equation in the second to third lines, by acting the Hamiltonian as an operator on the state $|\varphi_n\rangle$, we replaced Hamiltonian with the c-number $H(\varphi_n)$. In the deformation of the equation in the fourth to fifth lines, we used the field theoretic version

$$\langle\varphi_{n+1}|\varphi_n\rangle=\int\prod_{\boldsymbol{x}}\frac{d\pi_n}{2\pi}\exp\left[i\int d^3x\,\pi_n\{\varphi_{n+1}-\varphi_n\}\right]\tag{5.16}$$

of orthogonality

$$\langle x_{n+1}|x_n\rangle=\int\frac{dp_n}{2\pi}\exp\left[ip_n(x_{n+1}-x_n)\right]\tag{5.17}$$

in quantum mechanics. Therefore, if the integral measure are written collectively as

$$\mathcal{D}\varphi\mathcal{D}\pi:=\lim_{N\to\infty}\prod_{n=0}^{N-1}\prod_{\boldsymbol{x}}\frac{d\varphi(\eta_n,\boldsymbol{x})\,d\pi(\eta_n,\boldsymbol{x})}{2\pi}=\prod_{\eta_0\le\eta\le\eta_f}\prod_{\boldsymbol{x}}\frac{d\varphi(\eta_n,\boldsymbol{x})\,d\pi(\eta_n,\boldsymbol{x})}{2\pi}\tag{5.18}$$

and we assign the subscript + to the field, the time-ordered factors are denoted as

$$\begin{aligned}\langle O_\alpha(\eta_f)|Q(\eta)|\Omega(\eta_0)\rangle &= \int \mathcal{D}\varphi_+\mathcal{D}\pi_+ \exp\left[i\int_{\eta_0}^{\eta_f} d\eta d^3x\ \left(\pi_{+A}\varphi_+^{\prime A} - \mathscr{H}[\pi_+,\varphi_+]\right)\right] \\ &\times\ \varphi_+^{A_1}(\eta,\mathrm{x}_1)\cdots\varphi_+^{A_N}(\eta,\mathrm{x}_N)\langle O_\alpha(\eta_0,\boldsymbol{x})|\varphi_+(\eta_f)\rangle\langle\varphi_+(\eta_0)|\Omega\rangle.\end{aligned} \tag{5.19}$$

Similarly, we can obtain

$$\begin{aligned}&\langle\Omega(\eta_0)|O_\alpha(\eta_f,\boldsymbol{x})\rangle \\ =\ & \left\langle\Omega\left|e^{-iH\eta_0}e^{iH\eta_f}\right|O_\alpha(\eta_0,\boldsymbol{x})\right\rangle \\ =\ & \int\prod_{\boldsymbol{x}} d\varphi(\eta_0,\boldsymbol{x})\langle\Omega|\varphi(\eta_0,\boldsymbol{x})\rangle\langle\varphi(\eta_0,\boldsymbol{x})|e^{-iH\eta_0}e^{iH\eta_f}|O_\alpha(\eta_0,\boldsymbol{x})\rangle \\ =\ & \int\prod_{\boldsymbol{x}} d\varphi(\eta_0,\boldsymbol{x})d\varphi(\eta_1,\boldsymbol{x})\langle\Omega|\varphi(\eta_0,\boldsymbol{x})\rangle\langle\varphi(\eta_0,\boldsymbol{x})|e^{iH(\eta_1-\eta_0)}|\varphi(\eta_1,\boldsymbol{x})\rangle \\ &\times\langle\varphi(\eta_1,\boldsymbol{x})|e^{iH\left(\eta_f-\eta_1\right)}|O_\alpha(\eta_0,\boldsymbol{x})\rangle \\ =\ & \int\prod_{\boldsymbol{x}} d\varphi(\eta_0,\boldsymbol{x})d\varphi(\eta_1,\boldsymbol{x})\langle\Omega|\varphi(\eta_0,\boldsymbol{x})\rangle G[\varphi_0,\varphi_1;-(\eta_0-\eta_1)] \\ &\times\langle\varphi(\eta_1,\boldsymbol{x})|e^{iH\left(\eta_f-\eta_1\right)}|O_\alpha(\eta_0,\boldsymbol{x})\rangle \\ =\ & \cdots \\ =\ & \lim_{N\to\infty}\int\prod_{n=0}^{N-1}\prod_{\boldsymbol{x}} d\varphi(\eta_n,\boldsymbol{x})\langle\Omega|\varphi(\eta_0,\boldsymbol{x})\rangle G[\varphi_n,\varphi_{n+1};-\varepsilon]\langle\varphi(\eta_f,\boldsymbol{x})|O_\alpha(\eta_0,\boldsymbol{x})\rangle \\ =\ & \int\mathcal{D}\varphi_-\mathcal{D}\pi_-\exp\left[-i\int_{\eta_0}^{\eta_f} d\eta d^3x\ \left(\pi_{-A}\varphi_-^{\prime A} - \mathscr{H}[\pi_-,\varphi_-]\right)\right] \\ &\times\langle\varphi_-(\eta_f)|O_\alpha(\eta_0,\boldsymbol{x})\rangle\langle\Omega|\varphi_-(\eta_0)\rangle\end{aligned} \tag{5.20}$$

as anti-time-ordered factors.□

Substituting the two factors (5.10) and (5.11) into the expectation value $\langle Q(\eta)\rangle$

(5.7), we obtain

$$\begin{aligned}\langle Q\rangle = &\int \mathcal{D}\varphi_+ \mathcal{D}\pi_+ \mathcal{D}\varphi_- \mathcal{D}\pi_- \varphi_+^{A_1}(\eta, \boldsymbol{x}_1)\cdots\varphi_+^{A_N}(\eta, \boldsymbol{x}_N) \\ &\times \exp\left[i\int_{\eta_0}^{\eta_f} d\eta d^3x\, \left(\pi_{+A}\varphi_+'^A - \mathscr{H}[\pi_+, \varphi_+]\right)\right] \\ &\times \exp\left[-i\int_{\eta_0}^{\eta_f} d\eta d^3x\, \left(\pi_{-A}\varphi_-'^A - \mathscr{H}[\pi_-, \varphi_-]\right)\right] \\ &\times \langle\Omega|\varphi_-(\eta_0)\rangle\langle\varphi_+(\eta_0)|\Omega\rangle \prod_{A,\boldsymbol{x}}\delta\left(\varphi_+^A(\eta_f, \boldsymbol{x}) - \varphi_-^A(\eta_f, \boldsymbol{x})\right).\end{aligned} \tag{5.21}$$

Note that the path integral is unbounded at two times $\eta = \eta_0$ and $\eta = \eta_f$, which means that it must integrate over all possible states $|\varphi_-(\eta_0)\rangle$ and $\langle\varphi_+(\eta_0)|$ that appear in the integrand. As a result, two copies of the path integral are obtained. One is in the forward direction of time and the other is in the backward direction of time, and both coincide in the limit η_f of future time by the condition

$$\varphi_+^A(\eta_f) = \varphi_-^A(\eta_f). \tag{5.22}$$

The integration over momentum $\pi_{\pm A}$ can be performed directly in theories without higher-order derivatives, i.e. Lagrangian (5.1)

$$\mathscr{L}_{\rm cl} = \frac{1}{2}\mathcal{U}_{AB}\varphi'^A\varphi'^B + \mathcal{V}_A(\varphi)\varphi'^A + \mathcal{W}(\varphi) \tag{5.23}$$

theory. This is because in such cases, the Hamiltonian (5.3)

$$\mathscr{H}[\pi, \varphi] = \frac{1}{2}\pi_A\pi^A - \mathcal{V}_A\pi^A + \frac{1}{2}\mathcal{V}_A\mathcal{V}^A - \mathcal{W} \tag{5.24}$$

is second-order in momentum and the momentum integral of Eq. (5.21) is Gaussian. Hence, if the Hamiltonian is (5.3) the path integral is

$$\begin{aligned}&\int \mathcal{D}\pi_+ \exp\left[i\int_{\eta_0}^{\eta_f} d\eta d^3x\, \left(\pi_{+A}\varphi_+'^A - \mathscr{H}[\pi_+, \varphi_+]\right)\right] \\ = \quad &\exp\left[i\int_{\eta_0}^{\eta_f} d\eta d^3x\, \left(\frac{1}{2}\mathcal{U}_{AB}\varphi_+'^A\varphi_+'^B + \mathcal{V}_A(\varphi_+)\varphi_+'^A + \mathcal{W}(\varphi_+)\right)\right],\end{aligned} \tag{5.25}$$

and the integrand is nothing but a classical Lagrangian, but the argument is written in φ_{+A}. Similarly, the integral over π_- can be also performed and the factor whose argument is written in φ_{-A} is obtained. Therefore, the expectation value (5.21) is simplified as

$$\langle Q\rangle = \int \mathcal{D}\varphi_+ \mathcal{D}\varphi_- \varphi_+^{A_1}(\eta, \boldsymbol{x}_1)\cdots\varphi_+^{A_N}(\eta, \boldsymbol{x}_N)\exp\left[i\int_{\eta_0}^{\eta_f} d\eta d^3x\, \left(\mathscr{L}_{\rm cl}[\varphi_+] - \mathscr{L}_{\rm cl}[\varphi_-]\right)\right] \\ \times \langle\Omega|\varphi_-(\eta_0)\rangle\langle\varphi_+(\eta_0)|\Omega\rangle\prod_{A,\boldsymbol{x}}\delta\left(\varphi_+^A(\eta_f,\boldsymbol{x}) - \varphi_-^A(\eta_f,\boldsymbol{x})\right). \tag{5.26}$$

However, if the theory involves higher-order derivatives, the momentum integral cannot be performed in a closed form. It has been shown that such momentum integrals can be performed perturbatively in [105] and the results are consistent with classical Lagrangian up to the fourth order of the field. Therefore, in the following, we assume that equation (5.26) also holds for the case involving higher-order derivatives.

The first line of the expression (5.26) is weighted by the exponential function of the action

$$S[\varphi_\pm] = \int d\eta d^3x\, \left(\mathscr{L}[\varphi_+] - \mathscr{L}[\varphi_-]\right) \tag{5.27}$$

and takes the form of a path integral. The second line is a factor absent in the conventional path integral, consisting of two inner products of states and a delta functional. The meaning of the delta functional is clear: the path integrals of φ_+^A and φ_-^A should coincide at the end time slice $\eta = \eta_f$. On the other hand, as we will show in the next section, the two inner products $\langle\Omega|\varphi_-(\eta_0)\rangle$ and $\langle\varphi_+(\eta_0)|\Omega\rangle$ are responsible for giving the correct $i\epsilon$ prescription for time integration.

5.2 $i\epsilon$ prescription

Let us show that the two inner products $\langle\Omega|\varphi_-(\eta_0)\rangle$ and $\langle\varphi_+(\eta_0)|\Omega\rangle$ give the correct $i\epsilon$ prescription for the time integral. It is important to note that the wave functional of the vacuum is expressed in terms of a field basis, and the state $|\Omega\rangle$ satisfies the equation

$$b_A|\Omega\rangle = 0 \tag{5.28}$$

where b_A is the annihilation operator.

First, we denote the annihilation operator b_A by the field φ^A and its conjugate momentum π_A. In order to be more concrete, let us consider the case of inflation (generalizations can be made straightforwardly). Let H be the Hubble parameter and the scale factor be

$$a(\eta) \simeq \frac{1}{|H\eta|}. \tag{5.29}$$

As an example, let $\mathcal{U}_{AB} = \delta_{AB}$ for the field metric and consider Lagrangian

$$\mathscr{L}_{\rm cl}[\varphi] = \sum_A \left[\frac{1}{2}a^2(\eta)\varphi_A'^2(\eta,\boldsymbol{x}) - \frac{1}{2}a^2(\eta)\left[\partial_i\varphi_A(\eta,\boldsymbol{x})\right]^2 - \frac{1}{2}a^4(\eta)M_A^2\varphi_A^2(\eta,\boldsymbol{x})\right] + \cdots \tag{5.30}$$

of some massive scalar fields. Performing Fourier transform of the field $\varphi_A(\eta,\boldsymbol{x})$ in the spatial direction yields

$$\varphi_A(\eta,\boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3}\left[u_A(\eta,\boldsymbol{k})b_A(\boldsymbol{k}) + u_A^*(\eta,-\boldsymbol{k})b_A^\dagger(-\boldsymbol{k})\right]e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{5.31}$$

where $b_A^\dagger, b_A$ are the creation and annihilation operators and the equation of motion of the mode function $u_A(\eta,\boldsymbol{k})$ is

$$u_A''(\eta,\boldsymbol{k}) - \frac{2}{\eta}u_A'(\eta,\boldsymbol{k}) + \left(\boldsymbol{k}^2 + \frac{M_A^2}{H^2\eta^2}\right)u_A(\eta,\boldsymbol{k}) = 0. \tag{5.32}$$

We choose the Bunch-Davis vacuum as the initial condition, and normalizing with

$$a^2(\eta)\left[u_A(\eta,\boldsymbol{k})u_A^{*\prime}(\eta,-\boldsymbol{k}) - u_A'(\eta,\boldsymbol{k})u_A^*(\eta,-\boldsymbol{k})\right] = i \tag{5.33}$$

to satisfy the canonical commutation relation

$$[\varphi_A(\eta,\boldsymbol{x}),\pi_B(\eta,\boldsymbol{y})] = i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\delta_{AB}, \tag{5.34}$$

$$\left[b_A(\boldsymbol{k}_1), b_B^\dagger(\boldsymbol{k}_2)\right] = (2\pi)^3\delta^{(3)}(\boldsymbol{k}_1-\boldsymbol{k}_2)\delta_{AB} \tag{5.35}$$

for the field $\varphi_A(\eta,\boldsymbol{x})$, its conjugate momentum $\pi_A = a^2(\eta)\varphi_A'$, and the creation and annihilation operators, we find that the solution of this equation is

$$u_A(\eta,\boldsymbol{k}) = -\frac{i\sqrt{\pi}}{2}e^{i\pi(\nu/2+1/4)}H(-\eta)^{3/2}H_{\nu_A}^{(1)}(-k\eta), \tag{5.36}$$

where $H_{\nu_A}^{(1)}(-k\eta)$ is a Hankel function of the first kind and ν_A is

$$\nu_A := \sqrt{\frac{9}{4} - \left(\frac{M_A}{H}\right)^2}. \tag{5.37}$$

Now, using the asymptotic form

$$H_\nu^{(1)}(x) \to \sqrt{\frac{2}{\pi x}} e^{i(x-\pi/4-\pi\nu/2)} \tag{5.38}$$

of the Hankel function in $x \to \infty$, the mode function in the infinite past is

$$u_A(\eta_0, \boldsymbol{k}) = \frac{iH\eta}{\sqrt{2k}} e^{-ik\eta_0}. \tag{5.39}$$

This means that in the infinite past η_0, these fields are not only free fields but also massless. Then, from the Fourier transform

$$\varphi_A(\eta, \boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3} \left[u_A(\eta, \boldsymbol{k}) b_A(\boldsymbol{k}) + u_A^*(\eta, -\boldsymbol{k}) b_A^\dagger(-\boldsymbol{k}) \right] e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{5.40}$$

$$\pi_A(\eta, \boldsymbol{x}) = a^2(\eta) \int \frac{d^3k}{(2\pi)^3} \left[u_A'(\eta, \boldsymbol{k}) b_A(\boldsymbol{k}) + u_A^{*\prime}(\eta, -\boldsymbol{k}) b_A^\dagger(-\boldsymbol{k}) \right] e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{5.41}$$

the annihilation operator can be solved as

$$b_A(\boldsymbol{k}) = -i \int d^3x \left[a^2(\eta) u_A^{*\prime}(\eta, -\boldsymbol{k}) \varphi_A(\eta, \boldsymbol{x}) - u_A^*(\eta, -\boldsymbol{k}) \pi_A(\eta, \boldsymbol{x}) \right] e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{5.42}$$

(without summing over A). Furthermore, since the conjugate momentum in the field basis $|\varphi_+(\eta_0)\rangle$ is

$$\pi_A(\eta, \boldsymbol{x}) = -i \frac{\delta}{\delta \varphi^A(\eta, \boldsymbol{x})}, \tag{5.43}$$

note that the equation

$$b_A |\Omega\rangle = 0 \tag{5.44}$$

expressed in the field basis has the following form

$$\begin{aligned} 0 &= \int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \left[\frac{\delta}{\delta \varphi_+^A(\eta_0, \boldsymbol{x})} - \frac{i a^2(\eta_0)\, u_A^{*\prime}(\eta_0, -\boldsymbol{k})}{u_A^*(\eta_0, \boldsymbol{k})} \varphi_{+A}(\eta_0, \boldsymbol{x}) \right] \langle \varphi_+(\eta_0) | \Omega \rangle \\ &= \int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \left[\frac{\delta}{\delta \varphi_+^A(\eta_0, \boldsymbol{x})} + a^2(\eta_0)\, k \varphi_{+A}(\eta_0, \boldsymbol{x}) \right] \langle \varphi_+(\eta_0) | \Omega \rangle . \end{aligned} \tag{5.45}$$

We have used the fact that the mode function $u_A(\eta, \boldsymbol{k})$ is massless expression (5.39) in $\eta_0 \to -\infty$. Assuming that the solution of the above equation is

$$\begin{aligned} \langle \varphi_+(\eta_0) | \Omega \rangle &= \mathcal{N} \exp\left[-\frac{1}{2} \int d^3x d^3y\, \mathcal{E}_{AB}(\eta_0; \boldsymbol{x}, \boldsymbol{y})\, \varphi_+^A(\eta_0, \boldsymbol{x})\, \varphi_+^B(\eta_0, \boldsymbol{y}) \right] \\ &= \mathcal{N} \exp\left[-\frac{\epsilon}{2} \int_{\eta_0}^{\eta_f} d\eta \int d^3x d^3y\, \mathcal{E}_{AB}(\eta; \boldsymbol{x}, \boldsymbol{y}) \varphi_+^A(\eta, \boldsymbol{x}) \varphi_+^B(\eta, \boldsymbol{y}) e^{\epsilon\eta} \right], \end{aligned} \tag{5.46}$$

we determine $\mathcal{E}_{AB}(\eta;\boldsymbol{x},\boldsymbol{y})$ such that the equation is satisfied. We denote by ϵ the positive infinitesimal parameter and by $\mathcal{N}$ the normalization factor of the wavefunctional. $\mathcal{E}_{AB}(\eta;\boldsymbol{x},\boldsymbol{y})$ is obtained by substituting the expression (5.46) back into (5.45):

$$\mathcal{E}_{AB}(\eta;\boldsymbol{x},\boldsymbol{y}) = a^2(\eta)\int \frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}k\delta_{AB}. \tag{5.47}$$

Therefore, the solution of the equation (5.45) is

$$\langle\varphi_+(\eta_0)|\Omega\rangle = \mathcal{N}\exp\left[-\frac{\epsilon}{2}\int_{\eta_0}^{\eta_f}d\eta\, a^2(\eta)\int\frac{d^3k}{(2\pi)^3}k\varphi_{+A}(\eta,\boldsymbol{k})\varphi_+^A(\eta,-\boldsymbol{k})\right]. \tag{5.48}$$

We neglect $e^{\epsilon\eta}$, which is a higher-order correction to ϵ. Similarly, the other inner product $\langle\Omega,|\varphi_-(\eta_0)\rangle$ contained in the expectation value (5.26) can be expressed as

$$\langle\Omega|\varphi_-(\eta_0)\rangle = \mathcal{N}^*\exp\left[-\frac{\epsilon}{2}\int_{\eta_0}^{\eta_f}d\eta\, a^2(\eta)\int\frac{d^3k}{(2\pi)^3}k\varphi_{-A}(\eta,\boldsymbol{k})\varphi_-^A(\eta,-\boldsymbol{k})\right]. \tag{5.49}$$

The factors $\mathcal{N},\mathcal{N}^*$ are not important. This is because they do not affect the expectation value since they also appear in the vacuum $\langle\Omega|\Omega\rangle$.

Now, let us substitute the inner product (5.48) and (5.49) into the expected value (5.26)

$$\begin{aligned}\langle Q\rangle = &\int\mathcal{D}\varphi_+\mathcal{D}\varphi_-\varphi_+^{A_1}(\eta,\boldsymbol{x}_1)\cdots\varphi_+^{A_N}(\eta,\boldsymbol{x}_N)\exp\left[i\int_{\eta_0}^{\eta_i}d\eta d^3x\left(\mathscr{L}_{\rm cl}[\varphi_+]-\mathscr{L}_{\rm cl}[\varphi_-]\right)\right]\\ &\times\langle\Omega|\varphi_-(\eta_0)\rangle\langle\varphi_+(\eta_0)|\Omega\rangle\prod_{A,\boldsymbol{x}}\delta\left(\varphi_+^A(\eta_f,\boldsymbol{x})-\varphi_-^A(\eta_f,\boldsymbol{x})\right).\end{aligned} \tag{5.50}$$

Then, two extra terms are added to Lagrangian:

$$\mathscr{L}_{\rm cl}[\varphi_\pm]\to\mathscr{L}_{\rm cl}[\varphi_\pm]\pm\frac{i\epsilon}{2}\int d\eta\, a^2(\eta)\int\frac{d^3k}{(2\pi)^3}k\varphi_{\pm A}(\boldsymbol{k})\varphi_\pm^A(-\boldsymbol{k}). \tag{5.51}$$

Comparing this with Lagrangian (5.30)

$$\mathscr{L}_{\rm cl}[\varphi] = \sum_A\left[\frac{1}{2}a^2(\eta)\varphi_A'^2(\eta,\boldsymbol{x})-\frac{1}{2}a^2(\eta)\left[\partial_i\varphi_A(\eta,\boldsymbol{x})\right]^2-\frac{1}{2}a^4(\eta)M_A^2\varphi_A^2(\eta,\boldsymbol{x})\right]+\cdots, \tag{5.52}$$

we see that the extra term makes a correction

$$k\eta\to(1-i\epsilon)k\eta \tag{5.53}$$

to the mode function (5.39) of the time-ordered variable φ_+^A and a correction

$$k\eta \to (1 + i\epsilon)k\eta \tag{5.54}$$

to the mode function of the anti-time-ordered variable φ_-^A. This means that the time-ordered part is deformed as

$$\eta \to (1 - i\epsilon)\eta \tag{5.55}$$

in the time direction of the complex plane, and the anti-time-ordered part is deformed as

$$\eta \to (1 + i\epsilon)\eta. \tag{5.56}$$

Therefore, the combined contribution of the two inner products yields the correct $i\epsilon$ formulation in the path integral. Hence, from now on, we assume that the time integral is properly transformed and remove the two inner products from the expectation value (5.26). Then, the expected value can be written as

$$\begin{aligned}\langle Q\rangle \;=\; & \int \mathcal{D}\varphi_+ \mathcal{D}\varphi_- \varphi_+^{A_1}(\eta, \boldsymbol{x}_1) \cdots \varphi_+^{A_N}(\eta, \boldsymbol{x}_N) \exp\left[i \int_{\eta_0}^{\eta_f} d\eta d^3x \left(\mathscr{L}_{\rm cl}[\varphi_+] - \mathscr{L}_{\rm cl}[\varphi_-]\right)\right] \\ & \times \prod_{A,\boldsymbol{x}} \delta\left(\varphi_+^A(\eta_f, \boldsymbol{x}) - \varphi_-^A(\eta_f, \boldsymbol{x})\right).\end{aligned} \tag{5.57}$$

This is the path integral expression for the in-in formalism of the expectational value at a given time. The formula is intuitive, as can be explained very well in the following three steps:

Path integral in in-in formalism
1. Make the field φ^A in the classical Lagrangian doubled: $\varphi_\pm^A$
2. Assign Lagrangian $\mathscr{L}$ for φ_+^A and Lagrangian $-\mathscr{L}$ for φ_-^A.
3. Match the path integral of $\varphi_\pm^A$ at the time slice of the end time $\eta = \eta_f$ by the delta functional.

Since the expectation value (5.57) has almost the same form as the ordinary path integral, it is easy to derive the diagrammatic rules for the perturbation calculation. This will be done in the next section.

5.3 generating functionals

Similar to the field theory on flat spacetime, the expectation value (5.57) can be computed by the functional derivative of the generating functionals, and the Feynman rule can be constructed. For simplicity of notation, we consider only real scalar fields φ, but the generalization is straightforward.

First, we introduce the source $J_\pm(\eta, \boldsymbol{x})$ for the scalar field $\varphi_\pm(\eta, \boldsymbol{x})$ and define a generating functional

$$Z\left[J_+, J_-\right] = \int \mathcal{D}\varphi_+ \mathcal{D}\varphi_- \exp\left[i\int_{\eta_0}^{\eta_f} \mathrm{d}\eta d^3x \left(\mathscr{L}_{\mathrm{cl}}\left[\varphi_+\right] - \mathscr{L}_{\mathrm{cl}}\left[\varphi_-\right] + J_+\varphi_+ - J_-\varphi_-\right)\right]. \tag{5.58}$$

Then the general amplitude $\langle \varphi_{a_1}(\eta, \boldsymbol{x}_1) \cdots \varphi_{a_N}(\eta, \boldsymbol{x}_N)\rangle$ $(a_1, \cdots, a_N = \pm)$ can be computed by taking the functional derivative as usual. However, since the sign differs in differentiating + type fields and − type fields, $a_i = \pm$ is assigned to each derivatives:

$$\begin{aligned} & \langle \varphi_{a_1}(\eta, \boldsymbol{x}_1) \cdots \varphi_{a_N}(\eta, \boldsymbol{x}_N)\rangle \\ = & \int \mathcal{D}\varphi_+ \mathcal{D}\varphi_- \varphi_{a_1}(\eta, \boldsymbol{x}_1) \cdots \varphi_{a_N}(\eta, \boldsymbol{x}_N) \\ & \times \exp\left[i\int_{\eta_0}^{\eta_f} d\eta d^3x \left(\mathscr{L}_{\mathrm{cl}}\left[\varphi_+\right] - \mathscr{L}_{\mathrm{cl}}\left[\varphi_-\right] + J_+\varphi_+ - J_-\varphi_-\right)\right] \\ = & \frac{\delta}{ia_1\delta J_{a_1}(\eta, \boldsymbol{x}_1)} \cdots \frac{\delta}{ia_N\delta J_{a_N}(\eta, \boldsymbol{x}_N)} Z\left[J_+, J_-\right]\Big|_{J_\pm=0}. \end{aligned} \tag{5.59}$$

By separating the Lagrangian into free field part $\mathscr{L}_0$ and interaction part $\mathscr{L}_{\mathrm{int}}$, as in

$$\mathscr{L}_{\mathrm{cl}}[\varphi] = \mathscr{L}_0[\varphi] + \mathscr{L}_{\mathrm{int}}[\varphi], \tag{5.60}$$

we can calculate this amplitude perturbatively. Next, let the generating functional (5.58) separate the free field part $Z_0\left[J_+, J_-\right]$ as in

$$Z\left[J_+, J_-\right] = \exp\left[i\int_{\eta_0}^{\eta_f} d\eta d^3x \left(\mathscr{L}_{\mathrm{int}}\left[\frac{\delta}{i\delta J_+}\right] - \mathscr{L}_{\mathrm{int}}\left[-\frac{\delta}{i\delta J_-}\right]\right)\right] Z_0\left[J_+, J_-\right], \tag{5.61}$$

$$Z_0\left[J_+, J_-\right] \equiv \int \mathcal{D}\varphi_+ \mathcal{D}\varphi_- \exp\left[i\int_{\eta_0}^{\eta_f} d\eta d^3x \left(\mathscr{L}_0\left[\varphi_+\right] - \mathscr{L}_0\left[\varphi_-\right] + J_+\varphi_+ - J_-\varphi_-\right)\right]. \tag{5.62}$$

Note that the path integral of the free field part $Z_0[J_+, J_-]$ is Gaussian, and can be performed explicitly. Then, (5.60) can be perturbatively expanded and combined with

(5.59) to compute the expected value $\langle Q\rangle$. Let's see how the Feynman rule is obtained with examples.

5.4 Propagators

First, we consider the tree-level propagators. It is defined by the two-point function as

$$-i\Delta_{ab}\left(\eta_1,\boldsymbol{x}_1;\eta_2,\boldsymbol{x}_2\right):=\left.\frac{\delta}{ia\delta J_a\left(\eta_1,\boldsymbol{x}_1\right)}\frac{\delta}{ib\delta J_b\left(\eta_2,\boldsymbol{x}_2\right)}Z_0\left[J_+,J_-\right]\right|_{J_\pm=0} \tag{5.63}$$

where $a,b=\pm$. There are four types of propagators depending on the choice of subscript a,b. For example, the propagator of type $(+,+)$ has the form

$$\begin{aligned}
-i\Delta_{++}\left(\eta_1,\boldsymbol{x}_1;\eta_2,\boldsymbol{x}_2\right) &= \left.\frac{\delta}{i\delta J_+\left(\eta_1,\boldsymbol{x}_1\right)}\frac{\delta}{i\delta J_+\left(\eta_2,\boldsymbol{x}_2\right)}Z_0\left[J_+,J_-\right]\right|_{J_\pm=0}\\
&= \int\mathcal{D}\varphi_+\mathcal{D}\varphi_-\varphi_+\left(\eta_1,\boldsymbol{x}_1\right)\varphi_+\left(\eta_2,\boldsymbol{x}_2\right)e^{i\int d\eta d^3x\left(\mathscr{L}_0[\varphi_+]-\mathscr{L}_0[\varphi_-]\right)}\\
&= \left\langle\Omega\left|\mathrm{T}\left\{\varphi\left(\eta_1,\boldsymbol{x}_1\right)\varphi\left(\eta_2,\boldsymbol{x}_2\right)\right\}\right|\Omega\right\rangle .
\end{aligned} \tag{5.64}$$

Similarly, other propagators are obtained: The propagators of type $(-,-)$, $(+,-)$ and $(-,+)$ are

$$\begin{aligned}
-i\Delta_{--}\left(\eta_1,\boldsymbol{x}_1;\eta_2,\boldsymbol{x}_2\right) &= \left.\frac{-\delta}{i\delta J_-\left(\eta_1,\boldsymbol{x}_1\right)}\frac{-\delta}{i\delta J_-\left(\eta_2,\boldsymbol{x}_2\right)}Z_0\left[J_+,J_-\right]\right|_{J_\pm=0}\\
&= \int\mathcal{D}\varphi_+\mathcal{D}\varphi_-\varphi_-\left(\eta_1,\boldsymbol{x}_1\right)\varphi_-\left(\eta_2,\boldsymbol{x}_2\right)e^{i\int d\eta d^3x\left(\mathscr{L}_0[\varphi_+]-\mathscr{L}_0[\varphi_-]\right)}\\
&= \left\langle\Omega\left|\overline{\mathrm{T}}\left\{\varphi\left(\eta_1,\boldsymbol{x}_1\right)\varphi\left(\eta_2,\boldsymbol{x}_2\right)\right\}\right|\Omega\right\rangle ,
\end{aligned} \tag{5.65}$$

$$\begin{aligned}
-i\Delta_{+-}\left(\eta_1,\boldsymbol{x}_1;\eta_2,\boldsymbol{x}_2\right) &= \left.\frac{\delta}{i\delta J_+\left(\eta_1,\boldsymbol{x}_1\right)}\frac{-\delta}{i\delta J_-\left(\eta_2,\boldsymbol{x}_2\right)}Z_0\left[J_+,J_-\right]\right|_{J_\pm=0}\\
&= \int\mathcal{D}\varphi_+\mathcal{D}\varphi_-\varphi_+\left(\eta_1,\boldsymbol{x}_1\right)\varphi_-\left(\eta_2,\boldsymbol{x}_2\right)e^{i\int d\eta d^3x\left(\mathscr{L}_0[\varphi_+]-\mathscr{L}_0[\varphi_-]\right)}\\
&= \left\langle\Omega\left|\varphi\left(\eta_2,\boldsymbol{x}_2\right)\varphi\left(\eta_1,\boldsymbol{x}_1\right)\right|\Omega\right\rangle ,
\end{aligned} \tag{5.66}$$

$$\begin{aligned}
-i\Delta_{-+}\left(\eta_1,\boldsymbol{x}_1;\eta_2,\boldsymbol{x}_2\right) &= \left.\frac{-\delta}{i\delta J_-\left(\eta_1,\boldsymbol{x}_1\right)}\frac{\delta}{i\delta J_+\left(\eta_2,\boldsymbol{x}_2\right)}Z_0\left[J_+,J_-\right]\right|_{J_\pm=0}\\
&= \int\mathcal{D}\varphi_+\mathcal{D}\varphi_-\varphi_-\left(\eta_1,\boldsymbol{x}_1\right)\varphi_+\left(\eta_2,\boldsymbol{x}_2\right)e^{i\int d\eta d^3x\left(\mathscr{L}_0[\varphi_+]-\mathscr{L}_0[\varphi_-]\right)}\\
&= \left\langle\Omega\left|\varphi\left(\eta_1,\boldsymbol{x}_1\right)\varphi\left(\eta_2,\boldsymbol{x}_2\right)\right|\Omega\right\rangle
\end{aligned} \tag{5.67}$$

respectively. Because of the translational and rotational invariance on each time slice, it is possible to transform to the 3-dimensional momentum space. Describing the field φ in terms of the mode function $u(\eta, \boldsymbol{k})$ and the creation and annihilation operators for a given 3-dimensional momentum $\boldsymbol{k}$, we substitute it into the above four types of propagators. Then, we obtain the propagators in momentum space, which is connected to the coordinate space by the relation

$$G_{ab}\left(\eta_1, \eta_2, k\right) = -i \int d^3x e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \Delta_{ab}\left(\eta_1, \boldsymbol{x}; \eta_2, \boldsymbol{0}\right). \tag{5.68}$$

The reason why G_{ab} is appended with $-i$ is to prevent extra factors from appearing in the rules in momentum space. Furthermore, the momentum dependence of G_{ab} can be written as $k = |\boldsymbol{k}|$. This is because the propagators do not depend on the direction of 3-dimensional momentum but only on its magnitude, due to the rotational symmetry. Then, the propagators in 3-dimensional momentum space are easily obtained:

$$G_{++}\left(\eta_1, \eta_2, k\right) = G_{>}\left(\eta_1, \eta_2, k\right) \theta\left(\eta_1 - \eta_2\right) + G_{<}\left(\eta_1, \eta_2, k\right) \theta\left(\eta_2 - \eta_1\right), \tag{5.69}$$

$$G_{+-}\left(\eta_1, \eta_2, k\right) = G_{<}\left(\eta_1, \eta_2, k\right), \tag{5.70}$$

$$G_{-+}\left(\eta_1, \eta_2, k\right) = G_{>}\left(\eta_1, \eta_2, k\right), \tag{5.71}$$

$$G_{--}\left(\eta_1, \eta_2, k\right) = G_{<}\left(\eta_1, \eta_2, k\right) \theta\left(\eta_1 - \eta_2\right) + G_{>}\left(\eta_1, \eta_2, k\right) \theta\left(\eta_2 - \eta_1\right), \tag{5.72}$$

where we defined

$$G_{>}\left(\eta_1, \eta_2, k\right) := u\left(\eta_1, k\right) u^*\left(\eta_2, k\right), \tag{5.73}$$

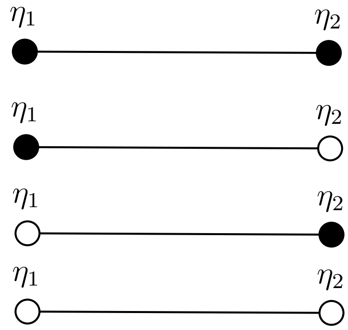
$$G_{<}\left(\eta_1, \eta_2, k\right) := u^*\left(\eta_1, k\right) u\left(\eta_2, k\right). \tag{5.74}$$

It is clear that these propagators are not completely independent. We can immediately see that only three of the four propagators are linearly independent. If complex conjugation is taken into account, further relations

$$G_{>}^* = G_{<}, \quad G_{++}^* = G_{--}, \quad G_{+-}^* = G_{-+} \tag{5.75}$$

are also obtained.

As a graph, black dots and white dots are assigned to represent + and −, respectively. Hence, the four propagators are denoted by



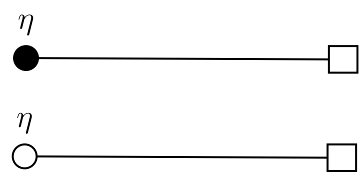
$$= G_{++}(\eta_1, \eta_2, k),$$

$$= G_{+-}(\eta_1, \eta_2, k),$$

$$= G_{-+}(\eta_1, \eta_2, k),$$

$$= G_{--}(\eta_1, \eta_2, k).$$

The propagators derived above apply to both the internal (bulk propagators) and external (bulk-to-boundary propagators) of the graph. We adopt η_f as an argument for the external line connecting the time slice (boundary point) of the end time $\eta = \eta_f$. Since there is no distinction between $+$ and $-$ for boundary points, there are only two types of bulk-to-boundary propagators. We assign a square to the Boundary point:



$$= G_{++}(\eta, \eta_f, k),$$

$$= G_{-+}(\eta, \eta_f, k).$$

In inflation (i.e., in a quasi-de Sitter background), the end-time slice η_f is an infinite future.

5.5 Vertices

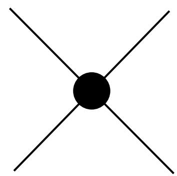
Next, let us consider vertices. For one interaction vertex in the original Lagrangian, we need to write down two vertices corresponding to $+, -$ types. Then, an extra minus sign is added to the $-$ type vertex. To understand this rule, we will look at simple examples of non-derivative couplings and derivative couplings.

First, we look at the rules for non-derivative couplings. It is almost the same as the quantum field theory of ordinary flat space-time. The only difference is that only the spatial coordinates is Fourier transformed, while the temporal coordinate is not. Therefore, the rules for vertex are a mixture of Feynman rules for coordinate space and

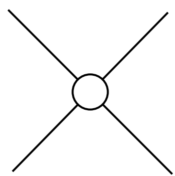
Feynman rules for momentum space. Let us consider the $\lambda\varphi^4$ theory as an example:

$$\mathscr{L}_{\text{int}} \supset -\frac{\lambda}{4!} a^4(\eta)\varphi^4 \tag{5.76}$$

where $a^4(\eta)$ comes from $\sqrt{-g}$ contained in the action. The following rules are obtained for verticies in 3-dimensional momentum space:



$$= -i\lambda \int_{\eta_0}^{\eta_f} d\eta \; a^4(\eta) \cdots$$



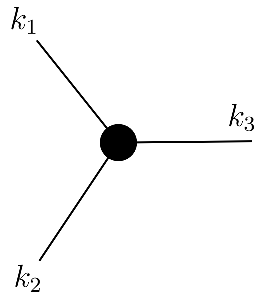
$$= +i\lambda \int_{\eta_0}^{\eta_f} d\eta \; a^4(\eta) \cdots$$

where $\cdots$ denotes all η-dependent parts of the graph coming from the propagators connected to the vertex.

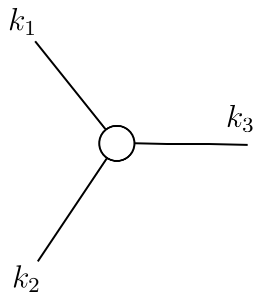
Next, we look at the rules for derivative couplings. Since the Fourier transform is performed only in spatial coordinates, it must be divided into spatial and time derivatives. The spatial derivative is easy since it becomes a momentum after the Fourier transform. For example, the rule for interaction

$$\mathscr{L}_{\text{int}} \supset -\frac{\lambda}{3!} a^2(\eta)\varphi\left(\partial_i\varphi\right)\left(\partial_i\varphi\right) \tag{5.77}$$

is given by



$$= +\frac{i\lambda}{3}(\boldsymbol{k}_1 \cdot \boldsymbol{k}_2 + \boldsymbol{k}_2 \cdot \boldsymbol{k}_3 + \boldsymbol{k}_3 \cdot \boldsymbol{k}_1) \ \int_{\eta_0}^{\eta_f} d\eta \ a^2(\eta) \cdots$$

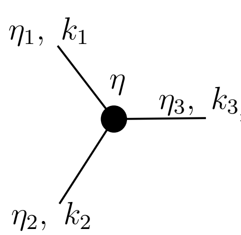


$$= -\frac{i\lambda}{3}(\boldsymbol{k}_1 \cdot \boldsymbol{k}_2 + \boldsymbol{k}_2 \cdot \boldsymbol{k}_3 + \boldsymbol{k}_3 \cdot \boldsymbol{k}_1) \ \int_{\eta_0}^{\eta_f} d\eta \ a^2(\eta) \cdots$$

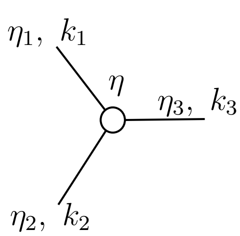
noting that the two $(\partial_i \varphi)$ are indistinguishable, which yields a symmetry factor 2. On the other hand, the time derivative is not Fourier transformed and must act directly on the propagators. For example, the rule for interaction

$$\mathscr{L}_{\text{int}} \supset -\frac{\lambda}{3!} a^2(\eta) \varphi \varphi'^2 \tag{5.78}$$

is given by



$$= -\frac{i\lambda}{3} \int_{\eta_0}^{\eta_f} d\eta \ a^2(\eta) [\partial_\eta G_{+a_1}(\eta, \eta_1, k_1)][\partial_\eta G_{+a_2}(\eta, \eta_2, k_2)] G_{+a_3}(\eta, \eta_3, k_3)$$
$$+2 \text{ permutations}$$



$$= +\frac{i\lambda}{3} \int_{\eta_0}^{\eta_f} d\eta \ a^2(\eta) [\partial_\eta G_{+a_1}(\eta, \eta_1, k_1)][\partial_\eta G_{+a_2}(\eta, \eta_2, k_2)] G_{+a_3}(\eta, \eta_3, k_3)$$
$$+2 \text{ permutations}$$

noting that the two φ'^2 are indistinguishable, which yields the symmetry factor 2, where $a_1, a_2, a_3 = \pm$.

5.6 Summary of Feynman rule

We summarize here the Feynman rule for calculating expectation values.

- We separate the classical Lagrangian $\mathscr{L}_{\rm cl}[\varphi]$ into the free field part $\mathscr{L}_0[\varphi]$ and the interaction part $\mathscr{L}_{\rm int}[\varphi]$. Solve the equation of motion
$$\frac{\delta \mathscr{L}_0[\varphi]}{\delta\varphi} = 0 \tag{5.79}$$
under the given initial condition $|\Omega\rangle$ to obtain the mode function of the field φ. Due to the asymmetry of space-time, Fourier transforms are performed for spatial coordinates, but not for temporal coordinates. Hence, the mode function is expressed as $u(\eta, \boldsymbol{k})$ with 3-dimensional momentum $\boldsymbol{k}$.

- For each $\varphi(\eta, \boldsymbol{k}_i)$ in $Q(\eta)$, draw a square and call it an external point. As in the usual flat space-time perturbation theory, write the vertices read from the interaction part $\mathscr{L}_{\rm int}[\varphi]$. Then, in all possible combinations, connect vertices and external point with a line (but do not connect two external point with a line). Up to this point, the rules are the same as those in ordinary flat spacetime field theories.

- In all possible combinations, assign each vertex a black dot (called a + type vertex) or a white dot (called a − type vertex). Therefore, for theories with V vertices, we obtain 2^V graphs.

- Assign propagators to the line connecting the two vertices:
$$G_{++}(\eta_1, \eta_2, k) = G_>(\eta_1, \eta_2, k)\,\theta(\eta_1 - \eta_2) + G_<(\eta_1, \eta_2, k)\,\theta(\eta_2 - \eta_1), \tag{5.80}$$
$$G_{+-}(\eta_1, \eta_2, k) = G_<(\eta_1, \eta_2, k), \tag{5.81}$$
$$G_{-+}(\eta_1, \eta_2, k) = G_>(\eta_1, \eta_2, k), \tag{5.82}$$
$$G_{--}(\eta_1, \eta_2, k) = G_<(\eta_1, \eta_2, k)\,\theta(\eta_1 - \eta_2) + G_>(\eta_1, \eta_2, k)\,\theta(\eta_2 - \eta_1). \tag{5.83}$$

There are four propagators depending on the type of the two vertices. Assign a boundary-to-bulk propagator to the line connecting the vertex and the external line. There are two boundary-to-bulk propagators $G_{++}(\eta, \eta_f, k)$, $G_{-+}(\eta, \eta_f, k)$ depending on the type of bulk vertex. The momentum of each propagators must be chosen such that the total momentum is conserved at each vertex.

- Assign appropriate factors to each vertex derived from Lagrangian. An extra minus sign is added for $-$ type graphs. Undetermined independent three-dimensional momenta are integrated. For each vertex, integrate over time from the initial slice $\eta = \eta_0$ to the final slice $\eta = \eta_f$.

- The symmetry factor is the same as the rule in ordinary flat spacetime field theories.

- The final result of the expectation value $\langle Q \rangle'$ is the sum of all graphs where $\langle Q \rangle'$ is

$$\langle Q \rangle = (2\pi)^3 \delta^3 \left(\sum_i \boldsymbol{k}_i \right) \langle Q \rangle'. \tag{5.84}$$

In the in-in formalism, the number of fields is doubled and the number of vertices in the graph is doubled, so the number of vertices in the internal line increases and the computation appears to be more complicated. However, it is not necessary to compute all the graphs. This is because there exists graphs in complex conjugate relation. As we can easily see from the rule that a graph of $-$ type should have an extra minus sign, the complex conjugate of the graph can be obtained by turning over the black and white dots. An arbitrary expectation value is finally given by the sum of the graphs of all possible black dots and white dots. Thus, from the rule of complex conjugation, we immediately see that the expectation value $\langle Q \rangle'$ is real.

Chapter 6

Set up of our model

This Chapters 6 to 8 are the original results of this doctoral thesis. We consider the FLRW spacetime with fluctuations and curvature $K = 0$ expressed in the ADM formalism as the inflationary spacetime.

$$ds^2 = -N^2 dt^2 + \tilde{h}_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right), \tag{6.1}$$

where $\tilde{h}_{ij}$ is a spatial components of the metric, N is the lapse function and N^i is the shift function. The tilde represents a physical quantity in the comoving gauge. The action we consider is

$$S = S_{\text{grav}} + S_{\text{SU(5) gauge}} + S_{\text{SU(5) Higgs}} + S_{\text{SU(5) fermion}}, \tag{6.2}$$

where $S_{\text{grav}}, S_{\text{SU(5) gauge}}, S_{\text{SU(5) Higgs}}$ and $S_{\text{SU(5) fermion}}$ describe the actions of the gravity, SU(5) gauge bosons, an SU(5) adjoint Higgs boson to break GUT gauge symmetry and SU(5) fermions respectively[1]. We will now examine each action in detail but omit the $S_{\text{SU(5) gauge}}$ and $S_{\text{SU(5) fermion}}$ since they are unnecessary for our computation of the graph Fig.1.2. From the viewpoint of the effective field theory [96, 99], the Einstein-Hilbert action and the inflaton action after the transformation of time coordinates

[1]In the following, we consider in this thesis the SU(5) GUT as an illustrating example, but it can be easily extended to other GUT gauge group.

$t \mapsto \tilde{t} = t - \pi(\tilde{t}, \boldsymbol{x})$ can be written as

$$
\begin{aligned}
S_{\rm grav} = \int d^4x\sqrt{-g}\,\Big[&\frac{1}{2}M_{\rm Pl}^2 R - M_{\rm Pl}^2\left(3H^2(\tilde{t}+\pi) + \dot{H}(\tilde{t}+\pi)\right) \\
&+M_{\rm Pl}^2\dot{H}(\tilde{t}+\pi)\left(\partial_\mu(\tilde{t}+\pi)\partial_\nu(\tilde{t}+\pi)g^{\mu\nu}\right) \\
&+\frac{M_2(\tilde{t}+\pi)^4}{2!}\left(\partial_\mu(\tilde{t}+\pi)\partial_\nu(\tilde{t}+\pi)g^{\mu\nu}+1\right)^2 \\
&+\frac{M_3(\tilde{t}+\pi)^4}{3!}\left(\partial_\mu(\tilde{t}+\pi)\partial_\nu(\tilde{t}+\pi)g^{\mu\nu}+1\right)^3+\ldots\Big],
\end{aligned} \tag{6.3}
$$

where g is the determinant of the metric $g_{\mu\nu}$, $M_{\rm Pl}$ is the Planck mass, R is the Ricci curvature in four dimensions, H is the Hubble parameter, π is a Nambu-Goldstone boson of time translation which is identified with the inflaton. $M_{2,3}$ are the coefficients of the high-dimensional operators.

The quadratic terms of the effective action of inflaton π is identified as [96, 100]

$$
I_2 = M_{\rm Pl}^2\int dtd^3x\, a^3\left[-\frac{\dot{H}}{c_s^2}\left(\dot{\pi}^2 - c_s^2\frac{(\partial_i\pi)^2}{a^2}\right)\right]. \tag{6.4}
$$

The sound speed c_s is a quantity that is not determined by the effective field theory, but is bounded by the fundamental theory and observational data. The mode function $w(\tau, k)$ of the inflaton π is obtained as

$$
w(\tau,k) = \frac{c_s}{\sqrt{2\epsilon}aHM_{\rm Pl}}\frac{1+ic_sk\tau}{\sqrt{2c_sk}c_sk\tau}e^{-ic_sk\tau}, \tag{6.5}
$$

where τ is the conformal time and ϵ is the slow roll parameter. Using the expression for the propagators by in-in formalism in Appendix B, the propagators of inflaton π is obtained as

$$
\begin{aligned}
\Delta_>(\tau_1,\tau_2,k) &= w(\tau_1,k)\,w^*(\tau_2,k) \\
&= \frac{1+ic_sk(\tau_1-\tau_2)+(c_sk)^2\tau_1\tau_2}{4\epsilon M_{\rm Pl}c_sk^3}e^{-ic_sk(\tau_1-\tau_2)},
\end{aligned} \tag{6.6}
$$

$$
\begin{aligned}
\Delta_<(\tau_1,\tau_2,k) &= w(\tau_2,k)\,w^*(\tau_1,k) \\
&= \frac{1-ic_sk(\tau_1-\tau_2)+(c_sk)^2\tau_1\tau_2}{4\epsilon M_{\rm Pl}c_sk^3}e^{ic_sk(\tau_1-\tau_2)}
\end{aligned} \tag{6.7}
$$

Next, we discuss the action of the Higgs boson for GUT gauge symmetry breaking to the SM gauge symmetry. Let us denote Σ' for the Higgs of the 24-dimensional adjoint representation of SU(5) and its renormalizable potential is introduced as

$$
V(\Sigma') = -M^2\mathrm{tr}(\Sigma'^2) + \lambda_1\left\{\mathrm{tr}(\Sigma'^2)\right\}^2 + \lambda_2\,\mathrm{tr}(\Sigma'^4), \tag{6.8}
$$

where M, λ_1 and λ_2 are constants, and M is an order of the GUT scale.

$$M \sim \mathcal{O}(M_{\rm GUT}). \tag{6.9}$$

We expand the adjoint Higgs Σ' around the expectation value

$$\langle\Sigma\rangle = v \ {\rm diag}(2,2,2,-3,-3), \quad v \sim \mathcal{O}(M_{\rm GUT}) \tag{6.10}$$

as

$$\Sigma' = \langle\Sigma\rangle + \Sigma. \tag{6.11}$$

Now, we calculate three point function of inflaton to extract non-Gaussianity by the existence of Σ, which is given by a graph of tree level exchange of Σ shown in Fig. 1.2. In order to extract the first-order term of Σ in the potential, we expand Σ'^2 around the expectation value yields

$$\begin{aligned} \Sigma'^2 &= (\langle\Sigma\rangle + \Sigma)^2 \\ &= \langle\Sigma\rangle^2 + \langle\Sigma\rangle\Sigma + \Sigma\langle\Sigma\rangle + \Sigma^2, \end{aligned} \tag{6.12}$$

and we obtain

$$\begin{aligned} 2{\rm tr}\left(\langle\Sigma\rangle\Sigma\right) &= 2 \times (2v\Sigma_{11} + 2v\Sigma_{22} + 2v\Sigma_{33} - 3v\Sigma_{44} - 3v\Sigma_{55}) \\ &= 2 \times 5v\left(\Sigma_{11} + \Sigma_{22} + \Sigma_{33}\right) \\ &= 10v\left(\Sigma_{11} + \Sigma_{22} + \Sigma_{33}\right) \end{aligned} \tag{6.13}$$

from the terms in ${\rm tr}(\Sigma'^2)$. Note that only diagonal components of Σ are taken into account and the traceless condition for Σ

$$\Sigma_{11} + \Sigma_{22} + \Sigma_{33} + \Sigma_{44} + \Sigma_{55} = 0. \tag{6.14}$$

is used in the second equality. Then, since the first term of (6.12) can be computed as

$${\rm tr}\left(\langle\Sigma\rangle^2\right) = 30v^2, \tag{6.15}$$

we have

$${\rm tr}(\Sigma'^2) = 30v^2 + 10v\left(\Sigma_{11} + \Sigma_{22} + \Sigma_{33}\right) + {\rm tr}(\Sigma^2). \tag{6.16}$$

Now we can obtain the linear term for Σ

$$\begin{aligned}\lambda_1\left\{\mathrm{tr}(\Sigma'^2)\right\}^2 &= \lambda_1\left[30v^2+10v\left(\Sigma_{11}+\Sigma_{22}+\Sigma_{33}\right)+\mathrm{tr}(\Sigma^2)\right]^2\\ &\supset 600\lambda_1 v^3\left(\Sigma_{11}+\Sigma_{22}+\Sigma_{33}\right)\end{aligned}\tag{6.17}$$

from the second term in potential (6.8). Similarly,

$$\lambda_2\,\mathrm{tr}(\Sigma'^4)\supset 140\lambda_2 v^3\left(\Sigma_{11}+\Sigma_{22}+\Sigma_{33}\right)\tag{6.18}$$

is obtained from the third term in (6.8).

Next, we consider cubic term for Σ in potential (6.8):

$$\begin{aligned}\lambda_1\left\{\mathrm{tr}(\Sigma'^2)\right\}^2 &= \lambda_1\left[30v^2+10v\left(\Sigma_{11}+\Sigma_{22}+\Sigma_{33}\right)+\mathrm{tr}(\Sigma^2)\right]^2\\ &\supset 20\lambda_1 v\left(\Sigma_{11}+\Sigma_{22}+\Sigma_{33}\right)\mathrm{tr}(\Sigma^2)\end{aligned}\tag{6.19}$$

The last term $\mathrm{tr}(\Sigma^2)$ is

$$\mathrm{tr}(\Sigma^2)=\Sigma_{11}^2+\Sigma_{22}^2+\cdots+\Sigma_{55}^2+2(\Sigma_{12}\Sigma_{21}+\Sigma_{13}\Sigma_{31}+\cdots+\Sigma_{45}\Sigma_{54}),\tag{6.20}$$

but the only meaningful term are only $\Sigma_{11}^2+\Sigma_{22}^2+\cdots+\Sigma_{55}^2$, since the $\pi\Sigma$ term has only a diagonal component $\Sigma_{11},\Sigma_{22},\Sigma_{33}$ when performing the graph calculation. $\Sigma_{44}^2+\Sigma_{55}^2$ can be rewritten as

$$\Sigma_{44}^2+\Sigma_{55}^2=\left(\Sigma_{11}+\Sigma_{22}+\Sigma_{33}\right)^2-2\Sigma_{44}\Sigma_{55}\tag{6.21}$$

using (6.14). Hence, the meaningful part is

$$\begin{aligned}\mathrm{tr}(\Sigma^2) &\supset \Sigma_{11}^2+\Sigma_{22}^2+\Sigma_{33}^2+\Sigma_{44}^2+\Sigma_{55}^2\\ &\supset \Sigma_{11}^2+\Sigma_{22}^2+\Sigma_{33}^2+\left(\Sigma_{11}+\Sigma_{22}+\Sigma_{33}\right)^2\\ &= 2\left(\Sigma_{11}^2+\Sigma_{22}^2+\Sigma_{33}^2\right)+2\left(\Sigma_{11}\Sigma_{22}+\Sigma_{22}\Sigma_{33}+\Sigma_{33}\Sigma_{11}\right)\end{aligned}\tag{6.22}$$

and substituting this into (6.19), we obtain cubic term for Σ

$$\begin{aligned}\lambda_1\left\{\mathrm{tr}(\Sigma'^2)\right\}^2 \supset\ & 40\lambda_1 v\left(\Sigma_{11}^3+\Sigma_{22}^3+\Sigma_{33}^3\right.\\ &+2\Sigma_{11}^2\Sigma_{22}+2\Sigma_{11}^2\Sigma_{33}+2\Sigma_{22}^2\Sigma_{11}+2\Sigma_{22}^2\Sigma_{33}+2\Sigma_{33}^2\Sigma_{22}+2\Sigma_{33}^2\Sigma_{11}\\ &\left.+3\Sigma_{11}\Sigma_{22}\Sigma_{33}\right).\end{aligned}\tag{6.23}$$

Although there are cross terms among diagonal components in the second and last line of (6.23), their contributions to non-Gaussianity are comparable, therefore we only need to consider $40\lambda_1 v\Sigma_{11}^3$ in effect. We also omitted $\Sigma_{44}\Sigma_{55}$, since they are functions of Σ_{11}, Σ_{22} and Σ_{33} and they give the same order of contribution as the cross-terms. Furthermore, the cubic term of Σ is also obtained from the term $\lambda_2\,\mathrm{tr}(\Sigma'^4)$ of potential (6.8), since it only yields the same order of contribution, we write them collectively as α later. Therefore, using these equations (6.8), (6.16), (6.17), (6.18) and (6.23)

$$\begin{aligned} V(\Sigma') &= -M^2\mathrm{tr}(\Sigma'^2) + \lambda_1\left\{\mathrm{tr}(\Sigma'^2)\right\}^2 + \lambda_2\,\mathrm{tr}(\Sigma'^4), && (6.24)\\ \mathrm{tr}(\Sigma'^2) &= 30v^2 + 10v\left(\Sigma_{11}+\Sigma_{22}+\Sigma_{33}\right) + \mathrm{tr}(\Sigma^2), && (6.25)\\ \lambda_1\left\{\mathrm{tr}(\Sigma'^2)\right\}^2 &\supset 600\lambda_1 v^3\left(\Sigma_{11}+\Sigma_{22}+\Sigma_{33}\right) + 40\lambda_1 v\left(\Sigma_{11}^3+\Sigma_{22}^3+\Sigma_{33}^3\right), && (6.26)\\ \lambda_2\,\mathrm{tr}(\Sigma'^4) &\supset 140\lambda_2 v^3\left(\Sigma_{11}+\Sigma_{22}+\Sigma_{33}\right), && (6.27) \end{aligned}$$

the necessary part of potential for the adjoint Higgs boson of SU(5) for calculation in Fig.1.2 is

$$\begin{aligned} V(\Sigma) &= -10M^2 v\left(\Sigma_{11}+\Sigma_{22}+\Sigma_{33}\right) + \left(600\lambda_1+140\lambda_2\right)v^3\left(\Sigma_{11}+\Sigma_{22}+\Sigma_{33}\right)\\ &\quad +40\lambda_1 v\left(\Sigma_{11}^3+\Sigma_{22}^3+\Sigma_{33}^3\right). \end{aligned} \tag{6.28}$$

The corresponding action is

$$\begin{aligned} S_{\mathrm{SU(5)\ Higgs}} &= \int d^4x\,\sqrt{-g}\left[\left(D_\mu\Sigma\right)^\dagger D^\mu\Sigma - V(\Sigma)\right]\\ &\supset \int d^4x\,\sqrt{-g}\Big[\left(\partial_\mu\Sigma\right)^\dagger\partial^\mu\Sigma + 10M^2 v\left(\Sigma_{11}+\Sigma_{22}+\Sigma_{33}\right)\\ &\quad - \left(600\lambda_1+140\lambda_2\right)v^3\left(\Sigma_{11}+\Sigma_{22}+\Sigma_{33}\right) - 40\lambda_1 v\left(\Sigma_{11}^3+\Sigma_{22}^3+\Sigma_{33}^3\right)\Big]. \end{aligned} \tag{6.29}$$

Let us write this action

$$S_{\mathrm{SU(5)\ Higgs}} \supset \int d^4x\,\sqrt{-g}\mathcal{L}_{\mathrm{SU(5)\ Higgs}}, \tag{6.30}$$

i.e., we define the Lagrangian

$$\begin{aligned} \mathcal{L}_{\mathrm{SU(5)\ Higgs}} &:= \Big[\left(\partial_\mu\Sigma\right)^\dagger\partial^\mu\Sigma + 10M^2 v\left(\Sigma_{11}+\Sigma_{22}+\Sigma_{33}\right)\\ &\quad - \left(600\lambda_1+140\lambda_2\right)v^3\left(\Sigma_{11}+\Sigma_{22}+\Sigma_{33}\right) - 40\lambda_1 v\left(\Sigma_{11}^3+\Sigma_{22}^3+\Sigma_{33}^3\right)\Big]. \end{aligned} \tag{6.31}$$

To calculate the graph in Fig.1.2, we have to extract the terms proportional to π and π^2 from the metric $\sqrt{-g}$. From appendix A, the metric $\sqrt{-g}$ is expanded as

$$\sqrt{-g} = a^3(\tilde{t}) + 3a^3 H\pi - a^3\dot{\pi} + \frac{2}{9}a^3 H^2\pi^2 - 3a^3 H\pi\dot{\pi} + a^3\dot{\pi}^2 \tag{6.32}$$

after the transformation

$$t \mapsto \tilde{t} = t - \pi(\tilde{t}, \boldsymbol{x}) \tag{6.33}$$

of time coordinates. Note that we neglect the terms including ϵ which are discussed in Appendix A in three point functions of inflaton, since they make only sub-leading contributions to non-Gaussianity. Thus, the interactions between the linear terms of Σ and the inflaton π with no derivative are found

$$\begin{aligned} & \sqrt{-g}\mathcal{L}_{\mathrm{SU(5)\ Higgs}} \\ = \; & \sqrt{-g}\left[\left(\partial_\mu\Sigma\right)^\dagger\partial^\mu\Sigma + 10M^2 v\left(\Sigma_{11}+\Sigma_{22}+\Sigma_{33}\right)\right. \\ & \left. - \left(600\lambda_1 + 140\lambda_2\right)v^3\left(\Sigma_{11}+\Sigma_{22}+\Sigma_{33}\right) - 40\lambda_1 v\left(\Sigma_{11}^3+\Sigma_{22}^3+\Sigma_{33}^3\right)\right] \\ \supset \; & \sqrt{-g}\left[10M^2 v\left(\Sigma_{11}(t)+\Sigma_{22}(t)+\Sigma_{33}(t)\right) - \left(600\lambda_1+140\lambda_2\right)v^3\left(\Sigma_{11}(t)+\Sigma_{22}(t)+\Sigma_{33}(t)\right)\right] \end{aligned} \tag{6.34}$$

where the linear terms of Σ are extracted. Substituting only the linear and quadratic terms of π from $\sqrt{-g}$, we can get the interactions between the linear terms of Σ and the inflaton π with no derivative

$$\begin{aligned}
&\sqrt{-g}\mathcal{L}_{\mathrm{SU(5)\ Higgs}} \\
\supset\ & \left(3a^3H\pi + \frac{2}{9}a^3H^2\pi^2\right)\left(10M^2v - (600\lambda_1 + 140\lambda_2)\,v^3\right)\left(\Sigma_{11}(t) + \Sigma_{22}(t) + \Sigma_{33}(t)\right) \\
=\ & \left(3a^3H\pi + \frac{2}{9}a^3H^2\pi^2\right)\left(10M^2v - (600\lambda_1 + 140\lambda_2)\,v^3\right) \\
& \times\left(\Sigma_{11}(\tilde{t}+\pi) + \Sigma_{22}(\tilde{t}+\pi) + \Sigma_{33}(\tilde{t}+\pi)\right) \\
=\ & \left(3a^3H\pi + \frac{2}{9}a^3H^2\pi^2\right)\left(10M^2v - (600\lambda_1 + 140\lambda_2)\,v^3\right) \\
& \times\left(\Sigma_{11}(\tilde{t}) + \pi\dot{\Sigma}_{11}(\tilde{t}) + \Sigma_{22}(\tilde{t}) + \pi\dot{\Sigma}_{22}(\tilde{t}) + \Sigma_{33}(\tilde{t}) + \pi\dot{\Sigma}_{33}(\tilde{t})\right) \\
=\ & 3\left(10M^2v - (600\lambda_1 + 140\lambda_2)\,v^3\right)a^3H\pi\left(\Sigma_{11}(\tilde{t}) + \Sigma_{22}(\tilde{t}) + \Sigma_{33}(\tilde{t})\right) \\
& +\frac{2}{9}\left(10M^2v - (600\lambda_1 + 140\lambda_2)\,v^3\right)a^3H^2\pi^2\left(\Sigma_{11}(\tilde{t}) + \Sigma_{22}(\tilde{t}) + \Sigma_{33}(\tilde{t})\right) \\
& +3\left(10M^2v - (600\lambda_1 + 140\lambda_2)\,v^3\right)a^3H\pi^2\left(\dot{\Sigma}_{11}(\tilde{t}) + \dot{\Sigma}_{22}(\tilde{t}) + \dot{\Sigma}_{33}(\tilde{t})\right).
\end{aligned} \tag{6.35}$$

On the other hand, the interactions with time derivative are found similarly,

$$\begin{aligned}
&\sqrt{-g}\mathcal{L}_{\mathrm{SU(5)\ Higgs}} \\
\supset\ & \sqrt{-g}\left[10M^2v\left(\Sigma_{11}(t) + \Sigma_{22}(t) + \Sigma_{33}(t)\right) - (600\lambda_1 + 140\lambda_2)\,v^3\left(\Sigma_{11}(t) + \Sigma_{22}(t) + \Sigma_{33}(t)\right)\right] \\
\supset\ & \left(-a^3\dot{\pi} - 3a^3H\pi\dot{\pi} + a^3\dot{\pi}^2\right)\left(10M^2v - (600\lambda_1 + 140\lambda_2)\,v^3\right)\left(\Sigma_{11}(t) + \Sigma_{22}(t) + \Sigma_{33}(t)\right) \\
=\ & \left(-a^3\dot{\pi} - 3a^3H\pi\dot{\pi} + a^3\dot{\pi}^2\right)\left(10M^2v - (600\lambda_1 + 140\lambda_2)\,v^3\right) \\
& \times\left(\Sigma_{11}(\tilde{t}+\pi) + \Sigma_{22}(\tilde{t}+\pi) + \Sigma_{33}(\tilde{t}+\pi)\right) \\
=\ & \left(-a^3\dot{\pi} - 3a^3H\pi\dot{\pi} + a^3\dot{\pi}^2\right)\left(10M^2v - (600\lambda_1 + 140\lambda_2)\,v^3\right) \\
& \times\left(\Sigma_{11}(\tilde{t}) + \pi\dot{\Sigma}_{11}(\tilde{t}) + \Sigma_{22}(\tilde{t}) + \pi\dot{\Sigma}_{22}(\tilde{t}) + \Sigma_{33}(\tilde{t}) + \pi\dot{\Sigma}_{33}(\tilde{t})\right) \\
=\ & -a^3\left\{10M^2v - (600\lambda_1 + 140\lambda_2)\,v^3\right\}\dot{\pi}\left(\Sigma_{11}(\tilde{t}) + \Sigma_{22}(\tilde{t}) + \Sigma_{33}(\tilde{t})\right) \\
& -3a^3H\left\{10M^2v - (600\lambda_1 + 140\lambda_2)\,v^3\right\}\pi\dot{\pi}\left(\Sigma_{11}(\tilde{t}) + \Sigma_{22}(\tilde{t}) + \Sigma_{33}(\tilde{t})\right) \\
& -a^3\left\{10M^2v - (600\lambda_1 + 140\lambda_2)\,v^3\right\}\pi\dot{\pi}\left(\dot{\Sigma}_{11}(\tilde{t}) + \dot{\Sigma}_{22}(\tilde{t}) + \dot{\Sigma}_{33}(\tilde{t})\right) \\
& +a^3\left\{10M^2v - (600\lambda_1 + 140\lambda_2)\,v^3\right\}\dot{\pi}^2\left(\Sigma_{11}(\tilde{t}) + \Sigma_{22}(\tilde{t}) + \Sigma_{33}(\tilde{t})\right)
\end{aligned} \tag{6.36}$$

where the linear terms of Σ are extracted in the second line and substituting only the linear and quadratic terms of $\dot{\pi}$ from $\sqrt{-g}$ in the third line. The terms without time derivative of π are left even after performing integration by parts. This is because the translational symmetry with respect to time is explicitly broken by inflation, and π is a pesudo NG boson. Since Σ_{11}, Σ_{22} and Σ_{33} are equivalent, we consider Σ_{11} in the following.

Next, we consider the propagator of Σ, since the interaction has been obtained. Using the expressions of Higgs boson

$$\begin{aligned} \Sigma'^2 &= \left(\langle\Sigma\rangle + \Sigma\right)^2 \\ &= \langle\Sigma\rangle^2 + 2\langle\Sigma\rangle\Sigma + \Sigma^2 \\ &= v^2 \ \mathrm{diag}(4,4,4,9,9) + 2v \ \mathrm{diag}(2,2,2,-3,-3) \ \Sigma + \Sigma^2 \end{aligned} \tag{6.37}$$

and

$$\mathrm{tr}(\Sigma'^2) = 30v^2 + 10v\left(\Sigma_{11} + \Sigma_{22} + \Sigma_{33}\right) + \mathrm{tr}(\Sigma^2), \tag{6.38}$$

the quadratic terms of Σ in potential (6.8) is

$$\begin{aligned} V(\Sigma') &= -M^2\mathrm{tr}(\Sigma'^2) + \lambda_1\left\{\mathrm{tr}(\Sigma'^2)\right\}^2 + \lambda_2\,\mathrm{tr}(\Sigma'^4) \\ &\supset -M^2\left(\Sigma_{11}\right)^2 + 160\lambda_1 v^2\left(\Sigma_{11}\right)^2 + 24\lambda_2 v^2\left(\Sigma_{11}\right)^2 . \end{aligned} \tag{6.39}$$

Hence, by setting

$$-m^2 := -M^2 + 160\lambda_1 v^2 + 24\lambda_2 v^2, \tag{6.40}$$

Σ_{11} is a scalar field with mass m^2 in de Sitter spacetime, therefore it has propagators

$$\begin{aligned} G_>\left(\tau_1,\tau_2,k\right) &= -i\frac{\sqrt{\pi}}{2}e^{i\pi(\nu/2+1/4)}H\left(-\tau_1\right)^{3/2}H_\nu^{(1)}(-k\tau_1) \\ &\quad \times i\frac{\sqrt{\pi}}{2}e^{i\pi(\nu^*/2+1/4)}H\left(-\tau_2\right)^{3/2}H_\nu^{(2)}(-k\tau_2) \\ &= -\frac{\pi}{4}e^{-\pi\mathrm{Im}(\nu)}H^2\left(\tau_1\tau_2\right)^{3/2}H_\nu^{(1)}(-k\tau_1)H_\nu^{(2)}(-k\tau_2), \qquad (6.41) \\ G_<\left(\tau_1,\tau_2,k\right) &= -\frac{\pi}{4}e^{-\pi\mathrm{Im}(\nu)}H^2\left(\tau_1\tau_2\right)^{3/2}H_\nu^{(1)}(-k\tau_2)H_\nu^{(2)}(-k\tau_1). \qquad (6.42) \end{aligned}$$

from appendix B. From the viewpoint of effective field theory, since m^2 should be around the Hubble scale, λ_1 and λ_2 must be an order of $\mathcal{O}(10^{-2})$.

Chapter 7

Calculation of non-Gaussianity

7.1 Σ exchange

7.1.1 interaction without time derivatives

We are now ready to calculate non-Gaussianity. Since the interaction of Σ_{11} and the interaction of $\dot{\Sigma}_{11}$ give the same order contributions to non-Gaussianity, only the contribution by Σ_{11} is considered and the result is doubled by taking into account the latter contributuion. First, consider interactions without time derivative. According to the rule of in-in formalism, there exist four types of graphs, $(+,+)$, $(+,-)$, $(-,+)$ and $(-,-)$ as shown in Fig.7.1, mediated by one Σ between the three inflatons π.

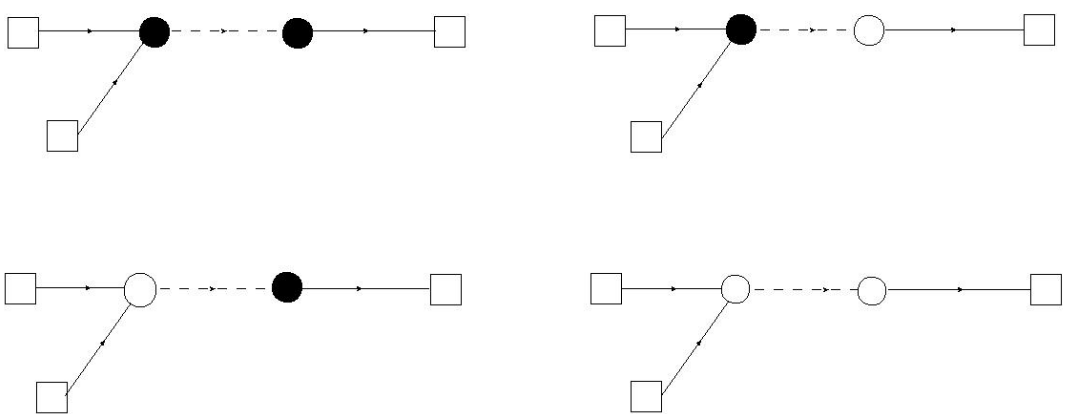


Figure 7.1: Four graphs in in-in formalism. Black circles indicate + and white circles indicate −.

As an example, let us consider the graphs of $(+,-)$ and $(-,+)$. Note that in the

Feynman rule for in-in formalism with conformal time τ, the time integral for conformal time, the scale factor $a(\tau)$, and the factor $\pm i$ for $\pm$ type appear. Using the interaction (6.35) without time derivatives

$$\begin{aligned}
&\sqrt{-g}\mathcal{L}_{\text{SU(5) Higgs}} \\
\supset\ & 3\left(10M^2v-(600\lambda_1+140\lambda_2)v^3\right)a^3H\pi\left(\Sigma_{11}(\tilde{t})+\Sigma_{22}(\tilde{t})+\Sigma_{33}(\tilde{t})\right) \\
&+\frac{2}{9}\left(10M^2v-(600\lambda_1+140\lambda_2)v^3\right)a^3H^2\pi^2\left(\Sigma_{11}(\tilde{t})+\Sigma_{22}(\tilde{t})+\Sigma_{33}(\tilde{t})\right), \quad (7.1)
\end{aligned}$$

the propagators of inflaton (6.6), (6.7)

$$\Delta_>(\tau_1,\tau_2,k) = \frac{1+ic_sk(\tau_1-\tau_2)+(c_sk)^2\tau_1\tau_2}{4\epsilon M_{\text{Pl}}c_sk^3}e^{-ic_sk(\tau_1-\tau_2)}, \quad (7.2)$$

$$\Delta_<(\tau_1,\tau_2,k) = \frac{1-ic_sk(\tau_1-\tau_2)+(c_sk)^2\tau_1\tau_2}{4\epsilon M_{\text{Pl}}c_sk^3}e^{ic_sk(\tau_1-\tau_2)} \quad (7.3)$$

and those of Higgs boson (6.41), (6.42)

$$G_>(\tau_1,\tau_2,k) = -\frac{\pi}{4}e^{-\pi\text{Im}(\nu)}H^2(\tau_1\tau_2)^{3/2}H_\nu^{(1)}(-k\tau_1)H_\nu^{(2)}(-k\tau_2), \quad (7.4)$$

$$G_<(\tau_1,\tau_2,k) = -\frac{\pi}{4}e^{-\pi\text{Im}(\nu)}H^2(\tau_1\tau_2)^{3/2}H_\nu^{(1)}(-k\tau_2)H_\nu^{(2)}(-k\tau_1), \quad (7.5)$$

the three point function of inflaton π contributed from the sum of two graphs is given as follows:

$$\begin{aligned}
\langle\pi\pi\pi\rangle =\ & \int_{-(1+i\epsilon)\infty}^{0}a(\tau_1)d\tau_1\int_{-(1+i\epsilon)\infty}^{0}a(\tau_2)d\tau_2(-i)\frac{2}{9}\left(10M^2v-(600\lambda_1+140\lambda_2)v^3\right) \\
&\times a^3(\tau_1)H^2\times 3i\left(10M^2v-(600\lambda_1+140\lambda_2)v^3\right)a^3(\tau_2)H \\
&\times G_{+-}(\tau_1,\tau_2,p_3)\Delta_{+-}(0,\tau_1,p_1)\Delta_{+-}(0,\tau_1,p_2)\Delta_{++}(\tau_2,0,p_3)+(p_1\leftrightarrow p_3) \quad (7.6) \\
&+(p_2\leftrightarrow p_3)+\text{c.c.} \\
=\ & \frac{2}{3}\left(10M^2v-(600\lambda_1+140\lambda_2)v^3\right)^2H^3 \\
&\times\int_{-(1+i\epsilon)\infty}^{0}d\tau_1\int_{-(1+i\epsilon)\infty}^{0}d\tau_2\,a^4(\tau_1)a^4(\tau_2)G_{+-}(\tau_1,\tau_2,p_3)\Delta_{+-}(0,\tau_1,p_1) \\
&\times\Delta_{+-}(0,\tau_1,p_2)\Delta_{++}(\tau_2,0,p_3)+(p_1\leftrightarrow p_3)+(p_2\leftrightarrow p_3)+\text{c.c.} \quad (7.7)
\end{aligned}$$

Now, we substitute the propagators (6.42) of Σ and the propagators (6.6) and (6.7) of π. As shown in detail in Appendix B, note that the notation for Δ_{++} etc. is given by

$$\Delta_{++}(\eta_1,\eta_2,k) = \Delta_>(\eta_1,\eta_2,k)\,\theta(\eta_1-\eta_2)+\Delta_<(\eta_1,\eta_2,k)\,\theta(\eta_2-\eta_1), \quad (7.8)$$

we have

$$\begin{aligned}\langle\pi\pi\pi\rangle &= -\frac{\pi}{2^7\cdot 3}e^{-\pi \mathrm{Im}(\nu)}\left(10M^2 v-\left(600\lambda_1+140\lambda_2\right)v^3\right)^2\frac{1}{\left(H\epsilon M_{\mathrm{Pl}}c_s p_1p_2p_3\right)^3}\\ &\times\int_{-(1+i\epsilon)\infty}^{0}d\tau_1\int_{-(1+i\epsilon)\infty}^{0}d\tau_2\ \left(\tau_1\tau_2\right)^{-5/2}H_\nu^{(1)}(-p_3\tau_1)H_\nu^{(2)}(-p_3\tau_2)\\ &\times\left(1+ic_sp_1\tau_1\right)\left(1+ic_sp_2\tau_1\right)\left(1+ic_sp_3\tau_2\right)e^{-ic_s(p_1+p_2)\tau_1}e^{-ic_sp_3\tau_2}\\ &+(p_1\leftrightarrow p_3)+(p_2\leftrightarrow p_3)+\text{c.c.}\end{aligned}\tag{7.9}$$

As discussed in [101] and [102], we use the approximation of the Hankel function

$$H_\nu^{(1)}(-p_3\tau_2)\to -i\frac{2^\nu}{\pi}(-p_3\tau_2)^{-\nu}\Gamma(\nu),\tag{7.10}$$

$$H_\nu^{(2)}(-p_3\tau_1)\to i\frac{2^\nu}{\pi}(-p_3\tau_1)^{-\nu}\Gamma(\nu)\tag{7.11}$$

with horizon exit $-p_3\tau_1, -p_3\tau_2\to 1$ to evaluate the integral. This is an approximation to extract the effect of the largest contribution as time evolves and the fluctuations freeze. Note that we consider the parameter region

$$0<\nu\leq\frac{3}{2},\tag{7.12}$$

in other words,

$$0<\frac{m^2}{H^2}\leq\frac{9}{4}\tag{7.13}$$

where the suppression factor $e^{-\pi\mathrm{Im}(\nu)}$ does not appear. Substituting approximations (7.10) and (7.11) of the Hankel functions into (7.9), three point function of inflaton $\langle\pi\pi\pi\rangle$ becomes

$$\begin{aligned}\langle\pi\pi\pi\rangle &= -\frac{\Gamma^2(\nu)}{2^{7-2\nu}\cdot 3\pi}\left(10M^2 v-\left(600\lambda_1+140\lambda_2\right)v^3\right)^2\frac{1}{\left(H\epsilon M_{\mathrm{Pl}}c_s p_1p_2p_3\right)^3}\frac{1}{p_3^{2\nu}}\\ &\times\int_{-(1+i\epsilon)\infty}^{0}d\tau_1\int_{-(1+i\epsilon)\infty}^{0}d\tau_2\ \left(\tau_1\tau_2\right)^{-5/2-\nu}e^{-ic_s(p_1+p_2)\tau_1}e^{-ic_sp_3\tau_2}\\ &\times\left(1+ic_sp_1\tau_1\right)\left(1+ic_sp_2\tau_1\right)\left(1+ic_sp_3\tau_2\right)+(p_1\leftrightarrow p_3)+(p_2\leftrightarrow p_3)+\text{c.c.}\end{aligned}\tag{7.14}$$

Note that the dependence on the external momentum p_3 is found to be non-local, because the non-local part of $\langle\pi\pi\pi\rangle$ is non-analytic in external momentum p_3 [22].

This implies that the contribution for the three point function actually comes from the Σ field. Non-Gaussianity is defined by

$$\langle\zeta\zeta\zeta\rangle = (2\pi)^7\delta^{(3)}(\boldsymbol{p}_1+\boldsymbol{p}_2+\boldsymbol{p}_3)\tilde{p}_\zeta^2\left(\frac{9}{10}f_{\rm NL}\right)\frac{1}{(p_1p_2p_3)^2} \tag{7.15}$$

in the case where the configuration of the external momentum are equilateral,

$$p := p_1 = p_2 = p_3 \tag{7.16}$$

in the following. Note that ζ is the curvature fluctuation, which is related to inflaton π by $\zeta = -H\pi$. Since the integral has only an effect up to the horizon exit $-p_3\tau_1, -p_3\tau_2 \to 1$, the upper limit of the integral should be [102]

$$\tau_{1*} = \tau_{2*} = -p^{-1}. \tag{7.17}$$

In this case, the higher order term of the factor $e^{-ic_s(p_1+p_2)\tau_1}e^{-ic_sp_3\tau_2}$ is a small quantity, then the integral of (7.14) can be calculated as follows:

$$\begin{aligned}
&\int_{-(1+i\epsilon)\infty}^{0}d\tau_1\int_{-(1+i\epsilon)\infty}^{0}d\tau_2\ (\tau_1\tau_2)^{-5/2-\nu}\,(1+ic_sp_1\tau_1)\,(1+ic_sp_2\tau_1)\,(1+ic_sp_3\tau_2)\\
&\times e^{-ic_s(p_1+p_2)\tau_1}e^{-ic_sp_3\tau_2}\\
=&\left(\frac{-2}{3+2\nu}-\frac{4ic_s}{1+2\nu}\right)\left(\frac{-2}{3+2\nu}-\frac{2ic_s}{1+2\nu}\right)p^{3+2\nu}.
\end{aligned} \tag{7.18}$$

Therefore, the three point function of inflaton π can be computed as

$$\begin{aligned}
\langle\pi\pi\pi\rangle &\simeq -\frac{\Gamma^2(\nu)}{2^{7-2\nu}\cdot 3\pi}\left(10M^2v-(600\lambda_1+140\lambda_2)\,v^3\right)^2\frac{1}{(H\epsilon M_{\rm Pl}c_sp_1p_2p_3)^3}\frac{1}{p_3^{2\nu}}\\
&\quad\times\left(\frac{-2}{3+2\nu}-\frac{4ic_s}{1+2\nu}\right)\left(\frac{-2}{3+2\nu}-\frac{2ic_s}{1+2\nu}\right)p^{3+2\nu}+(p_1\leftrightarrow p_3)+(p_2\leftrightarrow p_3)+\text{c.c.}\\
&= -\frac{\Gamma^2(\nu)}{2^{7-2\nu}\cdot 3\pi}\left(10M^2v+(600\lambda_1+140\lambda_2)\,v^3\right)^2\\
&\quad\times\frac{1}{(H\epsilon M_{\rm Pl}c_s)^3}p^{-6}\left(\frac{8}{(3+2\nu)^2}-\frac{16c_s^2}{(1+2\nu)^2}\right)\times 3
\end{aligned} \tag{7.19}$$

Recalling the relation between the curvature fluctuation ζ and inflaton π is $\zeta = -H\pi$, non-Gaussianity can be estimated as

$$\begin{aligned}
f_{\rm NL} &= \frac{10}{9}\frac{1}{(2\pi)^7\tilde{p}_\zeta^2}p^6M_{\rm Pl}^{-3}(-H)^3\\
&\quad\times\frac{-\Gamma^2(\nu)}{2^{7-2\nu}\cdot 3\pi}\left(10M^2v-(600\lambda_1+140\lambda_2)\,v^3\right)^2\\
&\quad\times\frac{1}{(H\epsilon M_{\rm Pl}c_s)^3}p^{-6}\left(\frac{8}{(3+2\nu)^2}-\frac{16c_s^2}{(1+2\nu)^2}\right)\times 3
\end{aligned} \tag{7.20}$$

by reviving $M_{\rm Pl}$ for ζ. This expression shows that non-Gaussianity $f_{\rm NL}$ is Planck suppressed but enhanced by the high scale of inflation. Here, the constant c_s has the relation $r = 16\epsilon c_s$ with the tensor-scalar ratio r and the upper bound $r < 0.036$ [103] gives

$$\epsilon c_s \sim \mathcal{O}(10^{-3\sim-4}). \tag{7.21}$$

Since the constant in (7.20) has the order of

$$M_{\rm Pl} \sim \mathcal{O}(10^{19})\,\text{GeV}, \quad v \sim M \sim \mathcal{O}(10^{15})\,\text{GeV}, \quad \tilde{p}_\zeta \sim 6.1 \times 10^{-9} \tag{7.22}$$

respectively, the non-Gaussianity is found as

$$f_{\rm NL} \sim \mathcal{O}(10^{-4\sim-1}) \times \alpha, \tag{7.23}$$

where α represents the similar contributions from other components such as Σ_{22}, Σ_{33} and $\dot{\Sigma}_{11}$ and the additional group theoretical numerical factor that appears in larger GUT gauge groups extension. The order is $\alpha \sim \mathcal{O}(10^{0\sim1})$.

7.1.2 interaction with time derivatives

Next, we consider interactions with time derivatives. To begin with, consider the case where all three inflatons are time differentiated. From equation (6.35), the corresponding interaction is

$$\begin{aligned}
&\sqrt{-g}\mathcal{L}_{\rm SU(5)\ Higgs} \\
\supset\ & -a^3\left\{10M^2 v - (600\lambda_1 + 140\lambda_2)\,v^3\right\}\dot{\pi}\left(\Sigma_{11}(\tilde{t}) + \Sigma_{22}(\tilde{t}) + \Sigma_{33}(\tilde{t})\right) \\
& +a^3\left\{10M^2 v - (600\lambda_1 + 140\lambda_2)\,v^3\right\}\dot{\pi}^2\left(\Sigma_{11}(\tilde{t}) + \Sigma_{22}(\tilde{t}) + \Sigma_{33}(\tilde{t})\right).
\end{aligned} \tag{7.24}$$

Using the Feynman rule for in-in formalism, the three point function of inflaton π from the sum of the graphs of $(+,-)$ and $(-,+)$ is given as follows:

$$\begin{aligned}
\langle\pi\pi\pi\rangle \ =\ & \int_{-(1+i\epsilon)\infty}^{0} a(\tau_1)d\tau_1 \int_{-(1+i\epsilon)\infty}^{0} a(\tau_2)d\tau_2(-i)a^3(\tau_1)Aia^3(\tau_2)A \\
& \times G_{+-}(\tau_1,\tau_2,p_3)\frac{1}{a(\tau_1)}\dot{\Delta}_{+-}(0,\tau_1,p_1)\frac{1}{a(\tau_1)}\dot{\Delta}_{+-}(0,\tau_1,p_2)\frac{1}{a(\tau_2)}\dot{\Delta}_{++}(\tau_2,0,p_3) \\
& +(p_1 \leftrightarrow p_3) + (p_2 \leftrightarrow p_3) + \text{c.c.}
\end{aligned} \tag{7.25}$$

where A is defined as

$$A := 10M^2 v - (600\lambda_1 + 140\lambda_2)\, v^3. \tag{7.26}$$

Note that in differentiating by conformal time, we get the inverse of the scale factor $a(\tau)$. Differentiating the propagator $\Delta_{++}(\tau_2, 0, p)$ with respect to τ_2 yields

$$\begin{aligned}
\dot{\Delta}_{++}(\tau_2, 0, p) &= \dot{\Delta}_{>}(\tau_2, 0, p)\,\theta(\tau_2 - 0) + \Delta_{>}(\tau_2, 0, p)\,\delta(\tau_2 - 0) \\
&\quad + \dot{\Delta}_{<}(\tau_2, 0, p)\,\theta(0 - \tau_2) - \Delta_{<}(\tau_2, 0, p)\,\delta(0 - \tau_2) \\
&= \frac{1 + ic_s p\tau_2}{4\epsilon M_{\mathrm{Pl}} c_s p^3} e^{-ic_s p\tau_2}\delta(\tau_2) \\
&\quad + \left(\frac{-ic_s p}{4\epsilon M_{\mathrm{Pl}} c_s p^3} e^{ic_s p\tau_2} + ic_s p\frac{1 - ic_s p\tau_2}{4\epsilon M_{\mathrm{Pl}} c_s p^3} e^{ic_s p\tau_2}\right)\theta(-\tau_2) \\
&\quad - \frac{1 - ic_s p\tau_2}{4\epsilon M_{\mathrm{Pl}} c_s p^3} e^{ic_s p\tau_2}\delta(-\tau_2) \\
&= \frac{1 + ic_s p\tau_2}{4\epsilon M_{\mathrm{Pl}} c_s p^3} e^{-ic_s p\tau_2}\delta(\tau_2) - \frac{1 - ic_s p\tau_2}{4\epsilon M_{\mathrm{Pl}} c_s p^3} e^{ic_s p\tau_2}\delta(-\tau_2) + \frac{c_s\tau_2}{4\epsilon M_{\mathrm{Pl}} p} e^{ic_s p\tau_2}\theta(-\tau_2),
\end{aligned} \tag{7.27}$$

which leads to a simple form

$$\dot{\Delta}_{++}(\tau_2, 0, p) = \frac{c_s\tau_2}{4\epsilon M_{\mathrm{Pl}} p} e^{ic_s p\tau_2} \tag{7.28}$$

because of the τ_2-integral. Similarly, differentiating the propagator $\Delta_{+-}(0, \tau_1, p)$ with respect to τ_1 yields

$$\begin{aligned}
\dot{\Delta}_{+-}(0, \tau_1, p) = \dot{\Delta}_{<}(0, \tau_1, p) &= \frac{ic_s p}{4\epsilon M_{\mathrm{Pl}} c_s p^3} e^{-ic_s p\tau_1} + (-ic_s p)\frac{1 + ic_s p\tau_1}{4\epsilon M_{\mathrm{Pl}} c_s p^3} e^{-ic_s p\tau_1} \\
&= \frac{c_s\tau_1}{4\epsilon M_{\mathrm{Pl}} p} e^{-ic_s p\tau_1}.
\end{aligned} \tag{7.29}$$

Substituting these results, $\langle\pi\pi\pi\rangle$ becomes

$$\begin{aligned}
\langle\pi\pi\pi\rangle &= \int_{-(1+i\epsilon)\infty}^{0} a(\tau_1)d\tau_1 \int_{-(1+i\epsilon)\infty}^{0} a(\tau_2)d\tau_2 (-i)a^3(\tau_1)Aia^3(\tau_2)AG_{+-}(\tau_1, \tau_2, p_3) \\
&\quad \times\frac{1}{a(\tau_1)}\dot{\Delta}_{+-}(0, \tau_1, p_1)\frac{1}{a(\tau_1)}\dot{\Delta}_{+-}(0, \tau_1, p_2)\frac{1}{a(\tau_2)}\dot{\Delta}_{++}(\tau_2, 0, p_3) \\
&\quad + (p_1 \leftrightarrow p_3) + (p_2 \leftrightarrow p_3) + \mathrm{c.c.} \\
&= \int d\tau_1 \int d\tau_2 \left(-\frac{1}{H\tau_1}\right)^2 \left(-\frac{1}{H\tau_2}\right)^3 A^2 \left(-\frac{\pi}{4}\right) e^{-\pi \mathrm{Im}(\nu)} H^2 (\tau_1\tau_2)^{3/2} \\
&\quad \times H_\nu^{(1)}(-p\tau_2)H_\nu^{(2)}(-p\tau_1)\frac{c_s\tau_1}{4\epsilon M_{\mathrm{Pl}} p_1} e^{-ic_s p_1\tau_1}\frac{c_s\tau_1}{4\epsilon M_{\mathrm{Pl}} p_2} e^{-ic_s p_2\tau_1}\frac{c_s\tau_2}{4\epsilon M_{\mathrm{Pl}} p_3} e^{ic_s p_3\tau_2} \\
&\quad + (p_1 \leftrightarrow p_3) + (p_2 \leftrightarrow p_3) + \mathrm{c.c.}
\end{aligned} \tag{7.30}$$

Furthermore, using the horizon exit approximations (7.10), (7.11) for Hankel functions

$$H_\nu^{(1)}(-p_3\tau_2) \to -i\frac{2^\nu}{\pi}(-p_3\tau_2)^{-\nu}\Gamma(\nu), \tag{7.31}$$

$$H_\nu^{(2)}(-p_3\tau_1) \to i\frac{2^\nu}{\pi}(-p_3\tau_1)^{-\nu}\Gamma(\nu), \tag{7.32}$$

we obtain

$$\begin{aligned}\langle\pi\pi\pi\rangle \;\sim\; & \int_{-(1+i\epsilon)\infty}^{\tau_*} d\tau_1 \int_{-(1+i\epsilon)\infty}^{\tau_*} d\tau_2\, \frac{-1}{H^5\tau_1^2\tau_2^3} A^2\left(-\frac{\pi}{4}\right) H^2\left(\tau_1\tau_2\right)^{3/2} \\ & \times(-i)\frac{2^\nu}{\pi}(-p_3\tau_2)^{-\nu}\Gamma(\nu)i\frac{2^\nu}{\pi}(-p_3\tau_1)^{-\nu}\Gamma(\nu)\left(\frac{c_s}{4\epsilon M_{\rm Pl}}\right)^3\frac{1}{p_1p_2p_3}\tau_1^2\tau_2 \\ & +(p_1\leftrightarrow p_3)+(p_2\leftrightarrow p_3)+\text{c.c.} \\ = \;& \frac{2^{2\nu}}{4\pi}\Gamma^2(\nu)\frac{1}{H^3}\left(\frac{c_s}{4\epsilon M_{\rm Pl}}\right)^3 A^2\frac{1}{p_1p_2p_3}p_3^{-2\nu} \\ & \times\int_{-(1+i\epsilon)\infty}^{\tau_*} d\tau_1\int_{-(1+i\epsilon)\infty}^{\tau_*} d\tau_2\,\tau_1^{3/2}\tau_2^{-1/2}(\tau_1\tau_2)^{-\nu}. \end{aligned}\tag{7.33}$$

Here, introducing the UV cutoff

$$\tau_\Lambda = -\Lambda^{-1}, \tag{7.34}$$

we can calculate as

$$\langle\pi\pi\pi\rangle \;\sim\; \frac{2^{2\nu}}{4\pi}\Gamma^2(\nu)\frac{1}{H^3}\left(\frac{c_s}{4\epsilon M_{\rm Pl}}\right)^3 A^2\frac{1}{p_1p_2p_3}p_3^{-2\nu}\;\frac{1}{5/2-\nu}\tau_1^{5/2-\nu}\Big|_{\tau_\Lambda}^{\tau_*}\frac{1}{1/2-\nu}\tau_2^{1/2-\nu}\Big|_{\tau_\Lambda}^{\tau_*} \tag{7.35}$$

Since the cutoff scale has a dimension of momentum, we can write

$$\Lambda = \gamma p, \tag{7.36}$$

using the dimensionless parameter γ. Therefore we can obtain

$$\frac{1}{5/2-\nu}\tau_1^{5/2-\nu}\Big|_{\tau_\Lambda}^{\tau_*}\frac{1}{1/2-\nu}\tau_2^{1/2-\nu}\Big|_{\tau_\Lambda}^{\tau_*} = \frac{4}{(5-2\nu)(1-2\nu)}(-p)^{2\nu-3}\left(1-\gamma^{\nu-1/2}-\gamma^{\nu-5/2}+\gamma^{2\nu-3}\right). \tag{7.37}$$

From the definition, non-Gaussianity is computed as

$$
\begin{aligned}
f_{\rm NL} &= \frac{10}{9}\frac{1}{(2\pi)^7\tilde{p}_\zeta^2}p^6 M_{\rm Pl}^{-3}(-H)^3\langle\pi\pi\pi\rangle \\
&= \frac{10}{9}\frac{1}{(2\pi)^7\tilde{p}_\zeta^2}p^6 M_{\rm Pl}^{-3}(-H)^3 \\
&\quad \times\frac{2^{2\nu}}{4\pi}\Gamma^2(\nu)\frac{1}{H^3}\left(\frac{c_s}{4\epsilon M_{\rm Pl}}\right)^3 A^2\frac{1}{p^3}p^{-2\nu} \\
&\quad \times\frac{4}{(5-2\nu)(1-2\nu)}(-p)^{2\nu-3}\left(1-\gamma^{\nu-1/2}-\gamma^{\nu-5/2}+\gamma^{2\nu-3}\right) \\
&\sim 10^{-12}\left(\frac{c_s}{\epsilon}\right)^3\left(1-\gamma^{\nu-1/2}-\gamma^{\nu-5/2}+\gamma^{2\nu-3}\right). \qquad (7.38)
\end{aligned}
$$

Now, recalling that

$$\epsilon c_s \sim 10^{-3\sim-4}, \qquad (7.39)$$

we have

$$f_{\rm NL} \sim 10^{-3\sim0}c_s^6\left(1-\gamma^{\nu-1/2}-\gamma^{\nu-5/2}+\gamma^{2\nu-3}\right). \qquad (7.40)$$

From the viewpoint of effective field theory, the cutoff is the inflationary scale, and γ is

$$\gamma \sim 1. \qquad (7.41)$$

In this case, non-Gaussianity has the width of

$$f_{\rm NL} \sim 10^{-3\sim0}c_s^6. \qquad (7.42)$$

c_s is a quantity such that it is 1 in the simplest model, and is not expected to vary significantly in order estimation. This result is comparable to that of case (7.11) where the time derivative is not included. The reasons are shown in the following table.

Table 1 shows the coefficients and propagator for three point function of inflaton with various number of time derivatives. In the table, the case where the three point function of inflaton without a time derivative is denoted by $\langle\pi\pi\pi\rangle$, with one, two and three time derivatives by $\langle\pi\pi\dot{\pi}\rangle$, $\langle\pi\dot{\pi}\dot{\pi}\rangle$, and $\langle\dot{\pi}\dot{\pi}\dot{\pi}\rangle$. For example, $\langle\pi\pi\pi\rangle$ represents equation (7.6) and $\langle\dot{\pi}\dot{\pi}\dot{\pi}\rangle$ represents equation (7.25). Taking into account that $a^{-1}=H\tau$ is added in taking the time derivative of the propagator, the coefficients are all of the form A^2H^3.

	factor	propagator
$\langle\pi\pi\pi\rangle$	$\frac{2}{9}AH^2 \times 3AH$	$G_{+-}\Delta_{+-}\Delta_{+-}\Delta_{++}$
$\langle\pi\pi\dot{\pi}\rangle$	$\frac{2}{9}AH^2 \times A$	$G_{+-}\Delta_{+-}\Delta_{+-}\dot{\Delta}_{++}$
	$3AH \times 3AH$	$G_{+-}\Delta_{+-}\dot{\Delta}_{+-}\Delta_{++}$
$\langle\pi\dot{\pi}\dot{\pi}\rangle$	$A \times 3AH$	$G_{+-}\dot{\Delta}_{+-}\dot{\Delta}_{+-}\Delta_{++}$
	$3AH \times A$	$G_{+-}\dot{\Delta}_{+-}\Delta_{+-}\dot{\Delta}_{++}$
$\langle\dot{\pi}\dot{\pi}\dot{\pi}\rangle$	$A \times A$	$G_{+-}\dot{\Delta}_{+-}\dot{\Delta}_{+-}\dot{\Delta}_{++}$

Table 7.1: The factors and propagator for three point function of inflaton with various number of time derivatives.

The coefficients except for A^2H^3 of the interaction are not so different in magnitude, and the difference among propagators is just $\mathcal{O}(1)$ factor as long as the cutoff is set to $\Lambda = \gamma p$. Thus, they all give the same contribution to non-Gaussianity.

7.2 cubic interaction of Σ

Next, we consider the graph generated by the cubic interaction of Σ (Fig.7.2).

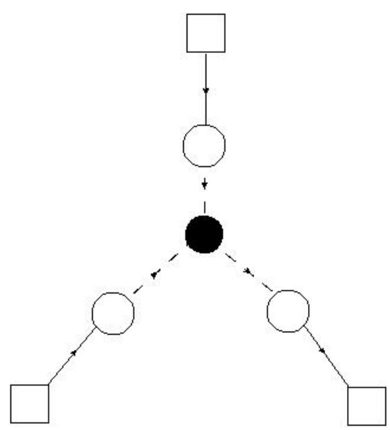


Figure 7.2: The graph generated by the cubic interaction of Σ. There are 2^4 graphs in total, and this is an example, representing the graph of $(-,-,-,+)$.

Let τ_4 be the time for the part of Σ^3 interaction and τ_1, τ_2, τ_3 be the time for the rest of $\pi\Sigma$ interaction. According to the rule of in-in formalism, there exist 2^4 types of graphs. Consider the $(-,-,-,+)$ graphs and its complex conjugation $(+,+,+,-)$ in the order $(\tau_1, \tau_2, \tau_3, \tau_4)$. From equations (6.31) and (6.35), the relevant interactions are

$$\begin{aligned}
&\sqrt{-g}\mathcal{L}_{\mathrm{SU(5)\ Higgs}} \\
\supset\ & 3\left(10M^2 v - (600\lambda_1 + 140\lambda_2)\,v^3\right) a^3 H\pi \left(\Sigma_{11}(\tilde{t}) + \Sigma_{22}(\tilde{t}) + \Sigma_{33}(\tilde{t})\right) \\
& -40\lambda_1 v a^3 \left(\Sigma_{11}^3(\tilde{t}) + \Sigma_{22}^3(\tilde{t}) + \Sigma_{33}^3(\tilde{t})\right). \qquad (7.43)
\end{aligned}$$

Using the Feynman rule for in-in formalism, the case where the interaction does not have time derivative is written down as

$$\begin{aligned}
\langle\pi\pi\pi\rangle =\ & \int_{-(1+i\epsilon)\infty}^{0} a(\tau_1)d\tau_1 \int_{-(1+i\epsilon)\infty}^{0} a(\tau_2)d\tau_2 \int_{-(1+i\epsilon)\infty}^{0} a(\tau_3)d\tau_3 \int_{-(1+i\epsilon)\infty}^{0} a(\tau_4)d\tau_4 \\
& \times(-i)a(\tau_1)3a^3(\tau_1)AH\Delta_{+-}(0,\tau_1,p_1)G_{-+}(\tau_1,\tau_4,p_1) \\
& \times(-i)a(\tau_2)3a^3(\tau_2)AH\Delta_{+-}(0,\tau_2,p_2)G_{-+}(\tau_2,\tau_4,p_2) \\
& \times ia(\tau_4)(-1)a^3(\tau_4)40\lambda_1 v G_{+-}(\tau_4,\tau_3,p_3) \\
& \times(-i)a(\tau_3)3a^3(\tau_3)AH\Delta_{-+}(\tau_3,0,p_3) + (p_1 \leftrightarrow p_3) + (p_2 \leftrightarrow p_3) + \mathrm{c.c.}
\end{aligned} \qquad (7.44)$$

Substituting the propagators of inflaton (6.6), (6.7)

$$\Delta_>(\tau_1,\tau_2,k) = \frac{1 + ic_s k(\tau_1-\tau_2) + (c_s k)^2\tau_1\tau_2}{4\epsilon M_{\mathrm{Pl}} c_s k^3} e^{-ic_s k(\tau_1-\tau_2)}, \qquad (7.45)$$

$$\Delta_<(\tau_1,\tau_2,k) = \frac{1 - ic_s k(\tau_1-\tau_2) + (c_s k)^2\tau_1\tau_2}{4\epsilon M_{\mathrm{Pl}} c_s k^3} e^{ic_s k(\tau_1-\tau_2)} \qquad (7.46)$$

and propagators of Higgs boson (6.41), (6.42)

$$G_>(\tau_1,\tau_2,k) = -\frac{\pi}{4} e^{-\pi\mathrm{Im}(\nu)} H^2 (\tau_1\tau_2)^{3/2} H_\nu^{(1)}(-k\tau_1)H_\nu^{(2)}(-k\tau_2), \qquad (7.47)$$

$$G_<(\tau_1,\tau_2,k) = -\frac{\pi}{4} e^{-\pi\mathrm{Im}(\nu)} H^2 (\tau_1\tau_2)^{3/2} H_\nu^{(1)}(-k\tau_2)H_\nu^{(2)}(-k\tau_1), \qquad (7.48)$$

we obtain

$$\begin{aligned}
\langle\pi\pi\pi\rangle =\ & 40\times 3^3\lambda_1 v A^3 H^{-7}\left(\frac{1}{4\epsilon M_{\mathrm{Pl}} c_s}\right)^3 \left(-\frac{\pi}{4}\right)^3 e^{-3\pi\mathrm{Im}(\nu)} \left(\frac{1}{p_1p_2p_3}\right)^3 \\
& \times\int_{-(1+i\epsilon)\infty}^{0} d\tau_1 \int_{-(1+i\epsilon)\infty}^{0} d\tau_2 \int_{-(1+i\epsilon)\infty}^{0} d\tau_3 \int_{-(1+i\epsilon)\infty}^{0} d\tau_4 \\
& \times(1+ic_s p_1\tau_1)(1+ic_s p_2\tau_2)(1+ic_s p_3\tau_3)\, e^{-ic_s p_1\tau_1} e^{-ic_s p_2\tau_2} e^{-ic_s p_3\tau_3} (\tau_1\tau_2\tau_3)^{-5/2}\tau_4^{1/2} \\
& \times H_\nu^{(1)}(-p_1\tau_1)H_\nu^{(1)}(-p_2\tau_2)H_\nu^{(1)}(-p_3\tau_3)H_\nu^{(2)}(-p_1\tau_4)H_\nu^{(2)}(-p_2\tau_4)H_\nu^{(2)}(-p_3\tau_4) \\
& +(p_1 \leftrightarrow p_3) + (p_2 \leftrightarrow p_3) + \mathrm{c.c.}
\end{aligned} \qquad (7.49)$$

Using the horizon exit approximations (7.10), (7.11) of Hankel functions

$$H_\nu^{(1)}(-p_3\tau_2) \to -i\frac{2^\nu}{\pi}(-p_3\tau_2)^{-\nu}\Gamma(\nu), \tag{7.50}$$

$$H_\nu^{(2)}(-p_3\tau_1) \to i\frac{2^\nu}{\pi}(-p_3\tau_1)^{-\nu}\Gamma(\nu), \tag{7.51}$$

we can compute

$$\begin{aligned}\langle\pi\pi\pi\rangle \;\simeq\; & -6\times 40\times 3^3\lambda_1 v A^3 H^{-7}\left(\frac{1}{4\epsilon M_{\rm Pl}c_s}\right)^3\left(\frac{\pi}{4}\right)^3\left(\frac{2^\nu}{\pi}\right)^6\Gamma^6(\nu)\left(\frac{1}{p_1p_2p_3}\right)^3\\ &\times\int_{-(1+i\epsilon)\infty}^{\tau_{1*}}d\tau_1\int_{-(1+i\epsilon)\infty}^{\tau_{2*}}d\tau_2\int_{-(1+i\epsilon)\infty}^{\tau_{3*}}d\tau_3\int_{-(1+i\epsilon)\infty}^{\tau_{4*}}d\tau_4\\ &\times(\tau_1\tau_2\tau_3)^{-5/2}\,\tau_4^{1/2}\,(-p_1\tau_1)^{-\nu}\,(-p_2\tau_2)^{-\nu}\,(-p_3\tau_3)^{-\nu}\,(-p_1\tau_4)^{-\nu}\,(-p_2\tau_4)^{-\nu}\,(-p_3\tau_4)^{-\nu}\end{aligned} \tag{7.52}$$

The first factor 6 comes from momentum exchange and contributions from complex conjugation. Now, if we insert the cutoff $\Lambda = \gamma p$ for τ_4, we obtain

$$\begin{aligned}\langle\pi\pi\pi\rangle \;\simeq\; & 6\times 40\times 3^3\lambda_1 v A^3 H^{-7}\left(\frac{1}{4\epsilon M_{\rm Pl}c_s}\right)^3\left(\frac{\pi}{4}\right)^3\left(\frac{2^\nu}{\pi}\right)^6\Gamma^6(\nu)\left(\frac{1}{p_1p_2p_3}\right)^3\\ &\times\left(\frac{2}{3+2\nu}\right)^3\frac{2}{3-6\nu}(p_1p_2)^{3/2-\nu}(-1)^{3\nu-3/2}p_3^{2\nu}\left(1-\gamma^{3\nu-3/2}\right),\end{aligned} \tag{7.53}$$

thus non-Gaussianity is given by

$$\begin{aligned}f_{\rm NL} \;=\; & \frac{10}{9}\frac{1}{(2\pi)^7\tilde{p}_\zeta^2}p^6M_{\rm Pl}^{-3}(-H)^3\langle\pi\pi\pi\rangle\\ =\; & \frac{10}{9}\frac{1}{(2\pi)^7\tilde{p}_\zeta^2}p^6M_{\rm Pl}^{-3}(-H)^3\\ &\times 6\times 40\times 3^3\lambda_1 v A^3 H^{-7}\left(\frac{1}{4\epsilon M_{\rm Pl}c_s}\right)^3\left(\frac{\pi}{4}\right)^3\left(\frac{2^\nu}{\pi}\right)^6\Gamma^6(\nu)\left(\frac{1}{p_1p_2p_3}\right)^3\\ &\times\left(\frac{2}{3+2\nu}\right)^3\frac{2}{3-6\nu}(p_1p_2)^{3/2-\nu}(-1)^{3\nu-3/2}p_3^{2\nu}\left(1-\gamma^{3\nu-3/2}\right).\end{aligned} \tag{7.54}$$

Substituting numerical values as before gives the result

$$|f_{\rm NL}| = \mathcal{O}(10^{-3\sim 0}). \tag{7.55}$$

The calculations for the case where the interaction involves time derivatives can be performed similarly to the discussion in Table 1, and they all yield the same contribution to non-Gaussianity.

7.3 results

After all, combining all the obtained non-Gaussianity results in

$$|f_{\rm NL}| = \mathcal{O}(10^{-3\sim 0}) \times \alpha. \tag{7.56}$$

α represents the contribution from components other than Σ_{11}, or the additional group theoretical numerical factor that appears in larger GUT gauge groups extension, and so on, which one the same order contribution to the non-Gaussianity. Roughly, α is expected to be an order of magnitude of $\mathcal{O}(10^{0\sim 1})$. Since the current observation limit (Fig.7.3) is

$$|f_{\rm NL}| \lesssim 1, \tag{7.57}$$

this result is consistent with the observation and it might be possible to detect the signature of the Higgs boson in GUT by 21cm spectrum, future LSS and future CMB depending on our model parameters. That is, the non-Gaussianity $f_{\rm NL}$ is suppressed by the Planck scale, but enhanced because of the large inflation scale. If $f_{\rm NL}$ is evaluated in a parameter region that is consistent with the currently observed upper bound of tensor-scalar ratio r, $f_{\rm NL}$ will be large enough to be observable in future experiments.

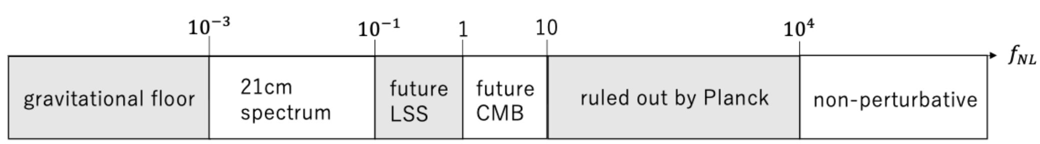


Figure 7.3: Schematic illustration of current and future constraints on the non-Gaussianity (Figure taken from [104]).

Chapter 8

Conclusion

The Standard Model of elementary particles, which has successfully explained many physical phenomena, is probably one of the most successful physical theories. However, the nature is full of rich phenomena that cannot be explained by the Standard Model alone. GUT is one of the attempts to describe these interesting phenomena. GUT is a fascinating theory that unifies the three interactions that exist in nature: strong interaction, electromagnetic interaction, and weak interaction. Although many researchers have tried to verify it, it has not yet been confirmed. For example, we have been looking for proton decay in Super Kamiokande, but have not observed that process. In addition, it is difficult to investigate by using accelerators because the GUT scale is very high energy (10^{15}GeV). For this reason, Cosmological Collider Physics has been the focus of much attention in recent years. Cosmological Collider Physics is a method to obtain information on elementary particles by using the effective field theory of inflation. Quantum fluctuations generated in the short time after the birth of the universe are stretched by inflation. It appears in the form of non-Gaussianity by observing the cosmic microwave background radiation. This means that Cosmological Collider Physics is a very interesting way to obtain information on high energy elementary particles that cannot be reached by terrestrial accelerators by means of precise observation of the universe.

In this thesis, we focus on the case where the energy scale of the inflation is close to the GUT scale, and discuss if the GUT can be verified by the non-Gaussianity due to the Higgs boson in GUT. Concretely, in addition to the effective action of inflation, we considered the action of the adjoint Higgs scalar field in SU(5) GUT. A characteristic feature of this model is that the Higgs boson has a vacuum expectation value due to spontaneous symmetry breaking, which leads to linear interactions of Higgs boson with the inflation. Therefore, the same argument can be applied not only to SU(5) GUT, but also to GUT such as SO(10) and E_6 where the Higgs boson has a vacuum expectation value and the symmetry is broken to the Standard Model, in which case the final result will be enhanced by a group theoretical factor compared to SU(5). Using these interactions, the three point function of the inflaton is generated by the tree level exchange of the GUT symmetry breaking Higgs boson. The graphs contributing to the inflaton three point function can be computed by performing horizon exit approximation, and non-Gaussianity is evaluated from the obtained values. As a result, we have shown

$$|f_{\rm NL}| \lesssim 1 \tag{8.1}$$

for non-Gaussianity without a drastic fine-tuning of parameters. This result is consistent with the current observed limit and suggests the existence of the GUT symmetry breaking Higgs boson and it might be possible to detect the signature of the GUT symmetry breaking Higgs boson by 21cm spectrum, future LSS and future CMB depending on our model parameters.

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Appendix A

Perturbations in uniformly isotropic spacetime

In appendix A to E, we follow [106] and review the perturbed metric in uniformly isotropic spacetime and gauge transformation. In this appendix, we show that the perturbed metric in uniformly isotropic spacetime is

$$ds^2 = a^2(\eta)\left[-(1+2A)d\eta^2 + 2(B_{|i} - S_i)d\eta dx^i + \left\{(1+2\psi)\gamma_{ij} + 2E_{|ij} + 2F_{i|j} + h_{ij}\right\}dx^i dx^j\right] \tag{A.1}$$

in the range of linear perturbations. The symbols appearing in the above equation are explained below.

In the universe with the FLRW metric ${}^{(b)}g_{\mu\nu}$ on the background, we consider the perturbation $\delta g_{\mu\nu}$ of the metric. This space-time metric can be written as

$$g_{\mu\nu} = {}^{(b)}g_{\mu\nu} + \delta g_{\mu\nu}. \tag{A.2}$$

For convenience, we define conformal time η by

$$\eta := \int \frac{1}{a} dt. \tag{A.3}$$

Furthermore, if the metric in 3-dimensional space with curvature K is $a^2\gamma_{ij}$, the FLRW metric ${}^{(b)}g_{\mu\nu}$ can be written as

$$\begin{aligned} {}^{(b)}ds^2 &= -dt^2 + a^2\gamma_{ij}dx^i dx^j \\ &= a^2\left(-d\eta^2 + \gamma_{ij}dx^i dx^j\right). \end{aligned} \tag{A.4}$$

As shown below, the perturbed metric $\delta g_{\mu\nu}$ can be separated into three contributions: scalar perturbation, vector perturbation, and tensor perturbation. From the symmetry of the metric tensor, there are 10 independent components in $\delta g_{\mu\nu}$.

(a) scalar perturbation

First, we consider scalar perturbations. The perturbed measure δg_{00} of the temporal-temporal component can be written

$$\delta g_{00}^{(S)} = -2a^2 A \tag{A.5}$$

with only a contribution from the scalar quantity A. The reason why a^2 is multiplied here is to correspond to the fact that the metric of the time component of the background spacetime (A.4) is ${}^{(b)}g_{00} = -a^2$. The contribution of a scalar quantity B to the perturbed metric of the temporal-spatial component δg_{0i} can be written as

$$\delta g_{0i}^{(S)} = a^2 B_{|i} \tag{A.6}$$

where $B_{|i}$ is the covariant derivative with respect to the 3-dimensional metric γ_{ij} of B. This is because the only vector that can be constracted naturally from a scalar is the derivative. The contribution of a scalar quantity to the perturbed metric δg_{ij} of the spatial-spatial component can be written as

$$\delta g_{ij}^{(S)} = 2a^2 \left(\psi\gamma_{ij} + E_{|ij}\right) \tag{A.7}$$

using two scalar quantities ψ and E. This is because the only quantity with tensor structure of the spatial part in the current theory is γ_{ij}, and the only tensor that can be constracted naturally from scalars is the second-order derivative.

In summary, the part of the metric that contains scalar perturbed metric are

$$\delta g_{00}^{(S)} = -2a^2 A, \tag{A.8}$$

$$\delta g_{0i}^{(S)} = a^2 B_{|i}, \tag{A.9}$$

$$\delta g_{ij}^{(S)} = 2a^2 \left(\psi\gamma_{ij} + E_{|ij}\right) \tag{A.10}$$

which consists of the four quantities A, B, ψ and E.

(b) vector perturbation

Next, we consider vector perturbations. Here, we use Helmholtz's theorem:

Helmholtz's theorem

Any 3-dimensional vector V_i can be written as

$$V_i = B_{|i} + S_i \tag{A.11}$$

by the sum of a vector $B_{|i}$ with zero rotation and a vector S_i with zero divergence:

$$B_{|[ij]} := \frac{1}{2}(B_{|ij} - B_{|ji}) = 0, \tag{A.12}$$
$$S_i{}^{|i} = 0 \tag{A.13}$$

This represents the decomposition of vector V_i into longitudinal and transverse wave components.

From Helmholtz's theorem, the perturbed metric $\delta g_{0i}^{(V)}$ consists of the sum of the purely vector part

$$\delta g_{0i}^{(V)} = -a^2 S_i \tag{A.14}$$

and the spatial derivative $B_{|i}$ of the scalar quantity B. This $B_{|i}$ is considered in (A.9). Similarly, the vector contribution $\delta g_{ij}^{(V)}$ to the perturbed metric δg_{ij} is the spatial derivative $F_{i|j}$ of the vector F_i with zero divergence satisfying the relation $F_i{}^{|i} = 0$, from Helmholtz's theorem. The scalar that comes out as a change of Helmholtz's theorem is E considered in the contribution (A.10) ($V_i = F_i + E_{|i}$). Using the symmetry of the metric $\delta g_{ij}^{(V)}$, we can write

$$\delta g_{ij}^{(V)} = a^2(F_{i|j} + F_{j|i}) \tag{A.15}$$

for perturbed metric originated by pure vector.

In summary, the perturbed metric component for vector perturbations are

$$\delta g_{0i}^{(V)} = -a^2 S_i, \tag{A.16}$$
$$\delta g_{ij}^{(V)} = a^2(F_{i|j} + F_{j|i}) \tag{A.17}$$

and the vector perturbations S_i and F_i satisfy the conditions

$$S_i{}^{|i} = 0, \quad F_i{}^{|i} = 0. \tag{A.18}$$

With these two restrictions, the vector perturbations S_i and F_i have $6 - 2 = 4$ degrees of freedom.

(c) tensor perturbation

Finally, we consider tensor perturbations. Here, we use the SVT (Scalar Vector Tensor) decomposition of the tensor:

SVT (Scalar Vector Tensor) decomposition of the tensor

Any second-order tensor T_{ij} can be written as

$$T_{ij}(x) = \underbrace{\left(D_i D_j - \frac{1}{3}\gamma_{ij}\nabla^2\right) S + \frac{1}{2}\left(D_i Y_j^{(L)} + D_j Y_i^{(R)}\right) + Y_{ij}}_{\text{traceless part}} + \underbrace{\frac{1}{3}\gamma_{ij} T^k{}_k}_{\text{trace part}} \qquad (A.19)$$

where $Y_i^{(L)}, Y_i^{(R)}, Y_{ij}$ satisfy

$$D^i Y_i^{(L)} = 0, \qquad (A.20)$$

$$D^i Y_i^{(R)} = 0, \qquad (A.21)$$

$$D^i Y_{ij} = D^j Y_{ij} = Y_i^{\ i} = 0 \qquad (A.22)$$

respectively. D^i represents the covariant derivative. If T_{ij} is a symmetric tensor, then $Y_i^{(L)} = Y_i^{(R)}$ and $Y_{ij} = Y_{ji}$. The above equation (A.19) is the decomposition of the vector into longitudinal and transverse wave components twice.

We write

$$\delta g_{ij}^{(T)} = a^2 T_{ij} \qquad (A.23)$$

for the contribution by the tensor of the perturbed metric δg_{ij}. From the SVT decomposition of the tensor, we have

$$T_{ij} = \left(D_i D_j - \frac{1}{3}\gamma_{ij}\nabla^2\right) S + \frac{1}{3}\gamma_{ij} Y + \frac{1}{2}\left(D_i Y_j^{(L)} + D_j Y_i^{(R)}\right) + Y_{ij}. \qquad (A.24)$$

The first term on the right-hand side is the traceless part, which is the contribution from scalar quantities. The second term is the trace part, which is also a contribution from scalar quantity. The third term is the traceless part, which is the contribution from vector quantities satisfying $D^i Y_i^{(L)} = 0, D^i Y_i^{(R)} = 0$. Therefore, to extract the fourth term, which is the contribution from the pure tensor part, T_{ij} must be divergenceless and traceless. We write T_{ij} satisfying these conditions as h_{ij}:

$$\delta g_{ij}^{(T)} = a^2 h_{ij}, \tag{A.25}$$

$$h_{ij}{}^{|j} = 0, \tag{A.26}$$

$$h_i{}^i = 0. \tag{A.27}$$

The independent degree of freedom of h_{ij} is $6 - 3 - 1 = 2$.

Summarizing the above scalar, vector and tensor perturbations, the perturbed metric with FLRW metric in the background from equations (A.8), (A.16) and (A.25) is as follows:

$$ds^2 = a^2(\eta)\left[-(1+2A)d\eta^2 + 2(B_{|i} - S_i)d\eta dx^i + \left\{(1+2\psi)\gamma_{ij} + 2E_{|ij} + 2F_{i|j} + h_{ij}\right\} dx^i dx^j\right] \tag{A.28}$$

Note that these perturbations satisfy the conditions

$$S_i{}^{|i} = 0, \tag{A.29}$$

$$F_i{}^{|i} = 0, \tag{A.30}$$

$$h_{ij}{}^{|j} = 0, \tag{A.31}$$

$$h_i{}^i = 0 \tag{A.32}$$

from equations (A.18), (A.26) and (A.27). All perturbations are functions of conformal time η and position $\boldsymbol{x}$. The physical meanings of each perturbation quantities are

A : fluctuations in the passage of time (Newton Potential),

$B_{|i} - S_i$: fluctuation of displacement vector,

ψ : fluctuation of spatial volume,

$2E_{|ij} + 2F_{i|j} + h_{ij}$: anisotropy of space (E is rescaled according to radius, h_{ij} is distortion)

respectively. The physical meaning of the conditions (A.18) and (A.25) is as follows. The conditions of the divergenceless

$$S_i{}^{|i} = 0, \quad F_i{}^{|i} = 0, \quad h_{ij}{}^{|j} = 0 \tag{A.33}$$

imply that the vectors S_i, F_i and the gravitational wave h_{ij} are transverse waves. Traceless condition

$$h_i{}^i = 0 \tag{A.34}$$

means that gravitational waves do not change the spatial volume.

Appendix B

Gauge transformation

Since the general relativity is a gauge theory with general coordinate transformations, the apparent degrees of freedom, called gauge degrees of freedom, appear in the theory. In linear perturbation theory, gauge degrees of freedom are understood as degrees of freedom of infinitesimal coordinate transformations. In this Appendix, we investigate how the perturbed quantities in metric (A.28)

$$ds^2 = a^2(\eta)\left[-(1+2A)d\eta^2 + 2(B_{|i} - S_i)d\eta dx^i + \left\{(1+2\psi)\gamma_{ij} + 2E_{|ij} + 2F_{i|j} + h_{ij}\right\} dx^i dx^j\right] \tag{B.1}$$

are transformed under such gauge transformations.

We consider the spacetime as a manifold that can take local coordinates at any point, and introduce the following two manifolds:

- background spacetime manifold $\mathcal{N}$
- physical (all spacetime) manifolds including perturbations $\mathcal{M}$

Set the coordinate system ${}^{(\mathrm{b})}x^\mu$ in the background spacetime $\mathcal{N}$ and set the coordinate system x^μ in the all spacetime $\mathcal{M}$. Consider a diffeomorphism map

$$\mathcal{D} : \mathcal{N} \to \mathcal{M}, \quad {}^{(\mathrm{b})}x^\mu \mapsto x^\mu \tag{B.2}$$

from $\mathcal{N}$ to $\mathcal{M}$ between these coordinate systems. The point P on $\mathcal{M}$ is transformed to the point $\mathcal{D}^{-1}(P)$ on $\mathcal{N}$ by the inverse map The perturbation δT(δT can be a scalar, vector, or tensor) at point P is defined by

$$\delta T\left(x^\mu(P)\right) := T\left(x^\mu(P)\right) - {}^{(\mathrm{b})}T\left({}^{(\mathrm{b})}x^\mu\left(\mathcal{D}^{-1}(P)\right)\right). \tag{B.3}$$

That is, the difference between the quantity on the total space-time $\mathcal{M}$ and the quantity on the background space-time $\mathcal{N}$ at the point P.

Next, consider a diffeomorphism map

$$\tilde{\mathcal{D}} : \mathcal{N} \to \mathcal{M}, \quad {}^{(\mathrm{b})}x^{\mu} \mapsto \tilde{x}^{\mu} \tag{B.4}$$

from the background space-time $\mathcal{N}$ to another coordinate system $\tilde{x}^{\mu}$ on the all space-time $\mathcal{M}$. In this coordinate system $\tilde{x}^{\mu}$, the value of T at the point P on $\mathcal{M}$ is defined as $\tilde{T}\left(\tilde{x}^{\mu}(P)\right)$. Since the value in the background spacetime $\mathcal{N}$ corresponding to the point P is ${}^{(b)}T\left(\tilde{\mathcal{D}}^{-1}(P)\right)$, the perturbation $\widetilde{\delta T}$ in the new coordinate system is

$$\widetilde{\delta T}\left(\tilde{x}^{\mu}(P)\right) = \tilde{T}\left(\tilde{x}^{\mu}(P)\right) - {}^{(\mathrm{b})}T\left({}^{(\mathrm{b})}\tilde{x}^{\mu}\left(\tilde{\mathcal{D}}^{-1}(P)\right)\right). \tag{B.5}$$

Now, consider the following infinitesimal coordinate transformations (gauge transformation):

$$\tilde{x}^{\mu} = x^{\mu} + \xi^{\mu}. \tag{B.6}$$

If T is a scalar quantity f, then the relation $\tilde{f}\left(\tilde{x}^{\mu}\right) = f\left(x^{\mu}\right)$ holds and the perturbation (B.5) at point P becomes

$$\begin{aligned}
\widetilde{\delta f}\left(\tilde{x}\right) &= f\left(x\right) - {}^{(\mathrm{b})}f\left({}^{(\mathrm{b})}\tilde{x}^{\mu}\left(\tilde{\mathcal{D}}^{-1}(P)\right)\right) \\
&= f\left(\tilde{x}\right) - \xi^{\mu}\frac{\partial f\left(\tilde{x}\right)}{\partial \tilde{x}^{\mu}} - {}^{(\mathrm{b})}f\left({}^{(\mathrm{b})}\tilde{x}^{\mu}\left(\tilde{\mathcal{D}}^{-1}(P)\right)\right) \quad \text{(first order expansion for } \xi\text{)} \\
&= \delta f\left(\tilde{x}\right) - \xi^{\mu}\frac{\partial f\left(\tilde{x}\right)}{\partial \tilde{x}^{\mu}}.
\end{aligned} \tag{B.7}$$

Since $\partial f\left(\tilde{x}\right)/\partial \tilde{x}^{\mu}$, which contributes to the last term in the first-order of infinitesimal quantities, is time-dependent quantity in the background space-time $\mathcal{N}$ and the scalar quantity f is ${}^{(\mathrm{b})}f\left(\tilde{x}^{0}\right)$ in uniformly isotropic spacetime,

$$\widetilde{\delta f} = \delta f - \xi^{0} f' \tag{B.8}$$

is obtained. Note that the prime represents the derivative by $\tilde{x}^{0} = \tilde{\eta}$.

Using Helmholtz's theorem on the vector $\tilde{x}^{i}$, the infinitesimal coordinate transformations of the temporal and spatial parts of the gauge transformation (B.6) can be written as

$$\begin{aligned}
\tilde{\eta} &= \eta + \xi^{0}, \\
\tilde{x}^{i} &= x^{i} + \xi_{|}{}^{i} + \zeta^{i}
\end{aligned} \tag{B.9}$$

respectively, where $\xi^0(\eta, x^i)$ and $\xi(\eta, x^i)$ are scalar functions and $\zeta^i(\eta, x^i)$ is vector with zero divergence ${\zeta^i}_{|i} = 0$. Since relations

$$\begin{aligned} \xi^0\left(\eta, x^i\right) &= \xi^0\left(\tilde{\eta}, \tilde{x}^i\right), && \text{(B.10)} \\ \xi\left(\eta, x^i\right) &= \xi\left(\tilde{\eta}, \tilde{x}^i\right), && \text{(B.11)} \\ \zeta^i\left(\eta, x^i\right) &= \zeta^i\left(\tilde{\eta}, \tilde{x}^i\right) && \text{(B.12)} \end{aligned}$$

hold in the first-order perturbation of infinitesimal coordinate transformations, we obtain

$$\begin{aligned} d\xi^0 &= \xi^{0\prime} d\tilde{\eta} + {\xi^0}_{|i} d\tilde{x}^i, && \text{(B.13)} \\ d\xi &= \xi' d\tilde{\eta} + \xi_{|j} d\tilde{x}^j, && \text{(B.14)} \\ d\zeta^i &= \zeta^{i\prime} d\tilde{\eta} + {\zeta^i}_{|j} d\tilde{x}^j && \text{(B.15)} \end{aligned}$$

for small changes. From these equations and the derivative of (B.9), we can obtain

$$\begin{aligned} d\eta &= d\tilde{\eta} - \xi^{0\prime} d\tilde{\eta} - {\xi^0}_{|i} d\tilde{x}^i && \text{(B.16)} \\ dx^i &= d\tilde{x}^i - \left({\xi'}_{|}{}^{i} + \zeta^{i\prime}\right) d\tilde{\eta} - \left({\xi_{|}}^{i}{}_{j} + {\zeta^i}_{|j}\right) d\tilde{x}^j. && \text{(B.17)} \end{aligned}$$

Also, a relation

$$a(\eta) = a\left(\tilde{\eta} - \xi^0\right) = a(\tilde{\eta}) - \xi^0 a'(\tilde{\eta}) \tag{B.18}$$

holds for the scale factor $a(\eta)$. Substituting equations (B.16), (B.17), and (B.18) into the perturbed metric (A.28)

$$ds^2 = a^2(\eta)\left[-(1+2A)d\eta^2 + 2(B_{|i} - S_i)d\eta dx^i + \left\{(1+2\psi)\gamma_{ij} + 2E_{|ij} + 2F_{i|j} + h_{ij}\right\} dx^i dx^j\right], \tag{B.19}$$

we can obtain gauge transformations of the perturbations: From equations (B.16), (B.17), and (B.18),

$$
\begin{aligned}
a^2(\eta) &= \left\{a(\tilde{\eta}) - \xi^0 a'(\tilde{\eta})\right\}^2 \\
&= a^2(\tilde{\eta}) - 2aa'\xi^0,
\end{aligned} \tag{B.20}
$$

$$
\begin{aligned}
d\eta^2 &= \left(d\tilde{\eta} - \xi^{0\prime} d\tilde{\eta} - \xi^0{}_{|i} d\tilde{x}^i\right)^2 \\
&= d\tilde{\eta}^2 - 2\xi^{0\prime} d\tilde{\eta}^2 - 2\xi^0{}_{|i} d\tilde{\eta} d\tilde{x}^i,
\end{aligned} \tag{B.21}
$$

$$
\begin{aligned}
d\eta dx^i &= \left(d\tilde{\eta} - \xi^{0\prime} d\tilde{\eta} - \xi^0{}_{|k} d\tilde{x}^k\right) \left\{d\tilde{x}^i - \left(\xi'{}_|{}^i + \zeta^{i\prime}\right) d\tilde{\eta} - \left(\xi_|{}^i{}_j + \zeta^i{}_{|j}\right) d\tilde{x}^j\right\} \\
&= d\tilde{\eta} d\tilde{x}^i - \left(\xi'{}_|{}^i + \zeta^{i\prime}\right) d\tilde{\eta}^2 - \left(\xi_|{}^i{}_j + \zeta^i{}_{|j}\right) d\tilde{\eta} d\tilde{x}^j - \xi^{0\prime} d\tilde{\eta} d\tilde{x}^i - \xi^0{}_{|k} d\tilde{x}^k d\tilde{x}^i \\
&= -\left(\xi'{}_|{}^i + \zeta^{i\prime}\right) d\tilde{\eta}^2 + (1 - \xi^{0\prime}) d\tilde{\eta} d\tilde{x}^i - \left(\xi_|{}^i{}_j + \zeta^i{}_{|j}\right) d\tilde{\eta} d\tilde{x}^j - \xi^0{}_{|j} d\tilde{x}^j d\tilde{x}^i,
\end{aligned} \tag{B.22}
$$

$$
\begin{aligned}
dx^i dx^j &= \left\{d\tilde{x}^i - \left(\xi'{}_|{}^i + \zeta^{i\prime}\right) d\tilde{\eta} - \left(\xi_|{}^i{}_k + \zeta^i{}_{|k}\right) d\tilde{x}^k\right\} \\
&\quad \times \left\{d\tilde{x}^j - \left(\xi'{}_|{}^j + \zeta^{j\prime}\right) d\tilde{\eta} - \left(\xi_|{}^j{}_l + \zeta^j{}_{|l}\right) d\tilde{x}^l\right\} \\
&= d\tilde{x}^i d\tilde{x}^j - \left(\xi'{}_|{}^j + \zeta^{j\prime}\right) d\tilde{\eta} d\tilde{x}^i \\
&\quad - \left(\xi_|{}^j{}_l + \zeta^j{}_{|l}\right) d\tilde{x}^l d\tilde{x}^i - \left(\xi'{}_|{}^i + \zeta^{i\prime}\right) d\tilde{\eta} d\tilde{x}^j - \left(\xi_|{}^i{}_k + \zeta^i{}_{|k}\right) d\tilde{x}^k d\tilde{x}^j
\end{aligned} \tag{B.23}
$$

are obtained, and substituting them into the perturbed metric (A.28), it is calculated as in

$$
\begin{aligned}
& ds^2 \\
= \ & \left\{a^2(\tilde{\eta}) - 2aa'\xi^0\right\} \left[-(1+2A)\left(d\tilde{\eta}^2 - 2\xi^{0\prime} d\tilde{\eta}^2 - 2\xi^0{}_{|i} d\tilde{\eta} d\tilde{x}^i\right)\right. \\
& +2(B_{|i} - S_i)\left\{-\left(\xi'_{|}{}^i + \zeta^{i\prime}\right) d\tilde{\eta}^2 + (1-\xi^{0\prime}) d\tilde{\eta} d\tilde{x}^i - \left(\xi_{|}{}^i{}_j + \zeta^i{}_{|j}\right) d\tilde{\eta} d\tilde{x}^j - \xi^0{}_{|j} d\tilde{x}^j d\tilde{x}^i\right\} \\
& + \left\{(1+2\psi)\gamma_{ij} + 2E_{|ij} + 2F_{i|j} + h_{ij}\right\} \left\{d\tilde{x}^i d\tilde{x}^j - \left(\xi'_{|}{}^j + \zeta^{j\prime}\right) d\tilde{\eta} d\tilde{x}^i - \left(\xi_{|}{}^j{}_l + \zeta^j{}_{|l}\right) d\tilde{x}^l d\tilde{x}^i\right. \\
& \qquad \left.\left. - \left(\xi'_{|}{}^i + \zeta^{i\prime}\right) d\tilde{\eta} d\tilde{x}^j - \left(\xi_{|}{}^i{}_k + \zeta^i{}_{|k}\right) d\tilde{x}^k d\tilde{x}^j\right\}\right] \\
= \ & \left\{a^2(\tilde{\eta}) - 2aa'\xi^0\right\} \left[-\left(1 + 2A - 2\xi^{0\prime}\right) d\tilde{\eta}^2 + 2(B_{|i} + \xi^0{}_{|i} - \xi'{}_{|i} - S_i - \zeta'{}_i) d\tilde{\eta} d\tilde{x}^i\right. \\
& \qquad \left. + \left\{(1+2\psi)\gamma_{ij} + 2E_{|ij} + 2F_{i|j} + h_{ij}\right\} d\tilde{x}^i d\tilde{x}^j - 2\gamma_{ij}\left(\xi_{|}{}^j{}_l + \zeta_{|}{}^j{}_l\right) d\tilde{x}^l d\tilde{x}^i\right] \\
= \ & a^2(\tilde{\eta}) \left[-\left(1 + 2A - 2\xi^{0\prime} - 2\frac{a'}{a}\xi^0\right) d\tilde{\eta}^2 + 2(B_{|i} + \xi^0{}_{|i} - \xi'{}_{|i} - S_i - \zeta'{}_i) d\tilde{\eta} d\tilde{x}^i\right. \\
& \qquad \left. + \left\{\left(1 + 2\psi - 2\frac{a'}{a}\xi^0\right)\gamma_{ij} + 2\left(E_{|ij} - \xi_{|ij}\right) + 2\left(F_{i|j} - \zeta_{i|j}\right) + h_{ij}\right\} d\tilde{x}^i d\tilde{x}^j\right].
\end{aligned}
\tag{B.24}
$$

Since the line element is a scalar invariant $ds^2 = d\tilde{s}^2$, it can be seen from equation (B.24) and

$$
d\tilde{s}^2 = a^2(\eta)\left[-(1+2\tilde{A})d\tilde{\eta}^2 + 2(\tilde{B}_{|i} - \tilde{S}_i) d\tilde{\eta} d\tilde{x}^i + \left\{(1+2\tilde{\psi})\gamma_{ij} + 2\tilde{E}_{|ij} + 2\tilde{F}_{i|j} + \tilde{h}_{ij}\right\} d\tilde{x}^i d\tilde{x}^j\right] \tag{B.25}
$$

that the perturbed quantities receives the following gauge transformation, respectively:

$$\tilde{A} = A - \mathcal{H}\xi^0 - \xi^{0\prime}, \tag{B.26}$$

$$\tilde{B} = B + \xi^0 - \xi', \tag{B.27}$$

$$\tilde{\psi} = \psi - \mathcal{H}\xi^0, \tag{B.28}$$

$$\tilde{E} = E - \xi, \tag{B.29}$$

$$\tilde{F}_i = F_i - \zeta_i, \tag{B.30}$$

$$\tilde{S}_i = S_i + \zeta_i', \tag{B.31}$$

$$\tilde{h}_{ij} = h_{ij}, \tag{B.32}$$

where $\mathcal{H}$ is defined as

$$\mathcal{H} := \frac{a'}{a} = aH = \dot{a}. \tag{B.33}$$

Appendix C

Fluctuation of matter density

In the previous appendix, we have considered perturbations of the metric that appears on the left-hand side of the Einstein equation. In this appendix, we consider the perturbation $\delta T^{\mu}{}_{\nu}$ of the energy momentum tensor $T^{\mu}{}_{\nu}$ that appears on the right-hand side of the Einstein equation. First, we compute the scalar, vector, and tensor components of the perturbation $\delta T^{\mu}{}_{\nu}$, respectively. Then, we derive the transformation laws for each component of the perturbation $\delta T^{\mu}{}_{\nu}$ by the gauge transformation (B.6)

$$\tilde{x}^{\mu} = x^{\mu} + \xi^{\mu}. \tag{C.1}$$

In the following, we will discuss the cases of fluid and scalar fields separately.

C.1 Fluctuation of fluid density

The energy-momentum tensor of a fluid with energy density ρ and pressure P can be written as

$$T^{\mu}{}_{\nu} = (\rho + P)u^{\mu}u_{\nu} + P\delta^{\mu}_{\nu} + \pi^{\mu}{}_{\nu} \tag{C.2}$$

in general. The $\pi^{\mu}{}_{\nu}$ is called an anisotropic stress and contributes to an anisotropy in the spatial direction. u^{μ} is 4-velocity satisfying a relation

$$g_{\mu\nu}u^{\mu}u^{\nu} = -1. \tag{C.3}$$

In perturbed metric (A.28)

$$ds^2 = a^2(\eta)\left[-(1+2A)d\eta^2 + 2(B_{|i} - S_i)d\eta dx^i + \left\{(1+2\psi)\gamma_{ij} + 2E_{|ij} + 2F_{i|j} + h_{ij}\right\}dx^i dx^j\right], \tag{C.4}$$

since the proper time τ is related to the conformal time η by

$$d\tau = a(1+A)d\eta, \tag{C.5}$$

the temporal component u^0 of the 4-velocity is

$$u^0 = \frac{d\eta}{d\tau} = \frac{1}{a}(1+A)^{-1} = \frac{1}{a}(1-A). \tag{C.6}$$

The spatial component of the 4-velocity u^μ is

$$u^i = \frac{dx^i}{d\tau} = \frac{dx^i}{d\eta}\frac{d\eta}{d\tau}, \tag{C.7}$$

while the vector $dx^i/d\eta$ can be written in terms of the contribution from a scalar $v_|{}^i$ and from a pure vector v^i, from Helmholtz's theorem, and thus

$$u^i = \frac{dx^i}{d\eta}\frac{d\eta}{d\tau} = \frac{1}{a}(1-A)(v_|{}^i + v^i) = \frac{1}{a}(v_|{}^i + v^i) \quad \left(|v_|{}^i|,\, |v^i| \ll 1\right). \tag{C.8}$$

The scalar quantity v is a velocity potential with zero rotation about the vector $v_|{}^i$. The vector v^i is a vector describing the rotation of the fluid with zero divergence, i.e. $v^i{}_{|i} = 0$. Therefore, the 4-velocity u^μ is given by

$$u^\mu = \frac{1}{a}\left(1-A, v_|{}^i + v^i\right). \tag{C.9}$$

The covariant component of the 4-velocity u_μ has temporal component

$$\begin{aligned} u_0 = g_{0\mu}u^\mu &= g_{00}u^0 + g_{0i}u^i \\ &= -a^2(1+2A)\frac{1}{a}(1-A) + a^2\left(B_{|i} - S_i\right)\frac{1}{a}\left(u_|{}^i + u^i\right) \\ &= -a(1+A), \end{aligned} \tag{C.10}$$

and spatial components

$$\begin{aligned} u_i = g_{i\mu}u^\mu &= g_{i0}u^0 + g_{ij}u^j \\ &= a^2\left(B_{|i} - S_i\right)\frac{1}{a}(1-A) + a^2\left\{(1+2\psi)\gamma_{ij} + 2E_{|ij} + 2F_{i|j} + h_{ij}\right\}\frac{1}{a}(v_|{}^j + v^j) \\ &= a(B_{|i} - S_i) + a\gamma_{ij}(v_|{}^j + v^j) \\ &= a(B_{|i} - S_i + v_{|i} + v_i) \end{aligned} \tag{C.11}$$

for the perturbed metric (A.28), which can be summarized as

$$u_\mu = a\left(-1-A, v_{|i} + B_{|i} + v_i - S_i\right). \tag{C.12}$$

The energy density and pressure of the fluid are divided into the background and perturbation parts and denoted as

$$\rho_{\text{total}} = \rho + \delta\rho, \quad P_{\text{total}} = P + \delta P. \tag{C.13}$$

ρ, P satisfy the continuous equation

$$\rho' + 3\mathcal{H}(\rho + P) = 0 \tag{C.14}$$

from the Einstein equation. Note that the anisotropic stress $\pi^{\mu}{}_{\nu}$ has only a spatial component $\pi^{i}{}_{j}$ and can be divided into a scalar part Π, a vector part π^{i}, a purely tensor part ${}^{(T)}\pi^{i}{}_{j}$ and a trace part $\pi^{k}{}_{k}\,(=0)$ as in

$$\pi^{i}{}_{j} = \left(\Pi_{|}{}^{i}{}_{j} - \frac{1}{3}\nabla^{2}\Pi\delta^{i}_{j}\right) + \frac{1}{2}\left(\pi^{i}{}_{|j} + \pi_{j|}{}^{i}\right) + {}^{(T)}\pi^{i}{}_{j} + \frac{1}{3}\pi^{k}{}_{k}\delta^{i}_{j} \tag{C.15}$$

from the SVT decomposition of the tensor, and substituting the 4-velocity (C.9), (C.12) into the energy momentum tensor (C.2)

$$T^{\mu}{}_{\nu} = (\rho_{\text{total}} + P_{\text{total}})u^{\mu}u_{\nu} + P_{\text{total}}\delta^{\mu}_{\nu} + \pi^{\mu}{}_{\nu}, \tag{C.16}$$

we can obtain

$$\begin{aligned} T^{0}{}_{0} &= (\rho + \delta\rho + P + \delta P)u^{0}u_{0} + P + \delta P \\ &= (\rho + \delta\rho + P + \delta P)(1 - A)(-1 - A) + P + \delta P \\ &= -(\rho + \delta\rho + P + \delta P)(1 - A^{2}) + P + \delta P \\ &= -\rho - \delta\rho, \end{aligned} \tag{C.17}$$

$$\begin{aligned} T^{0}{}_{i} &= (\rho + \delta\rho + P + \delta P)u^{0}u_{i} \\ &= (\rho + \delta\rho + P + \delta P)(1 - A)(v_{|i} + B_{|i} + v_{i} - S_{i}) \\ &= (\rho + P)(v_{|i} + B_{|i} + v_{i} - S_{i}), \end{aligned} \tag{C.18}$$

$$\begin{aligned} T^{i}{}_{0} &= (\rho + \delta\rho + P + \delta P)u^{i}u_{0} \\ &= (\rho + \delta\rho + P + \delta P)(v_{|}{}^{i} + v^{i})(-1 - A) \\ &= -(\rho + P)(v_{|}{}^{i} + v^{i}), \end{aligned} \tag{C.19}$$

$$
\begin{aligned}
T^i{}_j &= (\rho+\delta\rho+P+\delta P)u^i u_j + (P+\delta P)\delta^i_j + \pi^i{}_j \\
&= (\rho+\delta\rho+P+\delta P)(v_|{}^i + v^i)(v_{|i} + B_{|i} + v_i - S_i) + (P+\delta P)\delta^i_j + \pi^i{}_j \\
&= (P+\delta P)\delta^i_j + \pi^i{}_j, \\
&= (P+\delta P)\delta^i_j + \left(\Pi_|{}^i{}_j - \frac{1}{3}\nabla^2\Pi\delta^i_j\right) + \frac{1}{2}\left(\pi^i{}_{|j} + \pi_{j|}{}^i\right) +^{(T)} \pi^i{}_j. \qquad \text{(C.20)}
\end{aligned}
$$

From these expressions, the perturbations corresponding to scalar, vector, and tensor, respectively, are as follows:

- scalar perturbations

$$
\begin{aligned}
{}^{(S)}\delta T^0{}_0 &= -\delta\rho, \\
{}^{(S)}\delta T^0{}_i &= (\rho+P)\left(v_{|i} + B_{|i}\right), \\
{}^{(S)}\delta T^i{}_0 &= -(\rho+P)v_|{}^i, \\
{}^{(S)}\delta T^i{}_j &= \delta P\delta^i_j + \Pi_|{}^i{}_j - \frac{1}{3}\nabla^2\Pi\delta^i_j.
\end{aligned} \qquad \text{(C.21)}
$$

- vector perturbations

$$
\begin{aligned}
{}^{(V)}\delta T^0{}_0 &= 0, \\
{}^{(V)}\delta T^0{}_i &= (\rho+P)\left(v_i - S_i\right), \\
{}^{(V)}\delta T^i{}_0 &= -(\rho+P)v^i, \\
{}^{(V)}\delta T^i{}_j &= \frac{1}{2}\left(\pi^i{}_{|j} + \pi_{j|}{}^i\right).
\end{aligned} \qquad \text{(C.22)}
$$

- tensor perturbations

$$
\begin{aligned}
{}^{(T)}\delta T^0{}_0 &= 0, \\
{}^{(T)}\delta T^0{}_i &= 0, \\
{}^{(T)}\delta T^i{}_0 &= 0, \\
{}^{(T)}\delta T^i{}_j &=^{(T)} \pi^i{}_j.
\end{aligned} \qquad \text{(C.23)}
$$

Next, we determine the transformation law for each component of the perturbation $\delta T^\mu{}_\nu$ by the gauge transformation (B.6)

$$
\tilde{x}^\mu = x^\mu + \xi^\mu. \qquad \text{(C.24)}
$$

From the discussion in the previous appendix, since the scalar quantity f receives the transformation

$$\widetilde{\delta f} = \delta f - \xi^0 f', \tag{C.25}$$

the transformation with respect to the scalar perturbations $\delta\rho$ and δP become

$$\widetilde{\delta\rho} = \delta\rho - \rho'\xi^0, \qquad \widetilde{\delta P} = \delta P - P'\xi^0. \tag{C.26}$$

Next, we determine the transformations for the scalar part ${}^{(S)}\delta T^0{}_i$ and the vector part ${}^{(V)}\delta T^0{}_i$ of $\delta T^0{}_i$. Differentiating the gauge transformation (B.6)

$$\tilde{x}^i = x^i + \xi_|{}^i + \zeta^i \tag{C.27}$$

with respect to η yields

$$\begin{aligned} (\tilde{x}^i)' &= (x^i)' + (\xi_|{}^i)' + (\zeta^i)' \\ &= u^i + (\xi_|{}^i)' + (\zeta^i)' \\ &= v^i + v_|{}^i + (\xi_|{}^i)' + (\zeta^i)' \\ &= \left(v^i + (\zeta^i)'\right) + \left(v_|{}^i + (\xi_|{}^i)'\right) \\ &\equiv \tilde{v}^i(\tilde{x}) + \tilde{v}(\tilde{x}). \end{aligned} \tag{C.28}$$

Thus, the transformation of the scalar velocity potential v is

$$\tilde{v}(\tilde{x}) = v(x) + \xi'(x) \tag{C.29}$$

and that of the velocity vector v^i is

$$\tilde{v}^i(\tilde{x}) = v^i(x) + \left(\zeta^i\right)'(x). \tag{C.30}$$

Let us define two quantities to be used later. First, consider

$$\delta q_{|i} := a^{(S)}\delta T^0{}_i, \tag{C.31}$$

the density of energy flow generated by the scalar fluctuations. In the fluid case, this quantity is

$$\delta q = a(\rho + P)(v + B). \tag{C.32}$$

Note that the unperturbed scalar quantity f satisfies

$$\tilde{f}(\tilde{x}) = f(x), \tag{C.33}$$

from equation (B.27)

$$\tilde{B}(\tilde{x}) = B(x) + \xi^0(x) - \xi'(x) \tag{C.34}$$

and (C.29), δq receives a gauge transformation

$$\begin{aligned}
\widetilde{\delta q} &= \tilde{a}(\tilde{\eta}) \left\{ \tilde{\rho}(\tilde{\eta}) + \tilde{P}(\tilde{\eta}) \right\} \left\{ \tilde{v}(\tilde{x}) + \tilde{B}(\tilde{x}) \right\} \\
&= a(\eta) \left\{ \rho(\eta) + P(\eta) \right\} \left\{ v(x) + \xi'(x) + B(x) + \xi^0(x) - \xi'(x) \right\} \\
&= a(\eta) \left\{ \rho(\eta) + P(\eta) \right\} \left\{ v(x) + B(x) + \xi^0(x) \right\} \\
&= \delta q + a(\rho + P)\xi^0(x).
\end{aligned} \tag{C.35}$$

Similarly, we define

$$^{(V)}\delta q_i := a^{(V)}\delta T^0{}_i \tag{C.36}$$

to be the density of the energy flow generated by the vector fluctuations. In the fluid case, this quantity is

$$^{(V)}\delta q_i = a(\rho + P)\left(v_i - S_i\right). \tag{C.37}$$

From equation (B.31)

$$\tilde{S}_i = S_i + \zeta_i' \tag{C.38}$$

and (C.30)

$$\tilde{v}^i(\tilde{x}) = v^i(x) + \left(\zeta^i\right)'(x), \tag{C.39}$$

we see that $^{(V)}\delta q_i$ is gauge invariant

$$^{(V)}\widetilde{\delta q}_i = {}^{(V)}\delta q_i. \tag{C.40}$$

That is, $^{(V)}\delta q_i$ is invariant under the coordinate transformations.

C.2 Fluctuation of the scalar field density

Next, we consider density fluctuations of the scalar field ϕ. When the Lagrangian $\mathcal{L}$ can be written

$$\mathcal{L} = P(\phi, X) \tag{C.41}$$

as a function of ϕ and its kinetic energy

$$X = -\frac{1}{2}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi), \tag{C.42}$$

the field action is

$$S = \int d^4x\,\sqrt{-g}\mathcal{L}, \tag{C.43}$$

where g is the determinant of a 4×4 matrix with metric $g_{\mu\nu}$ as its component. The Lagrangian of a canonical scalar field with potential $V(\phi)$ is given by

$$\mathcal{L} = P(\phi, X) = X - V(\phi). \tag{C.44}$$

The model (C.43) is a general model that includes not only the potential $V(\phi)$ but also the case with nonlinear terms in X.

For the action S, we define the energy-momentum tensor to be the quantity that takes the variational with respect to $g^{\mu\nu}$:

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = -2\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}. \tag{C.45}$$

In the second equality, we used a relation

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}. \tag{C.46}$$

Since the relation

$$-2\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} = -2\frac{\partial P}{\partial X}\frac{\partial X}{\partial g^{\mu\nu}} = P_{,X}(\partial_\mu\phi)\partial_\nu\phi \tag{C.47}$$

holds for general Lagrangian

$$\mathcal{L} = P(\phi, X), \tag{C.48}$$

the energy-momentum tensor becomes

$$T_{\mu\nu} = P_{,X}(\partial_\mu\phi)\partial_\nu\phi + g_{\mu\nu}P. \tag{C.49}$$

We denote the derivative of the Lagrangian P with respect to the kinetic energy X as

$$P_{,X} := \frac{\partial P}{\partial X}. \tag{C.50}$$

If we raise the subscript one up, we get

$$T^\mu{}_\nu = P_{,X}(\partial^\mu\phi)\partial_\nu\phi + \delta^\mu_\nu P. \tag{C.51}$$

Separating the scalar field into a background part and a perturbed part as in

$$\phi_{\rm total}(\eta, \boldsymbol{x}) = \phi(\eta) + \delta\phi(\eta, \boldsymbol{x}), \tag{C.52}$$

the density ρ of the field for the background part is

$$\begin{aligned}
\rho = -{}^{(b)}T^0{}_0 &= -\left(P_{,X}(\partial^0\phi)\partial_0\phi + P\right) \\
&= -P_{,X}{}^{(b)}g^{0\mu}(\partial_\mu\phi)\partial_0\phi - P \\
&= \frac{1}{a^2}P_{,X}\phi'^2 - P \\
&= 2XP_{,X} - P,
\end{aligned} \tag{C.53}$$

where the kinetic energy of the background field

$$X = \frac{1}{2a^2}\phi'^2 \tag{C.54}$$

is used in the last equality. Note that the background field pressure $T^i{}_i/3$ is P itself. These ρ, P satisfy the continuous equation (C.14).

For the perturbed metric

$$ds^2 = a^2(\eta)\left[-(1+2A)d\eta^2 + 2(B_{|i} - S_i)d\eta dx^i + \left\{(1+2\psi)\gamma_{ij} + 2E_{|ij} + 2F_{i|j} + h_{ij}\right\}dx^i dx^j\right], \tag{C.55}$$

the perturbed part of the energy-momentum tensor (C.51) is as follows:

$$\delta\rho := -\delta T^0{}_0 = \left(P_{,X} + 2XP_{,XX}\right)\delta X - \left(P_{,\phi} - 2XP_{,X\phi}\right)\delta\phi, \tag{C.56}$$

$$\delta q_{|i} := a\delta T^0{}_i = -\frac{1}{a}P_{,X}\phi'\delta\phi_{|i}, \tag{C.57}$$

$$\delta P\delta^i_j := \delta T^i{}_j = \left(P_{,X}\,\delta X + P_{,\phi}\delta\phi\right)\delta^i_j, \tag{C.58}$$

where $P_{,X\phi}$ and δX are

$$P_{,X\phi} := \frac{\partial^2 P}{\partial\phi\partial X}, \quad \delta X = \frac{1}{a^2}\left(\phi'\delta\phi' - A\phi'^2\right) \tag{C.59}$$

respectively.

Here we derive the perturbed part of the energy-momentum tensor (C.56), (C.57), (C.58). First, by substituting $\phi + \delta\phi, X + \delta X$ into the $T^0{}_0$ component of the energy-momentum tensor, the energy density becomes

$$
\begin{aligned}
& \rho \\
= \; & -T^0{}_0(\phi+\delta\phi, X+\delta X) \\
= \; & -P_{,X}(\phi+\delta\phi, X+\delta X) g^{0\mu}\left(\partial_\mu\phi + \partial_\mu\delta\phi\right)\left(\partial_0\phi + \partial_0\delta\phi\right) - P(\phi+\delta\phi, X+\delta X) \\
= \; & \frac{1}{a^2}\left(P_{,X} + P_{,X\phi}\delta\phi + P_{,XX}\delta X\right)(1-2A)(\phi' + \delta\phi')^2 - P - P_{,\phi}\delta\phi - P_{,X}\delta X \\
= \; & \frac{1}{a^2}\left(P_{,X} + P_{,X\phi}\delta\phi + P_{,XX}\delta X\right)(1-2A)(\phi'^2 + 2\phi'\delta\phi') - P - P_{,\phi}\delta\phi - P_{,X}\delta X \\
= \; & \frac{1}{a^2}\left(P_{,X} + P_{,X\phi}\delta\phi + P_{,XX}\delta X\right)\left(\phi'^2 + 2\phi'\delta\phi' - 2A\phi'^2\right) - P - P_{,\phi}\delta\phi - P_{,X}\delta X \\
= \; & \frac{1}{a^2}P_{,X}\phi'^2 - P + \frac{1}{a^2}\left(2P_{,X}\phi'\delta\phi' - 2P_{,X}A\phi'^2 + P_{,X\phi}\phi'^2\delta\phi + P_{,XX}\phi^{2\prime}\delta X\right) \\
& \quad -P_{,\phi}\delta\phi - P_{,X}\delta X \\
= \; & 2P_{,X}X - P + \frac{1}{a^2}\left(2P_{,X}\phi'\delta\phi' - 2P_{,X}A\phi'^2\right) + (-P_{,X} + 2XP_{,XX})\delta X \\
& \quad -(P_{,\phi} - 2XP_{,X\phi})\delta\phi. \qquad \text{(C.60)}
\end{aligned}
$$

Note that since the total kinetic energy is

$$
\begin{aligned}
X_{\text{total}} &= -\frac{1}{2}g^{\mu\nu}\left(\partial_\mu\phi + \partial_\mu\delta\phi\right)\left(\partial_\nu\phi + \partial_\nu\delta\phi\right) \\
&= -\frac{1}{2}\left\{g^{00}\phi'^2 + 2g^{00}\phi'\delta\phi'\right\} \\
&= \frac{1}{2a^2}(1-2A)(\phi'^2 + 2\phi'\delta\phi') \\
&= \frac{1}{2a^2}\phi'^2 + \frac{1}{a^2}(\phi'\delta\phi' - A\phi'^2), \qquad \text{(C.61)}
\end{aligned}
$$

it follows that

$$\delta X = \frac{1}{a^2}\left(\phi'\delta\phi' - A\phi'^2\right) \qquad \text{(C.62)}$$

holds. Using this, the relation

$$\frac{1}{a^2}\left(2P_{,X}\phi'\delta\phi' - 2P_{,X}A\phi'^2\right) = 2P_{,X}\delta X \qquad \text{(C.63)}$$

holds, which implies that the energy density has the form

$$\rho = 2P_{,X}X - P + (P_{,X} + 2XP_{,XX})\delta X - (P_{,\phi} - 2XP_{,X\phi})\delta\phi, \qquad \text{(C.64)}$$

and its perturbed part is

$$\delta\rho = (P_{,X} + 2XP_{,XX})\delta X - (P_{,\phi} - 2XP_{,X\phi})\delta\phi. \tag{C.65}$$

Next, we derive the perturbed part of the density of energy flow. From the equation (C.51)

$$T^{\mu}{}_{\nu} = P_{,X}(\partial^{\mu}\phi)\partial_{\nu}\phi + \delta^{\mu}_{\nu}P, \tag{C.66}$$

its temporal-spatial component is

$$\begin{aligned} T^{0}{}_{i}(\phi+\delta\phi, X+\delta X) &= P_{,X}(\phi+\delta\phi, X+\delta X)g^{0\mu}\left(\partial_{\mu}\left(\phi+\delta\phi\right)\right)\partial_{i}\left(\phi+\delta\phi\right) \\ &= P_{,X}(\phi, X)g^{00}\phi'\partial_{i}\delta\phi \\ &= -\frac{1}{a^{2}}P_{,X}\phi'\delta\phi_{|i}. \end{aligned} \tag{C.67}$$

Since the background spacetime is uniformly isotropic, no zeroth-order terms remain. Thus, we find that the perturbed part of the density of energy flow is equation (C.57)

$$\delta q_{|i} := a\delta T^{0}{}_{i} = -\frac{1}{a}P_{,X}\phi'\delta\phi_{|i}. \tag{C.68}$$

The perturbed part of the pressure is clearly

$$\delta P\delta^{i}_{j} := \delta T^{i}{}_{j} = \left(P_{,X}\,\delta X + P_{,\phi}\delta\phi\right)\delta^{i}_{j}, \tag{C.69}$$

since we can immediately see that the first term in equation (C.51)

$$T^{i}{}_{j} = P_{,X}(\partial^{i}\phi)\partial_{j}\phi + \delta^{i}_{j}P \tag{C.70}$$

is of second or higher order. □

Recalling that the perturbed part of the scalar quantity f receives a transformation as in equation (B.8)

$$\widetilde{\delta f} = \delta f - \xi^{0}f' \tag{C.71}$$

under the gauge transformation

$$\tilde{x}^{\mu} = x^{\mu} + \xi^{\mu}, \tag{C.72}$$

we find that the perturbed part of the scalar field ϕ receives a transformation as in

$$\widetilde{\delta\phi} = \delta\phi - \phi'\xi^0. \tag{C.73}$$

For $\delta\rho$ and δP, we find the same transformation

$$\widetilde{\delta\rho} = \delta\rho - \rho'\xi^0, \qquad \widetilde{\delta P} = \delta P - P'\xi^0 \tag{C.74}$$

as in the fluid case. Given the equation (C.57)

$$\delta q_{|i} = a\delta T^0{}_i = -\frac{1}{a}P_{,X}\phi'\delta\phi_{|i} \tag{C.75}$$

and the fact that the part of ∂_i acting on $P_{,X}$ is an infinitesimal quantity, δq is

$$\delta q = -\frac{1}{a}P_{,X}\phi'\delta\phi. \tag{C.76}$$

Moreover, using equations (C.53)

$$\rho = \frac{1}{a^2}P_{,X}\phi'^2 - P \tag{C.77}$$

and (C.73), we see that the perturbed part δq receives the same transformation (C.35) as in the fluid case:

$$\begin{aligned}
\widetilde{\delta q} &= -\frac{1}{\tilde{a}}\tilde{P}_{,X}\tilde{\phi}'\widetilde{\delta\phi} \\
&= -\frac{1}{a}P_{,X}\phi'\left(\delta\phi - \phi'\xi^0\right) \\
&= -\frac{1}{a}P_{,X}\phi'\delta\phi + a\frac{1}{a^2}P_{,X}\phi'^2\xi^0 \\
&= \delta q + a(\rho + P)\xi^0.
\end{aligned} \tag{C.78}$$

Appendix D

Gauge invariants and gauge fixing

In Appendix B, we examined how the perturbations of the metric are transformed by the gauge transformation. In Appendix C, we examined how quantities related to matter fluctuations receive transformations. In this Appendix, we combine these quantities to form gauge invariants. Gauge fixing is also discussed.

D.1 Gauge invariants

The observables are invariant under the gauge transformation

$$\tilde{x}^\mu = x^\mu + \xi^\mu. \tag{D.1}$$

There exist

$$\Psi := A - \frac{1}{a}\left[a\left(E' - B\right)\right]', \quad \Phi := \psi - \mathcal{H}\left(E' - B\right) \tag{D.2}$$

as gauge invariant quantities composed of scalar perturbations A, B, E, ψ of the metric

$$ds^2 = a^2(\eta)\left[-(1+2A)d\eta^2 + 2(B_{|i} - S_i)d\eta dx^i + \left\{(1+2\psi)\gamma_{ij} + 2E_{|ij} + 2F_{i|j} + h_{ij}\right\}dx^i dx^j\right] \tag{D.3}$$

These two quantities are called Bardeen variables since they were introduced by Bardeen in 1980.

Check that the Bardeen variables Ψ, Φ are gauge invariants. First, Ψ can be written as

$$\Psi = A - \frac{1}{a}\left[a\left(E' - B\right)\right]' = A - \mathcal{H}(E' - B) - (E' - B)' \qquad \left(\mathcal{H} = \frac{a'}{a} = aH = \dot{a}\right). \tag{D.4}$$

From the transformation rules (B.26)∼(B.29)

$$\begin{aligned} \tilde{A} &= A - \mathcal{H}\xi^0 - \xi^{0\prime}, \\ \tilde{B} &= B + \xi^0 - \xi', \\ \tilde{\psi} &= \psi - \mathcal{H}\xi^0, \\ \tilde{E} &= E - \xi, \end{aligned} \tag{D.5}$$

we get

$$\begin{aligned} \tilde{\Psi} &= \tilde{A} - \mathcal{H}(\tilde{E}' - \tilde{B}) - (\tilde{E}' - \tilde{B})' \\ &= A - \mathcal{H}\xi^0 - \xi^{0\prime} - \mathcal{H}(E' - \xi' - B - \xi^0 + \xi') - (E' - \xi' - B - \xi^0 + \xi')' \\ &= A - \mathcal{H}\xi^0 - \xi^{0\prime} - \mathcal{H}(E' - B) + \mathcal{H}\xi^0 - (E' - B)' + \xi^{0\prime} \\ &= A - \mathcal{H}(E' - B) - (E' - B)' \end{aligned} \tag{D.6}$$

and certainly Ψ is gauge invariant:

$$\tilde{\Psi} = \Psi. \tag{D.7}$$

Next, the gauge transformation of Φ is

$$\begin{aligned} \tilde{\Phi} &= \tilde{\psi} - \mathcal{H}\left(\tilde{E}' - \tilde{B}\right) \\ &= \psi - \mathcal{H}\xi^0 - \mathcal{H}\left(E' - \xi' - B - \xi^0 + \xi'\right) \\ &= \psi - \mathcal{H}\left(E' - B\right), \end{aligned} \tag{D.8}$$

and indeed Φ is gauge invariant:

$$\tilde{\Phi} = \Phi. \quad \square \tag{D.9}$$

In addition, the following three quantities are also gauge invariants:

$$\zeta := \psi + \frac{H}{\rho + P}\delta q, \quad \Theta := \psi - \frac{\mathcal{H}}{\rho'}\delta\rho, \qquad \delta\phi_\psi := \delta\phi - \frac{\phi'}{\mathcal{H}}\psi. \tag{D.10}$$

Check that $\zeta, \Theta, \delta\phi_\psi$ are gauge invariants. The gauge transformations of the perturbed part of the energy density $\delta\rho$, the fluctuations of the spatial volume ψ, the perturbed part of the density of the energy flow δq, and the perturbed part of the scalar field $\delta\phi$ are respectively given by the expressions (C.26), (B.28) (C.35), (C.73)

$$\widetilde{\delta\rho} = \delta\rho - \rho'\xi^0, \quad \tilde{\psi} = \psi - \mathcal{H}\xi^0, \quad \widetilde{\delta q} = \delta q + a(\rho + P)\xi^0, \quad \widetilde{\delta\phi} = \delta\phi - \phi'\xi^0. \tag{D.11}$$

Using them, the gauge transformations each of these quantities are

$$\begin{aligned} \tilde{\zeta} &= \tilde{\psi} + \frac{\tilde{H}}{\tilde{\rho} + \tilde{P}}\widetilde{\delta q} \\ &= \psi - \mathcal{H}\xi^0 + \frac{H}{\rho + P}\left(\delta q + a\left(\rho + P\right)\xi^0\right) \\ &= \psi + \frac{H}{\rho + P}\delta q, \qquad \left(\mathcal{H} = \frac{a'}{a} = aH = \dot{a}\right) \end{aligned} \tag{D.12}$$

$$\begin{aligned} \tilde{\Theta} &= \tilde{\psi} - \frac{\tilde{\mathcal{H}}}{\tilde{\rho}'}\widetilde{\delta\rho} \\ &= \psi - \mathcal{H}\xi^0 - \frac{\mathcal{H}}{\rho'}\left(\delta\rho - \rho'\xi^0\right) \\ &= \psi - \frac{\mathcal{H}}{\rho'}\delta\rho, \end{aligned} \tag{D.13}$$

$$\begin{aligned} \widetilde{\delta\phi_\psi} &= \widetilde{\delta\phi} - \frac{\tilde{\phi}'}{\tilde{\mathcal{H}}}\tilde{\psi} \\ &= \delta\phi - \phi'\xi^0 - \frac{\phi'}{\mathcal{H}}\left(\psi - \mathcal{H}\xi^0\right) \\ &= \delta\phi - \frac{\phi'}{\mathcal{H}}\psi, \end{aligned} \tag{D.14}$$

which are indeed gauge invariant. □

Moreover, when we consider the scalar field ϕ, from the equations (C.53) and (C.76)

$$\rho = \frac{1}{a^2}P_{,X}\phi'^2 - P, \quad \delta q = -\frac{1}{a}P_{,X}\phi'\delta\phi, \tag{D.15}$$

we can write

$$\begin{aligned} \frac{H}{\rho+P}\delta q &= \frac{H}{\rho+P}\left(-\frac{1}{a}P_{,X}\phi'\delta\phi\right) \\ &= -\frac{H}{\frac{1}{a^2}P_{,X}\phi'^2}\left(\frac{1}{a}P_{,X}\phi'\delta\phi\right) \\ &= -\frac{\mathcal{H}}{\phi'}\delta\phi, \end{aligned} \tag{D.16}$$

then ζ can be also expressed as

$$\zeta = \psi - \frac{\mathcal{H}}{\phi'}\delta\phi. \tag{D.17}$$

From this expression and the equation (D.10)

$$\delta\phi_\psi = \delta\phi - \frac{\phi'}{\mathcal{H}}\psi, \tag{D.18}$$

it is clear that the relationship

$$\delta\phi_\psi = -\frac{\phi'}{\mathcal{H}}\zeta \tag{D.19}$$

holds. Using the equations (C.26) and (C.35) and the continuous equation (C.14)

$$\widetilde{\delta\rho} = \delta\rho - \rho'\xi^0, \quad \widetilde{\delta q} = \delta q + a(\rho+P)\xi^0, \quad \rho' + 3\mathcal{H}(\rho+P) = 0, \tag{D.20}$$

we find that the combination

$$\delta\rho_m := \delta\rho - 3H\delta q \tag{D.21}$$

is also gauge invariant.

Check that $\delta\rho_m$ is a gauge invariant:

$$\begin{aligned} \widetilde{\delta\rho_m} &= \widetilde{\delta\rho} - 3\tilde{H}\widetilde{\delta q} \\ &= \delta\rho - \rho'\xi^0 - 3H\left(\delta q + a(\rho+P)\xi^0\right) \\ &= \delta\rho - 3H\delta q - \left(\rho' + 3\mathcal{H}(\rho+P)\right)\xi^0 \\ &= \delta\rho - 3H\delta q. \quad \square \end{aligned} \tag{D.22}$$

Then, using this $\delta\rho_m$, we find that there exists a relation

$$\zeta = \Theta + \frac{\mathcal{H}}{\rho'}\delta\rho_m \tag{D.23}$$

between ζ and Θ.

Check that ζ can be written as in equation (D.23). From equation (D.10) and the continuous equation (C.14)

$$\zeta = \psi + \frac{H}{\rho + P}\delta q, \quad \Theta = \psi - \frac{\mathcal{H}}{\rho'}\delta\rho, \quad \rho' + 3\mathcal{H}(\rho + P) = 0 \tag{D.24}$$

we can obtain

$$\begin{aligned} \zeta &= \psi + \frac{H}{\rho + P}\delta q \\ &= \Theta + \frac{\mathcal{H}}{\rho'}\delta\rho + \frac{H}{\rho + P}\delta q \\ &= \Theta + \frac{\mathcal{H}}{\rho'}\delta\rho - 3\frac{\mathcal{H}H}{\rho'}\delta q \\ &= \Theta + \frac{\mathcal{H}}{\rho'}(\delta\rho - 3H\delta q) \\ &= \Theta + \frac{\mathcal{H}}{\rho'}\delta\rho_m. \quad \square \end{aligned} \tag{D.25}$$

For vector perturbations F_i and S_i, from equations (B.30), (B.31)

$$\begin{aligned} \tilde{F}_i &= F_i - \zeta_i, \\ \tilde{S}_i &= S_i + \zeta_i', \end{aligned} \tag{D.26}$$

it is clear that the quantity defined by

$$U_i := S_i + F_i' \tag{D.27}$$

is gauge invariant.

We summarize the above gauge invariants:

$$\Psi = A - \frac{1}{a}\left[a\left(E' - B\right)\right]' = A - \mathcal{H}(E' - B) - (E' - B)', \tag{D.28}$$

$$\Phi = \psi - \mathcal{H}\left(E' - B\right), \tag{D.29}$$

$$\zeta = \psi + \frac{H}{\rho + P}\delta q = \psi - \frac{\mathcal{H}}{\phi'}\delta\phi = \Theta + \frac{\mathcal{H}}{\rho'}\delta\rho_m = -\frac{\mathcal{H}}{\phi'}\delta\phi_\psi, \tag{D.30}$$

$$\Theta = \psi - \frac{\mathcal{H}}{\rho'}\delta\rho, \tag{D.31}$$

$$\delta\phi_\psi = \delta\phi - \frac{\phi'}{\mathcal{H}}\psi, \tag{D.32}$$

$$\delta\rho_m = \delta\rho - 3H\delta q, \tag{D.33}$$

$$U_i = S_i + F_i'. \tag{D.34}$$

D.2 Gauge fixing

In the gauge transformation (B.9),

$$\begin{aligned} \tilde{\eta} &= \eta + \xi^0, \\ \tilde{x}^i &= x^i + \xi_|{}^i + \zeta^i, \end{aligned} \tag{D.35}$$

note that the two scalar quantities ξ^0 and ξ and the three components of the vector quantity ζ^i are unrestricted and that ζ^i is subject to the restriction $\zeta^i{}_{|i} = 0$, the degrees of freedom for the gauge transformation are 4. These four degrees of freedom can be determined by fixing the gauge. For the vector ζ_i, for example, if we take the gauge $\tilde{F}_i = 0$ in equation (B.30)

$$\tilde{F}_i = F_i - \zeta_i, \tag{D.36}$$

the gauge is fixed to

$$F_i = \zeta_i \tag{D.37}$$

and the remaining two degrees of freedom to be fixed are ξ^0 and ξ. In the following, we consider gauge fixing for this scalar two degree of freedom. For convenience, we restate the perturbed metric and the gauge transformation for scalar perturbations:

$$d\tilde{s}^2 = a^2(\tilde{\eta})\left\{-(1+2\tilde{A})d\tilde{\eta}^2 + 2\left(\tilde{B}_{|i} - \tilde{S}_i\right)d\tilde{\eta}d\tilde{x}^i + \left[(1+2\tilde{\psi})\gamma_{ij} + 2\tilde{E}_{|ij} + 2\tilde{F}_{i|j} + \tilde{h}_{ij}\right]d\tilde{x}^i d\tilde{x}^j\right\}, \tag{D.38}$$

$$\begin{aligned}\tilde{A} &= A - \mathcal{H}\xi^0 - \xi^{0\prime}, &(D.39)\\ \tilde{B} &= B + \xi^0 - \xi', &(D.40)\\ \tilde{\psi} &= \psi - \mathcal{H}\xi^0, &(D.41)\\ \tilde{E} &= E - \xi &(D.42)\end{aligned}$$

(A) Newtonian gauge

Newtonian gauge is the gauge fixed such that

$$\tilde{B} = 0, \quad \tilde{E} = 0, \tag{D.43}$$

i.e., fixed to

$$\xi^0 = E' - B, \quad \xi = E. \tag{D.44}$$

At this point, recalling the Bardeen variable (D.2)

$$\Psi = A - \frac{1}{a}\left[a\left(E' - B\right)\right]' = A - \mathcal{H}(E' - B) - (E' - B)', \quad \Phi = \psi - \mathcal{H}\left(E' - B\right), \tag{D.45}$$

we find that it can be written as

$$\begin{aligned}\tilde{A} &= A - \mathcal{H}\xi^0 - \xi^{0\prime}\\ &= A - \mathcal{H}(E' - B) - (E' - B)'\\ &= \Psi, &(D.46)\end{aligned}$$

$$\begin{aligned}\tilde{\psi} &= \psi - \mathcal{H}\xi^0\\ &= \psi - \mathcal{H}(E' - B)\\ &= \Phi. &(D.47)\end{aligned}$$

Substituting these into the perturbed metric (D.38) yields

$$d\tilde{s}^2 = a^2(\tilde{\eta})\left\{-(1+2\tilde{\Psi})d\tilde{\eta}^2 - 2\tilde{S}_i d\tilde{\eta} d\tilde{x}^i + \left[(1+2\tilde{\Phi})\gamma_{ij} + \tilde{h}_{ij}\right] d\tilde{x}^i d\tilde{x}^j\right\} \tag{D.48}$$

for the Newtonian gauge. It can be seen that $\tilde{\Psi}$ corresponds to the gravitational potential in Newtonian mechanics and $\tilde{\Phi}$ corresponds to the fluctuation of the spatial volume. As we will see later, if anisotropic stress Π is zero, then $\Psi = -\Phi$. Since Ψ and Φ are gauge invariants, we can obtain expressions in general gauges by treating scalar perturbations in the Newtonian gauge and rewriting them as A, E, B, ψ, etc. using equation (D.2) after calculating the physical quantities.

(B) Spatially flat gauge

Spatially flat gauges are gauges for which the perturbation of the spatial part of the perturbed metric (D.38) (without considering the tensor perturbation $\tilde{h}_{ij}$) is zero, that is,

$$\tilde{\psi} = 0, \quad \tilde{E} = 0. \tag{D.49}$$

Since the equations (D.41) and (D.42) are

$$\tilde{\psi} = \psi - \mathcal{H}\xi^0, \ \tilde{E} = E - \xi, \tag{D.50}$$

we have

$$\xi^0 = \frac{\psi}{\mathcal{H}}, \quad \xi = E, \tag{D.51}$$

and the remaining two scalar perturbations can be written as

$$\tilde{A} = A - \psi - \left(\frac{\psi}{\mathcal{H}}\right)', \quad \tilde{B} = B + \frac{\psi}{\mathcal{H}} - E'. \tag{D.52}$$

Also, from the expressions (D.30), (D.31), and (D.32)

$$\zeta = -\frac{\mathcal{H}}{\phi'}\delta\phi_\psi,$$
$$\Theta = \psi - \frac{\mathcal{H}}{\rho'}\delta\rho,$$
$$\delta\phi_\psi = \delta\phi - \frac{\phi'}{\mathcal{H}}\psi,$$

these gauge invariants can be written as

$$\tilde{\zeta} = -\frac{\mathcal{H}}{\phi'}\widetilde{\delta\phi_\psi}, \tag{D.53}$$

$$\tilde{\Theta} = -\frac{\mathcal{H}}{\rho'}\widetilde{\delta\rho}, \tag{D.54}$$

$$\widetilde{\delta\phi_\psi} = \widetilde{\delta\phi}. \tag{D.55}$$

(C) Uniform density gauge

The uniform density gauge is a gauge that makes the fluctuation of the energy density ρ zero:

$$\widetilde{\delta\rho} = 0. \tag{D.56}$$

In this case, from equation (C.26)

$$\widetilde{\delta\rho} = \delta\rho - \rho'\xi^0, \tag{D.57}$$

ξ^0 is fixed to

$$\xi^0 = \frac{\delta\rho}{\rho'}. \tag{D.58}$$

Another scalar gauge degree of freedom ξ is determined by taking $\tilde{B} = 0$ or $\tilde{E} = 0$.

In the uniform density gauge, from equation (D.31)

$$\Theta = \psi - \frac{\mathcal{H}}{\rho'}\delta\rho, \tag{D.59}$$

we have

$$\tilde{\Theta} = \tilde{\psi}. \tag{D.60}$$

The three-dimensional spatial curvature on a constant-time hypersurface is denoted by

$${}^{(3)}\zeta = \frac{6}{a^2}(1 - 2\psi)K - \frac{4}{a^2}\nabla^2\psi. \tag{D.61}$$

Thus, the scalar perturbation ψ of the spatial part of the metric in flat spacetime ($K = 0$) is related to the 3-dimensional spatial curvature ζ as

$${}^{(3)}\zeta = -\frac{4}{a^2}\nabla^2\psi. \tag{D.62}$$

Therefore, Θ is called a curvature fluctuation on a constant-density hypersurface.

(D) Comoving gauge (Unitary gauge)

The expressions of the comoving gauges are different for the fluid case and the scalar field case.

- Fluid

  In the fluid case, the comoving gauge is the gauge for which the spatial components of the four-velocities (C.9) and (C.12)

$$\begin{aligned} u^\mu &= \frac{1}{a}\left(1 - A, v_|{}^i + v^i\right), \\ u_\mu &= a\left(-1 - A, v_{|i} + B_{|i} + v_i - S_i\right) \end{aligned}$$

are zero (assuming no vector perturbations are considered), i.e.,

$$\tilde{v} = 0, \quad \tilde{B} = 0. \tag{D.63}$$

Then, from equations (C.29) and (D.40)

$$\tilde{v}(\tilde{x}) = v(x) + \xi'(x), \quad \tilde{B} = B + \xi^0 - \xi', \tag{D.64}$$

we obtain

$$\xi = -\int v d\eta + Y(x^i), \quad \xi^0 = -v - B. \tag{D.65}$$

$Y(x^i)$ is an arbitrary scalar function that depends on the spatial coordinate x^i. It can be seen that $\tilde{E}$ depends on $Y(x^i)$ from equation (D.42)

$$\tilde{E} = E - \xi. \tag{D.66}$$

Since $\tilde{A}$ and $\tilde{\psi}$ are

$$\tilde{A} = A - \mathcal{H}\xi^0 - \xi^{0\prime}, \quad \tilde{\psi} = \psi - \mathcal{H}\xi^0, \tag{D.67}$$

and ξ^0 is fixed, we can see that scalar fluctuations $\tilde{A}$, $\tilde{B}$ and $\tilde{\psi}$ are fixed in this gauge except $\tilde{E}$. Also, since the time derivative $\tilde{E}'$ is independent of $Y(x^i)$, for example, the quantities Ψ, Φ written in equations (D.2)

$$\begin{aligned} \Psi &= A - \frac{1}{a}\left[a\left(E' - B\right)\right]' = A - \mathcal{H}(E' - B) - (E' - B)', \\ \Phi &= \psi - \mathcal{H}\left(E' - B\right) \end{aligned}$$

are fixed. Furthermore, transformations of scalar quantities such as $\delta\rho, \delta P, \delta\phi, \delta q, v$ involve ξ' and ξ^0 rather than ξ itself ((C.26) etc.):

$$\widetilde{\delta\rho} = \delta\rho - \rho'\xi^0, \tag{D.68}$$

$$\widetilde{\delta P} = \delta P - P'\xi^0, \tag{D.69}$$

$$\widetilde{\delta\phi} = \delta\phi - \phi'\xi^0, \tag{D.70}$$

$$\widetilde{\delta q} = \delta q + a(\rho + P)\xi^0(x), \tag{D.71}$$

$$\tilde{v} = v + \xi'. \tag{D.72}$$

Thus, the ambiguity of ξ coming from $Y(x^i)$ is not essentially a problem. Note that from equation (C.32), we have equation

$$\widetilde{\delta q} = a(\rho + P)(\tilde{v} + \tilde{B}) = 0 \tag{D.73}$$

(since the coordinate system moves with the fluid, the density of energy flow is 0), and from equation (D.10)

$$\zeta = \psi + \frac{H}{\rho + P}\delta q, \tag{D.74}$$

we can find

$$\tilde{\zeta} = \tilde{\psi}. \tag{D.75}$$

Combined with equation (D.62)

$${}^{(3)}\zeta = -\frac{4}{a^2}\nabla^2\psi, \tag{D.76}$$

we can obtain the relation

$${}^{(3)}\tilde{\zeta} = -\frac{4}{a^2}\nabla^2\tilde{\zeta}, \tag{D.77}$$

hence ζ is called the comoving curvature fluctuation.

- scalar field ϕ

The definition of a comoving gauge in the case of a scalar field ϕ is

$$\widetilde{\delta q} = 0. \tag{D.78}$$

In this case, the equation (D.77) is still valid. From equation (C.76)

$$\delta q = -\frac{1}{a}P_{,X}\phi'\delta\phi, \tag{D.79}$$

the above equation corresponds to

$$\widetilde{\delta\phi} = 0. \tag{D.80}$$

From this fact, we also refer to this gauge as the uniform field gauge. From the transformation (C.73)

$$\widetilde{\delta\phi} = \delta\phi - \phi'\xi^0, \tag{D.81}$$

the relation

$$\xi^0 = \frac{\delta\phi}{\phi'} \tag{D.82}$$

holds, and then, for example, if we choose $\tilde{E} = 0$, we can determine $\xi = E$ from $\tilde{E} = E - \xi$.

Note that the comoving gauge is sometimes referred to as unitary gauge.

If the gauge is fixed to one of the above (A) to (D), the physical contents to be solved are essentially the same. In other words, depending on the problem under consideration, we can proceed with the discussion using convenient gauge conditions.

In summary, we note the physical meaning of each gauge invariant:

Ψ (Bardeen variable) : in Newtonian gauge, gravitational potential in Newtonian mechanics

Φ (Bardeen variable) : in Newtonian gauge, fluctuation of spatial volume ($\Psi = -\Phi$ if anisotropic stress Π is zero)

Θ (the curvature fluctuation on a constant-density hypersurface) : in Uniform density gauge, three-dimensional spatial curvature $\left({}^{(3)}\tilde{\zeta} = -\frac{4}{a^2}\nabla^2\tilde{\Theta}\right)$

ζ (the comoving curvature fluctuation) : in comoving gauge, three-dimensional spatial curvature $\left({}^{(3)}\tilde{\zeta} = -\frac{4}{a^2}\nabla^2\tilde{\zeta}\right)$

$\delta\phi_\psi$: in spatially flat gauge, fluctuation of scalar field ϕ $\left(\widetilde{\delta\phi_\psi} = \widetilde{\delta\phi}\right)$

$\delta\rho_m$: in comoving gauge, fluctuations of the energy density ρ $\left(\widetilde{\delta\rho_m} = \widetilde{\delta\rho}\right)$

Appendix E

Perturbed Einstein equations

In this appendix, we consider the perturbed Einstein equations that linear fluctuations must satisfy for the metric

$$ds^2 = a^2(\eta)\left[-(1+2A)d\eta^2 + 2(B_{|i} - S_i)d\eta dx^i + \left\{(1+2\psi)\gamma_{ij} + 2E_{|ij} + 2F_{i|j} + h_{ij}\right\}dx^i dx^j\right]. \tag{E.1}$$

Since we proceed without fixing the gauge, the obtained equations can be used for arbitrary gauge fixings. The Einstein tensor $G^\mu{}_\nu$ and the energy-momentum tensor $T^\mu{}_\nu$ can be separated into the background spacetime part and the perturbation part and written as

$$G^\mu{}_\nu = {}^{(b)}G^\mu{}_\nu + \delta G^\mu{}_\nu, \quad T^\mu{}_\nu = {}^{(b)}T^\mu{}_\nu + \delta T^\mu{}_\nu. \tag{E.2}$$

From equation

$${}^{(b)}G^\mu{}_\nu = 8\pi G {}^{(b)}T^\mu{}_\nu \tag{E.3}$$

for the background spacetime, we get equations

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}, \tag{E.4}$$

$$3H^2 + 2\dot{H} = -8\pi G P - \frac{K}{a^2}, \tag{E.5}$$

where G is the universal gravitational constant. The perturbed Einstein equation can be written as

$$\delta G^\mu{}_\nu = 8\pi G \delta T^\mu{}_\nu. \tag{E.6}$$

The fluctuations $\delta T^\mu{}_\nu$ of the energy-momentum tensor on the right-hand side are concretely obtained by the equations (C.21), (C.22), (C.23), (C.56), (C.57) and (C.58) for fluid and the scalar field:

For fluid

- scalar perturbations

$$
\begin{aligned}
{}^{(S)}\delta T^0{}_0 &= -\delta\rho, \\
{}^{(S)}\delta T^0{}_i &= (\rho+P)\left(v_{|i}+B_{|i}\right), \\
{}^{(S)}\delta T^i{}_0 &= -(\rho+P)v_|{}^i, \\
{}^{(S)}\delta T^i{}_j &= \delta P\delta^i_j + \Pi_|{}^i{}_j - \frac{1}{3}\nabla^2\Pi\delta^i_j.
\end{aligned} \tag{E.7}
$$

- vector perturbations

$$
\begin{aligned}
{}^{(V)}\delta T^0{}_0 &= 0, \\
{}^{(V)}\delta T^0{}_i &= (\rho+P)\left(v_i - S_i\right), \\
{}^{(V)}\delta T^i{}_0 &= -(\rho+P)v^i, \\
{}^{(V)}\delta T^i{}_j &= \frac{1}{2}\left(\pi^i{}_{|j} + \pi_{j|}{}^i\right).
\end{aligned} \tag{E.8}
$$

- tensor perturbations

$$
\begin{aligned}
{}^{(T)}\delta T^0{}_0 &= 0, \\
{}^{(T)}\delta T^0{}_i &= 0, \\
{}^{(T)}\delta T^i{}_0 &= 0, \\
{}^{(T)}\delta T^i{}_j &= {}^{(T)}\pi^i{}_j.
\end{aligned} \tag{E.9}
$$

For scalar field

$$
\delta\rho = -\delta T^0{}_0 = \left(P_{,X} + 2XP_{,XX}\right)\delta X - \left(P_{,\phi} - 2XP_{,X\phi}\right)\delta\phi, \tag{E.10}
$$

$$
\delta q_{|i} = a\delta T^0{}_i = -\frac{1}{a}P_{,X}\phi'\delta\phi_{|i}, \tag{E.11}
$$

$$
\delta P\delta^i_j = \delta T^i{}_j = \left(P_{,X}\,\delta X + P_{,\phi}\delta\phi\right)\delta^i_j. \tag{E.12}
$$

Consider the fluctuation $\delta G^\mu{}_\nu$ of the Einstein tensor, which is the left-hand side of the perturbed Einstein equation. First, the perturbed Christoffel symbol is

$$
\delta\Gamma^\mu{}_{\nu\lambda} = \frac{1}{2}\delta g^{\mu\alpha}\left(g_{\alpha\nu,\lambda} + g_{\alpha\lambda,\nu} - g_{\nu\lambda,\alpha}\right) + \frac{1}{2}g^{\mu\alpha}\left(\delta g_{\alpha\nu,\lambda} + \delta g_{\alpha\lambda,\nu} - \delta g_{\nu\lambda,\alpha}\right). \tag{E.13}
$$

Then, the perturbations of the Ricci tensor $R_{\mu\nu}$ and the scalar curvature R are

$$\delta R_{\mu\nu} = \delta\Gamma^{\alpha}{}_{\mu\nu,\alpha} - \delta\Gamma^{\alpha}{}_{\mu\alpha,\nu} + \left(\delta\Gamma^{\alpha}{}_{\mu\nu}\right)\Gamma^{\beta}{}_{\alpha\beta} + \Gamma^{\alpha}{}_{\mu\nu}\delta\Gamma^{\beta}{}_{\alpha\beta} - \left(\delta\Gamma^{\alpha}{}_{\mu\beta}\right)\Gamma^{\beta}{}_{\alpha\nu} - \Gamma^{\alpha}{}_{\mu\beta}\delta\Gamma^{\beta}{}_{\alpha\nu}, \tag{E.14}$$

$$\delta R = \left(\delta g^{\mu\nu}\right) R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}, \tag{E.15}$$

respectively. From the above, the perturbed Einstein tensor is given by

$$\delta G^{\mu}{}_{\nu} = \delta R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}\delta R. \tag{E.16}$$

Using this, $\delta G^{\mu}{}_{\nu}$ for scalar, vector, and tensor perturbations, respectively, are as follows [106, 107]:

- scalar perturbations

$$^{(\mathrm{S})}\delta G^{0}{}_{0} = \frac{2}{a^2}\left[3\mathcal{H}\left(\mathcal{H}A - \psi'\right) + \nabla^2\left\{\psi - \mathcal{H}\left(E' - B\right)\right\} + 3K\psi\right], \tag{E.17}$$

$$^{(S)}\delta G^{0}{}_{i} = -\frac{2}{a^2}\left[\mathcal{H}A - \psi' + K\left(E' - B\right)\right]_{|i}, \tag{E.18}$$

$$^{(S)}\delta G^{i}{}_{j} = \frac{2}{a^2}\left[\left(\mathcal{H}^2 + 2\mathcal{H}'\right)A + \mathcal{H}A' - \psi'' - 2\mathcal{H}\psi' + K\psi\right]\delta^i_j + \frac{1}{a^2}\left(\nabla^2 D\delta^i_j - D_{|}{}^{i}{}_{j}\right), \tag{E.19}$$

where ∇^2 and D are

$$\nabla^2 = \delta^{ij}\partial_i\partial_j, \quad D := A + \psi - 2\mathcal{H}\left(E' - B\right) - \left(E' - B\right)'. \tag{E.20}$$

If we take the trace of the equation (E.19), we obtain

$$^{(S)}\delta G^{i}{}_{i} = \frac{6}{a^2}\left[\left(\mathcal{H}^2 + 2\mathcal{H}'\right)A + \mathcal{H}A' - \psi'' - 2\mathcal{H}\psi' + K\psi\right] + \frac{2}{a^2}\nabla^2 D. \tag{E.21}$$

- vector perturbations

$$^{(V)}\delta G^{0}{}_{0} = 0, \tag{E.22}$$

$$^{(V)}\delta G^{0}{}_{i} = -\frac{2K + \nabla^2}{2a^2}\left(S_i + F_i'\right), \tag{E.23}$$

$$^{(V)}\delta G^{i}{}_{j} = \frac{1}{2a^2}\left\{\left(\frac{\partial}{\partial\eta} + 2\mathcal{H}\right)\left[S^{i}{}_{|j} + S_{j|}{}^{i} + \left(F^{i}{}_{|j} + F_{j|}{}^{i}\right)'\right]\right\}. \tag{E.24}$$

- tensor perturbations

$$
\begin{aligned}
{}^{(T)}\delta G^0{}_0 &= 0, &\text{(E.25)}\\
{}^{(T)}\delta G^0{}_i &= 0, &\text{(E.26)}\\
{}^{(T)}\delta G^i{}_j &= \frac{1}{2a^2}\left[\left(h^i{}_j\right)'' + 2\mathcal{H}\left(h^i{}_j\right)' + \left(2K - \nabla^2\right) h^i{}_j\right]. &\text{(E.27)}
\end{aligned}
$$

In the following, we consider the perturbed equations separately for fluid and the scalar field.

E.1 Perturbed Einstein equation for fluid

Using $\mathcal{H} = aH = \dot{a}$ in the equation of FLRW background spacetime

$$
\begin{aligned}
H^2 &= \frac{8\pi G}{3}\rho - \frac{K}{a^2}, &\text{(E.28)}\\
3H^2 + 2\dot{H} &= -8\pi G P - \frac{K}{a^2}, &\text{(E.29)}
\end{aligned}
$$

we can write

$$
\begin{aligned}
3\mathcal{H}^2 &= 8\pi G a^2 \rho - 3K, &\text{(E.30)}\\
\mathcal{H}' - \mathcal{H}^2 &= -4\pi G a^2(\rho + P) + K. &\text{(E.31)}
\end{aligned}
$$

Using the fluctuations of the energy-momentum tensor of the fluid given by (E.7), (E.8), and (E.9), we obtain the basic equations for scalar, vector, and tensor perturbations, respectively:

- Scalar perturbation equations for fluid

  From equations (E.7), (E.17), and (E.18)

$$
\begin{aligned}
{}^{(S)}\delta T^0{}_0 &= -\delta\rho,\\
{}^{(S)}\delta T^0{}_i &= (\rho + P)\left(v_{|i} + B_{|i}\right),\\
{}^{(\mathrm{S})}\,\delta G^0{}_0 &= \frac{2}{a^2}\left[3\mathcal{H}\left(\mathcal{H}A - \psi'\right) + \nabla^2\left\{\psi - \mathcal{H}\left(E' - B\right)\right\} + 3K\psi\right],\\
{}^{(S)}\delta G^0{}_i &= -\frac{2}{a^2}\left[\mathcal{H}A - \psi' + K\left(E' - B\right)\right]_{|i},
\end{aligned}
$$

  the temporal-temporal (00) and temporal-spatial ($0i$) components of the perturbed Einstein equations

$$
{}^{(S)}\delta G^0{}_0 = 8\pi G^{(T)}\delta T^0{}_0, \quad {}^{(S)}\delta G^0{}_i = 8\pi G^{(S)}\delta T^0{}_i \qquad \text{(E.32)}
$$

can be expressed as

$$3\mathcal{H}\left(\psi' - \mathcal{H}A\right) - \left(\nabla^2 + 3K\right)\psi + \mathcal{H}\nabla^2\sigma \quad = \quad 4\pi G a^2 \delta\rho, \tag{E.33}$$

$$\psi' - \mathcal{H}A - K\sigma \quad = \quad 4\pi G a \delta q, \tag{E.34}$$

respectively. Here, we defined

$$\sigma := E' - B \tag{E.35}$$

and used equation (C.32)

$$\delta q = a(\rho + P)(v + B). \tag{E.36}$$

When $i \neq j$, using the fact that

$$^{(S)}\delta G^i{}_j = -\frac{1}{a^2} D_|{}^i{}_j, \quad ^{(S)}\delta T^i{}_j = \Pi_|{}^i{}_j, \tag{E.37}$$

the spatial-spatial components (ij) of the scalar perturbation Einstein equations of the fluid

$$^{(S)}\delta G^i{}_j = 8\pi G^{(S)}\delta T^i{}_j \tag{E.38}$$

give

$$D = -8\pi G a^2 \Pi \tag{E.39}$$

respectively, which can be written concretely as

$$\sigma' + 2\mathcal{H}\sigma - A - \psi = 8\pi G a^2 \Pi. \tag{E.40}$$

In addition, using an equation

$$^{(S)}\delta T^i{}_i = 3\delta P \tag{E.41}$$

with the trace of equation (E.7)

$$^{(S)}\delta T^i{}_j = \delta P \delta^i_j + \Pi_|{}^i{}_j - \frac{1}{3}\nabla^2 \Pi \delta^i_j \tag{E.42}$$

and equation (E.21)

$$^{(S)}\delta G^i{}_i = \frac{6}{a^2}\left[\left(\mathcal{H}^2 + 2\mathcal{H}'\right)A + \mathcal{H}A' - \psi'' - 2\mathcal{H}\psi' + K\psi\right] + \frac{2}{a^2}\nabla^2 D, \tag{E.43}$$

we obtain trace component of the perturbed Einstein equation:

$$\psi'' + 2\mathcal{H}\psi' - K\psi - \mathcal{H}A' - \left(\mathcal{H}^2 + 2\mathcal{H}'\right) A = -4\pi G a^2 \left(\delta P + \frac{2}{3}\nabla^2 \Pi\right). \quad \text{(E.44)}$$

Note that using Bardeen variables

$$\Psi = A - \frac{1}{a}\left[a\left(E' - B\right)\right]' = A - \mathcal{H}\sigma - \sigma', \quad \Phi = \psi - \mathcal{H}\left(E' - B\right) = \psi - \mathcal{H}\sigma, \quad \text{(E.45)}$$

the equation (E.40) can be written as

$$\Psi + \Phi = -8\pi G a^2 \Pi. \quad \text{(E.46)}$$

This indicates

$$\Psi = -\Phi \quad \text{(E.47)}$$

when the anisotropic stress Π is zero.

From the conservation law of energy and momentum

$$T^{\mu}{}_{0;\mu} = 0, \quad T^{\mu}{}_{i;\mu} = 0, \quad \text{(E.48)}$$

we can also obtain equations for the perturbations $\delta\rho$ and δq. $T^{\mu}{}_{0;\mu} = 0$ is written in concrete form as

$$T^{\mu}{}_{0;\mu} = \frac{\partial T^{0}{}_{0}}{\partial \eta} + \frac{\partial T^{i}{}_{0}}{\partial x^i} + \Gamma^{i}{}_{i0} T^{0}{}_{0} - \Gamma^{i}{}_{i0} T^{i}{}_{i} = 0, \quad \text{(E.49)}$$

and by using the equation

$$\rho' + 3\mathcal{H}(\rho + P) = 0 \quad \text{(E.50)}$$

for the continuous equation of background spacetime, the perturbed part becomes

$$\delta T^{\mu}{}_{0;\mu} = -\delta\rho' - 3\mathcal{H}(\delta\rho + \delta P) - (\rho + P)\left[3\psi' + \nabla^2\left(E' + v\right)\right]. \quad \text{(E.51)}$$

Thus, we obtain an equation

$$\delta\rho' + 3\mathcal{H}(\delta\rho + \delta P) = -(\rho + P)\left[3\psi' + \nabla^2\left(E' + v\right)\right] \quad \text{(E.52)}$$

for the perturbation $\delta\rho$. Similarly, from equation

$$T^{\mu}{}_{i;\mu} = \frac{1}{a}\left[\delta q' + 3\mathcal{H}\delta q + a\delta P + \frac{2}{3}a\left(\nabla^2 + 3K\right)\Pi + (\rho + P)aA\right]_{|i} = 0, \quad \text{(E.53)}$$

we can obtain

$$\delta q' + 3\mathcal{H}\delta q = -a\delta P - \frac{2}{3}a\left(\nabla^2 + 3K\right)\Pi - (\rho + P)aA \tag{E.54}$$

for the perturbation δq. Equations (E.52) and (E.54) can be also derived using (E.33)∼(E.44).

- Vector perturbation equations for fluid

  Since the temporal-temporal component (00) is a quantity for scalar perturbations, the equations for vector perturbations can only come from the temporal-spatial component ($0i$) and the spatial-spatial component (ij). From equations (E.8), (E.22), (E.23), (E.24)

$$\begin{aligned}
{}^{(V)}\delta T^0{}_0 &= 0, \\
{}^{(V)}\delta T^0{}_i &= (\rho + P)\left(v_i - S_i\right), \\
{}^{(V)}\delta T^i{}_0 &= -(\rho + P)v^i, \\
{}^{(V)}\delta T^i{}_j &= \frac{1}{2}\left(\pi^i{}_{|j} + \pi_{j|}{}^i\right), \\
{}^{(V)}\delta G^0{}_0 &= 0, \\
{}^{(V)}\delta G^0{}_i &= -\frac{2K + \nabla^2}{2a^2}\left(S_i + F_i'\right), \\
{}^{(V)}\delta G^i{}_j &= \frac{1}{2a^2}\left\{\left(\frac{\partial}{\partial\eta} + 2\mathcal{H}\right)\left[S^i{}_{|j} + S_{j|}{}^i + \left(F^i{}_{|j} + F_{j|}{}^i\right)'\right]\right\},
\end{aligned}$$

the vector perturbation Einstein equation for fluid ${}^{(V)}\delta G^i{}_j = 8\pi G^{(V)}\delta T^i{}_j$ are

$$\left(\nabla^2 + 2K\right)\left(F_i' + S_i\right) = -16\pi G a^{(V)}\delta q_i, \tag{E.55}$$

$$\tau^i{}_j{}' + 3\mathcal{H}\tau^i{}_j = 4\pi G a\left(\pi^i{}_{|j} + \pi_{j|}{}^i\right). \tag{E.56}$$

From the vector perturbation of the fluid (E.8)

$${}^{(V)}\delta T^0{}_i = (\rho + P)\left(v_i - S_i\right), \tag{E.57}$$

the density of energy flow is

$${}^{(V)}\delta q_i = a(\rho + P)\left(v_i - S_i\right). \tag{E.58}$$

We defined the quantity

$$\tau^i{}_j := \frac{1}{2a}\left[S^i{}_{|j} + S_{j|}{}^i + \left(F^i{}_{|j} + F_{j|}{}^i\right)'\right] \tag{E.59}$$

for convenience. Moreover, nothing is obtained from the temporal component of the conservation law of energy and momentum ${T^\mu}_{0;\mu} = 0$, but from the spatial component ${T^\mu}_{i;\mu} = 0$, we can obtain

$$^{(V)}\delta q_i' + 3\mathcal{H}^{(V)}\delta q_i = -a\left(\nabla^2 + 2K\right)\pi_i. \tag{E.60}$$

- Tensor perturbation equations for fluid
  The only equation for tensor perturbations are spatial-spatial components (ij). From equations (E.9) and (E.27)

$$\begin{aligned} ^{(T)}\delta {T^i}_j &= {}^{(T)}{\pi^i}_j, \\ ^{(T)}\delta {G^i}_j &= \frac{1}{2a^2}\left[\left({h^i}_j\right)'' + 2\mathcal{H}\left({h^i}_j\right)' + \left(2K - \nabla^2\right){h^i}_j\right], \end{aligned}$$

the tensor perturbation Einstein equation $^{(T)}\delta {G^i}_j = 8\pi G^{(T)}\delta {T^i}_j$ for a fluid is

$$\left({h^i}_j\right)'' + 2\mathcal{H}\left({h^i}_j\right)' + \left(2K - \nabla^2\right){h^i}_j = 16\pi G^{(T)}{\pi^i}_j a^2. \tag{E.61}$$

E.2 Perturbed Einstein equation for the scalar field

Next, we consider the perturbed Einstein equations in the case of a scalar field with the Lagrangian given by $P(\phi, X)$. In the FLRW background spacetime, the equations (E.30) and (E.31)

$$\begin{aligned} 3\mathcal{H}^2 &= 8\pi G a^2\rho - 3K, \\ \mathcal{H}' - \mathcal{H}^2 &= -4\pi G a^2(\rho + P) + K \end{aligned}$$

with energy density as equation (C.53)

$$\rho = 2XP_{,X} - P \tag{E.62}$$

hold, and the continuous equation can be expressed as follows:

$$(P_{,X} + 2XP_{,XX})\phi'' + 2(P_{,X} - XP_{,XX})\mathcal{H}\phi' + a^2\left(2XP_{,X\phi} - P_{,\phi}\right) = 0. \tag{E.63}$$

We show that the continuous equation can be expressed as (E.63). First, the time derivative of the energy density

$$\rho = 2XP_{,X}(\phi, X) - P(\phi, X) \tag{E.64}$$

is

$$\begin{aligned}\rho' &= 2X'P_{,X} + 2XP_{,XX}X' + 2XP_{,X\phi}\phi' - P_{,X}X' - P_{,\phi}\phi' \\ &= X'P_{,X} + 2XP_{,XX}X' + 2XP_{,X\phi}\phi' - P_{,\phi}\phi'.\end{aligned} \tag{E.65}$$

Next, the time derivative of the kinetic term for the background field

$$X = \frac{1}{2a^2}\phi'^2 \tag{E.66}$$

is

$$\begin{aligned}X' &= -\frac{a'}{a^3}\phi'^2 + \frac{1}{a^2}\phi'\phi'' \\ &= -2\frac{a'}{a}X + 2\frac{\phi''}{\phi'}X \\ &= 2X\left(-\mathcal{H} + \frac{\phi''}{\phi'}\right).\end{aligned} \tag{E.67}$$

Using these, the continuity equation becomes

$$2X\left(-\mathcal{H} + \frac{\phi''}{\phi'}\right)P_{,X} + 4X^2\left(-\mathcal{H} + \frac{\phi''}{\phi'}\right)P_{,XX} + 2XP_{,X\phi}\phi' - P_{,\phi}\phi' + 6\mathcal{H}XP_{,X} = 0. \tag{E.68}$$

Furthermore, multiplying by $\phi'/2X$ yields

$$-\mathcal{H}\phi'P_{,X} + \phi''P_{,X} - 2X\mathcal{H}\phi'P_{,XX} + 2X\phi''P_{,XX} + 2a^2XP_{,X\phi} - a^2P_{,\phi} + 3\mathcal{H}P_{,X}\phi' = 0, \tag{E.69}$$

and finally

$$(P_{,X} + 2XP_{,XX})\phi'' + 2(P_{,X} - XP_{,XX})\mathcal{H}\phi' + a^2\left(2XP_{,X\phi} - P_{,\phi}\right) = 0 \tag{E.70}$$

is obtained. □

The perturbed energy momentum tensor in the scalar field case is given by the equations (C.56), (C.57), (C.58)

$$\begin{aligned} \delta\rho &= -\delta T^0{}_0 = \left(P_{,X} + 2XP_{,XX}\right)\delta X - \left(P_{,\phi} - 2XP_{,X\phi}\right)\delta\phi, &\text{(E.71)}\\ \delta q_{|i} &= a\delta T^0{}_i = -\frac{1}{a}P_{,X}\phi'\delta\phi_{|i}, &\text{(E.72)}\\ \delta P\delta^i_j &= \delta T^i{}_j = \left(P_{,X}\,\delta X + P_{,\phi}\delta\phi\right)\delta^i_j. &\text{(E.73)}\end{aligned}$$

- Scalar perturbation equations for the scalar field

  The scalar perturbation equations for the scalar field are

$$\begin{aligned} 3\mathcal{H}\psi' &- \left[3\mathcal{H}^2 - 4\pi G\phi'^2\left(P_{,X} + 2XP_{,XX}\right)\right]A - \left(\nabla^2 + 3K\right)\psi + \mathcal{H}\nabla^2\sigma \\ &= -4\pi G\left[a^2\left(P_{,\phi} - 2XP_{,X\phi}\right)\delta\phi - \left(P_{,X} + 2XP_{,XX}\right)\phi'\delta\phi'\right], \quad \text{(E.74)}\end{aligned}$$

$$\begin{aligned} \psi' - \mathcal{H}A - K\sigma &= -4\pi GP_{,X}\,\phi'\delta\phi, &\text{(E.75)}\\ \sigma' + 2\mathcal{H}\sigma - A - \psi &= 0 \qquad (D = 0), &\text{(E.76)}\\ \psi'' + 2\mathcal{H}\psi' - K\psi - \mathcal{H}A' - \left(2\mathcal{H}^2 + \mathcal{H}' + K\right)A &= -4\pi G\left(P_{,X}\phi'\delta\phi' + a^2P_{,\phi}\delta\phi\right) \\ & &\text{(E.77)}\end{aligned}$$

  from the equations for the $(00), (0i), (ij)[i \neq j]$ components and the trace, i.e., equations (E.7), (E.17), (E.18), (E.19) and (E.21)

$$\begin{aligned} {}^{(S)}\delta T^0{}_0 &= -\delta\rho, \\ {}^{(S)}\delta T^0{}_i &= \left(\rho + P\right)\left(v_{|i} + B_{|i}\right), \\ {}^{(S)}\delta T^i{}_j &= \delta P\delta^i_j + \Pi_{|}{}^i{}_j - \frac{1}{3}\nabla^2\Pi\delta^i_j, \\ {}^{(S)}\delta T^i{}_i &= 3\delta P,\end{aligned}$$

$$\begin{aligned} {}^{(S)}\delta G^0{}_0 &= \frac{2}{a^2}\left[3\mathcal{H}\left(\mathcal{H}A - \psi'\right) + \nabla^2\left\{\psi - \mathcal{H}\left(E' - B\right)\right\} + 3K\psi\right], \\ {}^{(S)}\delta G^0{}_i &= -\frac{2}{a^2}\left[\mathcal{H}A - \psi' + K\left(E' - B\right)\right]_{|i}, \\ {}^{(S)}\delta G^i{}_j &= \frac{2}{a^2}\left[\left(\mathcal{H}^2 + 2\mathcal{H}'\right)A + \mathcal{H}A' - \psi'' - 2\mathcal{H}\psi' + K\psi\right]\delta^i_j + \frac{1}{a^2}\left(\nabla^2 D\delta^i_j - D_{|}{}^i{}_j\right), \\ {}^{(S)}\delta G^i{}_i &= \frac{6}{a^2}\left[\left(\mathcal{H}^2 + 2\mathcal{H}'\right)A + \mathcal{H}A' - \psi'' - 2\mathcal{H}\psi' + K\psi\right] + \frac{2}{a^2}\nabla^2 D.\end{aligned}$$

We used the kinetic energy of the perturbed part of the scalar field (C.62)

$$\delta X = \frac{1}{a^2}\left(\phi'\delta\phi' - A\phi'^2\right) \tag{E.78}$$

in obtaining the equations (E.74) and (E.77). σ and D are

$$\sigma = E' - B, \quad D = A + \psi - 2\mathcal{H}\left(E' - B\right) - \left(E' - B\right)' \tag{E.79}$$

as defined previously. In obtaining the equation (E.77), we used the equation for pressure perturbation (E.10)

$$\delta P \delta^i_j = \left(P_{,X}\,\delta X + P_{,\phi}\delta\phi\right)\delta^i_j. \tag{E.80}$$

In addition, we used the FLRW equation for the background field (E.31)

$$\mathcal{H}' - \mathcal{H}^2 = -4\pi G a^2(\rho + P) + K \tag{E.81}$$

and the relation

$$\mathcal{H}' - \mathcal{H}^2 - K = -4\pi G\phi'^2 P_{,X} \tag{E.82}$$

derived from the relation

$$\rho + P = 2XP_{,X} = \frac{1}{a^2}\phi'^2 P_{,X} \tag{E.83}$$

between energy density and pressure. The perturbed quantities $\delta\rho, \delta P$ for the scalar field satisfy the same equation (E.52)

$$\delta\rho' + 3\mathcal{H}(\delta\rho + \delta P) = -(\rho + P)\left[3\psi' + \nabla^2\left(E' + v\right)\right] \tag{E.84}$$

as in the fluid case, and transformed using the equation (C.32)

$$\delta q = a(\rho + P)(v + B), \tag{E.85}$$

we can write

$$\delta\rho' + 3\mathcal{H}(\delta\rho + \delta P) = -a^{-1}\nabla^2\delta q - (\rho + P)\left(3\psi' + \nabla^2\sigma\right). \tag{E.86}$$

Of course, this equation is not independent of equations (E.74)∼(E.77). Note that the equation (E.54)

$$\delta q' + 3\mathcal{H}\delta q = -a\delta P - \frac{2}{3}a\left(\nabla^2 + 3K\right)\Pi - (\rho + P)aA \tag{E.87}$$

for the perturbed momentum δq is automatically satisfied.

- Vector perturbation equations for the scalar field

  As mentioned before, since the vector quantities v_i, S_i, π_i are zero for scalar fields, we obtain

$$\begin{aligned} \left(\nabla^2 + 2K\right) F_i' &= 0, && \text{(E.88)} \\ {\tau^i}_j{}' + 3\mathcal{H}{\tau^i}_j &= 0, && \text{(E.89)} \\ {}^{(V)}\delta q_i' &= 0 && \text{(E.90)} \end{aligned}$$

  from the expressions (E.55), (E.56), and (E.60)

$$\begin{aligned} \left(\nabla^2 + 2K\right)\left(F_i' + S_i\right) &= -16\pi G a^{(V)}\delta q_i, \\ {\tau^i}_j{}' + 3\mathcal{H}{\tau^i}_j &= 4\pi G a\left({\pi^i}_{|j} + {\pi_{j|}}^i\right), \\ {}^{(V)}\delta q_i' + 3\mathcal{H}^{(V)}\delta q_i &= -a\left(\nabla^2 + 2K\right)\pi_i. \end{aligned}$$

  From equations (E.88) and (E.90) the perturbations F_i and ${}^{(V)}\delta q_i$ do not increase. From this fact, the gauge invariant

$$U_i = F_i' + S_i \tag{E.91}$$

  does not increase either. In addition, the spatial derivative of the vector perturbation

$${\tau^i}_j = \frac{1}{2a}\left[{S^i}_{|j} + {S_{j|}}^i + \left({F^i}_{|j} + {F_{j|}}^i\right)'\right] \tag{E.92}$$

  is reduced from equation (E.89) as

$${\tau^i}_j \propto \frac{1}{a^3}. \tag{E.93}$$

  Thus, without the vector component π^i of the anisotropic stress

$${\pi^i}_j = \left({\Pi_|}^i{}_j - \frac{1}{3}\nabla^2\Pi\delta^i_j\right) + \frac{1}{2}\left({\pi^i}_{|j} + {\pi_{j|}}^i\right) + {}^{(T)}{\pi^i}_j, \tag{E.94}$$

  the vector perturbation does not increase. For these reasons, vector perturbations are often not considered.

- Tensor perturbation equations for the scalar field

In the case of the scalar field, the transverse and traceless tensor component ${}^{(T)}\pi^i{}_j$ is zero, and thus from equation (E.61)

$$\left(h^i{}_j\right)'' + 2\mathcal{H}\left(h^i{}_j\right)' + \left(2K - \nabla^2\right) h^i{}_j = 16\pi G^{(T)}\pi^i{}_j a^2 \tag{E.95}$$

we obtain

$$\left(h^i{}_j\right)'' + 2\mathcal{H}\left(h^i{}_j\right)' + \left(2K - \nabla^2\right) h^i{}_j = 0. \tag{E.96}$$

Unlike vector perturbations, tensor perturbations generally have growing solutions. Since $\nabla^2 h^i{}_j \to 0$ (in Fourier components $|\boldsymbol{k}|^2 \to 0$) in the large scale limit, from equation (E.96), there exist solutions such that $h^i{}_j$ is constant in flat spacetime $K = 0$, as

$$\left(h^i{}_j\right)' = \frac{c_1}{a^2} \tag{E.97}$$

that is

$$h^i{}_j = -\frac{c_1}{a} + c_2 \to c_2 \tag{E.98}$$

where c_1, c_2 are integral constants.

In the case of the scalar field, since the momentum δq is given by equation (C.76)

$$\delta q = -\frac{1}{a} P_{,X}\phi'\delta\phi, \tag{E.99}$$

equation (E.75)

$$\psi' - \mathcal{H}A - K\sigma = -4\pi G P_{,X}\,\phi'\delta\phi \tag{E.100}$$

becomes

$$\psi' - \mathcal{H}A - K\sigma = 4\pi G a\delta q \tag{E.101}$$

when the right-hand side is denoted by δq, which is the same as the scalar perturbation equation (E.34) for fluid.

Appendix F

Calculation of the determinant of the metric

For the determinant of the metric, starting from the curvature fluctuation in unitary gauge $\zeta(t, \boldsymbol{x})$, we derive NG boson $\pi(\tilde{t}, \boldsymbol{x})$ terms in $\sqrt{-g}$. We can write the determinant of the metric in unitary gauge as

$$\sqrt{-g} = Na^3(t) = \left(1 + \frac{1}{H(t)} \frac{d}{dt} \zeta(t)\right) a^3(t) \tag{F.1}$$

using ADM formalism. Curvature fluctuations $\zeta(t, \boldsymbol{x})$ and inflaton $\pi(\tilde{t}, \boldsymbol{x})$ are related by the relation

$$\zeta(t, \boldsymbol{x}) = -H(\tilde{t})\pi(\tilde{t}, \boldsymbol{x}) \tag{F.2}$$

under the transformation of time

$$t \mapsto \tilde{t} = t - \pi(\tilde{t}, \boldsymbol{x}). \tag{F.3}$$

By rewriting the derivative of $\zeta(t, \boldsymbol{x})$ with respect to t into the derivative with respect to $\tilde{t}$, we obtain

$$\begin{aligned} \frac{d}{dt}\zeta(t, \boldsymbol{x}) &= \frac{d\tilde{t}}{dt} \frac{d}{d\tilde{t}} \left(-H(\tilde{t})\pi(\tilde{t}, \boldsymbol{x})\right) \\ &= -\left[1 - \frac{d\tilde{t}}{dt}\dot{\pi}(\tilde{t})\right] \left(\dot{H}\pi + H\dot{\pi}\right) \\ &= -\left(1 - \dot{\pi} + \dot{\pi}^2\right) \left(\dot{H}\pi + H\dot{\pi}\right) \\ &= \dot{H}(\tilde{t}) \left(-\pi + \pi\dot{\pi}\right) + H(\tilde{t}) \left(-\dot{\pi} + \dot{\pi}^2\right) \end{aligned} \tag{F.4}$$

up to the second order of π, where the dot represents the derivative with respect to $\tilde{t}$. After applying the time transformation to $H^{-1}(t)$, we obtain

$$H^{-1}(t) = H^{-1}(\tilde{t}+\pi) = \left(H(\tilde{t}) + \dot{H}(\tilde{t})\pi + \frac{1}{2}\ddot{H}(\tilde{t})\pi^2\right)^{-1} \tag{F.5}$$

which can be written as

$$\begin{aligned} H^{-1}(t) &\simeq \left(H(\tilde{t}) + \dot{H}(\tilde{t})\pi\right)^{-1} \\ &= \frac{1}{H(\tilde{t})}\left(1 + \frac{\dot{H}(\tilde{t})}{H(\tilde{t})}\pi\right)^{-1} \\ &= \frac{1}{H(\tilde{t})}\left(1 - H(\tilde{t})\epsilon(\tilde{t})\pi\right)^{-1} \\ &= \frac{1}{H(\tilde{t})} + \epsilon(\tilde{t})\pi + H\epsilon^2\pi^2. \end{aligned} \tag{F.6}$$

Note that $\ddot{H}$ is in the second order of $\epsilon = -\dot{H}/H^2$. Similarly, applying the time transformation to the scale factor $a(t)$, we obtain

$$\begin{aligned} a^3(t) = a^3(\tilde{t}+\pi) &= \left[a(\tilde{t}) + \dot{a}(\tilde{t})\pi + \frac{1}{2}\ddot{a}(\tilde{t})\pi^2\right]^3 \\ &= a^3(\tilde{t}) + 3a^2(\tilde{t})\dot{a}(\tilde{t})\pi(\tilde{t}) + \frac{3}{2}a^2\ddot{a}\pi^2 + 3a\dot{a}^2\pi^2. \end{aligned} \tag{F.7}$$

Putting them together, the determinant of the metric (F.1) expands to

$$\begin{aligned} \sqrt{-g} &= \left(1 + \frac{1}{H(t)}\frac{d}{dt}\zeta(t,\boldsymbol{x})\right)a^3(t) \\ &= \left[1 + \frac{\dot{H}(\tilde{t})}{H(\tilde{t})}\left\{-\pi + \pi\dot{\pi}\right\} - \dot{H}\epsilon\pi^2 - \dot{\pi} + \dot{\pi}^2 - H\epsilon\pi\dot{\pi}\right] \\ &\quad \times \left[a^3(\tilde{t}) + 3a^2(\tilde{t})\dot{a}(\tilde{t})\pi(\tilde{t}) + \frac{3}{2}a^2\ddot{a}\pi^2 + 3a\dot{a}^2\pi^2\right] \\ &= a^3(\tilde{t}) + 3a^2\dot{a}\pi + \frac{3}{2}a^2\ddot{a}\pi^2 + 3a\dot{a}^2\pi^2 - \frac{\dot{H}}{H}\left(a^3\pi + 3a^2\dot{a}\pi^2\right) + \frac{\dot{H}}{H}a^3\pi\dot{\pi} \\ &\quad -\dot{H}\epsilon a^3\pi^2 - a^3\dot{\pi} - 3a^2\dot{a}\pi\dot{\pi} + a^3\dot{\pi}^2 - H\epsilon a^3\pi\dot{\pi}. \end{aligned} \tag{F.8}$$

Using the relations

$$\dot{a} = aH, \quad \ddot{a} = aH^2 + a\dot{H} \tag{F.9}$$

and expressing the derivative of the scale factor a in terms of the Hubble H, we can obtain

$$\begin{aligned}
\sqrt{-g} &= a^3(\tilde{t}) + 3a^3 H\pi + \frac{3}{2}a^3\left(H^2 + \dot{H}\right)\pi^2 + 3a^3H^2\pi^2 \\
&\quad -\frac{\dot{H}}{H}a^3\pi - 3a^3\dot{H}\pi^2 + \frac{\dot{H}}{H}a^3\pi\dot{\pi} - \dot{H}\epsilon a^3\pi^2 - a^3\dot{\pi} - 3a^3H\pi\dot{\pi} + a^3\dot{\pi}^2 - H\epsilon a^3\pi\dot{\pi} \\
&= a^3(\tilde{t}) + 3a^3H\left(1 + \frac{\epsilon}{3}\right)\pi - a^3\dot{\pi} + a^3H^2\left(\frac{9}{2} + \frac{3}{2}\epsilon\right)\pi^2 - a^3H\left(3 + 2\epsilon\right)\pi\dot{\pi} + a^3\dot{\pi}^2.
\end{aligned} \tag{F.10}$$

Appendix G

Propagators for scalar fields with mass m in de Sitter spacetime

We consider the propagator of a complex scalar field ϕ with mass m in the de Sitter spacetime. By using conformal time η and the Minkowski metric $\eta^{\mu\rho}$ to write down the action, we have

$$S = \int d^4x \, \left[\eta^{\mu\rho}\phi^* \partial_\mu \partial_\rho \phi - a^2 m^2 |\phi|^2\right]. \tag{G.1}$$

From this action, we obtain the equation of motion

$$\left(\eta^{\mu\rho}\partial_\mu \partial_\rho - a^2 m^2\right)\phi = 0, \tag{G.2}$$

which is the Klein-Gordon equation with mass a^2m^2. Due to the rotational symmetry, we can transform the scalar field ϕ into three dimensional momentum space and write its mode function as $u(\eta, k)$, the Klein-Gordon equation (G.2) is

$$\frac{\partial^2 u}{\partial \eta^2}(\eta, k) + \left(k^2 + \frac{m^2}{H^2\eta^2}\right) u(\eta, k) = 0. \tag{G.3}$$

The solution to this Klein-Gordon equation is

$$u(\eta, k) = -i\frac{\sqrt{\pi}}{2} e^{i\pi(\nu/2+1/4)} \, (-\eta)^{\frac{1}{2}} \, H_\nu(-k\eta), \tag{G.4}$$

where $H_\nu(-k\eta)$ is a Hankel function of the first kind and we define the index

$$\nu \equiv \sqrt{\frac{1}{4} - \left(\frac{m}{H}\right)^2}. \tag{G.5}$$

Therefore, a scalar field with mass m^2 in de Sitter spacetime has propagators

$$\begin{aligned}
G_{>}(\eta_1,\eta_2,k) &= -i\frac{\sqrt{\pi}}{2}e^{i\pi(\nu/2+1/4)}H\left(-\eta_1\right)^{3/2}H_\nu^{(1)}(-k\eta_1) \\
&\quad \times i\frac{\sqrt{\pi}}{2}e^{i\pi(\nu^*/2+1/4)}H\left(-\eta_2\right)^{3/2}H_\nu^{(2)}(-k\eta_2) \\
&= -\frac{\pi}{4}e^{-\pi\mathrm{Im}(\nu)}H^2\left(\eta_1\eta_2\right)^{3/2}H_\nu^{(1)}(-k\eta_1)H_\nu^{(2)}(-k\eta_2), \qquad \text{(G.6)} \\
G_{<}(\eta_1,\eta_2,k) &= -\frac{\pi}{4}e^{-\pi\mathrm{Im}(\nu)}H^2\left(\eta_1\eta_2\right)^{3/2}H_\nu^{(1)}(-k\eta_2)H_\nu^{(2)}(-k\eta_1) \qquad \text{(G.7)}
\end{aligned}$$

from equations (5.73) and (5.74).

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