Exponential suppression of cosmological constant in non-supersymmetric heterotic string theories with general Z_2 twists

(一般の Z_2 ねじれを持つ非超対称ヘテロ型弦理論における宇宙定数の指数関数的抑制)

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Abstract

We investigated the non-supersymmetric string models constructed by freely acting general $\mathbb{Z}_2$ orbifolding. The main subject in this thesis is the heterotic model.

In preparation for this thesis, we first review two types of closed superstring models on d-dimensional torus: type II and heterotic ones.

We review the construction of non-supersymmetric string models, d-dimensionally compactified with the arbitrary number of freely acting $\mathbb{Z}_2$ twisted directions. We also construct the 10-dimensional non-supersymmetric models from type II and heterotic superstring theories, especially focusing on the tachyon-free $SO(16) \times SO(16)$ heterotic models.

We study the behavior of 9D and 8D models in the limits of the compactified radii to zero and infinity, so-called endpoint limits, and then classify the models in terms of the endpoints. In particular, we show that there are various patterns of interpolation between endpoints in 8-dimensional heterotic models.

We investigate the massless spectra of non-supersymmetric heterotic models toward the evaluation of cosmological constant. Considering the massless conditions, we clarify the relation between toroidally compactified superstrings and non-supersymmetric models.

Next, we evaluate the cosmological constant in the heterotic models with general $\mathbb{Z}_2$ twists. We show that even in the general setup, the leading behavior of the cosmological constant is controlled by the massless spectra, as in the case of the simplest 9-dimensional interpolating model.

Finally, we study the stability of Wilson-line moduli using the one-loop effective potential in the case with general $\mathbb{Z}_2$ twists. We show that the configuration of moduli, which gives to the suppressed cosmological constant, corresponds to the saddle point in the moduli space.

This thesis is mainly based on our work [1] and also based on [2, 3] in collaboration with Hiroshi Itoyama and Sota Nakajima.

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Contents

1 Introduction

The ultimate goal of particle physics is to construct a unified theory of all four interactions. String theory is one of the leading candidates for a unified theory. In particular, from a theoretical point of view, most of the research has focused on supersymmetric string theory. In addition, superstring theory determines the dimension of spacetime to be ten from the consistency of the theory. Therefore, it predicts an extra six-dimensional space in addition to the four dimensions we inhabit. Since supersymmetry can solve several problems faced by the Standard Model, when obtaining a 4-dimensional theory from a 10-dimensional one, compactifications such as Calabi-Yau manifolds [4] and orbifolds [5, 6] that preserve supersymmetry have been the main research subjects. However, recent accelerator experiments have found no evidence of supersymmetry on the expected energy scales. Therefore, the scenario in which supersymmetry is broken at very high energy scales has also attracted interest, and attempts have been made to construct a realistic model from non-supersymmetric string theory.

While most non-supersymmetric string theories include a tachyon in their spectrum, tachyon-free models without supersymmetry also exist: there are $SO(16) \times SO(16)$ heterotic models in 10 dimensions originally constructed in [7, 8] and tachyon-free string theories were constructed in 4 dimensions and in general dimensions [9, 10]. However, a top-down approach from a tachyon-free non-supersymmetric string theory faces a serious problem: the energy density of the vacuum, or cosmological constant, is very large. This very large cosmological constant is not only inconsistent with the observations but also induces the instability of vacua.

In this thesis, we focus on the string models constructed by orbifolding with freely acting $\mathbb{Z}_2$ twists in which the cosmological constant can be exponentially suppressed [1–3, 11–29] (other non-supersymmetric string models with small or vanishing cosmological constants have been proposed [30–41]). This construction is a stringy version of the Scherk-Scwharz compactifications, which breaks supersymmetry [42–44]. In the 9-dimensional model based on this, a 10-dimensional supersymmetric model can be obtained by setting the limit of the compactification radius to zero or infinity in a particular choice of $\mathbb{Z}_2$ twists. Of these, of particular interest are the so-called interpolating models, which connect the superstring and the non-supersymmetric string in 10 dimensions and whose interpolation properties are related to the target space duality of the non-supersymmetric string; in the interpolating heterotic model in 9 dimensions, as the dimensionless radius R of the circle increases (supersymmetry is asymptotically restored), It is shown that the leading contribution of the one-loop cosmological constant can be evaluated as follows. [11, 12]

$$\Lambda^{(9)} = (n_F - n_B)\xi R^{-9} + \mathcal{O}(e^{-R}), \tag{1.1}$$

where n_F and n_B represent the degrees of freedom of massless fermions and bosons, respectively, and ξ is a computable positive constant.

The formula (1.1) implies that the cosmological constant is exponentially suppressed in the 9-dimensional interpolating heterotic models when the degeneracy between boson and fermion is realized at the massless level. The heterotic interpolating models with Wilson-line moduli were investigated [13–15], where only one direction is twisted. In this thesis, we further generalize the d-dimensionally compactified (interpolating) models to the ones with the arbitrary number of $\mathbb{Z}_2$ twisted directions. In bosonic constructions, the $\mathbb{Z}_2$ twists in non-supersymmetric strings are characterized by the shift vector δ [2, 7, 9]. As discussed later, the components of δ determine the internal directions where supersymmetry can be restored.

This paper is organized as follows. In section 2, we review the toroidal compactification of closed superstrings. In section 3, we construct non-supersymmetric models d-dimensionally compactified with general $\mathbb{Z}_2$ twists. In section 4, we investigate the behavior of the partition functions at the limits of the compactified radii to zero and infinity (which we call the endpoint limits) and show some examples of interpolations in 9- and 8-dimensional models. In section 5, we study the massless spectra of the heterotic models with general $\mathbb{Z}_2$ twists. In section 6, we evaluate the one-loop cosmological constant in the region where supersymmetry is asymptotically restored. We can then obtain the formula generalized from (1.1) and find the configurations of Wilson lines that give the exponentially suppressed cosmological constant. In section 7, the stability of Wilson-line moduli on the cosmological constant is also analyzed. Finally, we conclude this thesis in Section 8. In appendix A, we summarize the $SO(2n)$ conjugacy classes and characters used to express the partition functions. In appendix B, we show the complete classification of 8-dimensional non-supersymmetric heterotic models.

2 Toroidal compactification of superstrings

In preparation for the construction of the non-supersymmetric string models, we first look at the closed superstrings toroidally compactified where supersymmetry is maximally preserved. Though the main subjects in this thesis are the heterotic models, the non-supersymmetric models from type II strings can be constructed in the same way as heterotic ones.

The one-loop partition function of the string models compactified on T^d can be written as follows:

$$Z^{T^d} = Z_B^{(8-d)} Z_F Z_{\Gamma^{d_L,d_R}}, \tag{2.1}$$

where each contribution to the partition function is given as[1]

$$Z_B^{(8-d)} = \tau_2^{-\frac{8-d}{2}} (\eta\bar{\eta})^{-(8-d)}, \tag{2.2a}$$

$$Z_F = \begin{cases} (V_8 - S_8)\left(\bar{V}_8 - \bar{S}_8\right) \text{ or } (V_8 - S_8)\left(\bar{V}_8 - \bar{C}_8\right) & \text{(type IIB or type IIA string)} \\ \bar{V}_8 - \bar{S}_8 & \text{(heterotic string)} \end{cases}, \tag{2.2b}$$

$$Z_{\Gamma^{d_L,d_R}} = \eta^{-d_L}\bar{\eta}^{-d_R} \sum_{P\in\Gamma^{d_L,d_R}} q^{\frac{1}{2}P_L^2}\bar{q}^{\frac{1}{2}P_R^2}, \quad \text{with} \begin{cases} d_L = d_R = d & \text{(type II string)} \\ d_L - 16 = d_R = d & \text{(heterotic string)} \end{cases}. \tag{2.2c}$$

Here $q = e^{2\pi i\tau}$, $\eta(\tau)$ is the Dedekind eta function and (O_8, V_8, S_8, C_8) denotes a set of $SO(8)$ characters defined in appendix A. Modular invariance of the one-loop partition function requires that the lattice Γ^{d_L,d_R} must be a Narain lattice, an even and self-dual one with Lorentzian signature (d_L, d_R). Using the generalized vierbein $\mathcal{E}(\lambda^a)$ of the Narain lattice which is expressed as a $(d_L + d_R) \times (d_L + d_R)$ matrix, an element P of Γ^{d_L,d_R} is written as

$$P = Z\mathcal{E}(\lambda^a), \tag{2.3}$$

where $Z \in \mathbb{Z}^{d_L+d_R}$ is a $(d_L + d_R)$-dimensional row vector with integer components, and λ^a represents a set of $d_L \times d_R$ parameters called moduli. This element is interpreted as an internal momentum. We define the inner product of $P_1 = Z_1\mathcal{E}$ and $P_2 = Z_2\mathcal{E}$ as

$$P_1 \cdot P_2 = Z_1\mathcal{E}\eta\mathcal{E}^t Z_2^t = Z_1 J Z_2^t, \tag{2.4}$$

where $\eta = diag\left(\mathbf{1}_{d_L}, -\mathbf{1}_{d_R}\right)$ and $J = \mathcal{E}\eta\mathcal{E}^t$ is called the Narain metric.

[1] We omit a modular parameter $\tau = \tau_1 + i\tau_2$ of the world-sheet torus.

2.1 The type II case

The type II models compactified on T^d have $d \times d$ moduli: a metric $G = ee^t$ of the compactification lattice and an anti-symmetric two-form B. These moduli can be combined into a $d \times d$ background matrix $E = G + B$. We can take the simple choice of a Narain metric in $\Gamma^{d,d}$ as

$$J = \begin{pmatrix} 0 & \mathbf{1}_d \\ \mathbf{1}_d & 0 \end{pmatrix}. \tag{2.5}$$

An element of the Narain lattice in the type II case is then written as

$$P = Z\mathcal{E}(e, B) = Z\tilde{\mathcal{E}}(e, B)\tilde{\mathcal{E}}_0, \tag{2.6}$$

where

$$\tilde{\mathcal{E}}(e, B) = \begin{pmatrix} e & Be^{-t} \\ 0 & e^{-t} \end{pmatrix}, \quad \tilde{\mathcal{E}}_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1}_d & -\mathbf{1}_d \\ \mathbf{1}_d & \mathbf{1}_d \end{pmatrix}. \tag{2.7}$$

We can check $\tilde{\mathcal{E}}_0 \eta \tilde{\mathcal{E}}_0^t = J$ and $\tilde{\mathcal{E}} J \tilde{\mathcal{E}}^t = J$ so that $\mathcal{E}\eta\mathcal{E}^t = J$. The internal momentum $P = (p_L; p_R)$ is written using the background matrix E as

$$P = \frac{1}{\sqrt{2}} \left(n + wE; n - wE^t\right) e^{-t}, \tag{2.8}$$

where $Z = (w; n) = \left(w^1, \cdots, w^d; n_1, \cdots, n_d\right)$. Here w^i is the i-th winding number while n_i is the i-th KK momentum.

2.2 The heterotic case

The heterotic models compactified on T^d have $(16 + d) \times d$ moduli: a metric $G = ee^t$ of the compactification lattice, an anti-symmetric two-form B and Wilson lines A. The standard choice of a Narain metric in $\Gamma^{16+d,d}$ as

$$J = \begin{pmatrix} g_{16} & 0 & 0 \\ 0 & 0 & \mathbf{1}_d \\ 0 & \mathbf{1}_d & 0 \end{pmatrix}, \quad g_{16} = \alpha_{16}\alpha_{16}^t \tag{2.9}$$

where α_{16} represents a set of the basis of a 16-dimensional even and self-dual Euclidean lattice Γ^{16}, discussed in detail below. An internal momentum $P \in \Gamma^{16+d,d}$ is then expressed as

$$P = Z\mathcal{E}(e,B,A) = Z\tilde{\mathcal{E}}(e,B,A)\tilde{\mathcal{E}}_0, \tag{2.10}$$

where $Z = (q,w,n) \in \mathbb{Z}^{16} \times \mathbb{Z}^d \times \mathbb{Z}^d$ and

$$\tilde{\mathcal{E}}(e,B,A) = \begin{pmatrix} \mathbf{1}_{16} & 0 & \alpha_{16}A^t e^{-t} \\ -A\alpha_{16}^{-1} & e & Ce^{-t} \\ 0 & 0 & e^{-t} \end{pmatrix}, \quad \tilde{\mathcal{E}}_0 = \begin{pmatrix} \alpha_{16} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}\mathbf{1}_d & -\frac{1}{\sqrt{2}}\mathbf{1}_d \\ 0 & \frac{1}{\sqrt{2}}\mathbf{1}_d & \frac{1}{\sqrt{2}}\mathbf{1}_d \end{pmatrix}, \quad C = B - \frac{1}{2}AA^t. \tag{2.11}$$

We can write down $P = (\ell_L, p_L; p_R)$ explicitly as [45]

$$\ell_L = \Pi - wA, \tag{2.12a}$$

$$p_L = \frac{1}{\sqrt{2}}\left[\Pi A^t + w\left(G + C\right) + n\right]e^{-t}, \tag{2.12b}$$

$$p_R = \frac{1}{\sqrt{2}}\left[\Pi A^t - w\left(G - C\right) + n\right]e^{-t}, \tag{2.12c}$$

where $\Pi = q\alpha_{16}$ lives in Γ^{16}. One can easily check that $\tilde{\mathcal{E}}_0\eta\tilde{\mathcal{E}}_0^t = J$ and $\tilde{\mathcal{E}}J\tilde{\mathcal{E}}^t = J$, and the inner product is independent of the moduli as follows:

$$P_1 \cdot P_2 = P_1\eta P_2^t = Z_1 J Z_2^t = \Pi_1\Pi_2^t + w_1 n_2^t + n_1 w_2^t. \tag{2.13}$$

In this thesis, we will consider the specific background that the internal circles are all perpendicular. Then, the internal metric G is diagonal and expressed as

$$G_{ij} = R_i^2\delta_{ij}, \tag{2.14}$$

where $i = 1, 2, \ldots, d$ and R_i is the radius of the i-th compactified circle. Then the internal momenta (2.12) are expressed as

$$\ell_L^I = \Pi^I - w^i A_i^I, \tag{2.15a}$$

$$p_{L,i} = \frac{1}{\sqrt{2}R_i}\left(\Pi^I A_i^I + n_i + w^j\left(G_{ij} + B_{ij} - \frac{1}{2}A_i^I A_j^I\right)\right), \tag{2.15b}$$

$$p_{R,i} = \frac{1}{\sqrt{2}R_i}\left(\Pi^I A_i^I + n_i - w^j\left(G_{ij} - B_{ij} + \frac{1}{2}A_i^I A_j^I\right)\right), \tag{2.15c}$$

where $I = 1, \ldots, 16,\ i = 1, \ldots, d$.

It is known that there are only two types of the 16-dimensional Narain lattice Γ^{16}: one is the $Spin(32)/\mathbb{Z}_2$ root lattice and the other is $E_8 \times E_8$ root lattice. We can write down these lattice using $SO(2n)$ conjugacy class (see appendix A) as follows:

$$\Gamma^{16}_{Spin(32)/\mathbb{Z}_2} = \Gamma^{(16)}_g + \Gamma^{(16)}_s, \tag{2.16}$$

$$\Gamma^{16}_{E_8\times E_8} = \left(\Gamma^{(8)}_g + \Gamma^{(8)}_s\right) \times \left(\Gamma^{(8)}_g + \Gamma^{(8)}_s\right). \tag{2.17}$$

Using the $SO(2n)$ characters, we can write $Z_{\Gamma^{16,0}} = Z_{\Gamma^{16}}$ explicitly as follows:

$$Z_{\Gamma^{16}_{Spin(32)/\mathbb{Z}_2}} = O_{32} + S_{32} = O_{16}O_{16} + V_{16}V_{16} + S_{16}S_{16} + C_{16}C_{16}, \tag{2.18}$$

$$Z_{\Gamma^{16}_{E_8\times E_8}} = (O_{16} + S_{16})(O_{16} + S_{16}) = O_{16}O_{16} + O_{16}S_{16} + S_{16}O_{16} + S_{16}S_{16}. \tag{2.19}$$

3 Non-supersymmetric string models

In this section, we review the non-supersymmetric string theories constructed from closed superstring models discussed in Section 2 by $\mathbb{Z}_2$ twisted compactifications. In this thesis, we focus on the bosonic construction, which was proposed in [7, 9].

3.1 Construction

The non-supersymmetric models are constructed by $\mathbb{Z}_2$ freely acting orbifolding the toroidal model (2.1). The $\mathbb{Z}_2$ shift action is expressed as $(-1)^F\alpha$, where F is the spacetime fermion number ($F = F_L F_R$ for Type II models and $F = F_R$ for heterotic ones), and α is an order two operator in the Narain lattice, which gives an eigenvalue $\alpha = e^{2\pi i P\cdot\delta}$ for a state with an internal momentum P. Here δ is called a shift-vector in the Narain lattice, satisfying $2\delta \in \Gamma^{d_L,d_R}$ which gives $P\cdot 2\delta \in \mathbb{Z}$. For convenience, Γ^{d_L,d_R} is splitted into two subsets $\Gamma_+^{d_L,d_R}$ and $\Gamma_-^{d_L,d_R}$ as follows:

$$\Gamma_+^{d_L,d_R}(\delta) = \left\{ P \in \Gamma^{d_L,d_R} \,\middle|\, P\cdot\delta \in \mathbb{Z} \right\}, \quad \Gamma_-^{d_L,d_R}(\delta) = \left\{ P \in \Gamma^{d_L,d_R} \,\middle|\, P\cdot\delta \in \mathbb{Z} + \frac{1}{2} \right\}. \tag{3.1}$$

Since $2\delta \in \Gamma^{d_L,d_R}$, the shift vector δ is expressed as

$$\delta = \frac{1}{2}\hat{Z}\mathcal{E}(\lambda^a), \tag{3.2}$$

with a certain integer vector $\hat{Z} \in \mathbb{Z}^{d_L+d_R}$. For type II string theories, we can write $\hat{Z} = (\hat{w}, \hat{n}) \in \mathbb{Z}^d \times \mathbb{Z}^d$, then the inner product $P\cdot\delta$ can be written from the definition (2.4) as

$$P\cdot\delta = \frac{1}{2}ZJ\hat{Z}^t = \frac{1}{2}\left(w\hat{n}^t + n\hat{w}^t\right). \tag{3.3}$$

For heterotic string theories, $\hat{Z} = (\hat{q}, \hat{w}, \hat{n}) \in \mathbb{Z}^{16} \times \mathbb{Z}^d \times \mathbb{Z}^d$ and $P\cdot\delta$ can be written as

$$P\cdot\delta = \frac{1}{2}\left(\Pi\cdot\hat{\Pi} + w\hat{n}^t + n\hat{w}^t\right). \tag{3.4}$$

Modular invariance of the one-loop partition function requires that δ^2 be an integer, as discussed later in footnote two from the transformation rule of the partition function under the T-transformation of $SL(2, Z)$. Thus $\hat{Z}$ must satisfy

$$\hat{Z}J\hat{Z}^t = 0 \pmod 4. \tag{3.5}$$

For type II string theories, the condition (3.5) reads

$$\hat{w}\hat{n}^t = 0 \pmod 2, \tag{3.6}$$

while for heterotic string theories, (3.5) reads

$$\hat{\Pi}^2 + 2\hat{w}\hat{n}^t = 0 \pmod 4, \tag{3.7}$$

where we define $\hat{\Pi} = \hat{q}\alpha_{16}$. Moreover, if two choices $\hat{Z}$ and $\hat{Z}'$ satisfy $\hat{Z} = \hat{Z}' \pmod 2$, these give the same splitting of the Narain lattice by definition of $\Gamma_{\pm}^{d_L,d_R}$ (3.1). We can thus choose $\hat{Z}$ whose components take either 0 or 1[2].

The action α to a state with an internal momentum P is expressed as

$$\alpha\,|P\rangle = \pm\,|P\rangle \quad \text{for } P \in \Gamma_{\pm}^{d_L,d_R}. \tag{3.8}$$

Projecting out the partition function of the toroidal models (2.1) by $\left(1 + (-1)^F \alpha\right)/2$, then we get

$$Z_B^{(8-d)} \left\{ \left(V_8\bar{V}_8 + S_8\bar{S}_8\right) Z_{\Gamma_+^{d,d}} - \left(V_8\bar{S}_8 + S_8\bar{V}_8\right) Z_{\Gamma_-^{d,d}} \right\}, \qquad \text{(type IIB string)} \tag{3.9a}$$

$$Z_B^{(8-d)} \left\{ \bar{V}_8 Z_{\Gamma_+^{16+d,d}} - \bar{S}_8 Z_{\Gamma_-^{16+d,d}} \right\}, \qquad \text{(heterotic string)} \tag{3.9b}$$

where we define $Z_{\Gamma_{\pm}^{d_L,d_R}}$ as

$$Z_{\Gamma_{\pm}^{d_L,d_R}} = \eta^{-d_L}\bar{\eta}^{-d_R} \sum_{P\in\Gamma_{\pm}^{d_L,d_R}} q^{\frac{1}{2}P_L^2}\bar{q}^{\frac{1}{2}P_R^2} = \eta^{-d_L}\bar{\eta}^{-d_R} \sum_{P\in\Gamma^{d_L,d_R}} \frac{1 \pm e^{2\pi i P\cdot\delta}}{2} q^{\frac{1}{2}P_L^2}\bar{q}^{\frac{1}{2}P_R^2}. \tag{3.10}$$

These are the contributions from the untwisted sectors. One can easily find that they are not modular invariant. We must impose the modular invariance on the one-loop partition function, which requires adding twisted sectors. The contributions from the twisted sectors can be obtained by the modular covariance of the partition functions as follows. First, we can rewrite (3.9) as

$$\frac{1}{2} Z_B^{(8-d)} \left\{ (V_8 - S_8)\left(\bar{V}_8 - \bar{S}_8\right) Z_{\Gamma^{d,d}} + (V_8 + S_8)\left(\bar{V}_8 + \bar{S}_8\right) Z_{\Gamma^{d,d}}^{(0,\delta)} \right\}, \qquad \text{(type IIB string)} \tag{3.11a}$$

$$\frac{1}{2} Z_B^{(8-d)} \left\{ \left(\bar{V}_8 - \bar{S}_8\right) Z_{\Gamma^{16+d,d}} + \left(\bar{V}_8 + \bar{S}_8\right) Z_{\Gamma^{16+d,d}}^{(0,\delta)} \right\}, \qquad \text{(heterotic string)} \tag{3.11b}$$

[2] $\hat{Z} = \left(0^{d_L}, 0^{d_R}\right)$ corresponds to the toroidal compactification.

where $Z^{(0,\delta)}_{\Gamma^{d_L,d_R}}$ is defined as

$$Z^{(0,\delta)}_{\Gamma^{d_L,d_R}} = Z_{\Gamma_+^{d_L,d_R}} - Z_{\Gamma_-^{d_L,d_R}} = \eta^{-d_L}\bar{\eta}^{-d_R} \sum_{P\in\Gamma^{d_L,d_R}} e^{2\pi i P\cdot\delta} q^{\frac{1}{2}P_L^2}\bar{q}^{\frac{1}{2}P_R^2}, \tag{3.12}$$

and (twice) the first and second terms in (3.11) are referred to as Z_+^+ and Z_-^+ respectively. Note that the first terms in (3.11) are independently modular invariant since they are half of the partition functions for toroidal models. By the formula

$$\sum_{P\in\Gamma^{d_L,d_R}} \delta\left(P'-P\right) = \sum_{P''\in\Gamma^{d_L,d_R}} \exp\left(2\pi i P'\cdot P''\right), \tag{3.13}$$

and S-transformation laws of the $SO(8)$ characters (A.11), we can find the S-transformation for the second terms in (3.11) as

$$\begin{aligned} &Z_B^{(8-d)}\left(V_8+S_8\right)\left(\bar{V}_8+\bar{S}_8\right) Z^{(0,\delta)}_{\Gamma^{d,d}} \\ &\qquad \overset{S}{\leftrightarrow} Z_B^{(8-d)}\left(O_8-C_8\right)\left(\bar{O}_8-\bar{C}_8\right) Z^{(\delta,0)}_{\Gamma^{d,d}}, \qquad \text{(type IIB string)} \end{aligned} \tag{3.14a}$$

$$Z_B^{(8-d)}\left(\bar{V}_8+\bar{S}_8\right) Z^{(0,\delta)}_{\Gamma^{16+d,d}} \overset{S}{\leftrightarrow} Z_B^{(8-d)}\left(\bar{O}_8-\bar{C}_8\right) Z^{(\delta,0)}_{\Gamma^{16+d,d}}, \qquad \text{(heterotic string)} \tag{3.14b}$$

where $Z^{(\delta,0)}_{\Gamma^{d_L,d_R}}$ is defined as

$$Z^{(\delta,0)}_{\Gamma^{d_L,d_R}} = Z_{\Gamma^{d_L,d_R}+\delta} = \eta^{-d_L}\bar{\eta}^{-d_R} \sum_{P\in\Gamma^{d_L,d_R}} q^{\frac{1}{2}(P_L+\delta_L)^2}\bar{q}^{\frac{1}{2}(P_R+\delta_R)^2}, \tag{3.15}$$

and right-hand sides of (3.14) are referred to as Z_+^-. Note that Z_+^- is obtained by S-transformation of Z_-^+ and vice versa (this is the reason why we use two-way arrows), so the sum of (3.11) and right-hand sides of (3.14) $\left(Z_+^+ + Z_-^+\right) + Z_+^-$ is invariant under S-transformation. We also find that under T-transformation, $\left(Z_+^+ + Z_-^+\right)$ is invariant and Z_+^- transforms as

$$\begin{aligned} &Z_B^{(8-d)}\left(O_8-C_8\right)\left(\bar{O}_8-\bar{C}_8\right) Z^{(\delta,0)}_{\Gamma^{d,d}} \\ &\qquad \overset{T}{\leftrightarrow} \mp Z_B^{(8-d)}\left(O_8+C_8\right)\left(\bar{O}_8+\bar{C}_8\right) Z^{(\delta,\delta)}_{\Gamma^{d,d}}, \qquad \text{(type IIB string)} \end{aligned} \tag{3.16a}$$

$$Z_B^{(8-d)}\left(\bar{O}_8-\bar{C}_8\right) Z^{(\delta,0)}_{\Gamma^{16+d,d}} \overset{T}{\leftrightarrow} \pm Z_B^{(8-d)}\left(\bar{O}_8+\bar{C}_8\right) Z^{(\delta,\delta)}_{\Gamma^{16+d,d}}, \qquad \text{(heterotic string)} \tag{3.16b}$$

where $Z^{(\delta,\delta)}_{\Gamma^{16+d,d}}$ is defines as

$$Z^{(\delta,\delta)}_{\Gamma^{d_L,d_R}} = Z_{\Gamma_+^{d_L,d_R}+\delta} - Z_{\Gamma_-^{d_L,d_R}+\delta} = \eta^{-d_L}\bar{\eta}^{-d_R} \sum_{P\in\Gamma^{d_L,d_R}} e^{2\pi i\delta\cdot P} q^{\frac{1}{2}(P_L+\delta_L)^2}\bar{q}^{\frac{1}{2}(P_R+\delta_R)^2}. \tag{3.17}$$

Here the upper and lower signs in (3.16) are applied for δ^2 odd and δ^2 even respectively[3], and right-hand sides of (3.16) are referred to as Z^-_-. Therefore, the partition function of the non-supersymmetric models is given as the following general formula:

$$Z^{\not{SUSY}}_{(\hat{Z})} = \frac{1}{2}\left(Z^+_+ + Z^+_- + Z^-_+ + Z^-_-\right). \tag{3.18}$$

For type IIB strings, (3.18) is expressed as

$$\begin{aligned} Z^{\not{SUSY}}_{(\hat{Z})} = Z_B^{(8-d)} \Big\{ &\left(V_8\bar{V}_8 + S_8\bar{S}_8\right) Z_{\Gamma^{d,d}_+} - \left(V_8\bar{S}_8 + S_8\bar{V}_8\right) Z_{\Gamma^{d,d}_-} \\ &+ \left(O_8\bar{O}_8 + C_8\bar{C}_8\right) Z_{\Gamma^{d,d}_\mp + \delta} - \left(O_8\bar{C}_8 + C_8\bar{O}_8\right) Z_{\Gamma^{d,d}_\pm + \delta} \Big\}, \end{aligned} \tag{3.19}$$

and for heterotic strings, (3.18) is expressed as

$$Z^{\not{SUSY}}_{(\hat{Z})} = Z_B^{(8-d)} \left\{ \bar{V}_8 Z_{\Gamma^{16+d,d}_+} - \bar{S}_8 Z_{\Gamma^{16+d,d}_-} + \bar{O}_8 Z_{\Gamma^{16+d,d}_\pm + \delta} - \bar{C}_8 Z_{\Gamma^{16+d,d}_\mp + \delta} \right\}. \tag{3.20}$$

In the following subsections, we look at the simplest case of the ten-dimensional models.

3.2 10D type 0 models

The 10D type II models have no internal direction, and hence the $\mathbb{Z}_2$ generator is simply written as $(-1)^F$. Orbifolding by this $\mathbb{Z}_2$ twist gives the ten-dimensional non-supersymmetric models, so-called the type 0B model and the type 0A model. Both their spectra include tachyon as the ground state. The partition functions of the type 0B and type 0A models are

$$Z^{(0\mathrm{B})} = Z_B^{(8)} \left(O_8\bar{O}_8 + V_8\bar{V}_8 + S_8\bar{S}_8 + C_8\bar{C}_8\right), \tag{3.21a}$$

$$Z^{(0\mathrm{A})} = Z_B^{(8)} \left(O_8\bar{O}_8 + V_8\bar{V}_8 + S_8\bar{C}_8 + C_8\bar{S}_8\right). \tag{3.21b}$$

We can get these functions as follows. Since the internal momentum $P = 0$ and the shift vector $\delta = 0$ in the case of $d = 0$, the splitted lattice $\Gamma^{d,d}_+$ is equivalent to $\Gamma^{d,d}$ while $\Gamma^{d,d}_-$ cannot be constructed from the definition (3.1). We can then get $Z_{\Gamma^{d,d}_+} = Z_{\Gamma^{d,d}_+ + \delta} = 1$ and $Z_{\Gamma^{d,d}_-} = Z_{\Gamma^{d,d}_- + \delta} = 0$, so the partiton functions (3.21) can be obtained from (3.19).

[3] T-transformation gives the phase factor $e^{\pi i\left(P^2+\delta^2+2P\cdot\delta\right)}$, where P^2 is even and $2P\cdot\delta$ is an integer. Thus, δ^2 must be an integer in order to make the partition function modular invariant.

Lattice	$\delta = \hat{\Pi}/2$	Gauge symmetry
$Spin(32)/\mathbb{Z}_2$	$(1, 0^{15})$	$SO(32)$
$Spin(32)/\mathbb{Z}_2$	$\left(\left(\frac{1}{2}\right)^4, 0^{12}\right)$	$SO(24) \times SO(8)$
$Spin(32)/\mathbb{Z}_2$	$\left(\left(\frac{1}{4}\right)^{16}\right)$	$SU(16) \times U(1)$
$Spin(32)/\mathbb{Z}_2$	$\left(\left(\frac{1}{2}\right)^8, 0^8\right)$	$SO(16) \times SO(16)$
$E_8 \times E_8$	$(1, 0^7; 0^8)$	$SO(16) \times E_8$
$E_8 \times E_8$	$\left(\left(\frac{1}{2}\right)^2, 0^6; \left(\frac{1}{2}\right)^2, 0^6\right)$	$(E_7 \times SU(2))^2$
$E_8 \times E_8$	$(1, 0^7; 1, 0^7)$	$SO(16) \times SO(16)$

Table 1: The 10-dimensional non-supersymmetric heterotic models constructed from the $Spin(32)/\mathbb{Z}_2$ and $E_8 \times E_8$ lattice by the shift vector δ and the realized gauge symmetries.

3.3 10D heterotic models

We can obtain the 10-dimensional non-supersymmetric heterotic models from the heterotic $Spin(32)/\mathbb{Z}_2$ and $E_8 \times E_8$ models by setting $d = 0$, as shown in Table 1 which was originally discussed in [7]. In this case, the shift vector $\delta = \hat{\Pi}/2$ is one-half of an element of the $Spin(32)/\mathbb{Z}_2$ or $E_8 \times E_8$ root lattice.

The partition function is

$$Z_{(\delta)}^{\not{SUSY}} = Z_B^{(8)} \left\{ \bar{V}_8 Z_{\Gamma_+^{16}} - \bar{S}_8 Z_{\Gamma_-^{16}} + \bar{O}_8 Z_{\Gamma_\pm^{16}+\delta} - \bar{C}_8 Z_{\Gamma_\mp^{16}+\delta} \right\}, \tag{3.22}$$

where

$$Z_{\Gamma_\pm^{16}} = \eta^{-16} \sum_{\Pi \in \Gamma_\pm^{16}} q^{\frac{1}{2}\Pi^2}, \tag{3.23}$$

$$Z_{\Gamma_\pm^{16}+\delta} = \eta^{-16} \sum_{\Pi \in \Gamma_\pm^{16}+\delta} q^{\frac{1}{2}\Pi^2} = \eta^{-16} \sum_{\Pi \in \Gamma_\pm^{16}} q^{\frac{1}{2}(\Pi+\delta)^2}. \tag{3.24}$$

In this case, the shift vector $\delta = \hat{\Pi}/2$ satisfies $2\delta \in \Gamma^{16}$, and $\Gamma_\pm^{16}$ is written as

$$\Gamma_+^{16}(\delta) = \left\{ \Pi \in \Gamma^{16} \,\middle|\, \delta \cdot \Pi \in \mathbb{Z} \right\}, \quad \Gamma_-^{16}(\delta) = \left\{ \Pi \in \Gamma^{16} \,\middle|\, \delta \cdot \Pi \in \mathbb{Z} + \frac{1}{2} \right\}. \tag{3.25}$$

Note that the inner product is taken by an Euclidian metric. In the following, we see these models in more detail.

3.3.1 $SO(16)\times SO(16)$ model

Among ten-dimensional non-supersymmetric heterotic string theories, the most interesting model is tachyon-free $SO(16)\times SO(16)$ one. This model is constructed by δ satisfying $\delta^2\in 2\mathbb{Z}$. First, we see the construction from the supersymmetric $Spin(32)/\mathbb{Z}_2$ model. Using the $SO(2n)$ conjugacy classes, the lattice (2.16) can be written as

$$\begin{aligned}\Gamma^{16}_{Spin(32)/\mathbb{Z}_2} &= \Gamma^{(16)}_g + \Gamma^{(16)}_s \\ &= \left(\Gamma^{(8)}_g\times\Gamma^{(8)}_g + \Gamma^{(8)}_v\times\Gamma^{(8)}_v\right) + \left(\Gamma^{(8)}_s\times\Gamma^{(8)}_s + \Gamma^{(8)}_c\times\Gamma^{(8)}_c\right). \end{aligned}\tag{3.26}$$

The $SO(16)\times SO(16)$ model can be obtained from $\delta=\left(\left(\frac{1}{2}\right)^8;0^8\right)$ in Table 1. From this δ, (3.25) is written as

$$\Gamma^{16}_+ = \left\{\Pi\in\Gamma^{16}_{Spin(32)/\mathbb{Z}_2}\ \middle|\ \sum_{I=1}^{8}\Pi^I\in 2\mathbb{Z}\right\},\quad \Gamma^{16}_- = \left\{\Pi\in\Gamma^{16}_{Spin(32)/\mathbb{Z}_2}\ \middle|\ \sum_{I=1}^{8}\Pi^I\in 2\mathbb{Z}+1\right\}.\tag{3.27}$$

Using the $SO(16)$ conjugacy classes, $\Gamma^{16}_\pm$ in untwisted sectors are written as

$$\Gamma^{16}_+ = \Gamma^{(8)}_g\times\Gamma^{(8)}_g + \Gamma^{(8)}_s\times\Gamma^{(8)}_s,\qquad \Gamma^{16}_- = \Gamma^{(8)}_v\times\Gamma^{(8)}_v + \Gamma^{(8)}_c\times\Gamma^{(8)}_c,\tag{3.28}$$

and $\Gamma^{16}_\pm+\delta$ in twisted sectors are written as

$$\Gamma^{16}_+ +\delta = \Gamma^{(8)}_g\times\Gamma^{(8)}_s + \Gamma^{(8)}_s\times\Gamma^{(8)}_g,\qquad \Gamma^{16}_- +\delta = \Gamma^{(8)}_v\times\Gamma^{(8)}_c + \Gamma^{(8)}_c\times\Gamma^{(8)}_v.\tag{3.29}$$

Then we can write down $Z_{\Gamma^{16}_\pm}$ and $Z_{\Gamma^{16}_\pm+\delta}$ in terms of $SO(16)$ characters (A.5) as follows:

$$\begin{aligned} &Z_{\Gamma^{16}_+} = O_{16}O_{16}+S_{16}S_{16},\qquad Z_{\Gamma^{16}_-} = V_{16}V_{16}+C_{16}C_{16},\\ &Z_{\Gamma^{16}_++\delta} = O_{16}S_{16}+S_{16}O_{16},\qquad Z_{\Gamma^{16}_-+\delta} = V_{16}C_{16}+C_{16}V_{16}. \end{aligned}\tag{3.30}$$

Therefore, the partition function is expressed as

$$\begin{aligned} Z^{\not{SUSY}}_{\left(\left(\frac{1}{2}\right)^8;0^8\right)} = Z^{(8)}_B\Big\{&\bar{V}_8\left(O_{16}O_{16}+S_{16}S_{16}\right)-\bar{S}_8\left(V_{16}V_{16}+C_{16}C_{16}\right)\\ &+\bar{O}_8\left(V_{16}C_{16}+C_{16}V_{16}\right)-\bar{C}_8\left(O_{16}S_{16}+S_{16}O_{16}\right)\Big\}. \end{aligned}\tag{3.31}$$

Next we consider the construction of $SO(16)\times SO(16)$ model from the supersymmetric $E_8\times E_8$ model. The lattice (2.17) is written as

$$\begin{aligned}\Gamma^{16}_{E_8\times E_8} &= \left(\Gamma^{(8)}_g+\Gamma^{(8)}_s\right)\times\left(\Gamma^{(8)}_g+\Gamma^{(8)}_s\right)\\ &= \Gamma^{(8)}_g\times\Gamma^{(8)}_g+\Gamma^{(8)}_s\times\Gamma^{(8)}_s+\Gamma^{(8)}_g\times\Gamma^{(8)}_s+\Gamma^{(8)}_s\times\Gamma^{(8)}_g. \end{aligned}\tag{3.32}$$

From Table 1, $SO(16)\times SO(16)$ model can be obtained by choosing $\delta=(1,0^7;1,0^7)$. The lattice (3.25) is written as

$$\Gamma^{16}_{+}=\left\{\Pi\in\Gamma^{16}_{E_8\times E_8}\ \middle|\Pi^1+\Pi^9\in\mathbb{Z}\right\},\quad \Gamma^{16}_{-}=\left\{\Pi\in\Gamma^{16}_{E_8\times E_8}\ \middle|\Pi^1+\Pi^9\in\mathbb{Z}+\frac{1}{2}\right\}. \tag{3.33}$$

Thus we can write $\Gamma^{16}_{\pm}$ as

$$\Gamma^{16}_{+}=\Gamma^{(8)}_{g}\times\Gamma^{(8)}_{g}+\Gamma^{(8)}_{s}\times\Gamma^{(8)}_{s},\qquad \Gamma^{16}_{-}=\Gamma^{(8)}_{g}\times\Gamma^{(8)}_{s}+\Gamma^{(8)}_{s}\times\Gamma^{(8)}_{g}, \tag{3.34}$$

and $\Gamma^{16}_{\pm}+\delta$ as

$$\Gamma^{16}_{+}+\delta=\Gamma^{(8)}_{v}\times\Gamma^{(8)}_{v}+\Gamma^{(8)}_{c}\times\Gamma^{(8)}_{c},\qquad \Gamma^{16}_{-}+\delta=\Gamma^{(8)}_{v}\times\Gamma^{(8)}_{c}+\Gamma^{(8)}_{c}\times\Gamma^{(8)}_{v}. \tag{3.35}$$

From these expression, $Z_{\Gamma^{16}_{\pm}}$ and $Z_{\Gamma^{16}_{\pm}+\delta}$ is written as follows:

$$\begin{aligned} &Z_{\Gamma^{16}_{+}}=O_{16}O_{16}+S_{16}S_{16},\qquad Z_{\Gamma^{16}_{-}}=O_{16}S_{16}+S_{16}O_{16},\\ &Z_{\Gamma^{16}_{+}+\delta}=V_{16}V_{16}+C_{16}C_{16},\qquad Z_{\Gamma^{16}_{-}+\delta}=V_{16}C_{16}+C_{16}V_{16}, \end{aligned} \tag{3.36}$$

The partition function is, therefore expressed as

$$\begin{aligned} Z^{\not{SUSY}}_{(1,0^7;1,0^7)}=Z^{(8)}_{B}\big\{&\bar{V}_8\left(O_{16}O_{16}+S_{16}S_{16}\right)-\bar{S}_8\left(O_{16}S_{16}+S_{16}O_{16}\right)\\ &+\bar{O}_8\left(V_{16}C_{16}+C_{16}V_{16}\right)-\bar{C}_8\left(V_{16}V_{16}+C_{16}C_{16}\right)\big\}. \end{aligned} \tag{3.37}$$

Note that $S_8=C_8$, so two partition functions (3.31) and (3.37) are equivalent (the difference is only chirality).

From (3.28),(3.29) and (2.17), we can find

$$\begin{aligned} \Gamma^{16}_{+}+\left(\Gamma^{16}_{+}+\delta\right)\big|_{Spin(32)/\mathbb{Z}_2}&=\Gamma^{(8)}_{g}\times\Gamma^{(8)}_{g}+\Gamma^{(8)}_{s}\times\Gamma^{(8)}_{s}+\Gamma^{(8)}_{g}\times\Gamma^{(8)}_{s}+\Gamma^{(8)}_{s}\times\Gamma^{(8)}_{g}\\ &=\Gamma^{16}_{E_8\times E_8}. \end{aligned} \tag{3.38}$$

Likewise from (3.34),(3.35) and (2.16), we can get

$$\begin{aligned} \Gamma^{16}_{+}+\left(\Gamma^{16}_{+}+\delta\right)\big|_{E_8\times E_8}&=\Gamma^{(8)}_{g}\times\Gamma^{(8)}_{g}+\Gamma^{(8)}_{s}\times\Gamma^{(8)}_{s}+\Gamma^{(8)}_{v}\times\Gamma^{(8)}_{v}+\Gamma^{(8)}_{c}\times\Gamma^{(8)}_{c}\\ &=\Gamma^{16}_{Spin(32)/\mathbb{Z}_2}. \end{aligned} \tag{3.39}$$

Equivalently, using $SO(2n)$ characters, we obtain

$$\begin{aligned} Z_{\Gamma^{16}_{+}}+Z_{\Gamma^{16}_{+}+\delta}\Big|_{Spin(32)/\mathbb{Z}_2}&=O_{16}O_{16}+S_{16}S_{16}+O_{16}S_{16}+S_{16}O_{16}=Z_{\Gamma^{16}_{E_8\times E_8}},\\ Z_{\Gamma^{16}_{+}}+Z_{\Gamma^{16}_{+}+\delta}\Big|_{E_8\times E_8}&=O_{16}O_{16}+S_{16}S_{16}+V_{16}V_{16}+C_{16}C_{16}=Z_{\Gamma^{16}_{Spin(32)/\mathbb{Z}_2}}. \end{aligned} \tag{3.40}$$

These mean that if $SO(16)\times SO(16)$ model is constructed from one supersymmetric heterotic string theory, the sum of $Z_{\Gamma^{16}_{+}}$ and $Z_{\Gamma^{16}_{+}+\delta}$ in the partition function of the $SO(16)\times SO(16)$ model is equal to $Z_{\Gamma^{16}}$ of the other supersymmetric heterotic string theory. This nice property will play a crucial role later when the supersymmetric string theory is restored by tuning the moduli of Wilson lines, discussed in Subsection 4.2.1. Note that $Z_{\Gamma^{16}_{+}}+Z_{\Gamma^{16}_{+}+\delta}$ in the $SO(16)\times SO(16)$ model is modular invariant since $Z_{\Gamma^{16}}$ in the partition function of the 10D SUSY string is modular invariant.

3.3.2 The other non-SUSY models

All of ten-dimensional non-supersymmetric heterotic models except $SO(16)\times SO(16)$ model are constructed by δ satisfying $\delta^2\in 2\mathbb{Z}+1$. The partition functions can be obtained as in the case of $\delta^2\in 2\mathbb{Z}$. For completeness, we list up the partition functions that can be expressed using $SO(2n)$ characters below, except for the $SU(16)\times U(1)$ model:

- Non-supersymmetric $SO(32)$ model:

$$Z^{\cancel{SUSY}}_{(1,0^{15})}=Z_B^{(8)}\left\{\bar{V}_8\left(O_{16}O_{16}+V_{16}V_{16}\right)-\bar{S}_8\left(S_{16}S_{16}+C_{16}C_{16}\right)\right.$$
$$\left.+\bar{O}_8\left(O_{16}V_{16}+V_{16}O_{16}\right)-\bar{C}_8\left(S_{16}C_{16}+C_{16}S_{16}\right)\right\}. \tag{3.41}$$

- $SO(24)\times SO(8)$ model:

$$Z^{\cancel{SUSY}}_{\left(\left(\frac{1}{2}\right)^4,0^{12}\right)}=Z_B^{(8)}\left\{\bar{V}_8\left(O_8O_{24}+S_8S_{24}\right)-\bar{S}_8\left(V_8V_{24}+C_8C_{24}\right)\right.$$
$$\left.+\bar{O}_8\left(O_8S_{24}+S_8O_{24}\right)-\bar{C}_8\left(V_8C_{24}+C_8V_{24}\right)\right\}. \tag{3.42}$$

- $SO(16)\times E_8$ model:

$$Z^{\cancel{SUSY}}_{(1,0^7;0^8)}=Z_B^{(8)}\left(\bar{V}_8O_{16}-\bar{S}_8S_{16}+\bar{O}_8V_{16}-\bar{C}_8C_{16}\right)\left(O_{16}+S_{16}\right). \tag{3.43}$$

- $\left(E_7\times SU(2)\right)^2$ model:

$$\begin{aligned}
&Z^{\cancel{SUSY}}_{\left(\left(\frac{1}{2}\right)^2,0^6;\left(\frac{1}{2}\right)^2,0^6\right)}\\
&=Z_B^{(8)}\left[\bar{V}_8\left\{\left(O_4O_{12}+S_4S_{12}\right)\left(O_4O_{12}+S_4S_{12}\right)+\left(V_4V_{12}+C_4C_{12}\right)\left(V_4V_{12}+C_4C_{12}\right)\right\}\right.\\
&\qquad-\bar{S}_8\left\{\left(O_4O_{12}+S_4S_{12}\right)\left(V_4V_{12}+C_4C_{12}\right)+\left(V_4V_{12}+C_4C_{12}\right)\left(O_4O_{12}+S_4S_{12}\right)\right\}\\
&\qquad+\bar{O}_8\left\{\left(C_4O_{12}+V_4S_{12}\right)\left(C_4O_{12}+V_4S_{12}\right)+\left(S_4V_{12}+O_4C_{12}\right)\left(S_4V_{12}+O_4C_{12}\right)\right\}\\
&\qquad\left.-\bar{C}_8\left\{\left(C_4O_{12}+V_4S_{12}\right)\left(S_4V_{12}+O_4C_{12}\right)+\left(S_4V_{12}+O_4C_{12}\right)\left(C_4O_{12}+V_4S_{12}\right)\right\}\right].
\end{aligned} \tag{3.44}$$

4 Endpoint limit and interpolation

In this section, we study the specific regions in moduli space of the non-supersymmetric models (3.19) and (3.20) with $d \geq 1$ satisfying the conditions (3.5). To be more specific, we investigate the behaviors of the partition functions in the limits of $R_i \to \infty$ and $R_i \to 0$, so-called endpoint limits. Here, we only focus on the $d = 1, 2$ cases, but the analysis can be easily generalized to the $d \geq 3$ case. One of the purposes of the section is to understand the meaning of the components of $\hat{Z}$: we can find that they determine the $\mathbb{Z}_2$-twisted directions in which the supersymmetry can be restored at endpoint limits. We are interested in the limit when the supersymmetries are restored in the lower dimensional non-supersymmetric heterotic models because the vanishing of the cosmological constant is closely related to the supersymmetries of the models, which is the main subject of this thesis.

4.1 Case of turned-off moduli

In this subsection, we consider the non-supersymmetric models with the anti-symmetric two-form B and the Wilson lines A turned off for simplicity.

4.1.1 9D type IIB models

9-dimensional type II models have only one modulus R, and there are no degrees of freedom of the anti-symmetric two forms B. From the condition on δ (3.6), we can find the 9D type IIB non-supersymmetric models are classified into the three classes as $\hat{Z} = (\hat{w}, \hat{n}) = (0;0), (1;0), (0;1)$ and the orbifold for IIA is similarly constructed. Note that $\hat{Z} = (1;1)$ is prohibited by the constraint of δ[4].

- $(\hat{w}; \hat{n}) = (0;0)$:
  In this case, $P \cdot \delta = 0 \in \mathbb{Z}$ for any P. Therefore $\Gamma_+^{1,1}$ and $\Gamma_+^{1,1} + \delta$ are equivalent to $\Gamma^{1,1}$ while $\Gamma_-^{d,d}$ and $\Gamma_-^{1,1} + \delta$ vanish. The partiton function (3.19) then reads

$$Z_{(0;0)}^{\not{SUSY}} = Z_B^{(7)} \left(V_8 \bar{V}_8 + S_8 \bar{S}_8 + O_8 \bar{O}_8 + C_8 \bar{C}_8 \right) Z_{\Gamma^{1,1}}. \tag{4.1}$$

  This is the partition function for the 9D models constructed from 10D type 0B ones by simply S^1 compactification.

[4] We cannot thus construct the 9D type II models in which supersymmetry is restored at both endpoints (see also the class (4) in the subsubsection 4.1.2).

- $(\hat{w};\hat{n})=(1;0),(0;1)$:

  From the constraint (3.6), the condition $(\hat{w};\hat{n})=(1;0)$ $((0;1))$ gives $P\cdot\delta=n$ (w). We can then find that $(\hat{w};\hat{n})=(1;0)$ $((0;1))$ makes $n(w)$ even in $\Gamma^{1,1}_{+}$ while odd in $\Gamma^{1,1}_{-}$. For $(\hat{w};\hat{n})=(1;0)$, we can write $Z_{\Gamma^{1,1}_{\pm}(+\delta)}$ as

$$Z_{\Gamma^{1,1}_{+}(+\delta)}=(\eta\bar{\eta})^{-1}\sum_{w\in\mathbb{Z}\left(+\frac{1}{2}\right)}\sum_{n\in 2\mathbb{Z}}e^{-\pi\tau_2\left\{\left(\frac{n}{R}\right)^2+(wR)^2\right\}}e^{2\pi i\tau_1 wn}, \tag{4.2}$$

$$Z_{\Gamma^{1,1}_{-}(+\delta)}=(\eta\bar{\eta})^{-1}\sum_{w\in\mathbb{Z}\left(+\frac{1}{2}\right)}\sum_{n\in 2\mathbb{Z}+1}e^{-\pi\tau_2\left\{\left(\frac{n}{R}\right)^2+(wR)^2\right\}}e^{2\pi i\tau_1 wn}. \tag{4.3}$$

  From the real part of $Z_{\Gamma^{1,1}_{\pm}(+\delta)}$, we can find that the limit of $R\to\infty$ $(R\to 0)$ makes the contributions from $w\neq 0$ $(n\neq 0)$ vanish exponentially. Using the asymptotic behavior

$$\sum_{m\in\mathbb{Z}}e^{-\pi\tau_2 k^2 m^2}\xrightarrow{k\to 0}\int_{\infty}^{\infty}dx e^{-\pi\tau_2 k^2 x^2}=\frac{1}{k\sqrt{\tau_2}}, \tag{4.4}$$

  then, we get the behavior in the endpoint limits as

$$Z_{\Gamma^{1,1}_{\pm}}\xrightarrow{R\to\infty}\frac{R}{2\sqrt{\tau_2}}(\eta\bar{\eta})^{-1}, \quad Z_{\Gamma^{1,1}_{\pm}+\delta}\xrightarrow{R\to\infty}0, \tag{4.5}$$

$$Z_{\Gamma^{1,1}_{+}(+\delta)}\xrightarrow{R\to 0}\frac{1}{R\sqrt{\tau_2}}(\eta\bar{\eta})^{-1}, \quad Z_{\Gamma^{1,1}_{-}(+\delta)}\xrightarrow{R\to 0}0, \tag{4.6}$$

  Therefore the partition function (3.19) with $\hat{Z}=(1;0)$ reads the 10D type IIB model in $R\to\infty$ while 10D type 0A model in $R\to 0$. We should mention that since we perform T-dual in order to open up the compactified dimension, the chirality of the right-mover is flipped in $R\to 0$. In the same way, one can check that the type IIB model with $\hat{Z}=(0;1)$ reads the 10D type 0B model and the 10D type IIA model in the endpoint limits.

4.1.2 9D heterotic models

The starting point of the supersymmetric heterotic string model can be either that of $Spin(32)/\mathbb{Z}_2$ or $E_8\times E_8$ by choosing Γ^{16} correctly. From the constraint (3.7), we then classify the 9-dimensional non-supersymmetric heterotic models with $A=(0^{16})$ into the following $2^2=4$ classes depending on the choice of $(\hat{w},\hat{n})$ as studied in [2].

(1) $\hat{\Pi}^2=0 \pmod 4$, $\quad(\hat{w};\hat{n})=(0;0)$:

  In this class, the original non-supersymmetric heterotic model should be reproduced in the endpoint point limit since we do not introduce any moduli. The inner product (3.4) is written as

$$P \cdot \delta = \frac{1}{2}\Pi \cdot \hat{\Pi}. \tag{4.7}$$

From the definition (3.1), $\Gamma^{17,1}_{\pm}$ and $\Gamma^{17,1}_{\pm} + \delta$ in this class are written as the following sets:

$$\Gamma^{17,1}_{\pm} = \left\{ P = Z\mathcal{E} \,\middle|\, (\Pi, w, n) \in \left(\Gamma^{16}_{\pm}, \mathbb{Z}, \mathbb{Z}\right) \right\}, \tag{4.8}$$

$$\Gamma^{17,1}_{\pm} + \delta = \left\{ P = Z\mathcal{E} \,\middle|\, (\Pi, w, n) \in \left(\Gamma^{16}_{\pm} + \frac{\hat{\Pi}}{2}, \mathbb{Z}, \mathbb{Z}\right) \right\}, \tag{4.9}$$

where $\Gamma^{16}_{+}(\hat{\Pi})$ and $\Gamma^{16}_{-}(\hat{\Pi})$ are defined as

$$\Gamma^{16}_{+}(\hat{\Pi}) = \left\{ \Pi \in \Gamma^{16} \,\middle|\, \Pi \cdot \hat{\Pi} \in 2\mathbb{Z} \right\}, \qquad \Gamma^{16}_{-}(\hat{\Pi}) = \left\{ \Pi \in \Gamma^{16} \,\middle|\, \Pi \cdot \hat{\Pi} \in 2\mathbb{Z} + 1 \right\}. \tag{4.10}$$

We can find that the states with $w^1 = 0$ ($n_1 = 0$) only contribute as $R_1 \to \infty$ ($R_1 \to 0$) in the case of $A = (0^{16})$ as follows. The internal momenta (2.15) are written as

$$\ell_L = \Pi, \quad p_{L/R} = \frac{1}{\sqrt{2}R_1}\left(n_1 \pm w^1 R_1^2\right). \tag{4.11}$$

In the partition function, the moduli-dependent parts are the $Z_{\Gamma^{17,1}_{\pm}}$ and $Z_{\Gamma^{17,1}_{\pm}+\delta}$. In this case, we can get

$$Z_{\Gamma^{17,1}_{\pm}(+\delta)} = \eta^{-17}\bar{\eta}^{-1} \sum_{\Pi \in \Gamma^{16}_{\pm}(+\frac{\hat{\Pi}}{2})} q^{\frac{1}{2}\Pi^2} \sum_{w^1, n_1 \in \mathbb{Z}} e^{-\pi\tau_2\left\{\left(\frac{n_1}{R_1}\right)^2 + \left(w^1 R_1\right)^2\right\}} e^{2\pi i \tau_1 w^1 n_1}. \tag{4.12}$$

From the real part of $Z_{\Gamma^{17,1}_{\pm}(+\delta)}$, the contributions from $w^1 \neq 0$ ($n_1 \neq 0$) vanish exponentially when $R_1 \to \infty$ ($R_1 \to 0$). We then find the behavior of $Z_{\Gamma^{17,1}_{\pm}}$ and $Z_{\Gamma^{17,1}_{\pm}+\delta}$ with $A = (0^{16})$ in the endpoint limits:

$$Z_{\Gamma^{17,1}_{\pm}} \xrightarrow{R_1 \to \infty} \frac{R_1}{\sqrt{\tau_2}} \left(\eta\bar{\eta}\right)^{-1} Z_{\Gamma^{16}_{\pm}}, \quad Z_{\Gamma^{17,1}_{\pm}+\delta} \xrightarrow{R_1 \to \infty} \frac{R_1}{\sqrt{\tau_2}} \left(\eta\bar{\eta}\right)^{-1} Z_{\Gamma^{16}_{\pm}+\frac{\hat{\Pi}}{2}}, \tag{4.13}$$

$$Z_{\Gamma^{17,1}_{\pm}} \xrightarrow{R_1 \to 0} \frac{1}{R_1\sqrt{\tau_2}} \left(\eta\bar{\eta}\right)^{-1} Z_{\Gamma^{16}_{\pm}}, \quad Z_{\Gamma^{17,1}_{\pm}+\delta} \xrightarrow{R_1 \to 0} \frac{1}{R_1\sqrt{\tau_2}} \left(\eta\bar{\eta}\right)^{-1} Z_{\Gamma^{16}_{\pm}+\frac{\hat{\Pi}}{2}}, \tag{4.14}$$

where $Z_{\Gamma^{16}_{\pm}}$ and $Z_{\Gamma^{16}_{\pm}+\frac{\hat{\Pi}}{2}}$ are defined as

$$Z_{\Gamma^{16}_{\pm}} = \eta^{-16} \sum_{\Pi \in \Gamma^{16}_{\pm}} q^{\frac{1}{2}\Pi^2}, \qquad Z_{\Gamma^{16}_{\pm}+\frac{\hat{\Pi}}{2}} = \eta^{-16} \sum_{\Pi \in \Gamma^{16}_{\pm}} q^{\frac{1}{2}\left(\Pi + \frac{\hat{\Pi}}{2}\right)^2}. \tag{4.15}$$

As expected, in this class, the same 10-dimensional non-supersymmetric models constructed by the shift-vector $\delta = \hat{\Pi}/2$ are obtained in both endpoint limits. Then, we can find that in this class, the 9-dimensional non-supersymmetric models can be obtained by the S^1 compactification of the 10-dimensional non-supersymmetric ones listed in Table 1.

(2) $\hat{\Pi}^2 = 0 \pmod 4$, $\quad (\hat{w};\hat{n}) = (1;0)$:

In this class, the inner product (3.4) is given by

$$P \cdot \delta = \frac{1}{2}\left(\Pi \cdot \hat{\Pi} + n_1\right). \tag{4.16}$$

We can write $\Gamma^{17,1}_{\pm}$ and $\Gamma^{17,1}_{\pm} + \delta$ as

$$\Gamma^{17,1}_{\pm} = \left\{P = Z\mathcal{E}\,\middle|(\Pi, w, n) \in \left(\Gamma^{16}_{\pm}, \mathbb{Z}, 2\mathbb{Z}\right)\right\} \oplus \left\{P = Z\mathcal{E}\,\middle|(\Pi, w, n) \in \left(\Gamma^{16}_{\mp}, \mathbb{Z}, 2\mathbb{Z}+1\right)\right\}, \tag{4.17}$$

$$\begin{aligned}\Gamma^{17,1}_{\pm} + \delta = &\left\{P = Z\mathcal{E}\,\middle|(\Pi, w, n) \in \left(\Gamma^{16}_{\pm} + \frac{\hat{\Pi}}{2}, \mathbb{Z} + \frac{1}{2}, 2\mathbb{Z}\right)\right\}\\ &\oplus \left\{P = Z\mathcal{E}\,\middle|(\Pi, w, n) \in \left(\Gamma^{16}_{\mp} + \frac{\hat{\Pi}}{2}, \mathbb{Z} + \frac{1}{2}, 2\mathbb{Z}+1\right)\right\}.\end{aligned} \tag{4.18}$$

In this case, $Z_{\Gamma^{17,1}_{\pm}(+\delta)}$ can be written as

$$\begin{aligned}Z_{\Gamma^{17,1}_{\pm}(+\delta)} = \eta^{-17}\bar{\eta}^{-1}\Bigg\{ &\sum_{\Pi \in \Gamma^{16}_{\pm}(+\frac{\hat{\Pi}}{2})} q^{\frac{1}{2}\Pi^2} \sum_{w^1 \in \mathbb{Z}\left(+\frac{1}{2}\right)} \sum_{n_1 \in 2\mathbb{Z}} e^{-\pi\tau_2\left\{\left(\frac{n_1}{R_1}\right)^2 + \left(w^1 R_1\right)^2\right\}} e^{2\pi i \tau_1 w^1 n_1}\\ &+ \sum_{\Pi \in \Gamma^{16}_{\mp}(+\frac{\hat{\Pi}}{2})} q^{\frac{1}{2}\Pi^2} \sum_{w^1 \in \mathbb{Z}\left(+\frac{1}{2}\right)} \sum_{n_1 \in 2\mathbb{Z}+1} e^{-\pi\tau_2\left\{\left(\frac{n_1}{R_1}\right)^2 + \left(w^1 R_1\right)^2\right\}} e^{2\pi i \tau_1 w^1 n_1}\Bigg\}.\end{aligned} \tag{4.19}$$

In the limit of $R_1 \to \infty$, the untwisted sectors $Z_{\Gamma^{17,1}_{\pm}}$ behave as

$$\sum_{\Pi \in \Gamma^{16}_{\pm}} q^{\frac{1}{2}\Pi^2} \sum_{n \in 2\mathbb{Z}} e^{-\frac{\pi\tau_2}{R^2}n^2} + \sum_{\Pi \in \Gamma^{16}_{\mp}} q^{\frac{1}{2}\Pi^2} \sum_{n \in 2\mathbb{Z}+1} e^{-\frac{\pi\tau_2}{R^2}n^2} \to \frac{R}{2\sqrt{\tau_2}} \sum_{\Pi \in \Gamma^{16}} q^{\frac{1}{2}\Pi^2}, \tag{4.20}$$

where we use

$$\sum_{\Pi \in \Gamma^{16}} q^{\frac{1}{2}\Pi^2} = \sum_{\Pi \in \Gamma^{16}_{+}} q^{\frac{1}{2}\Pi^2} + \sum_{\Pi \in \Gamma^{16}_{-}} q^{\frac{1}{2}\Pi^2}. \tag{4.21}$$

For the twisted sectors $Z_{\Gamma^{17,1}_{\pm}+\delta}$, the winding number w^1 runs for $\mathbb{Z}+1/2$, so there is no contribution from $w^1=0$. In the limit of $R_1 \to 0$, the second term of (??) vanishes for both the untwisted and twisted sectors since it does not contain the contribution of $n_1 = 0$.

The behavior of $Z_{\Gamma^{17,1}_{\pm}}$ and $Z_{\Gamma^{17,1}_{\pm}+\delta}$ in the endpoint limits are given by

$$Z_{\Gamma^{17,1}_{\pm}} \xrightarrow{R_1\to\infty} \frac{R_1}{\sqrt{\tau_2}} (\eta\bar{\eta})^{-1} Z_{\Gamma^{16}}, \quad Z_{\Gamma^{17,1}_{\pm}+\delta} \xrightarrow{R_1\to\infty} 0, \tag{4.22}$$

$$Z_{\Gamma^{17,1}_{\pm}} \xrightarrow{R_1\to 0} \frac{1}{R_1\sqrt{\tau_2}} (\eta\bar{\eta})^{-1} Z_{\Gamma^{16}_{\pm}}, \quad Z_{\Gamma^{17,1}_{\pm}+\delta} \xrightarrow{R_1\to 0} \frac{1}{R_1\sqrt{\tau_2}} (\eta\bar{\eta})^{-1} Z_{\Gamma^{16}_{\pm}+\frac{\hat{\Pi}}{2}}. \tag{4.23}$$

The limit $R_1 \to \infty$ makes supersymmetry asymptotically restored while the limit $R_1 \to 0$ gives the 10-dimensional non-supersymmetric models in Table 1.

(3) $\hat{\Pi}^2 = 0 \pmod 4$, $\quad (\hat{w};\hat{n}) = (0;1)$:

In this class, we can write the inner product (3.4) as

$$P\cdot\delta = \frac{1}{2}\left(\Pi\cdot\hat{\Pi} + w^1\right). \tag{4.24}$$

Then $\Gamma^{17,1}_{\pm}$ and $\Gamma^{17,1}_{\pm}+\delta$ are written as

$$\Gamma^{17,1}_{\pm} = \left\{P = Z\mathcal{E}\,\middle|(\Pi,w,n)\in\left(\Gamma^{16}_{\pm},2\mathbb{Z},\mathbb{Z}\right)\right\} \oplus \left\{P = Z\mathcal{E}\,\middle|(\Pi,w,n)\in\left(\Gamma^{16}_{\mp},2\mathbb{Z}+1,\mathbb{Z}\right)\right\}, \tag{4.25}$$

$$\begin{aligned}\Gamma^{17,1}_{\pm}+\delta = &\left\{P = Z\mathcal{E}\,\middle|(\Pi,w,n)\in\left(\Gamma^{16}_{\pm}+\frac{\hat{\Pi}}{2},2\mathbb{Z},\mathbb{Z}+\frac{1}{2}\right)\right\}\\ &\oplus\left\{P = Z\mathcal{E}\,\middle|(\Pi,w,n)\in\left(\Gamma^{16}_{\mp}+\frac{\hat{\Pi}}{2},2\mathbb{Z}+1,\mathbb{Z}+\frac{1}{2}\right)\right\}.\end{aligned} \tag{4.26}$$

As in the case of class (2), the behaviors of $Z_{\Gamma^{17,1}_{\pm}}$ and $Z_{\Gamma^{17,1}_{\pm}+\delta}$ in the endpoint limits are obtained as

$$Z_{\Gamma^{17,1}_{\pm}} \xrightarrow{R_1\to\infty} \frac{R_1}{\sqrt{\tau_2}} (\eta\bar{\eta})^{-1} Z_{\Gamma^{16}_{\pm}}, \quad Z_{\Gamma^{17,1}_{\pm}+\delta} \xrightarrow{R_1\to\infty} \frac{R_1}{\sqrt{\tau_2}} (\eta\bar{\eta})^{-1} Z_{\Gamma^{16}_{\pm}+\frac{\hat{\Pi}}{2}}, \tag{4.27}$$

$$Z_{\Gamma^{17,1}_{\pm}} \xrightarrow{R_1\to 0} \frac{1}{R_1\sqrt{\tau_2}} (\eta\bar{\eta})^{-1} Z_{\Gamma^{16}}, \quad Z_{\Gamma^{17,1}_{\pm}+\delta} \xrightarrow{R_1\to 0} 0. \tag{4.28}$$

In this class, the limit $R_1 \to \infty$ gives the 10-dimensional non-supersymmetric heterotic models while the 10-dimensional heterotic superstring ones can be obtained in the limit $R_1 \to 0$.

(4) $\hat{\Pi}^2 = 2 \pmod 4$, $\quad (\hat{w};\hat{n}) = (1;1)$:

In this class, the inner product (3.4) is written as

$$P \cdot \delta = \frac{1}{2}\left(\Pi \cdot \hat{\Pi} + w^1 + n_1\right). \tag{4.29}$$

We can get $\Gamma^{17,1}_{\pm}$ and $\Gamma^{17,1}_{\pm} + \delta$ as

$$\begin{aligned}\Gamma^{17,1}_{\pm} = &\left\{P = Z\mathcal{E}\,\middle|(\Pi, w, n) \in \left(\Gamma^{16}_{\pm}, 2\mathbb{Z}, 2\mathbb{Z}\right)\right\}\\ &\oplus \left\{P = Z\mathcal{E}\,\middle|(\Pi, w, n) \in \left(\Gamma^{16}_{\pm}, 2\mathbb{Z}+1, 2\mathbb{Z}+1\right)\right\}\\ &\oplus \left\{P = Z\mathcal{E}\,\middle|(\Pi, w, n) \in \left(\Gamma^{16}_{\mp}, 2\mathbb{Z}, 2\mathbb{Z}+1\right)\right\}\\ &\oplus \left\{P = Z\mathcal{E}\,\middle|(\Pi, w, n) \in \left(\Gamma^{16}_{\mp}, 2\mathbb{Z}+1, 2\mathbb{Z}\right)\right\},\end{aligned} \tag{4.30}$$

$$\begin{aligned}\Gamma^{17,1}_{\pm} + \delta = &\left\{P = Z\mathcal{E}\,\middle|(\Pi, w, n) \in \left(\Gamma^{16}_{\pm} + \frac{\hat{\Pi}}{2}, 2\mathbb{Z}+\frac{1}{2}, 2\mathbb{Z}+\frac{1}{2}\right)\right\}\\ &\oplus \left\{P = Z\mathcal{E}\,\middle|(\Pi, w, n) \in \left(\Gamma^{16}_{\pm} + \frac{\hat{\Pi}}{2}, 2\mathbb{Z}-\frac{1}{2}, 2\mathbb{Z}-\frac{1}{2}\right)\right\}\\ &\oplus \left\{P = Z\mathcal{E}\,\middle|(\Pi, w, n) \in \left(\Gamma^{16}_{\mp} + \frac{\hat{\Pi}}{2}, 2\mathbb{Z}+\frac{1}{2}, 2\mathbb{Z}-\frac{1}{2}\right)\right\}\\ &\oplus \left\{P = Z\mathcal{E}\,\middle|(\Pi, w, n) \in \left(\Gamma^{16}_{\mp} + \frac{\hat{\Pi}}{2}, 2\mathbb{Z}-\frac{1}{2}, 2\mathbb{Z}+\frac{1}{2}\right)\right\}.\end{aligned} \tag{4.31}$$

In this case, $Z_{\Gamma^{17,1}_{\pm}(+\delta)}$ can be written as

$$\begin{aligned}Z_{\Gamma^{17,1}_{\pm}(+\delta)} = &\,\eta^{-17}\bar{\eta}^{-1}\sum_{\Pi \in \Gamma^{16}_{\pm}(+\frac{\hat{\Pi}}{2})} q^{\frac{1}{2}\Pi^2}\left\{\sum_{w^1 \in 2\mathbb{Z}\left(+\frac{1}{2}\right)}\ \sum_{n_1 \in 2\mathbb{Z}\left(+\frac{1}{2}\right)} e^{-\pi\tau_2\left\{\left(\frac{n_1}{R_1}\right)^2 + \left(w^1 R_1\right)^2\right\}} e^{2\pi i \tau_1 w^1 n_1}\right.\\ &\left. + \sum_{w^1 \in 2\mathbb{Z}+1\left(+\frac{1}{2}\right)}\ \sum_{n_1 \in 2\mathbb{Z}+1\left(+\frac{1}{2}\right)} e^{-\pi\tau_2\left\{\left(\frac{n_1}{R_1}\right)^2 + \left(w^1 R_1\right)^2\right\}} e^{2\pi i \tau_1 w^1 n_1}\right\}\\ &+ \eta^{-17}\bar{\eta}^{-1}\sum_{\Pi \in \Gamma^{16}_{\mp}(+\frac{\hat{\Pi}}{2})} q^{\frac{1}{2}\Pi^2}\left\{\sum_{w^1 \in 2\mathbb{Z}\left(+\frac{1}{2}\right)}\ \sum_{n_1 \in 2\mathbb{Z}+1\left(+\frac{1}{2}\right)} e^{-\pi\tau_2\left\{\left(\frac{n_1}{R_1}\right)^2 + \left(w^1 R_1\right)^2\right\}} e^{2\pi i \tau_1 w^1 n_1}\right.\\ &\left. + \sum_{w^1 \in 2\mathbb{Z}+1\left(+\frac{1}{2}\right)}\ \sum_{n_1 \in 2\mathbb{Z}\left(+\frac{1}{2}\right)} e^{-\pi\tau_2\left\{\left(\frac{n_1}{R_1}\right)^2 + \left(w^1 R_1\right)^2\right\}} e^{2\pi i \tau_1 w^1 n_1}\right\}.\end{aligned} \tag{4.32}$$

For the untwisted sectors, the limit of $R_1 \to \infty$ makes the second and fourth terms of (4.32) vanish while the limit of $R_1 \to 0$ makes the second and third terms of (4.32) vanish. We can also find that the twisted sectors vanish in both endpoint limits.

In the endpoint limits, $Z_{\Gamma^{17,1}_{\pm}}$ and $Z_{\Gamma^{17,1}_{\pm}+\delta}$ behave as follows:

$$Z_{\Gamma^{17,1}_{\pm}} \xrightarrow{R_1\to\infty} \frac{R_1}{\sqrt{\tau_2}} (\eta\bar{\eta})^{-1} Z_{\Gamma^{16}}, \quad Z_{\Gamma^{17,1}_{\pm}+\delta} \xrightarrow{R_1\to\infty} 0, \tag{4.33}$$

$$Z_{\Gamma^{17,1}_{\pm}} \xrightarrow{R_1\to 0} \frac{1}{R_1\sqrt{\tau_2}} (\eta\bar{\eta})^{-1} Z_{\Gamma^{16}}, \quad Z_{\Gamma^{17,1}_{\pm}+\delta} \xrightarrow{R_1\to 0} 0. \tag{4.34}$$

In this class, supersymmetry can asymptotically be restored in both endpoints, though it is broken at finite values of R_1.

The non-supersymmetric models in class (2) and class (3) are so-called interpolating models since they interpolate between two different higher-dimensional vacua. These are originally constructed in [11, 12]. We find that $\hat{w}, \hat{n} = 1$ implies that there is a $\mathbb{Z}_2$ twist on a compactified circle: the limit of $R_1 \to \infty$ ($R_1 \to 0$) gives the restoration of supersymmetry if $\hat{w} = 1$ ($\hat{n} = 1$).

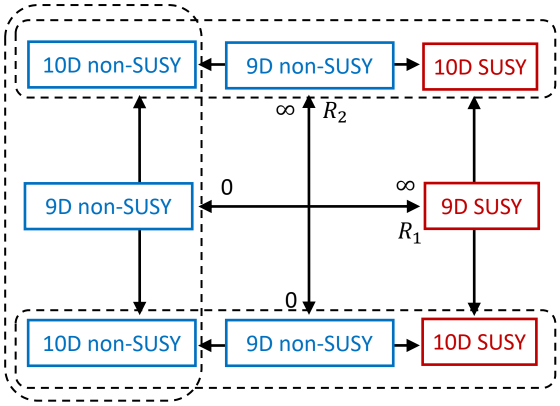
In preparation for the next subsubsection, we define $\Gamma^{17,1}_{\pm}|_{(k)}$ and $\Gamma^{17,1}_{\pm} + \delta|_{(k)}$ as the $\Gamma^{17,1}_{\pm}$ and $\Gamma^{17,1}_{\pm} + \delta$ in the class $(k) = (1), (2), (3), (4)$ by classifying $(\hat{w}, \hat{n})$, and $\Gamma^{17,1}$ is written as

$$\Gamma^{17,1} = \left\{ P = Z\mathcal{E} \,\middle|\, (\Pi, w, n) \in \left(\Gamma^{16}, \mathbb{Z}, \mathbb{Z}\right) \right\}. \tag{4.35}$$

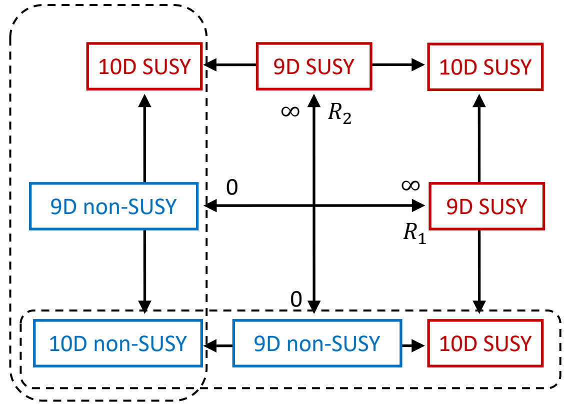
4.1.3 8D heterotic models

The 8-dimensional non-supersymmetric heterotic models are classified into $2^4 = 16$ classes. With $d = 2$, there are more patterns of interpolation (endpoint limits) than those in the $d = 1$ case, in the limits of compactified radii to zero and infinity. Here, we illustrate four examples of the endpoint limits of these non-supersymmetric heterotic models in Fig. 1 [1], and the complete classification is made in Appendix B. In each class, the interpolation patterns of the $d = 2$ case are identified as the combinations of those of $d = 1$, but we also find a non-trivial case.

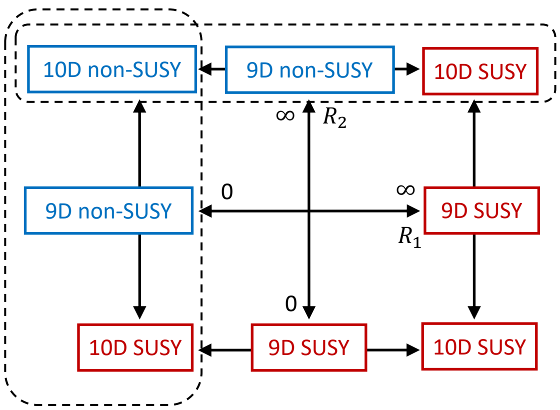
Let i_1 and i_2 be the numbers that label the 9D classes to which $(\hat{w}^1; \hat{n}_1)$ and $(\hat{w}^2; \hat{n}_2)$ belong, respectively. We label each 8-dimensional model with $[i_1 : i_2]$. We can take the sum on w^i, n_i for each i-th direction independently since we consider the background that the internal metric is diagonal.



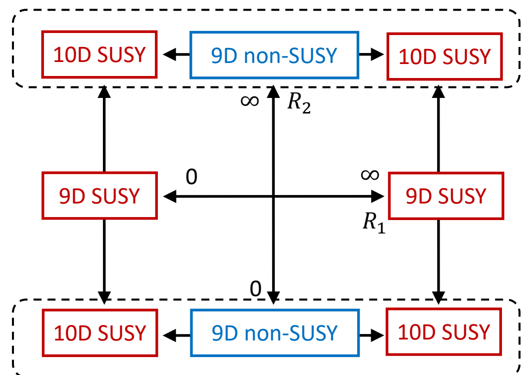
(a) $[2:1]$ $\hat{\Pi}^2 = 0 \pmod 4$, $(\hat{w};\hat{n}) = (1,0;0,0)$



(b) $[2:2]$ $\hat{\Pi}^2 = 0 \pmod 4$, $(\hat{w};\hat{n}) = (1,1;0,0)$



(c) $[2:3]$ $\hat{\Pi}^2 = 0 \pmod 4$, $(\hat{w};\hat{n}) = (1,0;0,1)$



(d) $[4:1]$ $\hat{\Pi}^2 = 2 \pmod 4$, $(\hat{w};\hat{n}) = (1,0;1,0)$

Fig. 1: The endpoints of the 8-dimensional non-supersymmetric heterotic models. The dotted lines refer to the 9D models classified into four ones.

[2:1] $\hat{\Pi}^2 = 0 \pmod 4$, $(\hat{w};\hat{n}) = (1,0;0,0)$:

In this class, the inner product (3.4) is written as

$$P \cdot \delta = \frac{1}{2}\left(\Pi \cdot \hat{\Pi} + n_1\right). \tag{4.36}$$

From this, $\Gamma_{\pm}^{18,2}$ and $\Gamma_{\pm}^{18,2} + \delta$ are written as

$$\begin{aligned}\Gamma_{\pm}^{18,2} = &\left\{P = Z\mathcal{E}\,\middle|\,(\Pi, w, n) \in \left(\Gamma_{\pm}^{16}, \mathbb{Z}^2, 2\mathbb{Z}\times\mathbb{Z}\right)\right\}\\ &\oplus \left\{P = Z\mathcal{E}\,\middle|\,(\Pi, w, n) \in \left(\Gamma_{\mp}^{16}, \mathbb{Z}^2, (2\mathbb{Z}+1)\times\mathbb{Z}\right)\right\},\end{aligned} \tag{4.37}$$

$$\Gamma^{18,2}_{\pm}+\delta=\left\{P=Z\mathcal{E}\,\middle|\,(\Pi,w,n)\in\left(\Gamma^{16}_{\pm}+\frac{\hat{\Pi}}{2},\left(\mathbb{Z}+\frac{1}{2}\right)\times\mathbb{Z},2\mathbb{Z}\times\mathbb{Z}\right)\right\}$$
$$\oplus\left\{P=Z\mathcal{E}\,\middle|\,(\Pi,w,n)\in\left(\Gamma^{16}_{\mp}+\frac{\hat{\Pi}}{2},\left(\mathbb{Z}+\frac{1}{2}\right)\times\mathbb{Z},(2\mathbb{Z}+1)\times\mathbb{Z}\right)\right\}. \tag{4.38}$$

There are four ten-dimensional endpoints obtained from $R_i\to\infty$ and $R_i\to 0$. The important point is that in the case of turning-off moduli $A=B=0$, the states with $w^i=0$ ($n_i=0$) only contribute to the partition function as $R_i\to\infty$ ($R_i\to 0$) for each i-th direction, as discussed before. From these $\Gamma^{18,2}_{\pm}$ and $\Gamma^{18,2}_{\pm}+\delta$, we find the behavior of $Z_{\Gamma^{18,2}_{\pm}}$ is

$$Z_{\Gamma^{18,2}_{\pm}}\xrightarrow{R_1\to\infty}\frac{R_1}{\sqrt{\tau_2}}\,(\eta\bar{\eta})^{-1}\,Z_{\Gamma^{17,1}}\xrightarrow{R_2\to\infty}\frac{R_1R_2}{\tau_2}\,(\eta\bar{\eta})^{-2}\,Z_{\Gamma^{16}},$$
$$Z_{\Gamma^{18,2}_{\pm}}\xrightarrow{R_2\to\infty}\frac{R_2}{\sqrt{\tau_2}}\,(\eta\bar{\eta})^{-1}\,Z_{\Gamma^{17,1}_{\pm}|_{(2)}}\xrightarrow{R_1\to\infty}\frac{R_1R_2}{\tau_2}\,(\eta\bar{\eta})^{-2}\,Z_{\Gamma^{16}},$$
$$Z_{\Gamma^{18,2}_{\pm}}\xrightarrow{R_1\to 0}\frac{1}{R_1\sqrt{\tau_2}}\,(\eta\bar{\eta})^{-1}\,Z_{\Gamma^{17,1}_{\pm}|_{(1)}}\xrightarrow{R_2\to 0}\frac{1}{R_1R_2\tau_2}\,(\eta\bar{\eta})^{-2}\,Z_{\Gamma^{16}_{\pm}},$$
$$Z_{\Gamma^{18,2}_{\pm}}\xrightarrow{R_2\to 0}\frac{1}{R_2\sqrt{\tau_2}}\,(\eta\bar{\eta})^{-1}\,Z_{\Gamma^{17,1}_{\pm}|_{(2)}}\xrightarrow{R_1\to 0}\frac{1}{R_1R_2\tau_2}\,(\eta\bar{\eta})^{-2}\,Z_{\Gamma^{16}_{\pm}},$$
$$Z_{\Gamma^{18,2}_{\pm}}\xrightarrow{R_1\to\infty,R_2\to 0}\frac{R_1}{R_2\tau_2}\,(\eta\bar{\eta})^{-2}\,Z_{\Gamma^{16}},$$
$$Z_{\Gamma^{18,2}_{\pm}}\xrightarrow{R_1\to 0,R_2\to\infty}\frac{R_2}{R_1\tau_2}\,(\eta\bar{\eta})^{-2}\,Z_{\Gamma^{16}_{\pm}},$$

while that of $Z_{\Gamma^{18,2}_{\pm}+\delta}$ is

$$Z_{\Gamma^{18,2}_{\pm}+\delta}\xrightarrow{R_1\to\infty}0,$$
$$Z_{\Gamma^{18,2}_{\pm}+\delta}\xrightarrow{R_2\to\infty}\frac{R_2}{\sqrt{\tau_2}}\,(\eta\bar{\eta})^{-1}\,Z_{\Gamma^{17,1}_{\pm}+\delta|_{(2)}}\xrightarrow{R_1\to\infty}0,$$
$$Z_{\Gamma^{18,2}_{\pm}+\delta}\xrightarrow{R_1\to 0}\frac{1}{R_1\sqrt{\tau_2}}\,(\eta\bar{\eta})^{-1}\,Z_{\Gamma^{17,1}_{\pm}+\delta|_{(1)}}\xrightarrow{R_2\to 0}\frac{1}{R_1R_2\tau_2}\,(\eta\bar{\eta})^{-2}\,Z_{\Gamma^{16}_{\pm}+\frac{\hat{\Pi}}{2}},$$
$$Z_{\Gamma^{18,2}_{\pm}+\delta}\xrightarrow{R_2\to 0}\frac{1}{R_2\sqrt{\tau_2}}\,(\eta\bar{\eta})^{-1}\,Z_{\Gamma^{17,1}_{\pm}+\delta|_{(2)}}\xrightarrow{R_1\to 0}\frac{1}{R_1R_2\tau_2}\,(\eta\bar{\eta})^{-2}\,Z_{\Gamma^{16}_{\pm}+\frac{\hat{\Pi}}{2}},$$
$$Z_{\Gamma^{18,2}_{\pm}+\delta}\xrightarrow{R_1\to 0,R_2\to\infty}\frac{R_2}{R_1\tau_2}\,(\eta\bar{\eta})^{-2}\,Z_{\Gamma^{16}_{\pm}+\frac{\hat{\Pi}}{2}}.$$

In the limit $R_1\to 0$, we can obtain the 9-dimensional non-supersymmetric model belonging to the class (1). Both $R_2\to\infty$ and $R_2\to 0$ limits give the 9-dimensional non-supersymmetric model belonging to the class (2), so-called interpolating model, in which a 10D superstring is obtained in $R_1 \to \infty$ while a 10-dimensional non-supersymmetric string in $R_1 \to 0$.

[2:2] $\hat{\Pi}^2 = 0 \pmod 4$, $(\hat{w}; \hat{n}) = (1,1;0,0)$:

In this class, the inner product (3.4) is written as

$$P \cdot \delta = \frac{1}{2}\left(\Pi \cdot \hat{\Pi} + n_1 + n_2\right). \tag{4.39}$$

Then $\Gamma^{18,2}_{\pm}$ and $\Gamma^{18,2}_{\pm} + \delta$ are written as

$$\begin{aligned}\Gamma^{18,2}_{\pm} = &\left\{P = Z\mathcal{E}\,\middle|\,(\Pi, w, n) \in \left(\Gamma^{16}_{\pm}, \mathbb{Z}^2, 2\mathbb{Z}\times 2\mathbb{Z}\right)\right\}\\ &\oplus \left\{P = Z\mathcal{E}\,\middle|\,(\Pi, w, n) \in \left(\Gamma^{16}_{\pm}, \mathbb{Z}^2, (2\mathbb{Z}+1)\times (2\mathbb{Z}+1)\right)\right\},\\ &\oplus \left\{P = Z\mathcal{E}\,\middle|\,(\Pi, w, n) \in \left(\Gamma^{16}_{\mp}, \mathbb{Z}^2, 2\mathbb{Z}\times (2\mathbb{Z}+1)\right)\right\},\\ &\oplus \left\{P = Z\mathcal{E}\,\middle|\,(\Pi, w, n) \in \left(\Gamma^{16}_{\mp}, \mathbb{Z}^2, (2\mathbb{Z}+1)\times 2\mathbb{Z}\right)\right\},\end{aligned} \tag{4.40}$$

$$\begin{aligned}\Gamma^{18,2}_{\pm} + \delta = &\left\{P = Z\mathcal{E}\,\middle|\,(\Pi, w, n) \in \left(\Gamma^{16}_{\pm} + \frac{\hat{\Pi}}{2}, \left(\mathbb{Z}+\frac{1}{2}\right)^2, 2\mathbb{Z}\times 2\mathbb{Z}\right)\right\}\\ &\oplus \left\{P = Z\mathcal{E}\,\middle|\,(\Pi, w, n) \in \left(\Gamma^{16}_{\pm} + \frac{\hat{\Pi}}{2}, \left(\mathbb{Z}+\frac{1}{2}\right)^2, (2\mathbb{Z}+1)\times (2\mathbb{Z}+1)\right)\right\},\\ &\oplus \left\{P = Z\mathcal{E}\,\middle|\,(\Pi, w, n) \in \left(\Gamma^{16}_{\mp} + \frac{\hat{\Pi}}{2}, \left(\mathbb{Z}+\frac{1}{2}\right)^2, 2\mathbb{Z}\times (2\mathbb{Z}+1)\right)\right\},\\ &\oplus \left\{P = Z\mathcal{E}\,\middle|\,(\Pi, w, n) \in \left(\Gamma^{16}_{\mp} + \frac{\hat{\Pi}}{2}, \left(\mathbb{Z}+\frac{1}{2}\right)^2, (2\mathbb{Z}+1)\times 2\mathbb{Z}\right)\right\},\end{aligned} \tag{4.41}$$

The behavior of $Z_{\Gamma^{18,2}_{\pm}}$ in the endpoint limits is given as

$$\begin{aligned}
Z_{\Gamma^{18,2}_{\pm}} &\xrightarrow{R_1\to\infty} \frac{R_1}{\sqrt{\tau_2}}\,(\eta\bar{\eta})^{-1}\,Z_{\Gamma^{17,1}} \xrightarrow{R_2\to\infty} \frac{R_1R_2}{\tau_2}\,(\eta\bar{\eta})^{-2}\,Z_{\Gamma^{16}},\\
Z_{\Gamma^{18,2}_{\pm}} &\xrightarrow{R_2\to\infty} \frac{R_2}{\sqrt{\tau_2}}\,(\eta\bar{\eta})^{-1}\,Z_{\Gamma^{17,1}} \xrightarrow{R_1\to\infty} \frac{R_1R_2}{\tau_2}\,(\eta\bar{\eta})^{-2}\,Z_{\Gamma^{16}},\\
Z_{\Gamma^{18,2}_{\pm}} &\xrightarrow{R_1\to 0} \frac{1}{R_1\sqrt{\tau_2}}\,(\eta\bar{\eta})^{-1}\,Z_{\Gamma^{17,1}_{\pm}|_{(2)}} \xrightarrow{R_2\to 0} \frac{1}{R_1R_2\tau_2}\,(\eta\bar{\eta})^{-2}\,Z_{\Gamma^{16}_{\pm}},\\
Z_{\Gamma^{18,2}_{\pm}} &\xrightarrow{R_2\to 0} \frac{1}{R_2\sqrt{\tau_2}}\,(\eta\bar{\eta})^{-1}\,Z_{\Gamma^{17,1}_{\pm}|_{(2)}} \xrightarrow{R_1\to 0} \frac{1}{R_1R_2\tau_2}\,(\eta\bar{\eta})^{-2}\,Z_{\Gamma^{16}},
\end{aligned}$$

$$Z_{\Gamma^{18,2}_{\pm}} \xrightarrow{R_1\to\infty, R_2\to 0} \frac{R_1}{R_2\tau_2}\left(\eta\bar{\eta}\right)^{-2} Z_{\Gamma^{16}},$$
$$Z_{\Gamma^{18,2}_{\pm}} \xrightarrow{R_1\to 0, R_2\to\infty} \frac{R_2}{R_1\tau_2}\left(\eta\bar{\eta}\right)^{-2} Z_{\Gamma^{16}},$$

and that of $Z_{\Gamma^{18,2}_{\pm}+\delta}$ as

$$Z_{\Gamma^{18,2}_{\pm}+\delta} \xrightarrow{R_1\to\infty} 0,$$
$$Z_{\Gamma^{18,2}_{\pm}+\delta} \xrightarrow{R_2\to\infty} 0,$$
$$Z_{\Gamma^{18,2}_{\pm}+\delta} \xrightarrow{R_1\to 0} \frac{1}{R_1\sqrt{\tau_2}}\left(\eta\bar{\eta}\right)^{-1} Z_{\Gamma^{17,1}_{\pm}+\delta|_{(2)}} \xrightarrow{R_2\to 0} \frac{1}{R_1R_2\tau_2}\left(\eta\bar{\eta}\right)^{-2} Z_{\Gamma^{16}_{\pm}+\frac{\hat{\Pi}}{2}},$$
$$Z_{\Gamma^{18,2}_{\pm}+\delta} \xrightarrow{R_2\to 0} \frac{1}{R_2\sqrt{\tau_2}}\left(\eta\bar{\eta}\right)^{-1} Z_{\Gamma^{17,1}_{\pm}+\delta|_{(2)}} \xrightarrow{R_1\to 0} \frac{1}{R_1R_2\tau_2}\left(\eta\bar{\eta}\right)^{-2} Z_{\Gamma^{16}_{\pm}+\frac{\hat{\Pi}}{2}},$$
$$Z_{\Gamma^{18,2}_{\pm}+\delta} \xrightarrow{R_1\to 0, R_2\to\infty} 0,$$
$$Z_{\Gamma^{18,2}_{\pm}+\delta} \xrightarrow{R_2\to 0, R_1\to\infty} 0.$$

We can find that the limits of $R_1 \to 0$ and $R_2 \to 0$ give the same 9-dimensional interpolating model belonging to the class (2). A 10-dimensional superstring model can be obtained in the limit of $R_1 \to \infty$ or $R_2 \to \infty$.

[2:3] $\hat{\Pi}^2 = 0 \pmod 4, \quad (\hat{w};\hat{n}) = (1,0;0,1)$:
In this class, the inner product (3.4) is written as

$$P\cdot\delta = \frac{1}{2}\left(\Pi\cdot\hat{\Pi} + w^2 + n_1\right). \tag{4.42}$$

Then $\Gamma^{18,2}_{\pm}$ and $\Gamma^{18,2}_{\pm}+\delta$ are written as

$$\begin{aligned}\Gamma^{18,2}_{\pm} = &\left\{P = Z\mathcal{E}\,\middle|(\Pi,w,n)\in\left(\Gamma^{16}_{\pm},\mathbb{Z}\times 2\mathbb{Z}, 2\mathbb{Z}\times\mathbb{Z}\right)\right\}\\ &\oplus\left\{P = Z\mathcal{E}\,\middle|(\Pi,w,n)\in\left(\Gamma^{16}_{\pm},\mathbb{Z}\times(2\mathbb{Z}+1),(2\mathbb{Z}+1)\times\mathbb{Z}\right)\right\}\\ &\oplus\left\{P = Z\mathcal{E}\,\middle|(\Pi,w,n)\in\left(\Gamma^{16}_{\mp},\mathbb{Z}\times 2\mathbb{Z},(2\mathbb{Z}+1)\times\mathbb{Z}\right)\right\}\\ &\oplus\left\{P = Z\mathcal{E}\,\middle|(\Pi,w,n)\in\left(\Gamma^{16}_{\mp},\mathbb{Z}\times(2\mathbb{Z}+1),2\mathbb{Z}\times\mathbb{Z}\right)\right\},\end{aligned} \tag{4.43}$$

$$
\begin{aligned}
\Gamma^{18,2}_{\pm}+\delta = &\left\{ p = Z\mathcal{E} \,\middle|\, (\Pi, w, n) \in \left(\Gamma^{16}_{\pm} + \frac{\hat{\Pi}}{2}, \left(\mathbb{Z}+\frac{1}{2}\right) \times 2\mathbb{Z}, 2\mathbb{Z} \times \left(\mathbb{Z}+\frac{1}{2}\right)\right) \right\} \\
&\oplus \left\{ P = Z\mathcal{E} \,\middle|\, (\Pi, w, n) \in \left(\Gamma^{16}_{\pm} + \frac{\hat{\Pi}}{2}, \left(\mathbb{Z}+\frac{1}{2}\right) \times (2\mathbb{Z}+1), (2\mathbb{Z}+1) \times \left(\mathbb{Z}+\frac{1}{2}\right)\right) \right\} \\
&\oplus \left\{ P = Z\mathcal{E} \,\middle|\, (\Pi, w, n) \in \left(\Gamma^{16}_{\mp} + \frac{\hat{\Pi}}{2}, \left(\mathbb{Z}+\frac{1}{2}\right) \times 2\mathbb{Z}, (2\mathbb{Z}+1) \times \left(\mathbb{Z}+\frac{1}{2}\right)\right) \right\} \\
&\oplus \left\{ P = Z\mathcal{E} \,\middle|\, (\Pi, w, n) \in \left(\Gamma^{16}_{\mp} + \frac{\hat{\Pi}}{2}, \left(\mathbb{Z}+\frac{1}{2}\right) \times (2\mathbb{Z}+1), 2\mathbb{Z} \times \left(\mathbb{Z}+\frac{1}{2}\right)\right) \right\}.
\end{aligned}
\tag{4.44}
$$

In the endpoint limits, $Z_{\Gamma^{18,2}_{\pm}}$ behave as

$$
\begin{aligned}
&Z_{\Gamma^{18,2}_{\pm}} \xrightarrow{R_1\to\infty} \frac{R_1}{\sqrt{\tau_2}}\,(\eta\bar{\eta})^{-1}\,Z_{\Gamma^{17,1}} \xrightarrow{R_2\to\infty} \frac{R_1R_2}{\tau_2}\,(\eta\bar{\eta})^{-2}\,Z_{\Gamma^{16}}, \\
&Z_{\Gamma^{18,2}_{\pm}} \xrightarrow{R_2\to\infty} \frac{R_2}{\sqrt{\tau_2}}\,(\eta\bar{\eta})^{-1}\,Z_{\Gamma^{17,1}_{\pm}|_{(2)}} \xrightarrow{R_1\to\infty} \frac{R_1R_2}{\tau_2}\,(\eta\bar{\eta})^{-2}\,Z_{\Gamma^{16}}, \\
&Z_{\Gamma^{18,2}_{\pm}} \xrightarrow{R_1\to 0} \frac{1}{R_1\sqrt{\tau_2}}\,(\eta\bar{\eta})^{-1}\,Z_{\Gamma^{17,1}_{\pm}|_{(3)}} \xrightarrow{R_2\to 0} \frac{1}{R_1R_2\tau_2}\,(\eta\bar{\eta})^{-2}\,Z_{\Gamma^{16}}, \\
&Z_{\Gamma^{18,2}_{\pm}} \xrightarrow{R_2\to 0} \frac{1}{R_2\sqrt{\tau_2}}\,(\eta\bar{\eta})^{-1}\,Z_{\Gamma^{17,1}} \xrightarrow{R_1\to 0} \frac{1}{R_1R_2\tau_2}\,(\eta\bar{\eta})^{-2}\,Z_{\Gamma^{16}}, \\
&Z_{\Gamma^{18,2}_{\pm}} \xrightarrow{R_1\to\infty, R_2\to 0} \frac{R_1}{R_2\tau_2}\,(\eta\bar{\eta})^{-2}\,Z_{\Gamma^{16}}, \\
&Z_{\Gamma^{18,2}_{\pm}} \xrightarrow{R_1\to 0, R_2\to\infty} \frac{R_2}{R_1\tau_2}\,(\eta\bar{\eta})^{-2}\,Z_{\Gamma^{16}_{\pm}},
\end{aligned}
$$

and $Z_{\Gamma^{18,2}_{\pm}+\delta}$ as

$$
\begin{aligned}
&Z_{\Gamma^{18,2}_{\pm}+\delta} \xrightarrow{R_1\to\infty} 0, \\
&Z_{\Gamma^{18,2}_{\pm}+\delta} \xrightarrow{R_2\to\infty} \frac{R_2}{\sqrt{\tau_2}}\,(\eta\bar{\eta})^{-1}\,Z_{\Gamma^{17,1}_{\pm}+\delta|_{(2)}} \xrightarrow{R_1\to\infty} 0, \\
&Z_{\Gamma^{18,2}_{\pm}+\delta} \xrightarrow{R_1\to 0} \frac{1}{R_1\sqrt{\tau_2}}\,(\eta\bar{\eta})^{-1}\,Z_{\Gamma^{17,1}_{\pm}+\delta|_{(3)}} \xrightarrow{R_2\to 0} 0, \\
&Z_{\Gamma^{18,2}_{\pm}+\delta} \xrightarrow{R_2\to 0} 0, \\
&Z_{\Gamma^{18,2}_{\pm}+\delta} \xrightarrow{R_1\to 0, R_2\to\infty} \frac{R_2}{R_1\tau_2}\,(\eta\bar{\eta})^{-2}\,Z_{\Gamma^{16}_{\pm}+\frac{\hat{\Pi}}{2}}.
\end{aligned}
$$

By $R_2 \to \infty$ limit, we obtain the 9-dimensional interpolating model belonging to the class (2), in which we get a 10-dimensional supersymmetric model in $R_1 \to \infty$ while a 10-dimensional non-supersymmetric model in $R_1 \to 0$. By the limit $R_1 \to 0$, we also obtain the 9-dimensional interpolating model belonging to the class (3), in which $R_2 \to \infty$ gives a 10-dimensional non-supersymmetric model while $R_2 \to 0$ gives a 10-dimensional superstring one.

[4:1] $\hat{\Pi}^2 = 2 \pmod 4, \quad (\hat{w};\hat{n}) = (1,0;1,0)$:

In this class, the inner product (3.4) is written as

$$P \cdot \delta = \frac{1}{2}\left(\Pi \cdot \hat{\Pi} + w^1 + n_1\right). \tag{4.45}$$

We can write $\Gamma^{18,2}_{\pm}$ and $\Gamma^{18,2}_{\pm} + \delta$ as

$$\begin{aligned}\Gamma^{18,2}_{\pm} = &\left\{P = Z\mathcal{E}\,\middle|(\Pi,w,n) \in \left(\Gamma^{16}_{\pm}, 2\mathbb{Z}\times\mathbb{Z}, 2\mathbb{Z}\times\mathbb{Z}\right)\right\}\\ &\oplus\left\{P = Z\mathcal{E}\,\middle|(\Pi,w,n) \in \left(\Gamma^{16}_{\pm}, (2\mathbb{Z}+1)\times\mathbb{Z}, (2\mathbb{Z}+1)\times\mathbb{Z}\right)\right\}\\ &\oplus\left\{P = Z\mathcal{E}\,\middle|(\Pi,w,n) \in \left(\Gamma^{16}_{\mp}, 2\mathbb{Z}\times\mathbb{Z}, (2\mathbb{Z}+1)\times\mathbb{Z}\right)\right\}\\ &\oplus\left\{P = Z\mathcal{E}\,\middle|(\Pi,w,n) \in \left(\Gamma^{16}_{\mp}, (2\mathbb{Z}+1)\times\mathbb{Z}, 2\mathbb{Z}\times\mathbb{Z}\right)\right\},\end{aligned} \tag{4.46}$$

$$\begin{aligned}\Gamma^{18,2}_{\pm} + \delta = &\left\{P = Z\mathcal{E}\,\middle|(\Pi,w,n) \in \left(\Gamma^{16}_{\pm} + \frac{\hat{\Pi}}{2}, \left(2\mathbb{Z}+\frac{1}{2}\right)\times\mathbb{Z}, \left(2\mathbb{Z}+\frac{1}{2}\right)\times\mathbb{Z}\right)\right\}\\ &\oplus\left\{P = Z\mathcal{E}\,\middle|(\Pi,w,n) \in \left(\Gamma^{16}_{\pm} + \frac{\hat{\Pi}}{2}, \left(2\mathbb{Z}-\frac{1}{2}\right)\times\mathbb{Z}, \left(2\mathbb{Z}-\frac{1}{2}\right)\times\mathbb{Z}\right)\right\}\\ &\oplus\left\{P = Z\mathcal{E}\,\middle|(\Pi,w,n) \in \left(\Gamma^{16}_{\mp} + \frac{\hat{\Pi}}{2}, \left(2\mathbb{Z}+\frac{1}{2}\right)\times\mathbb{Z}, \left(2\mathbb{Z}-\frac{1}{2}\right)\times\mathbb{Z}\right)\right\}\\ &\oplus\left\{P = Z\mathcal{E}\,\middle|(\Pi,w,n) \in \left(\Gamma^{16}_{\mp} + \frac{\hat{\Pi}}{2}, \left(2\mathbb{Z}-\frac{1}{2}\right)\times\mathbb{Z}, \left(2\mathbb{Z}+\frac{1}{2}\right)\times\mathbb{Z}\right)\right\}.\end{aligned} \tag{4.47}$$

Therefore the behavior of $Z_{\Gamma^{18,2}_{\pm}}$ is

$$\begin{aligned}Z_{\Gamma^{18,2}_{\pm}} &\xrightarrow{R_1\to\infty} \frac{R_1}{\sqrt{\tau_2}}\,(\eta\bar{\eta})^{-1}\,Z_{\Gamma^{17,1}} \xrightarrow{R_2\to\infty} \frac{R_1R_2}{\tau_2}\,(\eta\bar{\eta})^{-2}\,Z_{\Gamma^{16}},\\ Z_{\Gamma^{18,2}_{\pm}} &\xrightarrow{R_2\to\infty} \frac{R_2}{\sqrt{\tau_2}}\,(\eta\bar{\eta})^{-1}\,Z_{\Gamma^{17,1}_{\pm}|_{(4)}} \xrightarrow{R_1\to\infty} \frac{R_1R_2}{\tau_2}\,(\eta\bar{\eta})^{-2}\,Z_{\Gamma^{16}},\\ Z_{\Gamma^{18,2}_{\pm}} &\xrightarrow{R_1\to 0} \frac{1}{R_1\sqrt{\tau_2}}\,(\eta\bar{\eta})^{-1}\,Z_{\Gamma^{17,1}} \xrightarrow{R_2\to 0} \frac{1}{R_1R_2\tau_2}\,(\eta\bar{\eta})^{-2}\,Z_{\Gamma^{16}},\\ Z_{\Gamma^{18,2}_{\pm}} &\xrightarrow{R_2\to 0} \frac{1}{R_2\sqrt{\tau_2}}\,(\eta\bar{\eta})^{-1}\,Z_{\Gamma^{17,1}_{\pm}|_{(4)}} \xrightarrow{R_1\to 0} \frac{1}{R_1R_2\tau_2}\,(\eta\bar{\eta})^{-2}\,Z_{\Gamma^{16}},\end{aligned}$$

Limits of (R_1, R_2)	(∞,∞)	$(\infty,0)$	$(0,\infty)$	$(0,0)$
10D SUSY model	$\hat{w}^1+\hat{w}^2>0$	$\hat{w}^1+\hat{n}_2>0$	$\hat{n}_1+\hat{w}^2>0$	$\hat{n}_1+\hat{n}_2>0$
10D Non-SUSY model	$\hat{w}^1+\hat{w}^2=0$	$\hat{w}^1+\hat{n}_2=0$	$\hat{n}_1+\hat{w}^2=0$	$\hat{n}_1+\hat{n}_2=0$

Table 2: The conditions on $\hat{w}^i, \hat{n}_i$ $(i=1,2)$ which give the 10D (non-)supersymmetric endpoints.

$$
\begin{aligned}
Z_{\Gamma_\pm^{18,2}} &\xrightarrow{R_1\to\infty, R_2\to 0} \frac{R_1}{R_2\tau_2}\,(\eta\bar{\eta})^{-2}\, Z_{\Gamma^{16}},\\
Z_{\Gamma_\pm^{18,2}} &\xrightarrow{R_1\to 0, R_2\to\infty} \frac{R_2}{R_1\tau_2}\,(\eta\bar{\eta})^{-2}\, Z_{\Gamma^{16}},
\end{aligned}
$$

and that of $Z_{\Gamma_\pm^{18,2}+\delta}$ is

$$
\begin{aligned}
Z_{\Gamma_\pm^{18,2}+\delta} &\xrightarrow{R_1\to\infty} 0,\\
Z_{\Gamma_\pm^{18,2}+\delta} &\xrightarrow{R_2\to\infty} \frac{R_2}{\sqrt{\tau_2}}\,(\eta\bar{\eta})^{-1}\, Z_{\Gamma_\pm^{17,1}+\delta|_{(4)}} \xrightarrow{R_1\to\infty} 0,\\
Z_{\Gamma_\pm^{18,2}+\delta} &\xrightarrow{R_1\to 0} 0,\\
Z_{\Gamma_\pm^{18,2}+\delta} &\xrightarrow{R_2\to 0} \frac{1}{R_2\sqrt{\tau_2}}\,(\eta\bar{\eta})^{-1}\, Z_{\Gamma_\pm^{17,1}+\delta|_{(4)}} \xrightarrow{R_1\to\infty\ \text{or}\ 0} 0.
\end{aligned}
$$

We can find that the 9-dimensional supersymmetric models can be obtained in both $R_1\to\infty$ and $R_1\to 0$ limits, and both $R_2\to\infty$ and $R_2\to 0$ limits give the 9-dimensional non-supersymmetric model belonging to the class (4). In this class, the same four 10D endpoint models are obtained.

Note that we construct the model of this class by the condition, which can be recognized as the combination between 9D class (4) and (1), but there is no structure of 9D class (1), while the 9D model in class (4) can be obtained. This non-trivial phenomenon occurs because once the supersymmetry is restored by taking a limit along one direction, we cannot obtain the non-supersymmetric model in the limit of the other direction.

The 8-dimensional non-supersymmetric heterotic models with the moduli turned off are completely classified in Appendix B.

We would like to comment on the behavior in endpoint limits. We find that the 9-dimensional supersymmetric model can be obtained by the limit of $R_i\to\infty$ (0) if $\hat{w}^i$ $(\hat{n}_i)=1$ for $i=1,2$. This is the same behavior as in the $d=1$ case: the limit of $R_1\to\infty$ (0) gives the 10-dimensinal supersymmetric model with $\hat{w}^1$ $(\hat{n}_1) = 1$. In addition, the sum of $\hat{w}^i, \hat{n}_i$ determines whether the 10-dimensional endpoints are supersymmetric or not. We show the conditions to obtain the 10-dimensional (non-)supersymmetric models in Table 2. These discussions can be easily extended to the case of $d \geq 3$.

4.2 Case of turned-on moduli

In this subsection, we focus on the 9D non-supersymmetric heterotic models in class (1) and (2), with the non-zero Wilson line A. We can see the supersymmetry restoration at the specific value of A.

4.2.1 Non-supersymmetric heterotic model on S^1

Here we focus on the 9D $SO(16) \times SO(16)$ model obtained by S^1 compactification from (3.37). That is, consider the $d = 1$ case of the following 9D class (1) conditions:

$$\Pi \in \Gamma^{16}_{E_8 \times E_8}, \quad \delta_{16} \equiv \frac{\hat{\Pi}}{2} = (1, 0^7; 1, 0^7), \quad \hat{w} = \hat{n} = 0. \tag{4.48}$$

The partition function is expressed as

$$Z^{\not{SUSY}}_{(1,0^7;1,0^7;0;0)} = Z_B^{(7)} \left\{ \bar{V}_8 Z_{\Gamma^{17,1}_+} - \bar{S}_8 Z_{\Gamma^{17,1}_-} + \bar{O}_8 Z_{\Gamma^{17,1}_- + \delta} - \bar{C}_8 Z_{\Gamma^{17,1}_+ + \delta} \right\}, \tag{4.49}$$

where $\Gamma^{17,1}_\pm$ and $\Gamma^{17,1}_\pm + \delta$ is written as

$$\begin{aligned} \Gamma^{17,1}_\pm &= \left\{ P \in \Gamma^{17,1} \,\middle|\, (\Pi, w, n) \in \left(\Gamma^{16}_\pm, \mathbb{Z}, \mathbb{Z}\right) \right\}, \\ \Gamma^{17,1}_\pm + \delta &= \left\{ P \in \Gamma^{17,1} \,\middle|\, (\Pi, w, n) \in \left(\Gamma^{16}_\pm + \delta_{16}, \mathbb{Z}, \mathbb{Z}\right) \right\}. \end{aligned} \tag{4.50}$$

Here $\Gamma^{16}_\pm$ and $\Gamma^{16}_\pm + \delta_{16}$ are given in (3.34) and (3.35). The internal momenta (2.15) are expressed in $d = 1$ as

$$\ell^I_L = \Pi^I - w A^I, \tag{4.51a}$$

$$p_L = \frac{1}{\sqrt{2}R} \left(\Pi \cdot A + n + w \left(R^2 - \frac{1}{2}|A|^2 \right) \right), \tag{4.51b}$$

$$p_R = \frac{1}{\sqrt{2}R} \left(\Pi \cdot A + n - w \left(R^2 + \frac{1}{2}|A|^2 \right) \right). \tag{4.51c}$$

We can then write $q^{\frac{1}{2}P_L^2} \bar{q}^{\frac{1}{2}P_R^2}$ in $Z_{\Gamma^{17,1}_\pm (+\delta)}$ as

$$q^{\frac{1}{2}P_L^2} \bar{q}^{\frac{1}{2}P_R^2} = e^{-\pi\tau_2 \left(\ell_L^2 + p_L^2 + p_R^2\right)} e^{i\pi\tau_1 \left(\ell_L^2 + p_L^2 - p_R^2\right)}, \tag{4.52}$$

where

$$P_L^2+P_R^2=\sum_{I=1}^{16}\left(\Pi^I-wA^I\right)^2+\left\{\frac{1}{R^2}\left(\Pi\cdot A+n-\frac{1}{2}w|A|^2\right)^2+R^2w^2\right\}, \tag{4.53}$$

$$P_L^2-P_R^2=\sum_{I=1}^{16}\left(\Pi^I\right)^2+2wn. \tag{4.54}$$

In the following, we study the behavior of the partition function in endpoint limits.

- $R=\infty$ limit

  From (4.53), the limit of $R\to\infty$ behaves as follows:

$$e^{-\pi\tau_2\left(P_L^2+P_R^2\right)}\sim e^{-\pi\tau_2R^2w^2}, \tag{4.55}$$

  Thus, in $Z_{\Gamma_\pm^{17,1}}$ and $Z_{\Gamma_\pm^{17,1}+\delta}$, the contribution from the sum of $w\neq0$ vanishes exponentially, and the only contribution in this limit is when $w=0$. From (4.50), both $Z_{\Gamma_\pm^{17,1}}$ and $Z_{\Gamma_\pm^{17,1}+\delta}$ contain $w=0$, so in $R\to\infty$ we have

$$\begin{aligned} Z_{\Gamma_\pm^{17,1}(+\delta)} &\to \eta^{-17}\bar{\eta}^{-1}\sum_{\Pi\in\Gamma_\pm^{16}(+\delta_{16})}q^{\frac{1}{2}\Pi^2}\sum_{n\in\mathbb{Z}}e^{-\frac{\pi\tau_2}{R^2}(n+\Pi\cdot A)^2} \\ &\sim \eta^{-17}\bar{\eta}^{-1}\sum_{\Pi\in\Gamma_\pm^{16}(+\delta_{16})}q^{\frac{1}{2}\Pi^2}\int_{\infty}^{\infty}dx\ e^{-\frac{\pi\tau_2}{R^2}(x+\Pi\cdot A)^2} \\ &= \frac{R}{\sqrt{\tau_2}}\left(\eta\bar{\eta}\right)^{-1}Z_{\Gamma_\pm^{16}(+\delta_{16})}, \end{aligned} \tag{4.56}$$

  Therefore, in the limit of $R\to\infty$, the partition function (4.49) reads 10D $SO(16)\times SO(16)$ model (3.37).

- $R=0$ limit and SUSY restoration

  Let us see that the non-trivial phenomenon of supersymmetry restoration can occur in the limit of $R\to0$. From (4.53), in the limit of $R\to0$ we get

$$e^{-\pi\tau_2\left(P_L^2+P_R^2\right)}\sim e^{-\frac{\pi\tau_2}{R^2}\left(\Pi\cdot A+n-\frac{1}{2}w|A|^2\right)^2}, \tag{4.57}$$

  Thus, in $Z_{\Gamma_\pm^{17,1}}$ and $Z_{\Gamma_\pm^{17,1}+\delta}$, the contribution from the sum of $\Pi\cdot A+n-\frac{1}{2}w|A|^2\neq0$ vanishes exponentially. In this limit, the contribution comes only from the states satisfying

$$\Pi\cdot A+n-\frac{1}{2}w|A|^2=0. \tag{4.58}$$

The difference in the $R \to \infty$ case is that the terms contributing to $Z_{\Gamma_{\pm}^{17,1}}$ and $Z_{\Gamma_{\pm}^{17,1}+\delta}$ are dependent on the Wilson line A value. In the configurations of A such that there exists (Π, w, n) satisfying the condition (4.58), the behavior of $Z_{\Gamma_{\pm}^{17,1}(+\delta)}$ in the limit of $R \to 0$ as

$$
\begin{aligned}
Z_{\Gamma_{\pm}^{17,1}(+\delta)} &\to \eta^{-17}\bar{\eta}^{-1} \sum_{\Pi \in \Gamma_{\pm}^{16}(+\delta_{16})} \sum_{w,n\in\mathbb{Z}} e^{-\pi\tau_2\left[(\Pi-wA)^2+R^2w^2\right]} \, e^{i\pi\tau_1\left(\Pi^2+2nw\right)}\Big|_{\Pi\cdot A+n-\frac{1}{2}w|A|^2=0} \\
&= \eta^{-17}\bar{\eta}^{-1} \sum_{\Pi \in \Gamma_{\pm}^{16}(+\delta_{16})} \sum_{w\in\mathbb{Z}} e^{-\pi\tau_2\left[(\Pi-wA)^2+R^2w^2\right]} e^{i\pi\tau_1\left[\Pi^2+2w\left(\frac{1}{2}w|A|^2-\Pi\cdot A\right)\right]}\Big|_{\frac{1}{2}w|A|^2-\Pi\cdot A\in\mathbb{Z}} \\
&= \eta^{-17}\bar{\eta}^{-1} \sum_{w\in\mathbb{Z}} e^{-\pi\tau_2 R^2 w^2} \sum_{\Pi \in \Gamma_{\pm}^{16}(+\delta_{16})} q^{\frac{1}{2}(\Pi-wA)^2}\Big|_{(\Pi-wA)^2\in 2\mathbb{Z}} ,
\end{aligned}
\tag{4.59}
$$

Note that after the second line, we only sum over (Π, w) which satisfies $\frac{1}{2}w|A|^2 - \Pi\cdot A \in \mathbb{Z}$, that is, $(\Pi - wA)^2 \in 2\mathbb{Z}$. In the following, we consider the case where the Wilson line A takes particular configurations.

In the case of $A \in \Gamma^{16}$, the following equation is valid:

$$
\sum_{\Pi \in \Gamma_{\pm}^{16}(+\delta_{16})} q^{\frac{1}{2}(\Pi-wA)^2} = \sum_{\Pi \in \Gamma_{\pm}^{16}(+\delta_{16})} q^{\frac{1}{2}\Pi^2}. \tag{4.60}
$$

We then get the behavior of $Z_{\Gamma_{\pm}^{17,1}(+\delta)}$ as

$$
\begin{aligned}
Z_{\Gamma_{\pm}^{17,1}(+\delta)} &\to \eta^{-17}\bar{\eta}^{-1} \sum_{\Pi \in \Gamma_{\pm}^{16}(+\delta_{16})} q^{\frac{1}{2}\Pi^2} \sum_{w\in\mathbb{Z}} e^{-\pi\tau_2 R^2 w^2} \\
&\sim \eta^{-17}\bar{\eta}^{-1} \sum_{\Pi \in \Gamma_{\pm}^{16}(+\delta_{16})} q^{\frac{1}{2}\Pi^2} \int_{\infty}^{\infty} dx \; e^{-\pi\tau_2 R^2 x^2} \\
&= \frac{1}{R\sqrt{\tau_2}} \left(\eta\bar{\eta}\right)^{-1} Z_{\Gamma_{\pm}^{16}(+\delta_{16})}.
\end{aligned}
\tag{4.61}
$$

Therefore, the partition function (4.49) with $A \in \Gamma^{16}$, in the limit of $R \to 0$ becomes 10D $SO(16) \times SO(16)$model (3.37).

As a non-trivial case, let us consider the case $A = \delta_{16}$ $\left(2A = 2\delta_{16} \in \Gamma^{16}\right)$. In this case, the condition (4.58) is written as

$$
\Pi_1 + \Pi_9 + n - w = 0. \tag{4.62}
$$

Since $w, n \in \mathbb{Z}$ and from (3.33), we get $Z_{\Gamma_-^{17,1}}, Z_{\Gamma_-^{17,1}+\delta} \to 0$ in the limit of $R \to 0$. We also get the behavior of $Z_{\Gamma_+^{17,1}}$ as

$$\begin{aligned} Z_{\Gamma_+^{17,1}} &\to \eta^{-17}\bar{\eta}^{-1} \sum_{w \in \mathbb{Z}} e^{-\pi\tau_2 R^2 w^2} \sum_{\Pi \in \Gamma_+^{16}} q^{\frac{1}{2}(\Pi - w\delta_{16})^2} \\ &= \eta^{-17}\bar{\eta}^{-1} \left(\sum_{w \in 2\mathbb{Z}} e^{-\pi\tau_2 R^2 w^2} \sum_{\Pi \in \Gamma_+^{16}} q^{\frac{1}{2}\Pi^2} + \sum_{w \in 2\mathbb{Z}+1} e^{-\pi\tau_2 R^2 w^2} \sum_{\Pi \in \Gamma_+^{16}} q^{\frac{1}{2}(\Pi+\delta_{16})^2} \right) \\ &\sim \frac{1}{2R\sqrt{\tau_2}} (\eta\bar{\eta})^{-1} \left(Z_{\Gamma_+^{16}} + Z_{\Gamma_+^{16}+\delta_{16}} \right), \end{aligned} \tag{4.63}$$

and that of $Z_{\Gamma_+^{17,1}+\delta}$ as

$$\begin{aligned} Z_{\Gamma_+^{17,1}+\delta} &\to \eta^{-17}\bar{\eta}^{-1} \sum_{w \in \mathbb{Z}} e^{-\pi\tau_2 R^2 w^2} \sum_{\Pi \in \Gamma_+^{16}+\delta_{16}} q^{\frac{1}{2}(\Pi - w\delta_{16})^2} \\ &= \eta^{-17}\bar{\eta}^{-1} \left(\sum_{w \in 2\mathbb{Z}} e^{-\pi\tau_2 R^2 w^2} \sum_{\Pi \in \Gamma_+^{16}+\delta_{16}} q^{\frac{1}{2}\Pi^2} + \sum_{w \in 2\mathbb{Z}+1} e^{-\pi\tau_2 R^2 w^2} \sum_{\Pi \in \Gamma_+^{16}} q^{\frac{1}{2}\Pi^2} \right) \\ &\sim \frac{1}{2R\sqrt{\tau_2}} (\eta\bar{\eta})^{-1} \left(Z_{\Gamma_+^{16}+\delta_{16}} + Z_{\Gamma_+^{16}} \right), \end{aligned} \tag{4.64}$$

where we use the equation (4.60). From (3.40), we can find

$$Z_{\Gamma_+^{16}} + Z_{\Gamma_+^{16}+\delta_{16}} \Big|_{E_8 \times E_8} = Z_{\Gamma^{16}_{Spin(32)/\mathbb{Z}_2}}. \tag{4.65}$$

For $A = \delta_{16}$, the partition function (4.49) in the limit of $R \to 0$ is 10D $Spin(32)/\mathbb{Z}_2$ model. That is, supersymmetry is restored at the configuration of $A = \delta_{16}$ and $R = 0$.[5]

In the following, we investigate the T-dual region of $A = \delta_{16}, R = 0$. For radius R and Wilson A, the dual radius $\tilde{R}$ and dual Wilson line $\tilde{A}$ are given by

$$\tilde{R} = \frac{R}{R^2 + \frac{1}{2}|A|^2}, \qquad \tilde{A} = -\frac{A}{R^2 + \frac{1}{2}|A|^2}. \tag{4.66}$$

Equivalently we can obtain

$$R = \frac{\tilde{R}}{\tilde{R}^2 + \frac{1}{2}|\tilde{A}|^2}, \qquad A = -\frac{\tilde{A}}{\tilde{R}^2 + \frac{1}{2}|\tilde{A}|^2}. \tag{4.67}$$

[5]See also [9, 46].

Substituting these into (4.53), we can get

$$P_L^2 + P_R^2 = \sum_{I=1}^{16} \left(\Pi^I - n\tilde{A}^I\right)^2 + \left\{ \frac{1}{\tilde{R}^2} \left(\Pi \cdot \tilde{A} + w - \frac{1}{2} n |\tilde{A}|^2\right)^2 + \tilde{R}^2 n^2 \right\}. \tag{4.68}$$

Therefore, when we move to dual moduli, the positions of w and n are interchanged.

When the Wilson line $A = 0$, (4.66) becomes $\tilde{R} = 1/R$, so the tranformation $R \to \tilde{R}$ represents the usual T-dual transformation. When the Wilson line is fixed to a finite $A \neq 0$, the limit of $R \to \infty$ gives $\tilde{R} \to 0$, but the limit of $R \to 0$ also gives $\tilde{R} \to 0$. We can get $\tilde{R} = 1/R$ when considering limits on both R and A at the same time as

$$\frac{|A|^2}{R} \to 0. \tag{4.69}$$

Let us see whether supersymmetry can be restored in the T-dual region. The restoration of supersymmetry occurs when $R = 0,\ A = \delta_{16} = (1, 0^7; 1, 0^7)$, and the corresponding dual moduli are as follows from (4.66):

$$\tilde{R} = 0, \qquad \tilde{A} = -\delta_{16} = (-1, 0^7; -1, 0^7). \tag{4.70}$$

From this and (4.68), we can find that supersymmetry can be restored in the T-dual region, as discussed before.

The above discussions can be made for the other models than $SO(16) \times SO(16)$. However, we find that supersymmetry is not restored in those models even if $A = \delta_{16}$. The endpoint limits for the other models with moduli $A \neq 0$ are left for future work.

4.2.2 9D interpolating model

Next, we consider the interpolating model belonging to 9D class (2) obtained by twisted S^1 compactification from a superstring model. Here we focus on a 9D model that interpolates a 10D supersymmetric $E_8 \times E_8$ model and a 10D non-supersymmetric $SO(16) \times SO(16)$ model. In particular, we consider the case of $d = 1$ of the following conditions:

$$\Pi \in \Gamma^{16}_{E_8 \times E_8}, \quad \delta_{16} = \frac{\hat{\Pi}}{2} = (1, 0^7; 1, 0^7), \quad \hat{w} = 1,\ \hat{n} = 0. \tag{4.71}$$

In this case, $\delta^2 = \delta_{16}^2 + \hat{w}\hat{n}/2 = 2$ and then the partition function is

$$Z^{\cancel{SUSY}}_{(1,0^7;1,0^7;1;0)} = Z_B^{(7)} \left\{ \bar{V}_8 Z_{\Gamma_+^{17,1}} - \bar{S}_8 Z_{\Gamma_-^{17,1}} + \bar{O}_8 Z_{\Gamma_-^{17,1}+\delta} - \bar{C}_8 Z_{\Gamma_+^{17,1}+\delta} \right\}. \tag{4.72}$$

In the following, we study the behavior of the partition function in endpoint limits.

- $R \to \infty$ limit

  In the limit of $R \to \infty$, only $w = 0$ contributes to the partition function. However we find that $w \in \mathbb{Z} + \frac{1}{2}$ in $\Gamma_{\pm}^{17,1} + \delta$, so $Z_{\Gamma_{\pm}^{17,1}+\delta} \to 0$. The behavior of $Z_{\Gamma_{\pm}^{17,1}}$ is given as

$$
\begin{aligned}
Z_{\Gamma_{\pm}^{17,1}} &\to \eta^{-17}\bar{\eta}^{-1}\left\{\sum_{\Pi\in\Gamma_{\pm}^{16}} q^{\frac{1}{2}\Pi^2}\sum_{n\in 2\mathbb{Z}} e^{-\frac{\pi\tau_2}{R^2}(n+\Pi\cdot A)^2} + \sum_{\Pi\in\Gamma_{\mp}^{16}} q^{\frac{1}{2}\Pi^2}\sum_{n\in 2\mathbb{Z}+1} e^{-\frac{\pi\tau_2}{R^2}(n+\Pi\cdot A)^2}\right\} \\
&\sim \frac{R}{2\sqrt{\tau_2}}\eta^{-17}\bar{\eta}^{-1}\left\{\sum_{\Pi\in\Gamma_{\pm}^{16}} q^{\frac{1}{2}\Pi^2} + \sum_{\Pi\in\Gamma_{\mp}^{16}} q^{\frac{1}{2}\Pi^2}\right\} \\
&= \frac{R}{2\sqrt{\tau_2}}\left(\eta\bar{\eta}\right)^{-1} Z_{\Gamma^{16}},
\end{aligned}
\tag{4.73}
$$

  where $\Gamma^{16} = \Gamma^{16}_{E_8\times E_8}$. In the limit as $R \to \infty$, the partition function (4.72) becomes

$$
\begin{aligned}
Z^{\not{SUSY}}_{(1,0^7;1,0^7;1;0)} &= Z_B^{(7)}\left\{\bar{V}_8 Z_{\Gamma_{+}^{17,1}} - \bar{S}_8 Z_{\Gamma_{-}^{17,1}} + \bar{O}_8 Z_{\Gamma_{-}^{17,1}+\delta} - \bar{C}_8 Z_{\Gamma_{+}^{17,1}+\delta}\right\} \\
&\to \frac{R}{2} Z_B^{(8)}\left(\bar{V}_8 - \bar{S}_8\right) Z_{\Gamma^{16}_{E_8\times E_8}},
\end{aligned}
\tag{4.74}
$$

  Therefore, the limit of $R \to \infty$ gives 10D $E_8 \times E_8$ superstring model, and supersymmetry is restored at this limit. Note that in class (2) of the 9-dimensional model, the supersymmetric model can be obtained in the limit of $R \to \infty$ even when Wilson line A takes any value.

- $R \to 0$ limit

  In the limit of $R \to 0$, only Π, w, n that satisfy the condition (4.58) contribute to the partition function. From $\Gamma_{\pm}^{17,1}(+\delta)$ in the class (2) of 9D model, we get

$$
\begin{aligned}
&Z_{\Gamma_{\pm}^{17,1}(+\delta)} \\
&\to \eta^{-17}\bar{\eta}^{-1}\left\{\sum_{\Pi\in\Gamma_{\pm}^{16}(+\delta_{16})}\sum_{w\in\mathbb{Z}(+\frac{1}{2})}\sum_{n\in 2\mathbb{Z}} e^{-\pi\tau_2\left[(\Pi-wA)^2+R^2w^2\right]} e^{i\pi\tau_1\left(\Pi^2+2nw\right)}\right. \\
&\qquad \left.\left. + \sum_{\Pi\in\Gamma_{\mp}^{16}(+\delta_{16})}\sum_{w\in\mathbb{Z}(+\frac{1}{2})}\sum_{n\in 2\mathbb{Z}+1} e^{-\pi\tau_2\left[(\Pi-wA)^2+R^2w^2\right]} e^{i\pi\tau_1\left(\Pi^2+2nw\right)}\right\}\right|_{\Pi\cdot A+n-\frac{1}{2}w|A|^2=0}
\end{aligned}
$$

$$
\begin{aligned}
= \eta^{-17}\bar{\eta}^{-1} \Bigg\{ & \sum_{w\in\mathbb{Z}(+\frac{1}{2})} e^{-\pi\tau_2 R^2 w^2} \sum_{\Pi\in\Gamma^{16}_{\pm}(+\delta_{16})} q^{\frac{1}{2}(\Pi-wA)^2}\Bigg|_{\frac{1}{2}w|A|^2-\Pi\cdot A\in 2\mathbb{Z}} \\
& + \sum_{w\in\mathbb{Z}(+\frac{1}{2})} e^{-\pi\tau_2 R^2 w^2} \sum_{\Pi\in\Gamma^{16}_{\mp}(+\delta_{16})} q^{\frac{1}{2}(\Pi-wA)^2}\Bigg|_{\frac{1}{2}w|A|^2-\Pi\cdot A\in 2\mathbb{Z}+1} \Bigg\}. \qquad (4.75)
\end{aligned}
$$

In this case, we can check that supersymmetry is not restored even if $A = \delta_{16}$, as in the case of the other heterotic models than $SO(16)\times SO(16)$. It is left to future work to find a configuration $A \neq 0, R = 0$ in which supersymmetry is restored in the 9D model belonging to class (2).

5 Massless spectrum

In preparation for evaluating the cosmological constant in the next section 6, we investigate the spectra in the non-supersymmetric heterotic model d-dimensionally compactified. Since we will show that the leading behavior of cosmological constant is controlled by massless states and the contributions from twisted sectors are exponentially suppressed, we focus on the massless spectra in the untwisted sector in this section.

5.1 Untwisted sectors

The mass formulae for supersymmetric models are valid for the non-supersymmetric ones. Some of the massless states exist independent of the details of the moduli, while others depend. We will refer to the former sector 1 and the latter sector 2.

The left- and right-moving mass formulae for heterotic models in the untwisted sectors are expressed as

$$M_L^2 = \ell_L^2 + p_L^2 + 2(N_L - 1), \tag{5.1a}$$

$$M_R^2 = p_R^2 + 2(N_R - a_R), \tag{5.1b}$$

where $a_R = 0$ for R-sector and $a_R = 1/2$ for NS-sector. The physical states must satisfy the level-matching condition $M_L^2 = M_R^2$. There are two possibilities to obtain the massless states from (5.1). The first possibility is when the states satisfy the following condition

$$N_L = 1, \quad N_R = a_R, \quad \ell_L = p_L = p_R = 0. \tag{5.2}$$

We call the set of the massless states sector 1. We can find that from the internal momenta (2.15), the condition (5.2) reads

$$\Pi = w = n = 0, \tag{5.3}$$

which does not rely on the moduli. Thus, the massless states with (5.2) always exist at any point in the moduli space. These massless states consist of a gravity multiplet and gauge bosons transforming as $U(1)_L^{16} \times U(1)_l \times U(1)_r$. Their degrees of freedom are 8×8 and 8×16 respectively.

The other possibility is when the states satisfy the following condition

$$N_L = 0, \quad N_R = a_R, \quad \ell_L^2 + p_L^2 = 2, \quad p_R = 0. \tag{5.4}$$

We call the set of the massless states sector 2. In order to evaluate the cosmological constant in the next section, let us focus on the zero-winding massless states with $w^i = 0$. For such states, the massless condition can be expressed as

$$n_i = -\Pi \cdot A_i, \quad \Pi^2 = 2, \tag{5.5}$$

where the first condition holds for all $i = 1, \ldots, d$, and the second one is the condition for the nonzero roots of a subgroup g' of $g = SO(32)$ or $E_8 \times E_8$. Here, we define $\Delta_{g'}$ as a set of nonzero roots of g'. Δ_g is written as

$$\Delta_{SO(32)} = \left\{ \left(\underline{\pm, \pm, 0^{14}}\right) \right\}, \tag{5.6}$$

$$\begin{aligned}\Delta_{E_8\times E_8} = &\left\{ \left(\underline{\pm, \pm, 0^6}; 0^8\right), \frac{1}{2}\left(\underline{\pm, \pm, \pm, \pm, \pm, \pm, \pm, \pm}_{+}; 0^8\right) \right\} \\ &+ \left\{ \left(0^8; \underline{\pm, \pm, 0^6}\right), \frac{1}{2}\left(0^8; \underline{\pm, \pm, \pm, \pm, \pm, \pm, \pm, \pm}_{+}\right) \right\},\end{aligned} \tag{5.7}$$

where the underline indicates permutations of the components, and the subscript $+$ $(/-)$ denotes the number of $+$ is even (/odd).

We clarify the difference between the superstring theories on T^d and the non-supersymmetric models. In the superstring theories, $\Pi \in \Gamma^{16}$, $w \in \mathbb{Z}^d$ and $n \in \mathbb{Z}^d$ for both spacetime vectors and spinors. From the condition (5.5), we find that in toroidal models, a non-Abelian part of the gauge group is g' if the Wilson lines $A_{i(T)} = A_{i(T)}^{(g')}$ satisfy

$$\begin{cases} \Pi \cdot A_{i(T)}^{(g')} \in \mathbb{Z} & \text{for} \quad \Pi \in \Delta_{g'}, \\ \Pi \cdot A_{i(T)}^{(g')} \notin \mathbb{Z} & \text{for} \quad \Pi \in \Delta_g \backslash \Delta_{g'}, \end{cases} \tag{5.8}$$

where the subscript T implies the toroidal models.

In the non-supersymmetric models, however, it is more complicated to identify the massless states since the Narain lattice is split into $\Gamma_+^{17,1}$ where bosonic states live and $\Gamma_-^{17,1}$ where fermionic states live. First, we consider the sector 1. The momentum with (5.3) lives in $\Gamma_+^{17,1}$ independent of the choice of $\hat{Z}$. So, there are no fermionic massless states in sector 1. Below, we study the massless states satisfying (5.5) for sector 2 in the specific models.

5.2 Specific models and Wilson-line relations

In the following of this thesis, we focus on a specific non-supersymmetric heterotic model d-dimensinally compactified with $(\hat{w};\hat{n}) \in \mathbb{Z}^{2d}$ as follows:

$$(\hat{w}^{a_{(2)}}, \hat{n}_{a_{(2)}}) = (1,0), \tag{5.9a}$$

$$(\hat{w}^{a_{(4)}}, \hat{n}_{a_{(4)}}) = (1,1), \tag{5.9b}$$

$$(\hat{w}^{b_{(3)}}, \hat{n}_{b_{(3)}}) = (0,1), \tag{5.9c}$$

$$(\hat{w}^{b_{(1)}}, \hat{n}_{b_{(1)}}) = (0,0). \tag{5.9d}$$

Here $a_{(2)}, a_{(4)}, b_{(3)}, b_{(1)}$ are defined to run over as follows:

$$\begin{aligned} &a = 1,\dots,D; \quad && a_{(2)} = 1,\dots,D_2, && a_{(4)} = D_2+1,\dots,D, \\ &b = D+1,\dots,d; \quad && b_{(3)} = D+1,\dots,D+D_3, \quad && b_{(1)} = D+D_3+1,\dots,d, \end{aligned}$$

The subscripts $(1),\dots,(4)$ are the numbers of 9D heterotic classes. From (5.9), $(\hat{w};\hat{n})$ is explicitly written as

$$(\hat{w};\hat{n}) = (1^D, 0^{d-D}; 0^{D_2}, 1^{D+D_3-D_2}, 0^{d-D-D_3}). \tag{5.10}$$

In this model, the inner product $P \cdot \delta$ is expressed as

$$P \cdot \delta = \frac{1}{2}\left(\Pi \cdot \hat{\Pi} + \sum_{a=1}^{D} n_a + \sum_{i=D_2+1}^{D+D_3} w^i\right). \tag{5.11}$$

Then $\Gamma_{\pm}^{16+d,d}$ can be written as the following sets:

$$\begin{aligned} \Gamma_{\pm}^{16+d,d} = &\left\{ p = Z\mathcal{E} \,\middle|\, \left(\Pi, \sum_{i=D_2+1}^{D+D_3} w^i, \sum_{a=1}^{D} n_a\right) \in \left(\Gamma_{\pm}^{16}, 2\mathbb{Z}, 2\mathbb{Z}\right) \right\} \\ &\oplus \left\{ p = Z\mathcal{E} \,\middle|\, \left(\Pi, \sum_{i=D_2+1}^{D+D_3} w^i, \sum_{a=1}^{D} n_a\right) \in \left(\Gamma_{\pm}^{16}, 2\mathbb{Z}+1, 2\mathbb{Z}+1\right) \right\} \\ &\oplus \left\{ p = Z\mathcal{E} \,\middle|\, \left(\Pi, \sum_{i=D_2+1}^{D+D_3} w^i, \sum_{a=1}^{D} n_a\right) \in \left(\Gamma_{\mp}^{16}, 2\mathbb{Z}, 2\mathbb{Z}+1\right) \right\} \\ &\oplus \left\{ p = Z\mathcal{E} \,\middle|\, \left(\Pi, \sum_{i=D_2+1}^{D+D_3} w^i, \sum_{a=1}^{D} n_a\right) \in \left(\Gamma_{\mp}^{16}, 2\mathbb{Z}+1, 2\mathbb{Z}\right) \right\}. \end{aligned} \tag{5.12}$$

Note that n_a is even or odd, which implies that (5.5) leads to $\Pi \cdot A_a \in 2\mathbb{Z}$ or $2\mathbb{Z}+1$.

Let us study the massless condition (5.5) for sector 2 in this model. We define $\Delta_g^{\pm}$ as subsets of $\Gamma_{\pm}^{16}$ with $\Pi^2 = 2$. From the partition function (3.20), we can see that the massless vectors and spinors live in $\Gamma_{+}^{16+d,d}$ and $\Gamma_{-}^{16+d,d}$ respectively. Therefore the condition (5.5) implies that for massless vectors,

$$\sum_{a=1}^{D} \Pi \cdot A_a \in 2\mathbb{Z},\ \Pi \cdot A_b \in \mathbb{Z} \text{ for } \Pi \in \Delta_g^{+} \text{ and/or } \sum_{a=1}^{D} \Pi \cdot A_a \in 2\mathbb{Z}+1,\ \Pi \cdot A_b \in \mathbb{Z} \ \text{ for } \Pi \in \Delta_g^{-}, \tag{5.13}$$

while for massless spinors,

$$\sum_{a=1}^{D} \Pi \cdot A_a \in 2\mathbb{Z},\ \pi \cdot A_b \in \mathbb{Z} \text{ for } \Pi \in \Delta_g^{-} \text{ and/or } \sum_{a=1}^{D} \Pi \cdot A_a \in 2\mathbb{Z}+1,\ \Pi \cdot A_b \in \mathbb{Z} \ \text{ for } \Pi \in \Delta_g^{+}, \tag{5.14}$$

From the conditions (5.8) and (5.13), we find that $A_i^{(g')}$ which realize a gauge group g' in the non-supersymmetric models are expressed in terms of $A_{i(T)}^{(g')}$ in the toroidal models as follows:

$$\sum_{a=1}^{D} A_a^{(g')} = 2\sum_{a=1}^{D} A_{a(T)}^{(g')} + \hat{\Pi}, \quad A_b^{(g')} = A_{b(T)}^{(g')}. \tag{5.15}$$

This relation makes the conditions (5.13) for the massless vectors expressed in terms of $A_{i(T)}$ as

$$\sum_{a=1}^{D} \Pi \cdot A_{a(T)} \in \mathbb{Z},\ \Pi \cdot A_{b(T)} \in \mathbb{Z} \ \text{ for } \Pi \in \Delta_{g'}, \tag{5.16}$$

while the condition (5.14) for the massless spinors as

$$\sum_{a=1}^{D} \Pi \cdot A_{a(T)} \in \mathbb{Z} + \frac{1}{2},\ \Pi \cdot A_{b(T)} \in \mathbb{Z} \ \text{ for } \Pi \in \Delta_{g'}. \tag{5.17}$$

Since these two conditions do not depend on $\hat{\Pi}$, we can identify the massless spectra with $w = 0$ in the non-supersymmetric models without specifying the choice of $\hat{\Pi}$, using $A_{i(T)}$ but not A_i.

Although we can get the massless spectra with $w = 0$ by the conditions discussed above, we further discuss the massless conditions (5.13), (5.14) and (5.16), (5.17), and the Wilson-line relation (5.15) in order to evaluate the cosmological constant.

Recall that the massless condition (5.5) gives $\Pi \cdot A_a \in 2\mathbb{Z}$ or $2\mathbb{Z}+1$. In addtion to $\Pi \cdot A_b \in \mathbb{Z}$, we can write the condition for massless vectors (5.13) as

$$\sum_{a=1}^{D}(2n_a-1)(\Pi \cdot A_a) \in 2\mathbb{Z} \text{ for } \Pi \in \Delta_g^+ \text{ and/or } \sum_{a=1}^{D}(2n_a-1)(\pi \cdot A_a) \in 2\mathbb{Z}+1 \text{ for } \pi \in \Delta_g^-, \tag{5.18}$$

while for massless spinors (5.14) as

$$\sum_{a=1}^{D}(2n_a-1)(\Pi \cdot A_a) \in 2\mathbb{Z} \text{ for } \Pi \in \Delta_g^- \text{ and/or } \sum_{a=1}^{D}(2n_a-1)(\Pi \cdot A_a) \in 2\mathbb{Z}+1 \text{ for } \Pi \in \Delta_g^+. \tag{5.19}$$

We can also obtain the relation on the Wilson lines from (5.15) as follows:

$$\sum_{a=1}^{D}(2n_a-1)A_a = 2\sum_{a=1}^{D}(2n_a-1)A_{a(T)} + \hat{\Pi}, \quad A_b = A_{b(T)}. \tag{5.20}$$

From this relation, we can write the condition expressed with $A_{i(T)}$ for massless vectors as

$$\sum_{a=1}^{D}(2n_a-1)(\Pi \cdot A_{a(T)}) \in \mathbb{Z},\ \Pi \cdot A_{b(T)} \in \mathbb{Z} \ \text{ for } \Pi \in \Delta_{g'}, \tag{5.21}$$

while for the massless spinors, as

$$\sum_{a=1}^{D}(2n_a-1)(\Pi \cdot A_{a(T)}) \in \mathbb{Z}+\frac{1}{2},\ \Pi \cdot A_{b(T)} \in \mathbb{Z} \ \text{ for } \Pi \in \Delta_{g'}. \tag{5.22}$$

6 Cosmological constant

In this section, we evaluate the one-loop cosmological constant of non-supersymmetric heterotic models with general $\mathbb{Z}_2$ twists and show it is exponentially suppressed in the region where supersymmetry is asymptotically restored if there is a Bose-Fermi degeneracy in the massless level. We find that this holds generally even independently of the details of the models. We also show that the formula of the cosmological constant depends on only a 10D supersymmetric endpoint and find points in the Wilson-line moduli space where the cosmological constant is exponentially suppressed.

6.1 The formula of the cosmological constant

The one-loop cosmological constant (vacuum energy density) is defined as the integral of the partition function:

$$\Lambda^{(10-d)} = -\frac{1}{2}(2\pi\sqrt{\alpha'})^{-(10-d)} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z^{\not{SUSY}}_{(\hat{Z})}, \tag{6.1}$$

where $\mathcal{F}$ is a fundamental domain of the modular group:

$$\mathcal{F} = \left\{ \tau = \tau_1 + i\tau_2 \in \mathbb{C} \,\middle|\, -\frac{1}{2} \leq \tau_1 \leq \frac{1}{2},\ |\tau| \geq 1 \right\}. \tag{6.2}$$

Let us evaluate the one-loop cosmological constant of the models with (5.10) and $D \geq 1$ constructed in section 5, in the region that all $R_i \gg 1$ $(i = 1, \ldots, d)$ where supersymmetry is asymptotically restored. In this region, only zero-winding states with $w^i = 0$ make the contribution to the cosmological constant. The contributions from the states in the twisted sectors are exponentially suppressed.

Using Jacobi's abstruse identity $V_8 - S_8 = 0$, the partition function of the non-supersymmetric heterotic model (3.20) can be expressed as

$$Z^{\not{SUSY}}_{(\hat{Z})}(R; A) \sim Z_B^{(8-d)} \bar{V}_8 \left\{ Z_{\Gamma_+^{16+d,d}}(R; A) - Z_{\Gamma_-^{16+d,d}}(R; A) \right\}, \tag{6.3}$$

where $(R; A)$ denotes two types of the moduli, radii of the internal circles and Wilson lines: $(R_1, \ldots, R_d; A_1, \ldots, A_d)$[6]. Here we define the internal part of the partition function I as

[6] The B-dependence of the partition function vanish in these limits since B only couples to the winding number w in the momentum P (2.15).

$I = Z_{\Gamma_+^{16+d,d}}(R;A) - Z_{\Gamma_-^{16+d,d}}(R;A)$. Then I can be expressed as follows:

$$I \sim \eta^{-16}(\eta\bar{\eta})^{-d} \sum_{\epsilon=\pm} \epsilon \sum_{\Pi \in \Gamma_\epsilon^{16}} q^{\frac{\Pi^2}{2}} \prod_{a=1}^{D} \left(\sum_{n_a \in 2\mathbb{Z}} - \sum_{n_a \in 2\mathbb{Z}+1} \right) e^{-\frac{\pi\tau_2}{R_a^2}(n_a + \Pi\cdot A_a)^2} \prod_{b=D+1}^{d} \sum_{n_b \in \mathbb{Z}} e^{-\frac{\pi\tau_2}{R_b^2}(n_b+\Pi\cdot A_b)^2}.$$

Using Poisson's resummation formula,

$$\left(\sum_{n_a \in 2\mathbb{Z}} - \sum_{n_a \in 2\mathbb{Z}+1} \right) e^{-\frac{\pi\tau_2}{R_a^2}(n_a+\Pi\cdot A_a)^2} = \frac{R_a}{\sqrt{\tau_2}} \sum_{n_a \in \mathbb{Z}} e^{-\frac{\pi R_a^2}{4\tau_2}(2n_a-1)^2} e^{\pi i (2n_a-1)(\Pi\cdot A_a)}.$$

$$\sum_{n_b \in \mathbb{Z}} e^{-\frac{\pi\tau_2}{R_b^2}(n_b+\Pi\cdot A_b)^2} = \frac{R_b}{\sqrt{\tau_2}} \sum_{n_b\in\mathbb{Z}} e^{-\frac{\pi R_b^2}{\tau_2} n_b^2} e^{2\pi i n_b (\Pi\cdot A_b)}.$$

Then, the partition function is expressed as

$$\begin{aligned} Z_{(\hat{Z})}^{\cancel{SUSY}}(R;A) \sim \prod_{i=1}^{d} R_i \times \frac{1}{\tau_2^4} \eta^{-24} \bar{\eta}^{-8} \bar{V}_8 \sum_{\epsilon=\pm} \epsilon \sum_{\Pi\in\Gamma_\epsilon^{16}} q^{\frac{\Pi^2}{2}} \\ \times \prod_{i=1}^{d} \sum_{n_i=-\infty}^{\infty} \exp\left[\pi i \left\{ \sum_{a=1}^{D} (2n_a-1)(\Pi\cdot A_a) + \sum_{b=D+1}^{d} (2n_b)(\Pi\cdot A_b) \right\} \right] \\ \times \exp\left[-\frac{\pi}{4\tau_2} \left\{ \sum_{a=1}^{D} (2n_a-1)^2 R_a^2 + \sum_{b=D+1}^{d} (2n_b)^2 R_b^2 \right\} \right]. \end{aligned} \tag{6.4}$$

Decomposing the integral domain $\mathcal{F}$ into $\mathcal{F}_{<1}$ and $\mathcal{F}_{\geq 1}$ above and below $\tau_2 = 1$, we can show that since $\mathcal{F}_{<1}$ is a finite domain and contains no singular points, the contribution from this integral is exponentially suppressed. Therefore, the leading term is obtained by integrating (6.4) in the region of $\mathcal{F}_{\geq 1}$. Furthermore, expanding η^{-24} and $\bar{\eta}^{-8}\bar{V}_8$ by q and $\bar{q}$ respectively

$$\eta^{-24} = q^{-1} + 24 + \mathcal{O}(q), \tag{6.5a}$$

$$\bar{\eta}^{-8}\bar{V}_8 = 8 + \mathcal{O}(\bar{q}), \tag{6.5b}$$

we can then rewrite (6.4) as

$$\begin{aligned} \left(\prod_{i=1}^{d} R_i \right) \tau_2^{-4} \times 8 \sum_{\epsilon=\pm} \epsilon \sum_{\Pi\in\Gamma_\epsilon^{16}} \left(24 q^{\frac{1}{2}\Pi^2} + q^{\frac{1}{2}\Pi^2 - 1} + \mathcal{O}(q,\bar{q}) \right) \\ \times \prod_{i=1}^{d} \sum_{n_i=-\infty}^{\infty} \exp\left[\pi i \left\{ \sum_{a=1}^{D} (2n_a-1)\left(\Pi\cdot A_a\right) + \sum_{b=D+1}^{d} (2n_b)\left(\Pi\cdot A_b\right) \right\} \right] \\ \times \exp\left[-\frac{\pi}{4\tau_2} \left\{ \sum_{a=1}^{D} (2n_a-1)^2 R_a^2 + \sum_{b=D+1}^{d} (2n_b)^2 R_b^2 \right\} \right]. \end{aligned} \tag{6.6}$$

Integrating this in the $\mathcal{F}_{\geq 1}$ region, we can show that only the part of $(q\bar{q})^0$ in the $q, \bar{q}$ expansion terms, that is, the massless states, contribute to the leading term[7]. From the above equation, $(q\bar{q})^0$ is obtained when $\Pi = 0$ and $\Pi^2 = 2$, which correspond to sector 1 and sector 2 discussed in Section 5, respectively. Therefore, the leading behavior of the cosmological constant is expressed as

$$\Lambda^{(10-d)} \sim -\frac{1}{2}(2\pi\sqrt{\alpha'})^{-(10-d)} \left(\prod_{i=1}^{d} R_i\right) \prod_{i=1}^{d} \sum_{n_i=-\infty}^{\infty} \times 8 \left\{ 24 + \sum_{\epsilon=\pm} \sum_{\Pi \in \Delta_g^\epsilon} \epsilon \exp\left[\pi i \left\{ \sum_{a=1}^{D} (2n_a - 1)(\Pi \cdot A_a) + \sum_{b=D+1}^{d} (2n_b)(\Pi \cdot A_b) \right\}\right]\right\} \times \int_1^\infty \frac{d\tau_2}{\tau_2^6} \exp\left[-\frac{\pi}{4\tau_2}\left\{ \sum_{a=1}^{D} (2n_a-1)^2 R_a^2 + \sum_{b=D+1}^{d} (2n_b)^2 R_b^2 \right\}\right].$$

Here, the following formula can be used for the integral of τ_2:

$$\int_1^\infty \frac{dx}{x^d}\, e^{-\frac{a}{x}} = \frac{1}{a^{d-1}}\Gamma(d-1) - \frac{1}{a^{d-1}}\Gamma(d-1, a). \tag{6.7}$$

where $\Gamma(x)$ is the gamma function and $\Gamma(r, a)$ is the incomplete gamma function of the second kind. Let us consider that r, a are both positive real numbers. Using the property $\Gamma(r + 1, a) = r\Gamma(r, a) + a^r e^{-a}$, then we get

$$\Gamma(r, a) = a^{r-1} e^{-a} \left(1 + \sum_{k=1}^{r-1} \frac{(r-1)(r-2)\cdots(r-k)}{a^k}\right). \tag{6.8}$$

Therefore, we can obtain

$$\int_1^\infty \frac{dx}{x^d}\, e^{-\frac{a}{x}} = \frac{1}{a^{d-1}}\Gamma(d-1) - \frac{e^{-a}}{a} \left(1 + \sum_{k=1}^{d-2} \frac{(d-2)(d-3)\cdots(d-k-1)}{a^k}\right). \tag{6.9}$$

Now a is large enough so that the second term in the right-hand side of (6.7) is exponentially suppressed. Therefore, the cosmological constant can be obtained up to exponential

[7] More precisely when the integration in the τ_1 direction is performed, the contribution from states satisfying the level-matching(-like) condition is 0 except for $(q\bar{q})^N$. Furthermore, integration in the τ_2 direction suppresses exponentially the contribution of the massive states $N \neq 0$.

suppressed terms as follows:

$$\Lambda^{(10-d)} \sim -\frac{4!\cdot 2^{d-1}}{\pi^{15-d}(\sqrt{\alpha'})^{10-d}}\left(\prod_{i=1}^{d}R_i\right)\sum_{\boldsymbol{n}}\left\{\sum_{a=1}^{D}(2n_a-1)^2R_a^2+\sum_{b=D+1}^{d}(2n_b)^2R_b^2\right\}^{-5}$$
$$\times 8\left(24+\sum_{\epsilon=\pm}\sum_{\Pi\in\Delta_g^{\epsilon}}\epsilon\exp\left[\pi i\left\{\sum_{a=1}^{D}(2n_a-1)(\Pi\cdot A_a)+\sum_{b=D+1}^{d}(2n_b)(\Pi\cdot A_b)\right\}\right]\right), \tag{6.10}$$

where we define $\sum_{\boldsymbol{n}}=\prod_{i=1}^{d}\sum_{n_i=-\infty}^{\infty}$. Here 8×24 comes from $\Pi=0$, which corresponds to the massless states in sector 1, and the second term in the parentheses corresponds to those in sector 2.

In the following, we show the cosmological constant (6.10) is proportional to n_F-n_B, where n_B and n_F are the degrees of freedom of massless bosons and fermions respectively. Let us assume that A_i satisfies $\Pi\cdot A_i\in\mathbb{Z}$ for all Π. We can then find that

$$\epsilon\exp\left[\pi i\left\{\sum_{a=1}^{D}(2n_a-1)(\Pi\cdot A_a)+\sum_{b=D+1}^{d}(2n_b)(\Pi\cdot A_b)\right\}\right]=\begin{cases}+1 & \text{for } A_i \text{ with (5.18)}\\ -1 & \text{for } A_i \text{ with (5.19)}\end{cases}.$$

As studied in the previous section, since the conditions for massless vectors and spinors in sector 2 are given by (5.18) and (5.19), respectively, we can find that this factor assigns $+1$ to massless vectors and -1 to massless spinors. Hence, we can obtain the formula of the cosmological constant, including the contribution 8×24 from sector 1, up to exponential suppressed terms as follows:

$$\Lambda^{(10-d)} \sim \frac{4!\cdot 2^{d-1}}{\pi^{15-d}(\sqrt{\alpha'})^{10-d}}(n_F-n_B)\left(\prod_{i=1}^{d}R_i\right)\sum_{\boldsymbol{n}}\left\{\sum_{a=1}^{D}(2n_a-1)^2R_a^2+\sum_{b=D+1}^{d}(2n_b)^2R_b^2\right\}^{-5}. \tag{6.11}$$

This equation (6.11) shows that if $n_F=n_B$, which implies that the Bose-Fermi degeneracy occurs at a massless level, the cosmological constant is exponentially suppressed in the region where supersymmetry is asymptotically restored. This is a generalized expression for a d-dimensional compactified model with a general $\mathbb{Z}_2$ twist.

The remarkable point is that the cosmological constant (6.10) is independent of $\hat{\Pi}$. We can find

$$\exp\left[\pi i(\Pi\cdot\hat{\Pi})\right]=\begin{cases}+1 & \text{for } \Pi\in\Delta_g^{+}\\ -1 & \text{for } \Pi\in\Delta_g^{-}\end{cases},$$

then using the relation on Wilson line (5.20), the cosmological constant (6.10) can be expressed as follows:

$$\Lambda^{(10-d)} \sim -\frac{4!\cdot 2^{d-1}}{\pi^{15-d}(\sqrt{\alpha'})^{10-d}}\left(\prod_{i=1}^{d} R_i\right)\sum_{\boldsymbol{n}}\left\{\sum_{a=1}^{D}(2n_a-1)^2R_a^2+\sum_{b=D+1}^{d}(2n_b)^2R_b^2\right\}^{-5}$$
$$\times 8\left(24+\sum_{\Pi\in\Delta_g}\exp\left[2\pi i\left\{\sum_{a=1}^{D}(2n_a-1)(\Pi\cdot A_{a(T)})+\sum_{b=D+1}^{d}n_b(\Pi\cdot A_{b(T)})\right\}\right]\right). \tag{6.12}$$

Note that the sum on Π runs over Δ_g, not $\Delta_g^{\pm}$. Thus, the formula (6.12) depends only on the endpoint given by $R_i\to\infty$ for all i. In the next section, we use the expression (6.12) to analyze the Wilson-line moduli stability.

Finally, we further comment on the formula of cosmological constant. The formula (6.11) can be obtained from the expression (6.12) using the conditons (5.21) and (5.22). Since $A_{i(T)}$ with $\sum_a \Pi\cdot A_{a(T)}\in\mathbb{Z}$ and $\mathbb{Z}+1/2$ give the contributions of $+1$ and -1 respectively, (6.11) is valid for $A_{i(T)}$ which satisfy

$$\Pi\cdot 2A_{a(T)}\in\mathbb{Z}\ \text{ and }\ \Pi\cdot A_{b(T)}\in\mathbb{Z}\quad\text{for all }\Pi. \tag{6.13}$$

6.2 Exponential suppression with Wilson line

Using the formula (6.12), we study the massless spectra with $n_F=n_B$, which gives the exponential suppression of the cosmological constant. Let us focus on the Wilson lines $A_{i(T)}$ satisfying $\Pi\cdot 2A_{a(T)}\in\mathbb{Z}$ and $\Pi\cdot A_{b(T)}\in\mathbb{Z}$ for all Π in order to use the formula (6.11) .

For convenience, we define $\Delta_{g'}^{(B)}$ and $\Delta_{g'}^{(F)}$ in terms of (5.16) and (5.17) as follows:

$$\Delta_{g'}^{(B)}=\left\{\Pi\in\Delta_g\ \middle|\ \sum_{a=1}^{D}\Pi\cdot A_{a(T)}\in\mathbb{Z}\ \text{ and }\ \Pi\cdot A_{b(T)}\in\mathbb{Z}\right\}, \tag{6.14a}$$
$$\Delta_{g'}^{(F)}=\left\{\Pi\in\Delta_g\ \middle|\ \sum_{a=1}^{D}\Pi\cdot A_{a(T)}\in\mathbb{Z}+\frac{1}{2}\ \text{ and }\ \Pi\cdot A_{b(T)}\in\mathbb{Z}\right\}. \tag{6.14b}$$

One can notice that $n_f=n_B$ is realized if the Wilson lines $A_{i(T)}$ give the massless states which satisfy the following condition:

$$\left|\Delta_{g'}^{(F)}\right|-\left|\Delta_{g'}^{(B)}\right|=24, \tag{6.15}$$

where $|\Delta|$ implies the number of elements belonging to a set Δ, and 24 comes from the left-moving massless states in sector 1.

In the following, we consider two types of Δ_g.

6.2.1 $Spin(32)/\mathbb{Z}_2$ supersymmetric endpoint model

In this model, $\Delta_g = \Delta_{SO(32)}$ given by (5.6). Here we consider the following simplest configurations of Wilson lines $A_{i(T)}$:

$$A_{a(T)} = \left(0^p, \left(\frac{1}{2}\right)^q\right) \quad (p+q=16)\,, \quad A_{b(T)} = \left(0^{16}\right), \tag{6.16}$$

which implies that the Wilson lines $A_{a(T)}$ $(a = 1, \ldots, D)$ are the same configuration and $A_{b(T)}$ $(b = D+1, \ldots, d)$ is taken to be 0. We can then rewrite this configuration as

$$\sum_{a=1}^{D} A_{a(T)} = \left(0^p, \left(\frac{D}{2}\right)^q\right) \quad (p+q=16)\,, \quad A_{b(T)} = \left(0^{16}\right). \tag{6.17}$$

Note that (6.16) satisfies the condition (6.13) for all $\Pi \in \Delta_{SO(32)}$. In the following, we consider the two cases according to whether D is even or odd.

(I) $D \in 2\mathbb{Z}$

From (5.6), $\Delta_{g'}^{(B)} = \Delta_{SO(32)}$ and $\Delta_{g'}^{(F)}$ is empty. Therefore we get

$$\left|\Delta_{g'}^{(F)}\right| - \left|\Delta_{g'}^{(B)}\right| = -480, \tag{6.18}$$

and we find that there is no solution of (6.15) in $D \in 2\mathbb{Z}$ case.

(II) $D \in 2\mathbb{Z}+1$

In this case, we can write (6.14) as follows:

$$\Delta_{g'}^{(B)} = \left\{\left(\underline{\pm, \pm, 0^{p-2}}, 0^q\right), \left(0^p, \underline{\pm, \pm, 0^{q-2}}\right)\right\}, \tag{6.19a}$$

$$\Delta_{g'}^{(F)} = \left\{\left(\underline{\pm, 0^{p-1}}, \underline{\pm, 0^{q-1}}\right)\right\}. \tag{6.19b}$$

We can find that there are massless gauge bosons transforming in the adjoint representation of $SO(2p) \times SO(2q)$ and massless spinor transforming in $(\boldsymbol{2p}, \boldsymbol{2q})$ of $SO(2p) \times SO(2q)$. We can then count the number of elements in $\Delta_{g'}^{(B)}$ and $\Delta_{g'}^{(F)}$ as follows:

$$\left|\Delta_{g'}^{(F)}\right| - \left|\Delta_{g'}^{(B)}\right| = 4pq - \left\{2p(p-1) + 2q(q-1)\right\}. \tag{6.20}$$

Using $p+q=16$, the solutions of (6.15) are $(p,q)=\left(\underline{7,9}\right)$. Therefore, the cosmological constant is exponentially suppressed when the gauge symmetry is $SO(18)\times SO(14)$ under the configurations of Wilson lines (6.16) in $D\in 2\mathbb{Z}+1$ case.

6.2.2 $E_8\times E_8$ supersymmetric endpoint model

In this model, $\Delta_g=\Delta_{E_8\times E_8}$ given by (5.7). $\Delta_{E_8\times E_8}$ can be decomposed into two copies of Δ_{E_8} as $\Delta_{E_8\times E_8}=\Delta_{E_8}\oplus\Delta_{E_8}$, and it is known that Δ_{E_8} is further decomposed as

$$\Delta_{E_8}=\Delta_{SO(16)}\oplus\Delta_{\mathbf{128}_+}, \tag{6.21}$$

where $\Delta_{SO(16)}$ and $\Delta_{\mathbf{128}_+}$ are defined as

$$\Delta_{SO(16)}=\left\{(\underline{\pm,\pm,0^6})\right\},\qquad \Delta_{\mathbf{128}_+}=\left\{\frac{1}{2}(\underline{\pm,\pm,\pm,\pm,\pm,\pm,\pm,\pm}_+)\right\}. \tag{6.22}$$

We also express $\Delta^{(B)}_{g'}$ and $\Delta^{(F)}_{g'}$ as the direct sums of two sets as follows:

$$\Delta^{(B)}_{g'}=\Delta^{(B)}_{g'_1}+\Delta^{(B)}_{g'_2},\quad \Delta^{(F)}_{g'}=\Delta^{(F)}_{g'_1}+\Delta^{(F)}_{g'_2}, \tag{6.23}$$

where we define $\Delta^{(B)}_{g'_m}$ and $\Delta^{(F)}_{g'_m}$ for $\Pi=(\Pi_1;\Pi_2)$ and the Wilson lines $A_{a(T)}=(A_1;A_2)$ and $A_{b(T)}=(A'_1;A'_2)$ as follows:

$$\Delta^{(B)}_{g'_m}=\left\{\Pi_m\in\Delta_{E_8}\;\middle|\;\sum_{a=1}^{D}\Pi_m\cdot A_m\in\mathbb{Z}\text{ and }\Pi_m\cdot A'_m\in\mathbb{Z}\right\}, \tag{6.24a}$$

$$\Delta^{(F)}_{g'_m}=\left\{\Pi_m\in\Delta_{E_8}\;\middle|\;\sum_{a=1}^{D}\Pi_m\cdot A_m\in\mathbb{Z}+\frac{1}{2}\text{ and }\Pi_m\cdot A'_m\in\mathbb{Z}\right\}. \tag{6.24b}$$

Then the condition (6.15) can be rewritten as

$$\sum_{m=1,2}\left(\left|\Delta^{(F)}_{g'_m}\right|-\left|\Delta^{(B)}_{g'_m}\right|\right)=24. \tag{6.25}$$

Therefore, it is sufficient to investigate only the first eight components of the Wilson lines and to identify $\Delta^{(B)}_{g'_1}$ and $\Delta^{(F)}_{g'_1}$.

Let us study the simplest configurations of Wilson lines $A_{a(T)}=(A_1;A_2)$ and $A_{b(T)}=(A'_1;A'_2)$ as

$$A_m=\left(0^{p_m},\left(\frac{1}{2}\right)^{q_m}\right)\ (p_m+q_m=8),\quad A'_m=\left(0^8\right),\quad\text{for } m=1,2, \tag{6.26}$$

where q_m is even so that $\Pi \cdot 2A_{a(T)} \in \mathbb{Z}$ for all Π. Since p_m and q_m are independent of a, the Wilson lines $A_{a(T)}$ $(a = 1, \ldots, D)$ are the same configuration. As in the $Spin(32)/\mathbb{Z}_2$ supersymmetric endpoint model, we consider two cases with D even or odd. However, when D is even, we can find that $\Delta^{(B)}_{g'_1} = \Delta_{E_8}$ and $\Delta^{(F)}_{g'_1}$ is empty, which gives

$$\left|\Delta^{(F)}_{g'_1}\right| - \left|\Delta^{(B)}_{g'_1}\right| = -240. \tag{6.27}$$

Then, we cannot find a solution of (6.15) in $D \in 2\mathbb{Z}$. Next, we consider the $D \in 2\mathbb{Z}+1$ case.

(1) $p_1 = 0, 8$

In this case, $\Delta^{(B)}_{g'_1} = \Delta_{E_8}$ and $\Delta^{(F)}_{g'_1}$ has no elements, so we get (6.27).

(2) $p_1 = 2, 6$

For $p_1 = 2$, we find that the following $\pi_1 \in \Delta_{SO(16)}$ are in $\Delta^{(B)}_{g'_1}$ and $\Delta^{(F)}_{g'_1}$:

$$\left(\underline{\pm, \pm}, 0^6\right), \left(0^2, \underline{\pm, \pm, 0^4}\right) \in \Delta^{(B)}_{g'_1}, \tag{6.28a}$$

$$\left(\underline{\pm, 0}, \underline{\pm, 0^5}\right) \in \Delta^{(F)}_{g'_1}, \tag{6.28b}$$

and the following $\pi_1 \in \Delta_{\mathbf{128}_+}$ are in $\Delta^{(B)}_{g'_1}$ and $\Delta^{(F)}_{g'_1}$:

$$\frac{1}{2}(\underline{\pm, \pm}_-, \underline{\pm, \pm, \pm, \pm, \pm, \pm}_-) \in \Delta^{(B)}_{g'_1}, \tag{6.29a}$$

$$\frac{1}{2}(\underline{\pm, \pm}_+, \underline{\pm, \pm, \pm, \pm, \pm, \pm}_+) \in \Delta^{(F)}_{g'_1}. \tag{6.29b}$$

Thus we find that $\Delta^{(B)}_{g'_1}$ gives nonzero roots of $SU(2) \times E_7$ and $\Delta^{(F)}_{g'_1}$ gives $(\mathbf{2}, \mathbf{56})$ of $SU(2) \times E_7$. The same results can also be obtained in $p_1 = 6$. We then get

$$\left|\Delta^{(F)}_{g'_1}\right| - \left|\Delta^{(B)}_{g'_1}\right| = -16. \tag{6.30}$$

(3) $p_1 = 4$

In this case, we find that the following $\pi_1 \in \Delta_{SO(16)}$ are in $\Delta^{(B)}_{g'_1}$ and $\Delta^{(F)}_{g'_1}$:

$$\left(\underline{\pm, \pm, 0^2}, 0^4\right), \left(0^4, \underline{\pm, \pm, 0^2}\right) \in \Delta^{(B)}_{g'_1}, \tag{6.31a}$$

$$\left(\underline{\pm, 0^3}, \underline{\pm, 0^3}\right) \in \Delta^{(F)}_{g'_1}, \tag{6.31b}$$

and the following $\pi_1 \in \Delta_{\mathbf{128}_+}$ are in $\Delta^{(B)}_{g'_1}$ and $\Delta^{(F)}_{g'_1}$:

$$\frac{1}{2}(\underline{\pm, \pm, \pm, \pm}_+, \underline{\pm, \pm, \pm, \pm}_+) \in \Delta^{(B)}_{g'_1}, \tag{6.32a}$$

$$\frac{1}{2}(\underline{\pm, \pm, \pm, \pm}_-, \underline{\pm, \pm, \pm, \pm}_-) \in \Delta^{(F)}_{g'_1}. \tag{6.32b}$$

Therefore we find that $\Delta^{(B)}_{g'_1}$ gives nonzero roots of $SO(16)$ and $\Delta^{(F)}_{g'_1}$ gives **128** of $SO(16)$. We then get

$$\left|\Delta^{(F)}_{g'_1}\right| - \left|\Delta^{(B)}_{g'_1}\right| = 16. \tag{6.33}$$

From (6.27), (6.30) and (6.33), we conclude that there is no solution of (6.25) in $D \in 2\mathbb{Z}+1$ case since no combination of -240, -16 and 16 gives 24.

7 Stability of Wilson-line moduli

In this section, we analyze the stability of the Wilson-line moduli using (6.12). We show that the configuration of Wilson lines, which give the suppressed cosmological constant studied in the previous section, corresponds to the saddle point in the moduli space.

7.1 $Spin(32)/\mathbb{Z}_2$ supersymmetric endpoint model

First, we study the case of models whose endpoint is $Spin(32)/\mathbb{Z}_2$ supersymmetric one. Inserting $\Delta_g = \Delta_{SO(32)}$ into (6.12), the Wilson line dependent part can obtained as

$$\sum_{\Pi\in\Delta_{SO(32)}} \exp\left[2\pi i\left\{\sum_{a=1}^{D}(2n_a-1)(\Pi\cdot A_a)+\sum_{b=D+1}^{d} n_b(\Pi\cdot A_b)\right\}\right] = 4\sum_{I>J}\cos\left[2\pi\theta^I(A)\right]\cos\left[2\pi\theta^J(A)\right], \tag{7.1}$$

where $I, J = 1, \ldots, 16$ and we define the fase $\theta^I(A)$ depending on Wilson lines as

$$\theta^I(A) = \sum_{a=1}^{D}(2n_a-1)A_a^I + \sum_{b=D+1}^{d} n_b A_b^I. \tag{7.2}$$

For simplicity, we already omitted the subscript T of Wilson lines. We can then obtain the first derivative of the cosmological constant (6.12) as follows:

$$\frac{\partial\Lambda^{(10-d)}}{\partial A_i^I} \sim 8\pi\sum_{\boldsymbol{n}} C_{\boldsymbol{n}} N_i \sin\left[2\pi\theta^I(A)\right]\sum_{J\neq I}\cos\left[2\pi\theta^J(A)\right], \tag{7.3}$$

where we define N_i as

$$N_i = \begin{cases} 2n_i - 1 & \text{for} \quad i = 1,\ldots,D, \\ n_i & \text{for} \quad i = D+1,\ldots,d, \end{cases} \tag{7.4}$$

and $C_{\boldsymbol{n}}$ is a positive prefactor independent of A defined as

$$C_{\boldsymbol{n}} = \frac{4!\cdot 2^{d+2}}{\pi^{15-d}(\sqrt{\alpha'})^{10-d}}\left(\prod_{i=1}^{d} R_i\right)\left\{\sum_{a=1}^{D}(2n_a-1)^2R_a^2 + \sum_{b=D+1}^{d}(2n_b)^2R_b^2\right\}^{-5}. \tag{7.5}$$

Let us consider the simple Wilson lines given in (6.16). Inserting (6.16) into (7.3), we can find that the Wilson lines (6.16) are critical points for both even and odd D:

$$\frac{\partial\Lambda^{(10-d)}}{\partial A_i^I} \sim 0 \quad (I = 1,\ldots,16,\ i = 1,\ldots,d). \tag{7.6}$$

Next, we evaluate the second derivative of the cosmological constant, which is the Hessian. From (7.3), we get

$$\frac{\partial^2 \Lambda^{(10-d)}}{\partial A_i^I \partial A_j^J} \sim \begin{cases} -16\pi^2 \sum_{\boldsymbol{n}} C_{\boldsymbol{n}} N_i N_j \sin\left[2\pi\theta^I(A)\right] \sin\left[2\pi\theta^J(A)\right] & (I \neq J), \\ 16\pi^2 \sum_{\boldsymbol{n}} C_{\boldsymbol{n}} N_i N_j \cos\left[2\pi\theta^I(A)\right] \sum_{K\neq I}(A) \cos\left[2\pi\theta^K(A)\right] & (I = J), \end{cases} \tag{7.7}$$

where $K = 1, \ldots, 16$. The second derivatives (7.7) with $I \neq J$ vanish in the configuration of Wilson lines (6.16). For the components with $I = J$, they give different results whether D is even or odd.

(I) $D \in 2\mathbb{Z}$

Using (6.16), the second derivatives (7.7) with $I = J$ are expressed as

$$\frac{\partial^2 \Lambda^{(10-d)}}{\partial A_i^I \partial A_j^I} \sim 240\pi^2 \sum_{\boldsymbol{n}} C_{\boldsymbol{n}} N_i N_j \quad (I = 1, \ldots, 16). \tag{7.8}$$

We can notice that $\sum_{\boldsymbol{n}} C_{\boldsymbol{n}} N_i N_j = 0$ for $i \neq j$, and $\sum_{\boldsymbol{n}} C_{\boldsymbol{n}} N_i N_j > 0$ for $i = j$. Thus, we can find the Hessian matrix is positive definite in the configuration of the Wilson lines (6.16). Note that $\Lambda^{(10-d)}$ takes a global minimum if the Wilson lines are given by (6.16), where the gauge group is $SO(32)$ and there are no massless fermions.

(II) $D \in 2\mathbb{Z} + 1$

In serting (6.16) into (7.7), then we get the second derivatives with $I = J$ as follows:

$$\frac{\partial^2 \Lambda^{(10-d)}}{\partial A_i^I \partial A_j^I} \sim \begin{cases} 16\pi^2 (2p - 17) \sum_{\boldsymbol{n}} C_{\boldsymbol{n}} N_i N_j & (I = 1, \ldots, p), \\ 16\pi^2 (-2p + 15) \sum_{\boldsymbol{n}} C_{\boldsymbol{n}} N_i N_j & (I = p+1, \ldots, 16). \end{cases} \tag{7.9}$$

We find that the Hessian matrix is positively definite if $p = 0, 16$ while negatively definite if $p = 8$. That is, $\Lambda^{(10-d)}$ takes a global minimum when the gauge group is $SO(32)$ while a local maximum if the gauge group is $SO(16) \times SO(16)$. We also find that the Wilson line lies in the saddle points in the case of $p = 7, 9$, which gives the exponentially suppressed cosmological constant.

7.2 $E_8 \times E_8$ supersymmetric endpoint model

Next, we consider the case of models whose endpoint is $E_8 \times E_8$ supersymmetric one. We obtain the Wilson line dependent part by inserting $\Delta_g = \Delta_{E_8\times E_8}$ into (6.12) as follows:

$$\sum_{\pi\in\Delta_{E_8\times E_8}} \exp\left[2\pi i\left\{\sum_{a=1}^{D}(2n_a-1)(\Pi\cdot A_a)+\sum_{b=D+1}^{d} n_b(\Pi\cdot A_b)\right\}\right]$$

$$=\sum_{m=1,2}\left\{4\sum_{I_m>J_m}\cos\left[2\pi\theta^{I_m}(A)\right]\cos\left[2\pi\theta^{J_m}(A)\right]\right.$$

$$\left.+128\left(\prod_{I_m}\cos\left[\pi\theta^{I_m}(A)\right]+\prod_{I_m}\sin\left[\pi\theta^{I_m}(A)\right]\right)\right\}, \qquad (7.10)$$

where $I_1, J_1 = 1, \ldots, 8$ and $I_2, J_2 = 9, \ldots, 16$, respectively. Note that the sum over $m = 1, 2$ implies that $\Delta_{E_8\times E_8}$ is decomposed into two copies of Δ_{E_8}, so it is sufficient to study the half part ($m = 1$) of $\Lambda^{(10-d)}$. The first derivative of $\Lambda^{(10-d)}$ can be obtained as

$$\frac{\partial\Lambda^{(10-d)}}{\partial A_i^{I_1}} \sim 8\pi\sum_{\boldsymbol{n}} C_{\boldsymbol{n}} N_i\left\{\sin\left[2\pi\theta^{I_1}(A)\right]\sum_{J_1\neq I_1}\cos\left[2\pi\theta^{J_1}(A)\right]\right.$$

$$\left.+16\left(\sin\left[\pi\theta^{I_1}(A)\right]\prod_{J_1\neq I_1}\cos\left[\pi\theta^{J_1}(A)\right]-\cos\left[\pi\theta^{I_1}(A)\right]\prod_{J_1\neq I_1}\sin\left[\pi\theta^{J_1}(A)\right]\right)\right\}. \qquad (7.11)$$

We can also obtain the second derivative of $\Lambda^{(10-d)}$. For $I_1 \neq J_1$,

$$\frac{\partial^2\Lambda^{(10-d)}}{\partial A_i^{I_1}\partial A_j^{J_1}} \sim -16\pi^2\sum_{\boldsymbol{n}} C_{\boldsymbol{n}} N_i N_j\left\{\sin\left[2\pi\theta^{I_1}(A)\right]\sin\left[2\pi\theta^{J_1}(A)\right]\right.$$

$$+8\left(\sin\left[\pi\theta^{I_1}(A)\right]\sin\left[\pi\theta^{J_1}(A)\right]\prod_{K_1\neq I_1,J_1}\cos\left[\pi\theta^{K_1}(A)\right]\right.$$

$$\left.\left.+\cos\left[\pi\theta^{I_1}\right]\cos\left[\pi\theta^{J_1}(A)\right]\prod_{K_1\neq I_1,J_1}\sin\left[\pi\theta^{K_1(A)}\right]\right)\right\}, \qquad (7.12)$$

and for $I_1 = J_1$,

$$\frac{\partial^2 \Lambda^{(10-d)}}{\partial A_i^{I_1} \partial A_j^{J_1}} \sim 16\pi^2 \sum_{\boldsymbol{n}} C_{\boldsymbol{n}} N_i N_j \Bigg\{ \cos\left[2\pi\theta^{I_1}(A)\right] \sum_{K_1 \neq I_1} \cos\left[2\pi\theta^{K_1}(A)\right] + 8\left(\prod_{I_1=1}^{8} \cos\left[\pi\theta^{I_1}(A)\right] + \prod_{I_1=1}^{8} \sin\left[\pi\theta^{I_1}(A)\right] \right) \Bigg\}. \tag{7.13}$$

From now on, we focus on the simple configuration of the Wilson lines (6.26).

(I) $D \in 2\mathbb{Z}$

In this case, the first derivative (7.11) and the second derivative with $I_1 \neq J_1$ (7.12) vanish in (6.26). The second derivative with $I_1 = J_1$ (7.13) can be expressed as

$$\frac{\partial^2 \Lambda^{(10-d)}}{\partial A_i^{I_1} \partial A_j^{I_1}} \sim 240\pi^2 \sum_{\boldsymbol{n}} C_{\boldsymbol{n}} N_i N_j \qquad (I_1 = 1, \ldots, 8). \tag{7.14}$$

The Hessian matrix is hence positive definite in the configuration of the Wilson lines (6.26). We can also find that the half part of $\Lambda^{(10-d)}$ takes a global minimum if the Wilson lines are given by (6.26), where the gauge group is E_8, and there are no massless fermions, as in $Spin(32)/\mathbb{Z}_2$ supersymmetric endpoint model with D even.

(II) $D \in 2\mathbb{Z} + 1$

By using (6.26), the first derivative (7.11) vanishes since p_1 is even, so the Wilson lines (6.26) are the critical points of $\Lambda^{(10-d)}$. The second derivative with $I_1 \neq J_1$ (7.12) vanishes for $I_1 = 1, \ldots, p_1, J_1 = p_1 + 1, \ldots, 8$ and $I_1 = p_1 + 1, \ldots, 8, J_1 = 1, \ldots, p_1$. We also obtain for $I_1, J_1 = 1, \ldots, p_1$,

$$\frac{\partial^2 \Lambda^{(10-d)}}{\partial A_i^{I_1} \partial A_j^{J_1}} \sim \begin{cases} 0 & (p_1 \neq 2), \\ -128\pi^2 \sum_{\boldsymbol{n}} C_{\boldsymbol{n}} N_i N_j & (p_1 = 2). \end{cases} \tag{7.15}$$

and for $I_1, J_1 = p_1 + 1, \ldots, 8$,

$$\frac{\partial^2 \Lambda^{(10-d)}}{\partial A_i^{I_1} \partial A_j^{J_1}} \sim \begin{cases} 0 & (p_1 \neq 6), \\ -128\pi^2 \sum_{\boldsymbol{n}} C_{\boldsymbol{n}} N_i N_j & (p_1 = 6). \end{cases} \tag{7.16}$$

The second derivative with $I_1 = J_1$ (7.13) can be written for $I_1 = 1, \ldots, p_1$ as

$$\frac{\partial^2 \Lambda^{(10-d)}}{\partial A_i^{I_1} \partial A_j^{I_1}} \sim \begin{cases} 16\pi^2 (2p_1 - 9) \sum_{\boldsymbol{n}} C_{\boldsymbol{n}} N_i N_j & (p_1 \neq 8), \\ 240\pi^2 \sum_{\boldsymbol{n}} C_{\boldsymbol{n}} N_i N_j & (p_1 = 8), \end{cases} \tag{7.17}$$

and for $I_1 = p_1 + 1, \ldots, 8$ as

$$\frac{\partial^2 \Lambda^{(10-d)}}{\partial A_i^{I_1} \partial A_j^{I_1}} \sim \begin{cases} 16\pi^2(-2p_1 + 7) \sum_{\boldsymbol{n}} C_{\boldsymbol{n}} N_i N_j & (p_1 \neq 0), \\ 240\pi^2 \sum_{\boldsymbol{n}} C_{\boldsymbol{n}} N_i N_j & (p_1 = 0). \end{cases} \tag{7.18}$$

In the configuration of Wilson lines (6.26), the Hessian matrix is positive definite if $p_1 = 0, 8$, while negative definite if $p_1 = 4$. The Wilson lines with $p_1 = 0, 8$ correspond to global minima where the gauge group is E_8, while ones with $p_1 = 4$ to a local maximum where the gauge group is $SO(16) \times SO(16)$. We can also find that the Wilson lines (6.26) with $p_1 = 2, 6$ give the saddle points of $\Lambda^{(10-d)}$.

8 Summary

In this thesis, we have studied non-supersymmetric string models constructed by general $\mathbb{Z}_2$ twisted compactification.

We have first investigated the behavior of the 9- and 8-dimensional non-supersymmetric models in the endpoint limits. We have found that the interpolation patterns between endpoints in $d = 2$ are recognized as the combinations of those in $d = 1$, but we have also found the specific model that does not show the 9-dimensional structure labeled by the condition. We also found the conditions to obtain the 10-dimensional (non-)supersymmetric models in the endpoint limits. These discussions can be easily extended to the d-dimensionally compactified models.

Next, we have investigated the endpoints of 9-dimensional heterotic models with non-zero Wilson lines. In class (1), both endpoints are non-supersymmetric when Wilson lines are turned off, but we have found that the supersymmetry can be restored even in class (1) when the Wilson line is equivalent to the shift vector.

Toward evaluating the cosmological constant, we have studied the massless spectra in the non-supersymmetric heterotic string models with general $\mathbb{Z}_2$ twists. Comparing the massless condition in superstring theories with that in the non-supersymmetric models, we have obtained the Wilson-line relation. This relation has been used in evaluating the one-loop cosmological constant in the non-supersymmetric heterotic string models.

We have then evaluated the cosmological constant in the region where the supersymmetry is asymptotically restored. It depends on the two types of moduli: the radii of the internal space and the Wilson lines. Assuming the configurations of the Wilson lines, we have shown the formula with the factor of $n_F - n_B$, where n_F (n_B) is the degrees of freedom of massless fermions (bosons, respectively). This is the formula for the general $\mathbb{Z}_2$ case generalized from the one for 9-dimensional models, first calculated in [11, 12].

Using this equation, we have found Wilson-line configurations that give an exponentially suppressed cosmological constant in the $Spin(32)/\mathbb{Z}_2$ supersymmetric endpoint model. On the other hand, in the $E_8 \times E_8$ supersymmetric endpoint model, we cannot find a configuration of Wilson lines that gives an exponentially suppressed cosmological constant. We have further analyzed the Wilson-line stability of the cosmological constant. We have found that the Wilson lines giving exponentially suppressed $\Lambda^{(10-d)}$ correspond to the saddle points of $\Lambda^{(10-d)}$ and that the global minima of $\Lambda^{(10-d)}$ is negative in both $Spin(32)/\mathbb{Z}_2$ and $E_8 \times E_8$

endpoint models.

At the end of this section, we discuss the future direction.

- The results of Section 7 imply that when $\Lambda^{(10-d)}$ is exponentially suppressed, the moduli are unstable, and the points of stable moduli live in anti-de Sitter (AdS) vacua. Though these situations are considered undesirable in the context of string phenomenology, it is possible to construct perturbatively stable solutions in the AdS background of non-supersymmetric $SO(16)\times SO(16)$ heterotic strings [47]. It may be possible to generalize the construction of stable AdS vacua in the case of general $\mathbb{Z}_2$ twists.

- Since the discussion and calculation in this thesis are based on the one-loop level, it is worth computing higher-order loop corrections to the cosmological constant. In [19], the two-loop correction of the cosmological constant was evaluated in fermionic construction. It is necessary to analyze the moduli stability at the two-loop level in the bosonic construction we have studied in this thesis.

- We have evaluated $\Lambda^{(10-d)}$ in the supersymmetry-restored region where all the radii are very large for simplicity. However, it is not necessary to take all R_i to be large for the restoration of supersymmetry. It is sufficient to take some radii to be so and we can leave the other ones finite. The cosmological constant could then get the moduli dependence of anti-symmetric tensor field B, and the analysis of moduli stability would change.

- As discussed before, we have not completely revealed the conditions that supersymmetry can be restored in the non-supersymmetric models with non-zero values of moduli. This is an important issue since the suppression of cosmological constant occurs in that region. It is interesting to study the configurations of moduli, which give the supersymmetric endpoints in the general setup.

A $SO(2n)$ conjugacy classes and characters

The irreducible representation of $SO(2n)$ (D_n) can be classified into four conjugacy classes:

- The trivial conjugacy class (the root lattice):

$$\Gamma_g^{(n)} = \left\{ (m_1, \cdots, m_n) \,\middle|\, m_i \in \mathbb{Z},\ \sum_{i=1}^{n} m_i \in 2\mathbb{Z} \right\}, \tag{A.1}$$

- The vector conjugacy class:

$$\Gamma_v^{(n)} = \left\{ (m_1, \cdots, m_n) \,\middle|\, m_i \in \mathbb{Z},\ \sum_{i=1}^{n} m_i \in 2\mathbb{Z}+1 \right\}, \tag{A.2}$$

- The spinor conjugacy class:

$$\Gamma_s^{(n)} = \left\{ (m_1 + \frac{1}{2}, \cdots, m_n + \frac{1}{2}) \,\middle|\, m_i \in \mathbb{Z},\ \sum_{i=1}^{n} m_i \in 2\mathbb{Z} \right\}, \tag{A.3}$$

- The conjugate spinor conjugacy class:

$$\Gamma_c^{(n)} = \left\{ (m_1 + \frac{1}{2}, \cdots, m_n + \frac{1}{2}) \,\middle|\, m_i \in \mathbb{Z},\ \sum_{i=1}^{n} m_i \in 2\mathbb{Z}+1 \right\}. \tag{A.4}$$

The $SO(2n)$ characters are defined as follows:

$$O_{2n} = \frac{1}{\eta^n} \sum_{\Pi \in \Gamma_g^{(n)}} q^{\frac{1}{2}|\Pi|^2} = \frac{1}{2\eta^n} \left(\vartheta^n \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0,\tau) + \vartheta^n \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} (0,\tau) \right), \tag{A.5}$$

$$V_{2n} = \frac{1}{\eta^n} \sum_{\Pi \in \Gamma_v^{(n)}} q^{\frac{1}{2}|\Pi|^2} = \frac{1}{2\eta^n} \left(\vartheta^n \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0,\tau) - \vartheta^n \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} (0,\tau) \right), \tag{A.6}$$

$$S_{2n} = \frac{1}{\eta^n} \sum_{\Pi \in \Gamma_s^{(n)}} q^{\frac{1}{2}|\Pi|^2} = \frac{1}{2\eta^n} \left(\vartheta^n \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (0,\tau) + \vartheta^n \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (0,\tau) \right), \tag{A.7}$$

$$C_{2n} = \frac{1}{\eta^n} \sum_{\Pi \in \Gamma_c^{(n)}} q^{\frac{1}{2}|\Pi|^2} = \frac{1}{2\eta^n} \left(\vartheta^n \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (0,\tau) - \vartheta^n \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (0,\tau) \right). \tag{A.8}$$

where the Dedekind eta function and the theta function with characteristics are defined as

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \tag{A.9}$$

$$\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z,\tau) = \sum_{n=-\infty}^{\infty} \exp\left(\pi i (n+\alpha)^2 \tau + 2\pi i (n+\alpha)(z+\beta) \right). \tag{A.10}$$

From this definition, we find the transformations of $SO(2n)$ characters under $T : \tau \to \tau + 1$ and $S : \tau \to -\frac{1}{\tau}$ as follows:

$$T : (O_{2n}, V_{2n}, S_{2n}, C_{2n})\, \mathcal{T}_{2n}, \tag{A.11}$$

$$S : (O_{2n}, V_{2n}, S_{2n}, C_{2n})\, \mathcal{S}_{2n}, \tag{A.12}$$

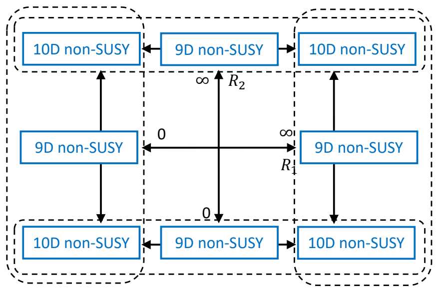
where $\mathcal{T}_{2n}$ and $\mathcal{S}_{2n}$ are represented as

$$\mathcal{T}_{2n} = \begin{pmatrix} e^{-\frac{i\pi n}{12}} & 0 & 0 & 0 \\ 0 & -e^{\frac{i\pi n}{12}} & 0 & 0 \\ 0 & 0 & e^{\frac{i\pi n}{6}} & 0 \\ 0 & 0 & 0 & e^{\frac{i\pi n}{6}} \end{pmatrix}, \quad \mathcal{S}_{2n} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & i^n & -i^n \\ 1 & -1 & -i^n & i^n \end{pmatrix}. \tag{A.13}$$

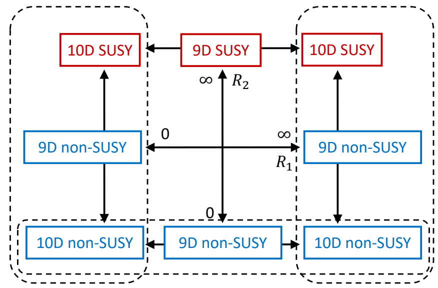
These transformations can be used to determine twisted sectors of the partition functions.

B 8D non-supersymmetric heterotic models

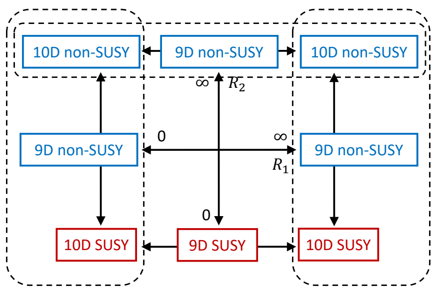
Here we give the complete classification of 8D non-supersymmetric heterotic models.



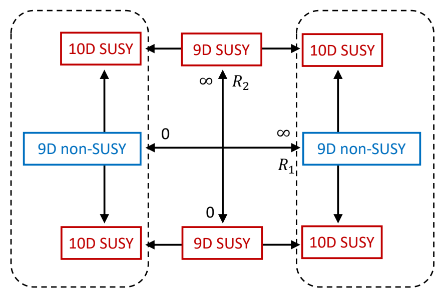
[1 : 1] $\hat{\Pi}^2 = 0 \pmod 4$, $(\hat{w}; \hat{n}) = (0, 0; 0, 0)$



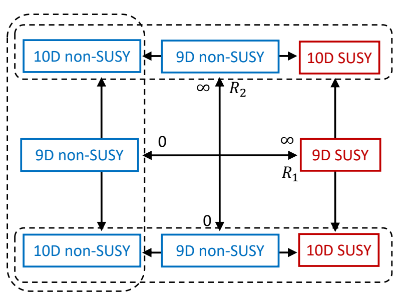
[1 : 2] $\hat{\Pi}^2 = 0 \pmod 4$, $(\hat{w}; \hat{n}) = (0, 1; 0, 0)$



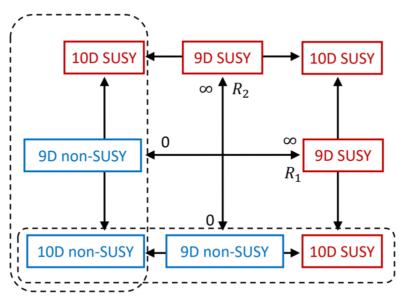
[1 : 3] $\hat{\Pi}^2 = 0 \pmod 4$, $(\hat{w}; \hat{n}) = (0, 0; 0, 1)$



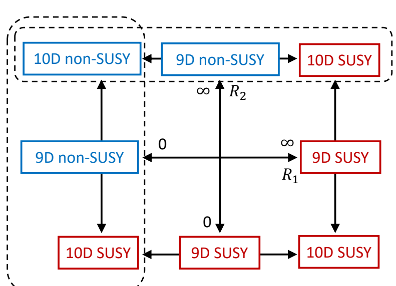
[1 : 4] $\hat{\Pi}^2 = 2 \pmod 4$, $(\hat{w}; \hat{n}) = (0, 1; 0, 1)$



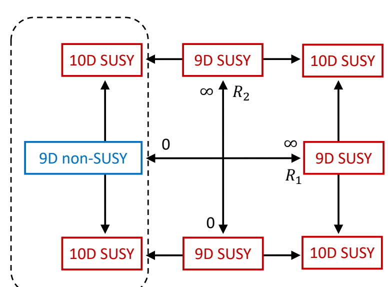
[2 : 1] $\hat{\Pi}^2 = 0 \pmod 4$, $(\hat{w}; \hat{n}) = (1, 0; 0, 0)$



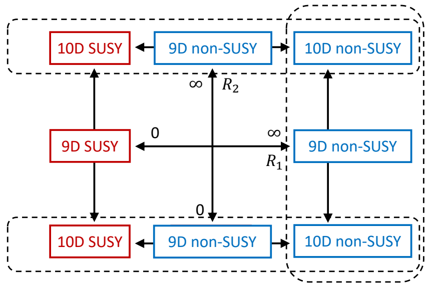
[2 : 2] $\hat{\Pi}^2 = 0 \pmod 4$, $(\hat{w}; \hat{n}) = (1, 1; 0, 0)$



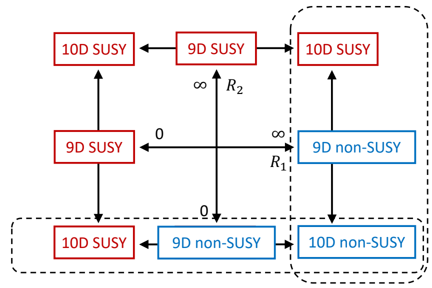
[2 : 3] $\hat{\Pi}^2 = 0 \pmod 4$, $(\hat{w}; \hat{n}) = (1, 0; 0, 1)$



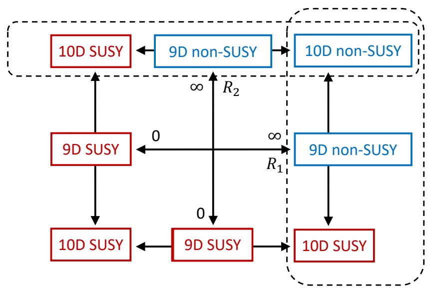
[2 : 4] $\hat{\Pi}^2 = 2 \pmod 4$, $(\hat{w}; \hat{n}) = (1, 1; 0, 1)$



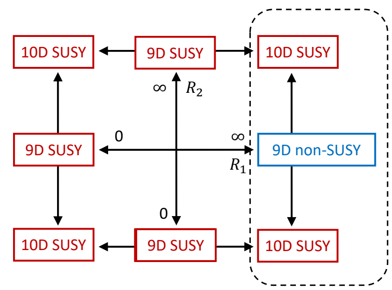
[3 : 1] $\hat{\Pi}^2 = 0 \pmod 4$, $(\hat{w}; \hat{n}) = (0, 0; 1, 0)$



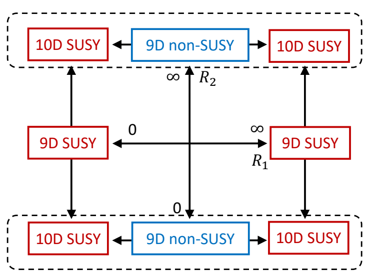
[3 : 2] $\hat{\Pi}^2 = 0 \pmod 4$, $(\hat{w}; \hat{n}) = (0, 1; 1, 0)$



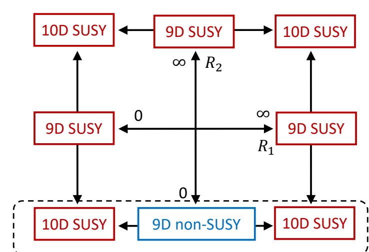
[3 : 3] $\hat{\Pi}^2 = 0 \pmod 4$, $(\hat{w}; \hat{n}) = (0, 0; 1, 1)$



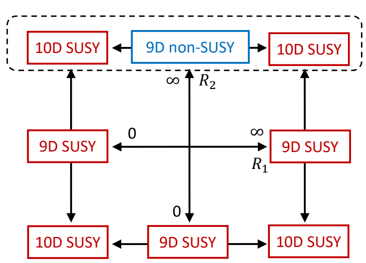
[3 : 4] $\hat{\Pi}^2 = 2 \pmod 4$, $(\hat{w}; \hat{n}) = (0, 1; 1, 1)$



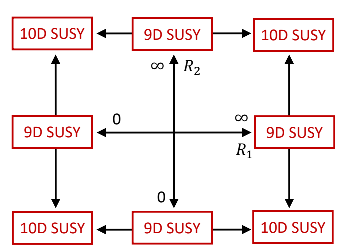
[4 : 1] $\hat{\Pi}^2 = 2 \pmod 4$, $(\hat{w}; \hat{n}) = (1, 0; 1, 0)$



[4 : 2] $\hat{\Pi}^2 = 2 \pmod 4$, $(\hat{w}; \hat{n}) = (1, 1; 1, 0)$



[4 : 3] $\hat{\Pi}^2 = 2 \pmod 4$, $(\hat{w}; \hat{n}) = (1, 0; 1, 1)$



[4 : 4] $\hat{\Pi}^2 = 0 \pmod 4$, $(\hat{w}; \hat{n}) = (1, 1; 1, 1)$

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