SPECTRA AND EIGENFORMS OF THE LAPLACIAN ON
\( S^n \) AND \( P^n(C) \)

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Introduction

Let \( M \) be a compact Riemannian manifold. We consider the Laplace
operator \( \Delta \) acting on the space of differential forms on \( M \). It is a strongly
elliptic self-adjoint differential operator, so it has discrete eigenvalues with
finite multiplicities. For a given Riemannian manifold, it may be an interesting
problem to determine explicitly the spectra and eigenforms of \( \Delta \) on \( M \). As
for the spectra and eigenfunctions of \( \Delta \) acting on the space of functions, they
are known for the cases where \( M \) are the following manifolds; flat tori, Klein
bottles [3], symmetric spaces [12] and the Hopf manifolds [1]. On the other
hand, as for the spectra and eigenforms of \( \Delta \) acting on the differential forms,
we have known no results except for flat tori. But, E. Calabi (unpublished)
and recently S. Gallot et D. Meyer [7] have computed the eigenvalues of differ-
etial forms on the standard sphere by using the harmonic polynomial forms
on \( \mathbb{R}^{n+1} \).

In this paper, applying the representation theory we compute the eigen-
values of \( \Delta \) and determine the spaces of eigenforms as representation spaces, on
the standard sphere \( S^n \) and the complex projective space \( P^n(C) \) with Fubini-Study
metric. Our method is as follows: Let \( M=\mathbb{G}/K \) be a Riemannian homogeneous
space with \( \mathbb{G} \) acting as transitive isometry group on \( M \). Then \( \Delta \) is a \( \mathbb{G} \)-invariant
differential operator, so its eigenspaces are \( \mathbb{G} \)-modules. First, we decompose the
space of differential forms on \( M \) into \( \mathbb{G} \)-irreducible modules. In the case where
\( M \) is \( S^n \), \( P^n(C) \), or more generally a symmetric space, roughly speaking \( \Delta=-\text{Casimir operator on } \mathbb{G} \). So from Freudenthal’s Formula, we can compute the
eigenvalues. But the first step of decomposing the space of differential forms
on \( M \) into \( \mathbb{G} \)-irreducible modules is generally not easy. In virtue of Frobenius
reciprocity law, the problem can be reduced to the following problem: For a
given irreducible \( \mathbb{G} \)-module, how does it decompose into irreducible \( K \)-modules?
For this problem, a few results are known (cf. H. Boerner [3] and D. P. Zelobenko
[15]), and in case \( M=S^n \), we apply the known results.

As for the Laplacian \( \Delta \) acting on the space of functions on \( S^n \) and \( P^n(C) \), its
eigenfunctions have been obtained by the restriction to $S^n$ and $P^n(C)$ of the harmonic homogeneous polynomials on $\mathbb{R}^{n+1}$ and $\mathbb{C}^{n+1}$ (cf. [12]). In 6 (Theorem 6.8) and 7 (Theorem 7.13), we give the analogy for differential forms on $S^n$ and $P^n(C)$ using harmonic polynomial forms.

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1. Preliminaries

Let $G$ be a compact connected Lie group, $K$ a closed subgroup and $M$ the quotient space $G/K$. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of $G$ and $K$ respectively. A $K$-invariant inner product on $\mathfrak{g}/\mathfrak{k}$ determines a $G$-invariant Riemannian metric on $M$. We fix a $G$-invariant Riemannian metric on $M$ and extend it canonically to a hermitian metric, denoted by $\langle \cdot, \cdot \rangle$, on $\Lambda^p M$, the $p$-th exterior power of the complexified cotangent bundle of $M$. For a smooth vector bundle $E$ over $M$, we denote by $\mathcal{C}^\infty(E)$ the vector space of smooth sections of $E$. When $E$ is a homogeneous vector bundle, $\mathcal{C}^\infty(E)$ is considered as a $G$-module by $(g \cdot s)(x) = g \cdot s(g^{-1}x)$ for $g \in G$, $s \in \mathcal{C}^\infty(E)$, and $x \in M$, in particular, $\mathcal{C}^\infty(\Lambda^p M)$ has a natural $G$-module structure over $C$. Now, we define the inner product $(\ , \ )_M$ on $\mathcal{C}^\infty(\Lambda^p M)$ by

$$
(\phi, \psi)_M = \int_M \langle \phi, \psi \rangle \ dm \quad (\phi, \psi \in \mathcal{C}^\infty(\Lambda^p M)),
$$

where $dm$ is the smooth measure on $M$ defined by the Riemannian metric. From the construction, the $G$-action preserves $(\ , \ )_M$, i.e.,

$$(g \cdot \phi, g \cdot \psi)_M = (\phi, \psi)_M \quad \text{for all } \phi, \psi \in \mathcal{C}^\infty(\Lambda^p M) \text{ and } g \in G.$$ 

By means of this inner product, we define the codifferentiation $\delta$ as the operator formally adjoint to the exterior differentiation $d$. We set $\Delta = d\delta + \delta d$, and call it the Laplace operator or Laplacian. $\Delta$ is a self-adjoint, strongly elliptic differential operator on $\mathcal{C}^\infty(\Lambda^p M)$ for each $p$, and commutes with the $G$-action on $\mathcal{C}^\infty(\Lambda^p M)$. The set of eigenvalues of $\Delta$ on $\mathcal{C}^\infty(\Lambda^p M)$ is a discrete set of non-negative real numbers; $0 \leq \lambda_1^p < \lambda_2^p < \cdots \to \infty$. Moreover, each eigenspace $E_{\lambda}^p$ is a finite dimensional $G$-submodule and the algebraic sum $\sum_{i=1}^{\infty} E_{\lambda_i}^p$ is a dense subspace of $\mathcal{C}^\infty(\Lambda^p M)$, the topology being defined by the inner product $(\ , \ )_M$.

First, we recall some effects of $d, \delta$ and $*$ on the spectra and the eigenspaces of Laplacian. Hodge decomposition theorem asserts that

$$
\mathcal{C}^\infty(\Lambda^p M) = E_p^* \oplus d\mathcal{C}^\infty(\Lambda^{p-1} M) \oplus \delta \mathcal{C}^\infty(\Lambda^{p+1} M),
$$

where $E_p^*$ is the space of harmonic $p$-forms. This is a direct sum as $G$-modules. For each eigenvalue $\lambda$ of Laplacian, we set
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\[ E^\lambda_k = \{ \phi \in E^\lambda_k; \ d\phi = 0 \} , \quad E^\lambda_k = \{ \phi \in E^\lambda_k; \ \delta\phi = 0 \} , \]

which are both \( G \)-submodules of \( E^\lambda_k \). We see

\[
\begin{align*}
(1.2) \quad \begin{cases} 
E^\lambda_k = '^E^\lambda_k \oplus ''E^\lambda_k & \text{for } \lambda \neq 0 , \\
E^\lambda_k = '^E^\lambda_k = ''E^\lambda_k & \text{for } \lambda = 0 , 
\end{cases}
\end{align*}
\]

and

\[
(1.3) \quad d : '^E^\lambda_k \cong '^E^{\lambda+1}_k , \quad \delta : ''E^\lambda_k \cong ''E^{\lambda-1}_k \quad \text{for } \lambda \neq 0 .
\]

Here and in the following, "\( \cong \)" means a \( G \)-isomorphism, unless specially mentioned.

Suppose that \( M \) is orientable and is oriented. Then, the so-called star operator

\[
* : \Lambda^pM \cong \Lambda^{n-p}M \quad (n = \dim M)
\]

is defined, and the codifferntiation is expressed as

\[
(1.4) \quad \delta = (-1)^{p(n-p+1)}d^* \quad \text{on } C^\infty(\Lambda^pM) .
\]

This together with \( *\Delta = \Delta^* \) implies

\[
(1.5) \quad \begin{cases} 
\lambda_i^\lambda = \lambda_i^{n-p} & (i = 1, 2, \ldots) \text{ and} \\
* : '^E^\lambda_k \cong ''E^{n-\lambda}_k \quad (p = 0, 1, \ldots, n) .
\end{cases}
\]

Now, for a finite dimensional vector space \( U \), we denote by \( C^\infty(G; U) \) the vector space of all smooth functions of \( G \) with values in \( U \), and consider it as a \( G \)-module by \( (g \cdot f)(x) = f(g^{-1}x) \) for \( g \in G, \ f \in C^\infty(G; U) \), and \( x \in G \). Further, when \( U \) is a \( K \)-module, we denote by \( C^\infty(G, K; U) \) the \( G \)-submodule consisting of \( \phi \in C^\infty(G; U) \) such that \( \phi(gk) = k^{-1}\phi(g) \) for any \( k \in K \) and \( g \in G \). On the other hand, the \( K \)-module \( U \) defines a homogeneous vector bundle \( G \times_K U \) over \( G/K \). Then, the \( G \)-module \( C^\infty(G, K; U) \) can be identified with \( C^\infty(G \times_K U) \), in particular, \( C^\infty(\Lambda^pM) \) is identified with \( C^\infty(G, K; \Lambda^p(\mathfrak{g}/\mathfrak{t})^*c) \), \( \mathfrak{g}/\mathfrak{t} \) being a \( K \)-module by the adjoint action of \( K \) in \( \mathfrak{g} \), since \( \Lambda^pM \) is canonically isomorphic to \( G \times_K \Lambda^p(\mathfrak{g}/\mathfrak{t})^*c \).

Let \( \mathcal{G} \) be a complete set of inequivalent irreducible representations of \( G \) over \( C \). For an element \( \rho \in \mathcal{G} \), we define a \( G \)-homomorphism

\[
\iota_\rho : \text{Hom}_C(V_\rho, C^\infty(\Lambda^pM)) \otimes_C V_\rho \to C^\infty(\Lambda^pM)
\]

by \( \phi \otimes u \mapsto \phi(u) \), where \( V_\rho \) is the representation space of \( \rho \). \( \iota_\rho \) is clearly injective. We set \( \mu_\rho = \dim_C \text{Hom}_C(V_\rho, C^\infty(\Lambda^pM)) \) and \( \Gamma_\rho^\rho = \text{the image of } \iota_\rho \). Then, it is easy to see that \( \mu_\rho \) and \( \Gamma_\rho^\rho \) depend only on the equivalence class of \( \rho \), and that \( \Gamma_\rho^\rho \) is isomorphic to the direct sum of \( \mu_\rho \)-copies of \( V_\rho \). To compute \( \mu_\rho \), we apply the following Frobenius' reciprocity law.
Proposition 1.1. Let $F$ be $\mathbb{R}$ or $\mathbb{C}$. For a finite dimensional $K$-module $U$ over $F$ and a finite dimensional $G$-module $V$ over $F$, we have a canonical isomorphism as vector spaces

$$\text{Hom}_K(V, C^\infty(G, K; U)) \simeq \text{Hom}_K(V, U).$$

For a proof, see R. Bott [4].

By this proposition, we get

$$(1.6) \quad \mu_p = \dim \text{Hom}_K(V_p, \Lambda^p(\mathfrak{g}/\mathfrak{t})^*) \quad \text{for any } \rho \in \mathcal{F}_G,$$

hence, in particular, $\mu_p$ is finite.

Proposition 1.2. Under the above notations, we have

$$(1.7) \quad \sum_{i} E_{\lambda_i} = \sum_{\rho \in \mathcal{F}_G} \Gamma_{\rho}^*$$

Proof. The left hand side of (1.7) is clearly included in the right. The converse is proved as follows. Since the inner product $(\cdot, \cdot)_\mathcal{M}$ on $C^\infty(\Lambda^p\mathcal{M})$ is invariant under $G$, $\Gamma_{\rho}^*$'s are orthogonal to each other. On the other hand, the left hand side of (1.7) is dense in $C^\infty(\Lambda^p\mathcal{M})$ with respect to the topology defined by $(\cdot, \cdot)_\mathcal{M}$, and included in the right. From these, we get easily the proposition.

q.e.d.

In the case that $(G, K)$ is a compact symmetric pair with a semisimple Lie group $G$, we shall see later that each $\Gamma_{\rho}^*$ is contained in a certain eigenspace $E_{\lambda_i}^\rho$.

2. The Laplace operator and the Casimir operator

In this section, we shall give some relations between $E_{\lambda_i}^\rho$'s and $\Gamma_{\rho}^*$'s, using the Casimir operator. Let $G$ be a compact semisimple Lie group and $B$ the Killing form of the Lie algebra $\mathfrak{g}$ of $G$. Let $\{X_1, \ldots, X_N\}$ be a basis of $\mathfrak{g}$, and put

$$(2.1) \quad \mathcal{C} = \sum_{i,j} C^{ij} X_i \cdot X_j, \quad \text{where } (C^{ij}) = (B(X_i, X_j))^{-1},$$

which is an element in the universal enveloping algebra $U_\mathfrak{g}$ of $\mathfrak{g}$. $\mathcal{C}$ is in the center of $U_\mathfrak{g}$ and is called the Casimir element. Since each $X_i$ may be regarded as a left invariant differential operator on $G$, $\mathcal{C}$ can be considered as a differential operator on $G$, and it is a two-sided invariant differential operator. Then, $\mathcal{C}$ is called the Casimir operator.

The following fact is easily proved.

Proposition 2.1. Let $G$ and $\mathcal{C}$ be as above, and $U$ a finite dimensional vector space over $\mathbb{C}$. Then, for any finite dimensional $G$-submodule $(V, \rho)$ of $C^\infty(G; U)$, we have
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\begin{equation}
Cf = \rho(C)f \quad \text{for all } f \text{ in } V,
\end{equation}

\(\rho\) being extended to the representation of \(U_q\).

For a proof, see M. Takeuchi [12].

**Corollary.** Each finite dimensional \(G\)-submodule of \(C^\infty(G; U)\) is stable under the Casimir operator.

In Proposition 2.1, if \(\rho\) is irreducible, then \(\rho(C)\) is a scalar operator by Schur's Lemma, and the value of \(\rho(C)\) is given by the following

**Proposition 2.2.** Let \(G\) and \(C\) be as before. We fix a lexicographic order on the dual space of a Cartan subalgebra of \(\mathfrak{g}\). Then, for any irreducible representation \((V, \rho)\) of \(G\) over \(C\) with the highest weight \(\lambda_p\), we have

\begin{equation}
\rho(C) = -4\pi^2 \langle \lambda_p + 2\delta_G, \lambda_p \rangle id_v,
\end{equation}

where \(\delta_G\) denotes the half sum of all positive roots of \(\mathfrak{g}\), and \(\langle \cdot, \cdot \rangle\) denotes the inner product on the dual space of the Cartan subalgebra of \(\mathfrak{g}\), induced from the Killing form sign changed.

For a proof, see N. Jacobson [9].

Now, we assume that \((G, K)\) is a compact symmetric pair with a compact connected semisimple Lie group \(G\). Let \(m\) be the orthogonal complement to \(\mathfrak{k}\) in \(\mathfrak{g}\) with respect to the Killing form. Then, we have the Cartan decomposition,

\[ \mathfrak{g} = \mathfrak{k} + m. \]

Restricting the Killing form sign changed to \(m\), we can define a \(G\)-invariant Riemannian metric on \(M=G/K\). Then, we have the following formula.

**Proposition 2.3.** Let \(G, K\) and \(M\) be as above. Let \(\Delta\) be the Laplace operator on \(M\) defined by the metric given above. Then, under the identification \(C^\infty(\Lambda^p M) \simeq C^\infty(G, K; \Lambda^p(\mathfrak{k} \cap \mathfrak{t})^* c)\), we have

\begin{equation}
\Delta = -C.
\end{equation}

Analogous formula is given by Y. Matsushima—S. Murakami [10], when \(G\) is of non-compact type. We shall prove the formula along the same way as in [10].

**Proof.** Let \(\pi: G \to M\) be the projection. The identification

\[ C^\infty(\Lambda^p M) \simeq C^\infty(G, K; \Lambda^p(\mathfrak{k} \cap \mathfrak{t})^* c), \]

denoted by \(\alpha \mapsto \tilde{\alpha}\), is given by

\[ (\tilde{\alpha}(g))(Y_1, \ldots, Y_p) = (\pi^*\alpha)(Y_1, \ldots, Y_p)(g), \]
for $g \in G$ and $Y_1, \cdots, Y_p \in \mathfrak{g}$, where $(\mathfrak{g}/\mathfrak{f})^*$ is considered as the set of linear forms on $\mathfrak{g}$ vanishing on $\mathfrak{f}$. Now, we choose an orthonormal basis $\{X_1, \cdots, X_n, X_{n+1}, \cdots, X_N\}$ of $\mathfrak{g}$, with respect to the inner product induced from the Killing form, in such a way that $\{X_1, \cdots, X_n\}$ forms a basis of $\mathfrak{m}$ and $\{X_{n+1}, \cdots, X_N\}$ a basis of $\mathfrak{f}$. An element $\eta$ in $C^\infty(G, K; \Lambda^*(\mathfrak{g}/\mathfrak{f})^*)$ is determined by the system of $C^\infty$-functions $\eta(X_{i_1}, \cdots, X_{i_p})$ $(1 \leq i_1 < \cdots < i_p \leq n)$.

Then, we define a linear map $D: C^\infty(G, K; \Lambda^*(\mathfrak{g}/\mathfrak{f})^*) \rightarrow C^\infty(G, K; \Lambda^{p+1}(\mathfrak{g}/\mathfrak{f})^*)$ by

\[(1) \quad (D\eta)(X_{i_1}, \cdots, X_{i_{p+1}}) = \sum_{\ell=1}^{N} (-1)^{p+1} X_{i_\ell} \eta(X_{i_1}, \cdots, \widehat{X_{i_\ell}}, \cdots, X_{i_{p+1}}) \quad (1 \leq i_1 < \cdots < i_{p+1} \leq n).\]

We have

\[\tilde{d}\alpha = D\alpha \quad (\alpha \in C^\infty(\Lambda^p M)).\]

This follows from the fact that $[X_{i_\ell}, X_{i_\ell}] \in \mathfrak{f}$ for $X_{i_\ell}, X_{i_\ell} \in \mathfrak{m}$.

Next, we define an inner product $(\ , \ )^*$ on $C^\infty(G, K; \Lambda^*(\mathfrak{g}/\mathfrak{f})^*)$, using a $K$-invariant inner product on $\Lambda^*(\mathfrak{g}/\mathfrak{f})^*$ by

\[(2) \quad (\xi, \eta)^* = \int_G \langle \xi(g), \eta(g) \rangle dg\]

for $\xi, \eta \in C^\infty(G, K; \Lambda^*(\mathfrak{g}/\mathfrak{f})^*)$, where $dg$ denotes a $G$-invariant smooth measure on $G$. Then, it is easy to see that

\[(3) \quad (\alpha, \beta)_M = c \cdot (\alpha, \beta)^* \quad (\alpha, \beta \in C^\infty(\Lambda^p M))\]

for some constant $c$.

The adjoint operator $D^*$ of $D$ is given by

\[(4) \quad (D^*\xi)(X_{i_1}, \cdots, X_{i_{p-1}}) = -\sum_{\ell=1}^{p-1} X_{i_\ell} \xi(X_{i_1}, X_{i_{\ell}}, \cdots, X_{i_{p-1}}), \quad (1 \leq i_1 < \cdots < i_{p-1} \leq n),\]

where $\xi \in C^\infty(G, K; \Lambda^{p+1}(\mathfrak{g}/\mathfrak{f})^*)$.

In fact, for $\eta \in C^\infty(G, K; \Lambda^{p+1}(\mathfrak{g}/\mathfrak{f})^*)$, we have

\[(\xi, D\eta)^* = \int_G \frac{1}{p!} \sum_{i_1, \cdots, i_p=1}^{N} \xi(X_{i_1}, \cdots, X_{i_p})
\cdot \sum_{\ell=1}^{p} (-1)^{p-1} X_{i_\ell} \eta(X_{i_1}, \cdots, \widehat{X_{i_\ell}}, \cdots, X_{i_{p}}) dg
\cdot \eta(X_{i_1}, \cdots, \widehat{X_{i_\ell}}, \cdots, X_{i_p})
\cdot (\text{cf. [10]})\]
We set $\Delta^0 = D D^* + D^* D$. Then, we see

$$ (\Delta \alpha, \beta)_M = c \cdot (\Delta^0 \alpha, \beta)^* \quad \text{for } \alpha, \beta \in \mathcal{C}^\infty(\Lambda^p M), $$

consequently, $\widetilde{\Delta} \alpha = \Delta^0 \alpha$.

To prove the proposition, it suffices to show that

$$ \widetilde{\Delta} \alpha = - \sum_{i=1}^N X_i^\alpha \quad \text{for } \alpha \in \mathcal{C}^\infty(\Lambda^p M). $$

From (1) and (4), for $X_{i_1}, \ldots, X_{i_p}$ with $1 \leq i_1 < \cdots < i_p \leq n$, we have

$$ (D D^* \alpha)(X_{i_1}, \ldots, X_{i_p}) = - \sum_{i=1}^p (-1)^{i-1} \sum_{k=1}^p X_{i_k} X_{i_k} \alpha(X_{i_1}, X_{i_2}, \ldots, \widehat{X}_{i_k}, \ldots, X_{i_p}) $$

and

$$ (D^* D \alpha)(X_{i_1}, \ldots, X_{i_p}) = - \sum_{i=1}^p X_i (D \alpha)(X_{i_1}, X_{i_2}, \ldots, X_{i_p}) $$

$$ = - \sum_{i=1}^p X_i \alpha(X_{i_1}, \ldots, X_{i_p}) $$

$$ - \sum_{i=1}^p \sum_{k=1}^p (-1)^{k-1} X_k X_{i_k} \alpha(X_{i_1}, X_{i_2}, \ldots, \widehat{X}_{i_k}, \ldots, X_{i_p}) $$

$$ - \sum_{i=1}^p \sum_{k=1}^p (-1)^{k+i} X_k \alpha([X_{i_k}, X_{i_k}], X_{i_1}, \ldots, \widehat{X}_{i_k}, \ldots, X_{i_p}) $$

$$ - \sum_{i=1}^p \sum_{k=1}^p (-1)^{k+i} X_k \alpha([X_{i_k}, X_{i_k}], X_{i_1}, \ldots, \widehat{X}_{i_k}, \ldots, \widehat{X}_{i_k}, \ldots, X_{i_p}). $$

Hence, we have

$$ (\Delta^0 \alpha)(X_{i_1}, \ldots, X_{i_p}) = - \sum_{i=1}^p X_i^\alpha(X_{i_1}, \ldots, X_{i_p}) $$

$$ - \sum_{i=1}^p \sum_{k=1}^p (-1)^{k+i} [X_{i_k}, X_{i_k}] \alpha(X_{i_1}, X_{i_2}, \ldots, \widehat{X}_{i_k}, \ldots, X_{i_p}). $$

The second term in the right hand side of the above equation will be denoted by II. Set $[X_i, X_j] = \sum_{j=1}^N b_{i,'}^j X_j$. Since $G$ is compact, the structure constants $b_{i,'}^j$ are skew symmetric with respect to $i, j, k$. Thus, we have

$$ II = - \sum_{i=1}^p \sum_{j=1}^p (-1)^{j+i} \sum_{k=1}^p b_{i,'}^j X_j \alpha(X_{i_1}, X_{i_2}, \ldots, \widehat{X}_{i_k}, \ldots, X_{i_p}) $$

by virtue of $[X_{i}, X_{j}] \in \mathfrak{f}$. On the other hand, for $j=n+1, \ldots, N$,...
\[ X \bar{\alpha}(X_{i_1}, \ldots, X_{i_p})(g) = \left. \frac{d}{dt} \right|_{t=0} \bar{\alpha}(g \cdot \exp tX_i)(X_{i_1}, \ldots, X_{i_p}) = \left. \frac{d}{dt} \right|_{t=0} \bar{\alpha}(g) (\exp tX_i \cdot X_{i_1}, \ldots, \exp tX_j \cdot X_{i_p}) = \sum_{\sigma=1}^p (-1)^{\sigma-1} \bar{\alpha}(g) ([X_j, X_{i_\sigma}], X_{i_1}, \ldots, X_{i_{\sigma-1}}, X_{i_\sigma}, \ldots, X_{i_p}) = \sum_{\sigma=1}^p (-1)^{\sigma-1} \sum_{\sigma=1}^p b_{j,i_\sigma} \bar{\alpha}(X_k, X_{i_1}, \ldots, \hat{X}_{i_\sigma}, \ldots, X_{i_p}) = \sum_{\sigma=1}^p (-1)^{\sigma-1} \sum_{\sigma=1}^p b_{j,i_\sigma} \bar{\alpha}(X_k, X_{i_1}, \ldots, \hat{X}_{i_\sigma}, \ldots, X_{i_p}). \]

Consequently, we have

\[ \Pi = - \sum_{j=1}^p X_j^2 \bar{\alpha}(X_{i_1}, \ldots, X_{i_p}), \]

which proves the desired formula.

q.e.d.

3. Some remarks in Kählerian cases

In this section, we assume that \( M = G/K \) is a homogeneous Kählerian manifold acted on by a compact Lie group \( G \). Denoting \( m = g/f \), we identify \( m \) with the tangent space of \( M \) at the origin \( e \cdot K \) by the projection \( G \to G/K \). We denote by \( J \) the complex structure on \( m \) and by \( \langle \cdot, \cdot \rangle \) the \( J \)-invariant inner product on \( m \) defined by the Kähler metric on \( M \). Then, the \( K \)-action on \( m \), which is identical to the isotropy representation at the origin, preserves \( J \) and \( \langle \cdot, \cdot \rangle \). Let \( U(m) \) be the group of linear automorphisms of \( m \) leaving \( J \) and \( \langle \cdot, \cdot \rangle \) invariant. Complexifying \( m \) and \( J \), we set \( m^c = m \otimes \mathbb{C} \) and \( m^\pm = \{ X \in m^c ; JX = \pm \sqrt{-1}X \} \). We have the direct sum as \( U(m) \)-modules and also as \( K \)-modules over \( \mathbb{C} \)

\[ m^c = m^+ \oplus m^- . \]

\( m^+ \) and \( m^- \) are conjugate-linearly \( K \)-isomorphic to each other by the conjugation \( X \to \overline{X} \) of \( m^c \) with respect to \( m \), and the complex bilinear extension \( \langle \cdot, \cdot \rangle^c \) of \( \langle \cdot, \cdot \rangle \) to \( m^c \) gives rise to canonical \( K \)-isomorphisms over \( \mathbb{C} \)

\[ (m^\pm)^* \cong m^\mp . \]

Now, we define a 2-form \( \omega \) of type (1.1) and a hermitian inner product \( \langle \cdot, \cdot \rangle^h \) on \( m^c \) by

\[ \omega(X, Y) = \langle X, JY \rangle^c \quad (X, Y \in m^c) , \]
\[ \langle X, Y \rangle^h = \langle X, \overline{Y} \rangle^c \quad (X, Y \in m^c) . \]

The \( U(m) \)-action and the hermitian inner product \( \langle \cdot, \cdot \rangle^h \) will be canonically ex-
tended to $\Lambda^q m \otimes \Lambda^p m^*$, and the later will be denoted simply by $\langle , \rangle$ in the sequel. $\Lambda^q m \otimes \Lambda^p m^*$ will be written by $\Lambda^{p,q} m$. $\omega$ is regarded as an element of $\Lambda^{1,1} M$ by (3.1).

We set $\Lambda^{0,q} = \{ u \in \Lambda^q m ; \langle u, \omega \wedge v \rangle = 0 \text{ for all } v \in \Lambda^{r-1,q-1} m \}$.

Then, the following fact is known (See A. Weil [13]).

**Proposition 3.1.**

(i) For $p$ and $q$ with $p+q \leq n$, the $U(m)$-irreducible decomposition of $\Lambda^{p,q} m$ is given by

\[
\Lambda^{p,q} m = \begin{cases}
\Lambda^p \otimes \omega \wedge \Lambda^{p-1,q-1} \otimes \cdots \otimes \omega^q \wedge \Lambda^{p,q} (p \geq q), \\
\Lambda^q \otimes \omega \wedge \Lambda^{q-1,p-1} \otimes \cdots \otimes \omega^p \wedge \Lambda^{q,p} (p \leq q).
\end{cases}
\tag{3.2}
\]

(ii) The mapping $u \mapsto \omega \wedge u$ of $\Lambda^{0,q}$ into $\omega^q \wedge \Lambda^{0,q}$ is a $K$-isomorphism, when max. $\{ p+k, q+k \} \leq n$.

The irreducibility of $\Lambda^{0,q}$ follows from Weyl's dimension formula.

Now, we denote by $\Lambda^{p,q} M$ the bundle of $(p+q)$-covectors on $M$ of type $(p, q)$.

Since the $G$-action preserves the complex structure on $M$, $\Lambda^{p,q} M$ is a homogeneous vector bundle;

\[
\Lambda^{p,q} M \cong G \times_k \Lambda^{1,1} M,
\tag{3.3}
\]

by virtue of (3.1). Under this isomorphism, the Kähler form of $M$ corresponds to the section of the bundle $G \times_k \Lambda^{1,1}$ given by

\[
\Omega_x = g \times_k \omega \quad (x = gK, x \in M),
\]

where $g \times_k \omega$ denotes the equivalence class of $(g, \omega)$. Then, we get immediately the corresponding decomposition of $\Lambda^{p,q} M$ for $p$ and $q$ with $p+q \leq n$,

\[
\Lambda^{p,q} M = \sum_i \Omega^i \wedge G \times_k \Lambda^{p+i,q-i}
\tag{3.4}
\]

where $\wedge$ is defined in an obvious manner.

Further, we define a bundle homomorphism $L$ of $\Lambda^{p,q} M$ into $\Lambda^{p+1,q+1} M$ by

\[
L \phi = \Omega_x \wedge \phi \quad \text{for } \phi \in \Lambda^{p,q} M_x
\]
on each fiber of $\Lambda^{p,q} M$. The $G$-invariance of $\Omega$ implies that $L$ commutes with the $G$-action and that the linear map induced from $L$ of $C^\infty(\Lambda^{p,q} M)$ into $C^\infty(\Lambda^{p+1,q+1} M)$ is a $G$-homomorphism, which we denote by the same letter $L$. By Proposition 3.1, the linear map $L^*$ of $C^\infty(\Lambda^{p,q} M)$ into $C^\infty(\Lambda^{p+k,q+k} M)$ is a $G$-isomorphism, when max. $\{ p+k, q+k \} \leq n$. Now, we denote by $L^*$ the bundle map adjoint to $L$ with respect to the fiber metric on $\Lambda^{p,q} M$.

\[
L^* : \Lambda^{p,q} M \to \Lambda^{p-1,q-1} M.
\]
The operator \( L^* \) commutes with the \( G \)-action and the subbundle \( \text{Ker} \ L^* \) of \( \Lambda^{p,q}M \) coincides with \( G \times \Lambda^{p,q}_s \). A section of \( G \times \Lambda^{p,q}_s \) is said to be a primitive form of type \( (p, q) \). It is known that the linear maps \( L \) and \( L^* \) commute with the Laplace operator. Laplacian preserves types of complex differential forms on \( M \). (See A. Weil [13].)

From the above arguments, in order to decompose \( C^\omega(\Lambda^{p,q}M) \) or \( C^\omega(\Lambda^{p+1,q}M) \) into the eigenspaces of the Laplace operator, or into the irreducible \( G \)-submodules, we may work on the space of primitive forms of type \( (r, s) \), \( C^\omega(G \times \Lambda^{r,s}_s) \).

Next, we recall effects of several operators acting on \( M \) on the spectra of \( \Delta \). More precisely, the exterior differentiation \( d \) decomposes
\[
d = \partial + \bar{\partial},
\]
where
\[
\partial : C^\omega(\Lambda^{p,q}M) \to C^\omega(\Lambda^{p+1,q}M)
\]
\[
\bar{\partial} : C^\omega(\Lambda^{p,q}M) \to C^\omega(\Lambda^{p+1,q+1}M).
\]

We denote by \( \partial^* \) and \( \bar{\partial}^* \) the operators formally adjoint to \( \partial \) and \( \bar{\partial} \) respectively. Then, the following formulas are known (A. Weil [13]);
\[
\partial^* = -\ast \bar{\partial}^*, \quad \bar{\partial}^* = -\ast \partial^*,
\]
\[
[L, \partial^*] = \sqrt{-1} \bar{\partial}, \quad [L, \bar{\partial}^*] = -\sqrt{-1} \partial,
\]
\[
[L^*, \partial] = \sqrt{-1} \partial^*, \quad [L^*, \bar{\partial}] = -\sqrt{-1} \bar{\partial}^*,
\]
where \([ , , ]\) denotes the commutator of two operators, for example,
\[
[L, \partial^*] = L \partial^* - \partial^* L.
\]

We have
\[
\Delta = 2(\partial \partial^* + \bar{\partial} \bar{\partial}) = 2(\bar{\partial} \partial^* + \bar{\partial}^* \partial).
\]

With respect to the splitting \( \Lambda^M = \sum_{p+q=r} \Lambda^{p,q}M \) and the above operators, we have the following decompositions of each eigenspace.

For each eigenvalue \( \lambda \) of \( \Delta \) on \( C^\omega(\Lambda^{p,q}M) \), we set
\[
E^{r,q}_\lambda = E^{r,q}_\lambda \cap C^\omega(\Lambda^{p,q}M),
\]
\[
'E^{r,q}_\lambda = \{ \phi \in E^{r,q}_\lambda ; \partial \phi = 0 \}, \quad ''E^{r,q}_\lambda = \{ \phi \in E^{r,q}_\lambda ; \partial^* \phi = 0 \},
\]
\[
',E^{r,q}_\lambda = \{ \phi \in E^{r,q}_\lambda ; \bar{\partial} \phi = 0 \}, \quad ,E^{r,q}_\lambda = \{ \phi \in E^{r,q}_\lambda ; \bar{\partial}^* \phi = 0 \}.
\]

These are all finite dimensional \( G \)-submodules of \( C^\omega(\Lambda^{p,q}M) \).

In our case, Hodge's decomposition theorem is expressed as
\[
C^\omega(\Lambda^{p,q}M) = E^{r,q}_\omega \bigoplus \partial C^\omega(\Lambda^{r-1,q}M) \bigoplus \partial^* C^\omega(\Lambda^{p+1,q}M).
\]
Hence, for \( \lambda \neq 0 \), we have

\[
E_\lambda^{p,q} = E_\lambda^{p,q} \oplus E_\lambda^{p,q-1}
\]

Further, we have the following \( G \)-isomorphisms,

\[
\begin{align*}
\vartheta : & E_\lambda^{p,q} \cong E_\lambda^{p+1,q}, \\
\vartheta^* : & E_\lambda^{p,q} \cong E_\lambda^{p-1,q}, \\
\bar{\vartheta} : & E_\lambda^{p,q} \cong \bar{E}_\lambda^{p,q+1}, \\
\bar{\vartheta}^* : & E_\lambda^{p,q} \cong \bar{E}_\lambda^{p,q-1},
\end{align*}
\]

where \( \lambda \neq 0 \), and

\[
\begin{align*}
*: & E_\lambda^{p,q} \cong E_\lambda^{n,q-p}, \\
*: & E_\lambda^{p,q} \cong E_\lambda^{n,q-\bar{p}}.
\end{align*}
\]

4. The spectra of Laplacian on the spheres

We retain the notations used in the preceding sections. In this section, we employ the following notations:

\[
G = SO(n+1), \quad K = \left\{(1 \ 0) \in M_{n+1}(R); \ A \in SO(n)\right\},
\]

\[
g = \{X \in M_{n+1}(R); \ X+X = 0\}, \quad \mathfrak{f} = \left\{(0 \ 0) \in M_{n+1}(R); \ 'Y+Y = 0\right\},
\]

\[
f = \left\{2\pi \left(\begin{array}{c}
0
\lambda_1
\vdots
0
\lambda_m
\end{array}\right); \ \lambda_1, \ldots, \lambda_m \in \mathbb{R}\right\},
\]

where \( n=2m-1 \) for an odd \( n \) and \( n=2m \) for an even \( n \). \( \mathfrak{f} \) is a Cartan subalgebra of \( g \). And \( \mathfrak{f} \) is also a Cartan subalgebra of \( \mathfrak{f} \) when \( n \) is even. If \( n \) is odd, then, the subspace \( \mathfrak{f}_i \) consisting of the elements of \( \mathfrak{f} \) with \( \lambda_i=0 \) forms a Cartan subalgebra of \( \mathfrak{f} \). We consider \( \lambda_1, \ldots, \lambda_m \) as linear forms on \( \mathfrak{f} \) and take a linear order on \( \mathfrak{f}^* \) such that \( \lambda_1 > \cdots > \lambda_m > 0 \). The Killing form \( B \) of \( g \) is given by

\[
B(X, Y) = (n-1) \text{tr} XY \quad (X, Y \in \mathfrak{g})
\]

and \( m \) is naturally identified with

\[
\left\{ \left(\begin{array}{c}
0
-x_1
\vdots
-x_n
\end{array}\right); \ x_1, \ldots, x_n \in \mathbb{R}\right\}.
\]

We adopt the usual metric on \( S^n \) imbedded in \( \mathbb{R}^{n+1} \) with radius one. We may identify \( S^n = G/K \) and \( TS^n = G \times \mathbb{R}^m \). Note that the metric induced from
the Killing form $B$ is $2(n-1)$-times the usual one and that the formulas recalled in 1 and 2 hold with certain constant multiples.

Any dominant integral form $\Lambda$ of $G$ with respect to $\mathfrak{f}$ is uniquely expressed as

$$\Lambda = k_1\lambda_1 + \cdots + k_m\lambda_m,$$

where $k_1, \ldots, k_m$ are integers satisfying

$$\begin{cases} k_1 \geq k_2 \geq \cdots \geq k_{m-1} \geq |k_m| & (n = 2m-1), \\ k_1 \geq k_2 \geq \cdots \geq k_{m-1} \geq k_m \geq 0 & (n = 2m). \end{cases}$$

When $n$ is odd, we set

$$z_j = \lambda_{j+1} | f_1 \quad (j = 1, \ldots, m-1),$$

where right hand side denotes the restriction of the linear form $\lambda_{j+1}$ on $\mathfrak{f}$ to the subspace $\mathfrak{f}_1$, and $z_1, \ldots, z_{m-1}$ are ordered as $z_1 > \cdots > z_{m-1}$.

**Proposition 4.1.** Let $(V, \rho)$ be an irreducible $G$-module over $\mathbb{C}$ with the highest weight $\lambda_0 = k_1\lambda_1 + \cdots + k_m\lambda_m$, where $k_1, \ldots, k_m$ satisfy (4.1). Then, as a $K$-module, $V$ decomposes into $K$-irreducible submodules as follows;

(i) In case $n=2m$,

$$V = \sum_{k_1, \ldots, k_m} V_{k_1\lambda_1 + \cdots + k_m\lambda_m},$$

where the summation runs over all integers $k_1, \ldots, k_m$ such that

$$k_1 \geq k_2 \geq \cdots \geq k_{m-1} \geq |k_m|,$$

and $V_{k_1\lambda_1 + \cdots + k_m\lambda_m}$ denotes the irreducible $K$-submodule of $V$ with the highest weight $k_1\lambda_1 + \cdots + k_m\lambda_m$.

(ii) In case $n=2m-1$,

$$V = \sum_{k_1, \ldots, k_{m-1}} V_{k_1\lambda_1 + \cdots + k_{m-1}\lambda_{m-1}},$$

where the summation runs over all integers $k_1, \ldots, k_{m-1}$ such that

$$k_1 \geq k_2 \geq \cdots \geq k_{m-1} \geq |k_m|,$$

and the meaning of $V_{k_1\lambda_1 + \cdots + k_{m-1}\lambda_{m-1}}$ is similar to the above.

For a proof, see H. Boerner [3].

Applying this proposition to our problem, we shall give explicitly the irreducible representations of $SO(n+1)$ intervening in $C^\infty(\Lambda^\bullet S^n)$.

The multiplicity $\mu_\rho$ of an irreducible representation $(V, \rho)$ of $G$ in $C^\infty(\Lambda^\bullet S^n)$ is equal to $\dim_{\mathbb{C}} \text{Hom}_G(V, \Lambda^n S^n)$ by Proposition 1.1, and the latter can be computed, applying Schur’s Lemma, from the $K$-irreducible decomposition of $\Lambda^n S^n$. 

and Proposition 4.1.

Now, identifying $K = SO(n)$, the $K$-module $m$ is isomorphic to $\mathbb{R}^n$ with the standard representation of $SO(n)$, and we know the following facts:

A. Suppose $n$ is odd, $n = 2m - 1$. Then, $C \otimes_R \Lambda^p \mathbb{R}^n$ is an irreducible $SO(n)$-module with the highest weight $\lambda_1 + \cdots + \lambda_p$ for each $p \leq m - 1$. For $0 \leq p \leq n$, $C \otimes_R \Lambda^p \mathbb{R}^n = C \otimes_R \Lambda^{n-p} \mathbb{R}^n$ as $SO(n)$-modules.

B. Suppose $n$ is even, $n = 2m$. Then, $C \otimes_R \Lambda^p \mathbb{R}^n$ is an irreducible $SO(n)$-module with the highest weight $\lambda_1 + \cdots + \lambda_p$ for each $p < m$, and $C \otimes_R \Lambda^p \mathbb{R}^n$ splits into two irreducible submodules with the highest weight $\lambda_1 + \cdots + \lambda_{m-1} - \lambda_m$ and $\lambda_1 + \cdots + \lambda_{m-1} + \lambda_m$ respectively. For $p > m$, $C \otimes_R \Lambda^p \mathbb{R}^n = C \otimes_R \Lambda^{n-p} \mathbb{R}^n$ as $SO(n)$-modules.

Now, we put

$$
\Lambda_0 = 0, \\
\Lambda_j = \lambda_1 + \cdots + \lambda_j \quad (j = 1, 2, \cdots, m-2), \\
\Lambda_{m-1} = \left\{ \begin{array}{ll}
\lambda_1 + \cdots + \lambda_{m-1} & (n = 2m), \\
\frac{1}{2} (\lambda_1 + \cdots + \lambda_{m-1} - \lambda_m) & (n = 2m-1),
\end{array} \right. \\
\Lambda_m = \frac{1}{2} (\lambda_1 + \cdots + \lambda_m).
$$

$\Lambda_1, \cdots, \Lambda_m$ are the so-called fundamental weights of $\mathfrak{g}$ and every dominant integral form of $G$ is uniquely expressed as a linear combination with non-negative integer coefficients of $\Lambda_1, \cdots, \Lambda_m, 2\Lambda_m$, when $n=2m$, and of $\Lambda_1, \cdots, \Lambda_{m-2}, 2\Lambda_{m-1}, \Lambda_{m-1} + \Lambda_m, 2\Lambda_m$, when $n=2m-1$. In our case, we remark $m^* = m$ (as $K$-modules). With these terminologies, by the above procedure we get easily the spectra of Laplacian on $S^n$ as given in the following theorem. Since $S^n$ is orientable, it would be sufficient to write them down for $p \leq \frac{n}{2}$.

**Theorem 4.2.**

(a) Suppose $p \leq \frac{n}{2}$. The highest weights $\lambda_\rho$ of the irreducible representations $\rho$ intervening in $C^\omega(\Lambda^p S^n)$, that is, $\rho$ in $\mathcal{J}_G$ with $\mu_\rho \geq 1$, are as follows;

(i) In case $n = 2m$,

$$
\lambda_\rho = \left\{ \begin{array}{ll}
k\Lambda_1 + \Lambda_p, & (0 \leq p \leq m-2), \\
k\Lambda_1 + \Lambda_{p+1}, & (p = m-2), \\
k\Lambda_1 + \Lambda_{m-1}, & (p = m-1), \\
k\Lambda_1 + 2\Lambda_m & (p = m),
\end{array} \right.
$$

where $k$ runs over all non-negative integers.

(ii) In case $n = 2m - 1$, 


Further, the multiplicity $\mu_p$ of the above $\rho$ in $C^\infty(\Lambda^p S^n)$ is exactly one except for the case $n=2m-1$ and $p=m$, and $\mu_p$ is two in this exceptional case.

(b) The Laplace operator has eigenvalue $4\pi^2 \langle \lambda_p + 2\delta_G, \lambda_p \rangle$ on an $SO(n+1)$-irreducible submodule of differential forms on $S^n$ with the highest weight $\lambda_p$, where $\langle , \rangle$ denotes the inner product on $\mathfrak{f}^*$ induced from $-\frac{1}{2(n-1)}$-times the Killing form.

The values $4\pi^2 \langle \lambda_p + 2\delta_G, \lambda_p \rangle$ are given in the following table.

<table>
<thead>
<tr>
<th>$\lambda_p$</th>
<th>$4\pi^2 \langle \lambda_p + 2\delta_G, \lambda_p \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k\Lambda_1$</td>
<td>$(k+n-1)$</td>
</tr>
<tr>
<td>$k\Lambda_1 + \Lambda_p$ ($1 \leq p \leq m-1$)</td>
<td>$(k+p)(k+n+1-p)$</td>
</tr>
<tr>
<td>$k\Lambda_1 + 2\Lambda_m$</td>
<td>$(k+m)(k+m+1)$</td>
</tr>
<tr>
<td>$k\Lambda_1$</td>
<td>$(k+n-1)$</td>
</tr>
<tr>
<td>$k\Lambda_1 + \Lambda_p$ ($1 \leq p \leq m-2$)</td>
<td>$(k+p)(k+n+1-p)$</td>
</tr>
<tr>
<td>$k\Lambda_1 + 2\Lambda_{m-1}$</td>
<td>$(k+m)^2$</td>
</tr>
<tr>
<td>$k\Lambda_1 + \Lambda_{m-1} + \Lambda_m$</td>
<td>$(k+m-1)(k+m+1)$</td>
</tr>
<tr>
<td>$k\Lambda_1 + 2\Lambda_m$</td>
<td>$(k+m)^2$</td>
</tr>
</tbody>
</table>

Remark. For the space of real differential forms on $S^n$, its irreducible decomposition can be obtained from Theorem 4.2 together with the results of N. Bourbaki [5] and N. Iwahori [8].

Except for the case $n=2m-1$ and $m$ is even, every irreducible submodule of $C^\infty(\Lambda^p S^n)$ is closed under the complex conjugation, and the space of real forms contained in this submodule is an irreducible $SO(n+1)$-module over $\mathbb{R}$. When $n=2m-1$ and $m$ is even, two irreducible submodules of differential forms with the highest weight $k\Lambda_1 + 2\Lambda_{m-1}$ and $k\Lambda_1 + 2\Lambda_m$ are transformed to each other under the complex conjugation, and the space of real forms in the sum of the two submodules is irreducible over $\mathbb{R}$. Conversely, any irreducible submodule of real forms on $S^n$ is obtained in this way.

5. The spectra of Laplacian on the complex projective spaces

In this section, we employ the following notations;

$$G = SU(n+1)$$
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\[ K = S(U(1) \times U(n)) = \left\{ \begin{pmatrix} \varepsilon & 0 \\ 0 & A \end{pmatrix} \in SU(n+1); \varepsilon \in U(1), A \in U(n) \right\}, \]

\[ g = \{X \in M_{n+1}(C); \frac{1}{4}X^2 + X = 0, \text{tr}X = 0\}, \]

\[ \mathfrak{f} = \left\{ \begin{pmatrix} \sqrt{-1}a & 0 \\ 0 & Y \end{pmatrix} \in M_{n+1}(C); \frac{1}{4}Y + Y = 0, a \in \mathbb{R}, \sqrt{-1}a + \text{tr}Y = 0 \right\}, \]

\[ B(X, Y) = 2(n+1)\text{tr}XY \quad (X, Y \in g), \]

\[ m = \left\{ \begin{pmatrix} \zeta_1 & -\bar{\zeta}_1 & \cdots & -\bar{\zeta}_n \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in M_{n+1}(C); \zeta_1, \ldots, \zeta_n \in C \right\}, \]

\[ f = \left\{ 2\pi \sqrt{-1} \begin{pmatrix} x_1 \\ 0 \\ \cdots \\ x_{n+1} \end{pmatrix}; x_1, \ldots, x_{n+1} \in \mathbb{R}, x_1 + \cdots + x_{n+1} = 0 \right\}. \]

The natural complex structure \( J \) on \( m \) is given by

\[ J \begin{pmatrix} \zeta_1 & -\bar{\zeta}_1 & \cdots & -\bar{\zeta}_n \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sqrt{-1}\zeta_1 & \cdots & -\sqrt{-1}\zeta_n \\ \sqrt{-1}\zeta_1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \sqrt{-1}\zeta_n & \cdots & \cdots & 0 \end{pmatrix}, \]

and hence \( m \) may be identified with \( \mathbb{C}^n \).

Then, we may identify \( P^n(C) = G/K \), and as complex vector bundle, we have \( TP^n(C) \cong G \times_\kappa m \).

We adopt the usual Fubini-Study metric on \( P^n(C) \). Note that the metric induced from the Killing form \( B \) is \((n+1)\)-times the Fubini-Study metric and the results of \( 1 \) and \( 2 \) hold with certain constant multiples.

Now, we consider the above \( x_1, \ldots, x_{n+1} \) as linear forms on \( f \) and introduce a linear order on \( f^* \) such that

\[ x_1 > x_2 > \cdots > x_n > 0 > x_{n+1}. \]

We note that the image of \( K \) in \( GL(m) \) under the adjoint representation in \( m \) is exactly \( U(m) \), the unitary group of \( m = \mathbb{C}^n \). Hence, the \( K \)-modules \( \Lambda^\kappa \) are all irreducible. The weights of \( m \) with respect to \( f \) are \( \{x_i - x_1; i=2, 3, \ldots, n+1\} \) and the highest weight of \( \Lambda^\kappa \) is \( (x_2 - x_1) + \cdots + (x_{n+1} - x_1) + (x_{n+1} - x_1) + \cdots + (x_1 - x_{n+2-p}) \).

Next, we shall give a general formula to decompose any irreducible \( G \)-module into irreducible \( K \)-modules. In our case, \( K \) is of maximal rank, and every dominant integral form \( \Lambda \) of \( G \) and \( K \) with respect to \( f \) is uniquely expressed as

\[ \Lambda = k_1 x_1 + k_2 x_2 + \cdots + k_n x_n, \]

where \( k_1, k_2, \ldots, k_n \) are integers satisfying \( k_1 \geq \cdots \geq k_n \geq 0 \) or \( k_2 \geq \cdots \geq k_n \geq 0 \).
according as the group is \( G \) or \( K \). We denote the Weyl groups of \( G \) and \( K \) by \( W_G \) and \( W_K \) respectively. \( W_G \) is the permutation group of \( x_1, \ldots, x_{n+1} \) and \( W_K \) is the permutation group of \( x_2, \ldots, x_{n+1} \).

**Proposition 5.1.** Let \((V, \rho)\) be an irreducible \( G \)-module with the highest weight \( \lambda = m_1 x_1 + \cdots + m_n x_n \) \((m_1 \geq \cdots \geq m_n \geq 0)\). Then, \((V, \rho)\) decomposes, as a \( K \)-module, into irreducible \( K \)-modules as follows:

\[
V = \sum V_{k_1 x_1 + k_2 x_2 + \cdots + k_n x_n}
\]

where the summation runs over all the integers \( k_1, \ldots, k_n \) for which there exists an integer \( k \) satisfying

\[
m_1 \geq k + k \geq m_2 + k \geq m_3 + k \geq \cdots \geq m_{n-1} + k \geq m_n \geq k \geq 0,
\]

\[
k_1 = \sum_{i=1}^{n} m_i - \sum_{i=2}^{n} k_i -(n+1)k.
\]

Proof. For an element \( x \in \mathfrak{f}^* \), we denote by \( \xi_G^x \) and \( \xi_K^x \) the principal alternating sum under the Weyl groups \( W_G \) and \( W_K \) respectively.

\[
\xi_G^x = \sum_{\sigma \in W_G} (-1)^{\sigma} e^x,
\]

\[
\xi_K^x = \sum_{\tau \in W_K} (-1)^{\tau} e^x,
\]

where \( e^x = \exp 2\pi \sqrt{-1} x \), and \((-1)^{\sigma}\) denotes the signature of permutation \( \sigma \). For irreducible representations \((V, \rho)\) and \((V', \rho')\) of \( G \) and \( K \) respectively, we denote their characters by \( \chi_\rho \) and \( \chi_\rho' \). Then, by the character formula of Weyl, we have

\[
(1) \quad \xi_G^x \chi_\rho = \xi_G^{\rho + \delta_G} \quad \text{on } \mathfrak{f}, \\
\xi_K^x \chi_{\rho'} = \xi_K^{\rho' + \delta_K} \quad \text{on } \mathfrak{f}.
\]

Now, we have

\[
\delta_G = \frac{1}{2} \sum_{i<j} (x_i - x_j) = \sum_{i=1}^{n+1} (n+1-i)x_i,
\]

\[
\delta_K = \frac{1}{2} \sum_{i<j} (x_i - x_j) = \sum_{i=2}^{n+1} (n+1-i)x_i + \frac{n-1}{2} x_1,
\]

and put

\[
\begin{aligned}
p_i &= m_i + n + 1 - i \quad (i = 1, 2, \ldots, n) \\
p_{n+1} &= 0, \\
\lambda + \delta_G &= p_1 x_1 + p_2 x_2 + \cdots + p_n x_n.
\end{aligned}
\]

Then, we have

\[
(2) \quad \xi_G^{\rho + \delta_G} = \det (e^{ip_i x_i})_{i,j=1,2,\ldots,n+1},
\]
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\[
\xi_{i_0}^R = \prod_{i < j} (e^{(x_j - x_i)x_1} - e^{(x_i - x_j)x_1}),
\]
\[
\xi_{i_0}^L = \prod_{z_i < i} (e^{x_i - x_jx_1} - e^{x_j - x_ix_1}).
\]

Hence, putting \(z_i = x_i - x_1 (i = 2, \ldots, n + 1)\), we have

\[
(3) \quad \xi_{i_0}^L = \prod_{i=2}^{n+1} (-1)^{e^{(x_i - x_1)x_1}}(e^{x_i - x_1x_1})^{-1}
\]

We compute the right hand side of (2) as follows;

\[
\begin{vmatrix}
\begin{array}{c}
p_{i_1}x_1, \ldots, p_{i_n}x_{n+1} \\
\vdots \\
p_{i_n}x_1, \ldots, p_{i_1}x_{n+1} \\
1, \ldots, 1
\end{array}
\end{vmatrix} = e^{(p_1 + \cdots + p_n)x_1}
\]

\[
= (-1)^n e^{p_1x_1} \prod_{j=2}^{n+1} (e^{x_j - 1} - 1, \ldots, e^{x_{n+1} - 1} - 1) \begin{vmatrix}
\begin{array}{c}
p_{i_1}x_1, \ldots, p_{i_n}x_{n+1} \\
\vdots \\
p_{i_n}x_1, \ldots, p_{i_1}x_{n+1} \\
1, \ldots, 1
\end{array}
\end{vmatrix} = (-1)^n e^{p_1x_1} \prod_{j=2}^{n+1} (e^{x_j - 1} - 1) \sum' \det (e^{x_jx_j})_{i=1,2,\ldots,n}
\]

where the summation \(\sum'\) runs over all integers \(q_1, \ldots, q_n\) satisfying

\[p_1 - 1 \geq q_1 \geq p_2, p_2 - 1 \geq q_2 \geq p_3, \ldots, p_n - 1 \geq q_n \geq 0.\]

Further, we have

\[\det (e^{x_jx_j})_{i,j} = e^{-(q_1 + \cdots + q_n)x_1} \cdot \det (e^{x_jx_j})_{i,j}.\]

Hence, from (1) and (3), we get

\[\xi_{i_0}^L \chi_p = \sum' e^{[p - (q_1 + \cdots + q_n)x_1]x_1 + 1/2(z_2 + \cdots + z_{n+1})} \cdot \det (e^{x_jx_j}).\]

Here we note that \(z_2 + \cdots + z_{n+1} = -(n+1)x_1\).

We put
Then, we get easily
\[ \varepsilon^K G \rho = \sum \varepsilon^K_{k_1 \cdots k_n} + s_{n+1} \]
where the summation runs over all integers \( k_1, \cdots, k_n \) satisfying
\[ m_1 \geq k_2 + q_k \geq m_2 \geq k_3 + q_k \geq \cdots \geq k_n + q_k \geq m_k \geq q_k \geq 0. \]

Therefore, by the character formula of Weyl, we get immediately the proposition.

q.e.d.

The highest weight of \( \Lambda^p \) with \( p+q \leq n \), which we have seen before, is described according to the type of \((p, q)\) as follows;

(i) \( p=0, q=0 \)
(ii) \( p=0, n>q>0, \quad -qx_2 - x_3 - \cdots - x_{q+1} \)
(iii) \( n>p>0, q=0 \)
(iv) \( p>0, q>0 \)
(v) \( p-q+1 \)
(vi) \( p=q\)
(vii) \( p=n, q=0 \)

By Proposition 5.1 and Frobenius' reciprocity, an irreducible \( G \)-module with the highest weight \( \chi = m_1 x_1 + m_2 x_2 + \cdots + m_n x_n \) appears in the irreducible decomposition of \( C^\omega(G \times K \Lambda^p \), if and only if \( m_i \)'s satisfy the condition given by Proposition 5.1. For instance, in case (iii), there exists an integer \( k \) such that
\[ m_1 \geq k \geq m_2 \geq k \geq \cdots \geq k \geq m_{n+1-p} \geq k \geq m_{n+2-p} \geq \cdots \geq m_n \geq k \geq 0. \]
The conditions are similar to the case above in the other cases.

We set \( \Lambda_0 = 0, \quad \Lambda_j = x_1 + x_2 + \cdots + x_j \)
and \( \Lambda_{n+1} = 0 \). Then, \( \Lambda_1, \cdots, \Lambda_n \) are the so-called fundamental weights of \( G \) so that every dominant integral form of \( G \) is uniquely written as a non-negative integral linear combination of them. We set
\[ \Lambda(k, r, s) = k(\Lambda_1 + \Lambda_r) + (r-s)\Lambda_1 + \Lambda_s + \Lambda_{n-r+1}, \]
where \( k \) satisfies the condition
\[ k + r - s \geq 0 \quad \text{and} \quad k \geq 0. \]

Then, the highest weights of irreducible \( G \)-modules appearing in \( C^\omega(G \times K \Lambda^p) \) are of the following forms
(\#, \( p, q \)) in case (i), \( \Lambda(k, 0, 0) \),
(\#, \( p, q \)) in case (ii), \( \Lambda(k, 0, q), \Lambda(k, 0, q+1) \),
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in case (iii), \( \Lambda(k, p, 0), \Lambda(k, p+1, 0) \),
in case (iv), \( \Lambda(k, p, q), \Lambda(k, p, q+1), \Lambda(k, p+1, q), \Lambda(k, p+1, q+1) \),
in case (v), \( \Lambda(k, p, q), \Lambda(k, p, q+1), \Lambda(k, p+1, q) \),
in case (vi), \( \Lambda(k, 0, n) \),
in case (vii), \( \Lambda(k, n, 0) \),
where \( k \)'s are integers satisfying (*)

Moreover, the multiplicity of the representation with the highest weight mentioned above in \( C^\infty(G \times_k \mathbb{A}^{s \otimes \delta}) \) is exactly one.

Now, we denote by \( \langle \, , \rangle \) the inner product on \( \mathfrak{f}^* \) induced from \( \frac{1}{n+1} B \). For \( \Lambda = k(\Lambda_1 + \Lambda_n) + (r-s)\Lambda_1 + \Lambda_s + \Lambda_{n-r+1} \), we have by simple computations,

\[
4\pi^2 \langle \Lambda + 2\delta_0, \Lambda \rangle = \begin{cases} 
(k+r) (k+n+2-s) & (1 \leq r, s \leq n), \\
(k+r) (k+n+1) & (1 \leq r \leq n, s=0), \\
k(k+n+1-s) & (r=0, 1 \leq s \leq n), \\
k(k+n) & (r=0, s=0), \\
(k+n)(k+n+1) & (r=n+1, s=0), \\
k(k+1) & (r=0, s=n+1).
\end{cases}
\]

Thus, we get

**Theorem 5.2.** Let \( P^\circ(C) = SU(n+1)/SU(1) \times U(n) \).

(a) Let \( p \) and \( q \) be non-negative integers with \( p+q \leq n \). The highest weight \( \lambda_p \) of the irreducible representation \( \rho \) of \( SU(n+1) \) intervening in \( C^\infty(G \times_k \mathbb{A}^{s \otimes \delta}) \), that is, \( \rho \) in \( \mathcal{C}_G \) with \( \mu_\mathfrak{h} \geq 1 \), are those of \((q, r, s) \) with \( r-s = p-q, r \leq p \) and \( s \leq q \).

For \( P^\circ(C) \) with the Fubini-Study metric, the multiplicity of the above representation \( \rho \) in the space of primitive forms of type \((r, s) \) is one.

(b) On the irreducible \( SU(n+1) \)-submodule of differential forms with the highest weight \( \Lambda(k, r, s) \), the Laplace operator defined by the Fubini-Study metric has the eigenvalue \( (k+r) (k+n+2-s) \) for \( 1 \leq r, s \leq n \), \( (k+r) (k+n+1) \) for \( 1 \leq r \leq n \) and \( s=0 \), \( k(k+n+1-s) \) for \( r=0 \) and \( 1 \leq s \leq n \), \( k(k+n) \) for \( r=s=0 \), \( (k+n)(k+n+1) \) for \( r=n+1 \) and \( s=0 \), and \( k(k+1) \) for \( r=0 \) and \( s=n+1 \).

**Remark.** For the space of real differential forms on \( P^\circ(C) \), its irreducible decomposition can be obtained from Theorem 5.2 together with the results of N. Bourbaki [5] and N. Iwahori [8]. Two irreducible representations of \( SU(n+1) \) over \( C \) with the highest weight \( \Lambda_j \) and \( \Lambda_{n-j+1} \) are anti-isomorphic to each other (\( j = 1, \ldots, n \)). We denote by \( h^sV_{r, s}^k \) the \( SU(n+1) \)-irreducible submodule of primitive forms of type \((p, q) \) with the highest weight \( k(\Lambda_1 + \Lambda_n) + (r-s)\Lambda_1 + \Lambda_s + \Lambda_{n-r+1} \). \( h^sV_{r, s}^k \) is complex conjugate of \( s^hV_{r, s}^{h^s \otimes r-s} \) and the space \( h^sU_{r, s}^k \) of real forms in \( h^sV_{r, s}^k + s^hV_{r, s}^{h^s \otimes r-s} \) is an irreducible \( SU(n+1) \)-module over \( R \). Conversely, any irreducible \( SU(n+1) \)-submodule over \( R \) in the space of real forms
on $P^h(C)$ is of the form $\Omega^h \wedge \Lambda^k U^h r, s$ for some $h, k, p, q, r$ and $s$.

6. Eigenforms on $S^n$ and the harmonic polynomial forms on $R^{n+1}$

In the following 6 and 7, we designate simply by $\Lambda^p M$ for $C^\infty(\Lambda^p M)$, the space of all differential $p$-forms on $M$.

Let $R^{n+1}$ be an $(n+1)$-dimensional real Euclidean space and $(x^0, x^1, \ldots, x^n)$ be the standard coordinate system on $R^{n+1}$. We put $r^2 = \sum_{i=0}^n (x^i)^2$, $r \frac{d}{dr} = \sum_{i=0}^n x^i \frac{d}{dx^i}$, $rdr = \sum_{i=0}^n x^i dx^i$. Let $d_\circ$ be the differential on $\Lambda^*(R^{n+1}) = \bigoplus_{p=0}^{n+1} \Lambda^p(R^{n+1})$, $\delta_\circ$ the codifferential and $*_\circ$ the Poincaré duality on $\Lambda^*(R^{n+1})$. We define a linear operator $e(rdr)$ of $\Lambda^p(R^{n+1})$ into itself by $e(rdr)\alpha = r dr \wedge \alpha$, for $\alpha \in \Lambda^p(R^{n+1})$, and denote by $i\left( r \frac{d}{dr} \right)$ the interior product by $r \frac{d}{dr}$ on $\Lambda^*(R^{n+1})$. Then the following lemma is easily verified.

**Lemma 6.1.** Let $\alpha$ be a differential $p$-form on $R^{n+1}$. Then we have the following formulas:

1. $i\left( r \frac{d}{dr} \right) *_\circ \alpha = (-1)^{p} *_\circ e(rdr) \alpha$,

2. $e(rdr) d_\circ \alpha = d_\circ e(rdr) \alpha$.

3. $d_\circ \left( r \frac{d}{dr} \right) \alpha + i\left( r \frac{d}{dr} \right) d_\circ \alpha = L_X (r \frac{d}{dr}) \alpha$.

4. $\delta_\circ e(rdr) \alpha + e(rdr) \delta_\circ \alpha = (-1)^{p+1} L_X (r \frac{d}{dr}) *_\circ \alpha$,

where $L_X$ denotes the Lie derivation by a vector field $X$ on $R^{n+1}$.

We denote by $\Delta = d_\circ \delta_\circ + \delta_\circ d_\circ$ the Laplace operator on the space of differential forms on $R^{n+1}$. Then the operators $d_\circ, \delta_\circ, \Delta, e(rdr)$ and $i\left( r \frac{d}{dr} \right)$ acting on $\Lambda^*(R^{n+1})$ commute with the natural action of $O(n+1)$, the orthogonal group of $R^{n+1}$, on $\Lambda^*(R^{n+1})$.

Now, let $P^h_k$ be the set of $\alpha \in \Lambda^h(R^{n+1})$ of the form

$\alpha = \sum_{0 \leq i_1 < \cdots < i_p \leq n} \alpha_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$,

where $\alpha_{i_1 \cdots i_p}$ are homogeneous polynomials of degree $k$. 
Proposition 6.2. Let $\alpha \in P^k_\delta$. Then we have
\begin{enumerate}
\item $d_\delta \left( r \frac{d}{dr} \right) \alpha + i \left( r \frac{d}{dr} \right) d_\delta \alpha = (k + p) \alpha$
\item $\delta_\delta (r dr) \alpha + e (r dr) \delta_\delta \alpha = -(n + 1 - p + k) \alpha$
\end{enumerate}

Proof. Let $\alpha \in P^k_\delta$. Then we can see easily $L_r (d_\delta \alpha) = (k + p) \alpha$. Combining this with (3), (4) in Lemma 6.1, we have Proposition 6.2.

Corollary 6.3. We have the following direct sum decompositions;
\begin{align*}
P^k_0 & = (\text{Ker } d_\delta \cap P^k_\delta) \oplus (\text{Ker } i \left( r \frac{d}{dr} \right) \cap P^k_\delta) \quad (k + p = 0), \\
P^k_p & = (\text{Ker } \delta_\delta \cap P^k_\delta) \oplus (\text{Ker } e (r dr) \cap P^k_\delta) \quad (n + 1 - p + k = 0)
\end{align*}

Remark. We have
\begin{align*}
P^0_0 & = \{ \text{constant functions on } \mathbb{R}^{*+1} \}, \\
P^0_{p+1} & = \{ a \; dx^0 \wedge \cdots \wedge dx^p : a \in C \}.
\end{align*}

Definition. A harmonic polynomial form is an element $\alpha \in P^k_\delta$ such that $\Delta \alpha = 0$ and $\delta_\delta \alpha = 0$. Let $H^k_\delta$ be the set of these forms;
\[ H^k_\delta = \text{Ker } \Delta \cap \text{Ker } \delta_\delta \cap P^k_\delta. \]

Lemma 6.4. We have the direct sum decomposition;
\[ P^k_\delta = H^k_\delta \oplus (r^2 P^k_{p-2} + e (r dr) P^k_{p-1}). \]

Proof. Let $S^k(\mathbb{R}^{*+1})$ and $\Lambda^k(\mathbb{R}^{*+1})$ be the real vector spaces of a symmetric tensors of degree $k$ and antisymmetric tensors of degree $p$ over $\mathbb{R}^{*+1}$. Then we have the natural isomorphism between $P^k_\delta$ and $S^k(\mathbb{R}^{*+1}) \otimes \Lambda^k(\mathbb{R}^{*+1}) \otimes C$ as $O(n+1)$-modules, where $O(n+1)$ acts identically on $C$. We consider $S^k(\mathbb{R}^{*+1}) \otimes \Lambda^k(\mathbb{R}^{*+1})$ as a subspace of $T^{k+p}(\mathbb{R}^{*+1})$, the real vector space of tensors of degree $k+p$. We define linear operators $s_i$ of $T^{k+p}(\mathbb{R}^{*+1})$ to $T^{k+p-2}(\mathbb{R}^{*+1})$ by
\[ s_i(e_1 \otimes \cdots \otimes e_{k+p}) = \langle e_i, e \rangle e_1 \otimes \cdots \otimes \hat{e}_i \otimes \cdots \otimes e_{k+p}, \]
where $\langle , \rangle$ denotes the standard inner product on $\mathbb{R}^{*+1}$. Put $\mathcal{H}^{k+p} = \bigcap_{1 \leq i \leq k+p} \text{Ker } s_i$ and $\mathcal{H}^{k+p}_\delta = \mathcal{H}^{k+p} \cap S^k(\mathbb{R}^{*+1}) \otimes \Lambda^k(\mathbb{R}^{*+1})$. We denote by $\mathcal{Q}^k_\delta$ the subspace orthogonal complement to $\mathcal{H}^{k+p}_\delta$ in $S^k(\mathbb{R}^{*+1}) \otimes \Lambda^k(\mathbb{R}^{*+1})$. Put $\mathcal{Q}^k_\delta = \mathcal{Q}^k_\delta \otimes \mathbb{R} \subset \mathcal{Q}^k_\delta \otimes C$ and $\mathcal{H}^{k+p}_\delta = \mathcal{H}^{k+p}_\delta \otimes C$. Then the subspaces in $P^k_\delta$ corresponding to $\mathcal{Q}^k_\delta$ and $\mathcal{H}^{k+p}_\delta$ by the natural isomorphism in $P^k_\delta$ are $Q^k_\delta = e (r dr) P^k_{p-1} + r^2 P^k_{p-2}$ and $H^k_\delta$.

q.e.d.

Proposition 6.5. Let $S^n$ be the unit sphere in $\mathbb{R}^{*+1}$ and $i$ be the inclusion of $S^n$ into $\mathbb{R}^{*+1}$. Then we have
(1) \(i^*: \text{Ker } \delta_0 \cap P_k^\delta \to \Lambda^p(S^n) \) is injective,

(2) \(i^*(P_k^\delta) = \sum_{\mu \geq 0} i^*(H^{\mu-2i}_k) \) (direct sum).

Proof. (2) follows easily by induction on \(k\) using Lemma 6.4, since \(i^*(r^2) = 1\) and \(i^*(rdr) = 0\). We shall show (1). Let \(\alpha \in \text{Ker } \delta_0 \cap P_k^\delta\). Suppose \(i^*\alpha = 0\). Then \(\alpha\) is in \(\text{Ker } e(rdr) \cap P_k^\delta\). On the other hand \(\delta_0\alpha = 0\), and so from Corollary 6.3, we have \(\alpha = 0\).

q.e.d.

**Corollary 6.6.** \(i^*: \sum H^i_k \to \Lambda^p(S^n) \) is injective and its image is dense in \(\Lambda^p(S^n)\).

Proof. By the polynomial approximation, \(i^*(\sum P^i_k)\) is dense in \(\Lambda^p(S^n)\). Proposition 6.5 implies that \(i^*(\sum P^i_k) = i^*(\sum H^i_k)\). Thus the density is proved.

Now, let \(\alpha_1 \in \sum H^i_k\) and \(\alpha_2 \in \sum H^{-i}_{k-1}\). Assume \(i^*(\alpha_1 + \alpha_2) = 0\). Then we shall show that \(\alpha_1 = \alpha_2 = 0\). By Proposition 6.5, we may take \(\bar{\alpha}_1, \bar{\alpha}_2 \) in \(P^i_{2k_0}\) and \(P^{-i}_{2k_0 - 1}\) respectively for some \(k_0\) satisfying \(i^*(\bar{\alpha}_1) = i^*(\alpha_1)\) and \(i^*(\bar{\alpha}_2) = i^*(\alpha_2)\). We have \((\bar{\alpha}_1 + \bar{\alpha}_2) \wedge rdr = 0\) on \(S^n\). To prove the corollary, it is now sufficient to show that if \(f_1, f_2\) are homogeneous polynomials of degree \(2k_0, 2k_0 - 1\) respectively, and if \(f_1 + f_2\) is zero on \(S^n\), then \(f_1 = f_2 = 0\) on \(R^{n+1}\). By the assumption, we see that \(f_1 + rf_2\) is zero on \(S^n\) and homogeneous on \(R^{n+1}\). Thus \(f_1 + rf_2 \equiv 0\) on \(R^{n+1}\). Substituting \(-x_i\) for \(x_i(0 \leq i \leq n)\), we have \(f_1 - rf_2 \equiv 0\) on \(R^{n+1}\). Thus we have \(f_1 = f_2 = 0\).

q.e.d.

**Lemma 6.7.** Let \(\alpha \in H^i_k\). Then we have \(d_0\alpha \in H^i_{k-1}\) and \(i\left(r \frac{d}{dr}\right)\alpha \in H^i_{k-1}\).

Proof. The first statement follows easily from the facts that \(\Delta = d_0\delta_0 + \delta_0 d_0\) and \(d_0\Delta = \Delta d_0\). We see \(i\left(r \frac{d}{dr}\right)\alpha \in H^i_{k-1}\), since \(\delta_0 i\left(r \frac{d}{dr}\right)\alpha = -i\left(r \frac{d}{dr}\right)\delta_0\alpha = 0\) by Lemma 6.1 (2) and we have

\[
\Delta i\left(r \frac{d}{dr}\right)\alpha = (\delta_0 d_0 + d_0 \delta_0) i\left(r \frac{d}{dr}\right)\alpha \\
= \delta_0 L_{r(d/dr)} \alpha - \delta_0 i\left(r \frac{d}{dr}\right) d_0 \alpha \\
= (k+p) \delta_0 \alpha + i\left(r \frac{d}{dr}\right) \delta_0 d_0 \alpha \\
= 0. 
\]

q.e.d.

Put \(H^i_k = \text{Ker } d_0 \cap H^i_k\) and \(H^i_k = \text{Ker } i\left(r \frac{d}{dr}\right) \cap H^i_k\). By Proposition 6.2 (1) and Lemma 6.7, we get
Theorem 6.8. (1) The modules $H^k_0 (k \geq 0)$ and $"H^k_0 (0 < p < n+1)$ are reduced to zero.

(2) The module $'H^k_0 (p > 0)$ is decomposed into the irreducible $SO(n+1)$-modules with the following highest weights and each of these appears with multiplicity one.

(i) case $n = 2m$,
$$k\Lambda_1 + \Lambda_p \quad (0 < p < m)$$
$$k\Lambda_1 + 2\Lambda_m \quad (p = m, m+1)$$
$$k\Lambda_1 + \Lambda_{n+1-p} \quad (m+1 < p \leq 2m).$$

(ii) case $n = 2m-1$
$$k\Lambda_1 + \Lambda_p \quad (0 < p < m-1)$$
$$k\Lambda_1 + \Lambda_{m-1} + \Lambda_m \quad (p = m-1, m+1)$$
$$k\Lambda_1 + 2\Lambda_{m-1}, k\Lambda_1 + 2\Lambda_m \quad (p = m)$$
$$k\Lambda_1 + \Lambda_{n+1-p} \quad (m+1 < p \leq 2m-1).$$

(3) The module $"H^k_0 (k \geq 1, p \leq n)$ is decomposed into the irreducible $SO(n+1)$-modules with the following highest weights and each of these appears with multiplicity one.

(i) case $n = 2m$,
$$(k-1)\Lambda_1 + \Lambda_{p+1} \quad (0 \leq p < m-1)$$
$$(k-1)\Lambda_1 + 2\Lambda_m \quad (p = m-1, m)$$
$$(k-1)\Lambda_1 + \Lambda_{n-p} \quad (m+1 \leq p < 2m-1).$$

(ii) case $n = 2m-1$,
$$(k-1)\Lambda_1 + \Lambda_{p+1} \quad (0 \leq p < m-2)$$
$$(k-1)\Lambda_1 + \Lambda_{m-1} + \Lambda_m \quad (p = m-2, m)$$
$$(k-1)\Lambda_1 + 2\Lambda_{m-1}, (k-1)\Lambda_1 + 2\Lambda_m \quad (p = m-1)$$
$$(k-1)\Lambda_1 + \Lambda_{n-p} \quad (m < p \leq 2m-2).$$

(4) The module $i^*(H^k_0)$ is consist of $d$-closed forms and the module $i^*("H^k_0)$ of $S$-closed forms.

Proof. We shall give the proof for the case $n=2m$, the case $n=2m-1$, being treated quite analogously. The statement (1) can be easily seen by Proposition 6.2. We shall show (2) and (3). The module $P^k_0$ is isomorphic to $S^d(R^{n+1}) \otimes \Lambda^d(R^{n+1}) \otimes C$ as $SO(n+1)$-modules. Therefore, the module $P^k_0$ contains the $SO(n+1)$-irreducible submodule with the highest weight $k\Lambda_1 + \Lambda_p$. 
We denote this submodule by $E_k$. By Lemma 6.4, we have $E_k \subset H_k$. The module $d_0 E_k$ is isomorphic to $E_k$ or reduced to zero. But the highest weight of the module $P_k$ is $(k-1)\Lambda_1 + \Lambda_p$, which is strictly lower than $k\Lambda_1 + \Lambda_p$. Thus the module $d_0 E_k$ is reduced to zero and this implies $E_k \subset H_k$. On the other hand, the module $i\left( \frac{d}{dr} \right) E_{k-1}$ is contained in $H_k$. By Proposition 6.2, the module $d_0 \left( \frac{d}{dr} \right) E_{k-1}$ is equal to $E_{k-1}$. Therefore, by (6.1) the module $i\left( \frac{d}{dr} \right) E_{k-1}$ is also isomorphic to the module $E_{k-1}$. Thus we have shown that the module $H_k$ contains the irreducible submodules with the highest weights in (3). Then by (6.1) and Theorem 4.2, no other submodules appear in $H_k$. Thus we have proved (2) and (3). Next we shall show (4). By the well known formula $d^* = i^* d_0$, we see that the module $i^*(H_k)$ is $d$-closed. When we note that $d$, $\delta$ and $*$ are $SO(n+1)$-homomorphisms, we see that the module $i^*(H_k)$ is $\delta$-closed by comparing the irreducible submodules appearing in $H_k$ and $\sum H_k$. 

q.e.d.

7. Eigenforms on $P^*(C)$ and the harmonic polynomial forms on $C^{n+1}$

Let $(z^0, z^1, \ldots, z^n)$ be a standard holomorphic coordinate on $C^{n+1}$ and $g = \sum_{i=0}^{n} dz^i \cdot dz^i$ be the flat Kähler metric on $C^{n+1}$. We denote by $\Lambda^{p,q}(C^{n+1})$ the space of differential forms of type $(p, q)$. We designate by $d_0$ the differential and $\delta_0$ the codifferential on $\Lambda^*(C^{n+1})$. The operators $d_0$ and $\delta_0$ are decomposed as follows:

$$d_0 = \partial_0 + \overline{\partial}_0$$

with

$$\partial_0 : \Lambda^{p,q}(C^{n+1}) \to \Lambda^{p+1,q}(C^{n+1})$$

$$\overline{\partial}_0 : \Lambda^{p,q}(C^{n+1}) \to \Lambda^{p-1,q}(C^{n+1}),$$

and

$$\delta_0 = \partial^*_0 + \overline{\partial}^*_0$$

with

$$\partial^*_0 : \Lambda^{p,q}(C^{n+1}) \to \Lambda^{p+1,q}(C^{n+1})$$

$$\overline{\partial}^*_0 : \Lambda^{p,q}(C^{n+1}) \to \Lambda^{p-1,q}(C^{n+1}).$$

Furthermore, we denote by $*_0$ the Poincaré duality and put

$$W_0 = \sum_{i=0}^{n} z_i \frac{\partial}{\partial z_i}, \ W_0^* = \sum_{i=0}^{n} \bar{z}_i \frac{\partial}{\partial z_i}, \ W_0^* = \sum_{i=0}^{n} z_i \bar{z}_i, \ \text{and} \ \bar{W}_0^* = \sum_{i=0}^{n} \bar{z}_i \bar{z}_i.$$

For $\alpha \in \Lambda^*(C^{n+1})$, we define $e(W_0^*) \alpha$ and $e(\bar{W}_0^*) \alpha$ by

$$e(W_0^*) \alpha = W_0^* \wedge \alpha$$

and

$$e(\bar{W}_0^*) \alpha = \bar{W}_0^* \wedge \alpha.$$
We designate by $i(W^\sharp)$ (resp. $i(W^\sharp)$) the interior product by $W_0$ (resp. $W^\sharp$). Then these operators $d_0, \delta_0, \partial_0, \partial_\delta, \delta^0, e(W^\sharp), e(W^\sharp)$, $i(W_0)$ and $i(W^\sharp)$ on $\Lambda^*(C^{*+1})$ commute with the action of $U(n+1)$ on $\Lambda^*(C^{*+1})$. Now the following lemma is easily verified.

Lemma 7.1. For any $\alpha \in \Lambda^{\leq q}(C^{*+1})$, we have

1. $i(W_0)_{*\alpha} = (-1)^{p+q} \partial_{W_0}(W^\sharp) \alpha$
   
   $i(W_0)_{*\alpha} = (-1)^{p+q} \delta(\partial_0)_{*\alpha}.$

2. $i(W_0)_{\delta_0 \alpha} + \partial_{\delta_0}(W_0)_{*\alpha} = L_{W_0} \alpha$
   
   $i(W_0)_{\delta_0 \alpha} + \partial_{\delta_0}(W_0)_{*\alpha} = L_{W_0} \alpha$

3. $\partial^0_{\delta_0}(W^\sharp)_{\alpha} + e(W^\sharp)_{*\partial^0_{\delta_0} \alpha} = (-1)^{p+q+1} L_{W^\sharp} \alpha$
   
   $\partial^0_{\delta_0}(W^\sharp)_{\alpha} + e(W^\sharp)_{*\partial^0_{\delta_0} \alpha} = (-1)^{p+q+1} L_{W^\sharp} \alpha$

Now, we put $\Omega_\alpha = -\sum dz^i \wedge dx^i$ and define linear maps $L_0, \Lambda_0$ of $\Lambda^*(C^{*+1})$ into itself by

$L_0 \alpha = \Omega_\alpha \wedge \alpha$

and

$\Lambda_0 \alpha = \ast \Omega_\alpha \wedge \alpha$

for $\alpha \in \Lambda^*(C^{*+1})$.

Then the operators $L_0$ and $\Lambda_0$ also commute with the action of $U(n+1)$ on $\Lambda^*(C^{*+1})$. We have the following

Lemma 7.2. For $\alpha \in \Lambda^*(C^{*+1})$, we have

1. $\partial_0 e(W^\sharp) \alpha + e(W^\sharp) \partial_0 \alpha = 0$
   
   $\partial_0 e(W^\sharp) \alpha + e(W^\sharp) \partial_0 \alpha = 0$

2. $e(W_0) \alpha + i(W_0) \partial_0 \alpha = 0$
   
   $e(W_0) \alpha + i(W_0) \partial_0 \alpha = 0$

Put $P_{\leq q}$ be the set of $\alpha \in \Lambda^{\leq q}(C^{*+1})$ of the form

$\alpha = \sum_{0 \leq i_1 < \cdots < i_j \leq n \atop 0 \leq j \leq n} \alpha_{i_1 \cdots i_j} dz^{i_1} \wedge \cdots \wedge dz^{i_j} \wedge dx^1 \wedge \cdots \wedge dx^j$, 

where $\alpha_{i_1 \cdots i_j}$ are $\alpha_{i_1 \cdots i_j}$.
where $\alpha_{i_1 \cdots i_p}$ are polynomials of degree $k$ with respect to $z^0, z^1, \ldots, z^n$ and of degree $l$ with respect to $z^0, z^1, \ldots, z^n$. Then, using Lemma 7.1, we have in the same way as for Proposition 6.2

**Proposition 7.3.** Let $\alpha \in P_{k,l}$. Then we have

1. $i(W)\partial \alpha + \partial i(W)\alpha = (k+p)\alpha$
2. $i(W)\overline{\partial} \alpha + \overline{\partial} i(W)\alpha = (l+q)\alpha$
3. $e(W)\partial^*_\alpha + \partial^*_e(W)\alpha = -(n+1-p+l)\alpha$
4. $e(W)\overline{\partial^*_\alpha} + \overline{\partial^*_e(W)}\alpha = -(n+1-q+k)\alpha$

By this proposition, we have the following direct sum decompositions;

$$P_{k,l} = (\text{Ker} \, \partial \cap \text{Ker} \, P_{k,l}^0) \oplus (\text{Ker} \, \partial \cap \text{Ker} \, i(W) \cap P_{k,l}^0)$$

$$= (\text{Ker} \, \overline{\partial} \cap \text{Ker} \, P_{k,l}^0) \oplus (\text{Ker} \, i(W) \cap \text{Ker} \, i(W) \cap P_{k,l}^0)$$

$$= (\text{Ker} \, \overline{\partial} \cap \text{Ker} \, P_{k,l}^0) \oplus (\text{Ker} \, e(W) \cap \text{Ker} \, P_{k,l}^0)$$

By Lemma 7.1, we have the following direct sum decompositions;

$$P_{k,l} = (\text{Ker} \, \partial \cap \text{Ker} \, \overline{\partial} \cap \text{Ker} \, P_{k,l}^0) \oplus (\text{Ker} \, \partial \cap \text{Ker} \, i(W) \cap \text{Ker} \, i(W) \cap P_{k,l}^0)$$

$$= (\text{Ker} \, \overline{\partial} \cap \text{Ker} \, e(W) \cap \text{Ker} \, e(W) \cap P_{k,l}^0)$$

**Remarks.** We have

1. $P_{k,l}^0 = \text{Ker} \, \partial \cap \text{Ker} \, P_{k,l}^0 = \text{Ker} \, i(W) \cap P_{k,l}^0$,
2. $P_{k,l}^0 = \text{Ker} \, \overline{\partial} \cap \text{Ker} \, P_{k,l}^0 = \text{Ker} \, i(W) \cap P_{k,l}^0$,
3. $P_{k,l}^0 = \text{Ker} \, \partial \cap \text{Ker} \, e(W) \cap \text{Ker} \, e(W) \cap P_{k,l}^0$,
4. $P_{k,l}^0 = \{\text{constant functions on } C^{n+1}\}$

Furthermore, in the same way as for Lemma 6.4, we have

(7.2) $P_{k,l} = (\text{Ker} \, \partial \cap \text{Ker} \, \overline{\partial} \cap \text{Ker} \, \square_0 \cap \text{Ker} \, P_{k,l}^0)$

$$\oplus (\text{Ker} \, e(W) \cap P_{k,l}^0)$$

where $\square_0 = \partial \overline{\partial} + \overline{\partial} \partial$.

Now, put
SPECTRA AND EIGENFORMS OF THE LAPLACIAN

\[ H^p_k; f = \text{Ker } \partial^p \cap \text{Ker } \bar{\partial}^p \cap \text{Ker } \square_0 \cap \text{Ker } \Lambda_0 \cap P^p_k; f \].

DEFINITION. A harmonic polynomial form on \( C^{n+1} \) is an element of the space \( H^p_k; f \).

The space \( H^p_k; f \) is an \( SU(n+1) \)-invariant subspace in the space \( P^p_k; f \). Assume \( p \neq 0, q \neq 0 \) and \( p + q \leq n + 1 \). Then by (7.2), the module \( H^p_k; f \) contains the highest weight vector in the module \( P^p_k; f \). The irreducible subspace in \( H^p_k; f \) which contains the highest weight vector of the module \( P^p_k; f \) shall be denoted by \( E^p_k; f \). Then the irreducible module \( E^p_k; f \) has the following highest weight;

\[ l\Lambda_1 + \Lambda_q + k\Lambda_n + \Lambda_{n+1-p} \cdot \]

The following Lemma 7.5 can be verified easily from the formulas in (3.5) and Lemma 7.1.

**Lemma 7.4.** Let \( \alpha \in H^p_k; f \). Then we have \( \partial_\theta \alpha \in H^{p-1}_k; f \), \( \bar{\partial}_\theta \alpha \in H^{p-1}_k; f \), \( i(\bar{W}_0) \alpha \in H^{p-1}_k; f \) and \( i(\bar{W}_0) \alpha \in H^{p-1}_k; f \).

Combining this lemma with (7.1), we have

\[ H^p_k; f = (\text{Ker } \partial_\theta \cap \text{Ker } \bar{\partial}_\theta \cap H^p_k; f) \oplus (\text{Ker } \partial_\theta \cap \text{Ker } i(\bar{W}_0) \cap H^p_k; f) \]
\[ \oplus (\text{Ker } i(\bar{W}_0) \cap \text{Ker } \partial_\theta \cap \text{Ker } \bar{W}_0 \cap H^p_k; f) \]
\[ (k + p = 0, l + q = 0) \]

**Proposition 7.5.** Suppose \( p + q \leq n \). Then the module \( H^p_k; f \) contains the irreducible \( SU(n+1) \)-modules with the following highest weights;

1. \( l\Lambda_1 + k\Lambda_n \) (\( p = q = 0 \))
2. \( l\Lambda_1 + k\Lambda_n + \Lambda_q, (l-1)\Lambda_1 + k\Lambda_n + \Lambda_{q+1} \) (\( p = 0, n > q > 0 \))
3. \( l\Lambda_1 + k\Lambda_n + \Lambda_{n+1-p}, l\Lambda_1 + (k-1)\Lambda_n + \Lambda_{n-p} \) (\( n > p > 0, q = 0 \))
4. \( l\Lambda_1 + k\Lambda_n + \Lambda_q + \Lambda_{n+1-p}, (l-1)\Lambda_1 + (k-1)\Lambda_n + \Lambda_{q+1} + \Lambda_{n+1-p} \)
\[ l\Lambda_1 + (k-1)\Lambda_n + \Lambda_q + \Lambda_{n-p}, (l-1)\Lambda_1 + (k-1)\Lambda_n + \Lambda_{q+1} + \Lambda_{n-p} \]
\[ (p > 0, q > 0, 1 < p + q < n) \]
5. \( l\Lambda_1 + k\Lambda_n + \Lambda_q + \Lambda_{n+1-p}, (l-1)\Lambda_1 + k\Lambda_n + \Lambda_{q+1} + \Lambda_{n+1-p} \)
\[ l\Lambda_1 + (k-1)\Lambda_n + \Lambda_q + \Lambda_{n-p} \]
\[ (p = 0, q = 0, p + q = n) \]
6. \( l\Lambda_1 + (k+1)\Lambda_n \) (\( p = 0, q = n \))
7. \( (l+1)\Lambda_1 + k\Lambda_n \) (\( p = n, q = 0 \))

Proof. Assume \( p \neq 0, q \neq 0 \) and \( p + q \leq n + 1 \). Then in the same way as for the proof of Theorem 6.8, we get \( \partial_\theta E^p_k; f = 0 \) and \( \bar{\partial}_\theta E^p_k; f = 0 \). We shall show that the module \( i(\bar{W}_0) i(\bar{W}_0) E^p_k; f \) is not reduced to zero. Let \( \alpha \in E^p_k; f \). Assume \( i(\bar{W}_0) i(\bar{W}_0) \alpha = 0 \). Then we have \( i(\bar{W}_0) \alpha \in \text{Ker } i(\bar{W}_0) \). On the other hand, we get \( \partial_\theta i(\bar{W}_0) \alpha = -i(\bar{W}_0) \partial_\theta \alpha = 0 \). Therefore we have \( i(\bar{W}_0) \alpha \in \text{Ker } \partial_\theta \). Thus we have
\[ i(W_o)\alpha = 0, \text{ i.e., } \alpha \in \text{Ker } i(W_o). \] Since \( \alpha \in \text{Ker } \partial \), we have \( \alpha = 0 \). Thus we see that the module \( i(W_o)i(W_o)E^p; \) is isomorphic to the module \( E^p; \). Moreover the modules \( i(W_o)E^p; \) and \( i(W_o)E^p; \) are also isomorphic to \( E^p; \). Combining these facts with Lemma 7.4, we get Proposition 7.5.

q.e.d.

Let \( \pi \) be the natural projection of \( C^{n+1} \) onto \( P^n(C) \) and \( \pi_s \) its restriction to \( S^{2n+1} \). For \( \rho \in S^{2n+1} \), we denote by \( T_{\rho}(S^{2n+1}) \) the tangent space at \( \rho \). Put \( F_{\rho} = \text{Ker } ((\pi_s)_\rho) \). Let \( F^\perp_{\rho} \) be the orthogonal complementary subspace to \( F_{\rho} \) in \( T_{\rho}(S^{2n+1}) \);

\[ T_{\rho}(S^{2n+1}) = F_{\rho} \oplus F^\perp_{\rho}. \]

We introduce the Riemannian metric on \( P^n(C) \) so that the restriction of \( (\pi_s)_* \) to \( F^\perp_{\rho} \) is an isometry onto \( T^\perp_{\rho}(P^n(C)) \). Let \( J_0 \) and \( J \) be the standard complex structures on \( C^{n+1} \) and \( P^n(C) \) respectively. Then for \( v \in F^\perp_{\rho} \), we have

\[ J((\pi_s)_*v) = (\pi_s)_*J_0v. \]

We denote by \( \Lambda^{p,q}_S(C^{n+1}) \) the set of all \( \alpha \in \Lambda^{p,q}(C^{n+1}) \) such that \( g^*\alpha = \alpha \) for any \( g \in S^1 = \{ (e^{2\pi i \theta}, 0, \ldots, 0, e^{2\pi i \theta}) \in U(n+1) \} \) and we define a linear map

\[ \phi: \Lambda^{p,q}_S(C^{n+1}) \rightarrow \Lambda^{p,q}(P^n(C)) \]

by

\[ \phi(\alpha)(X_1, \ldots, X_{p+q}) = \alpha(X_1, \ldots, X_{p+q}), \]

where \( X_i \) are tangent vectors at \( \pi(p) \) to \( P^n(C) \) and \( X_i \) are the lift to \( S^{2n+1} \) by the isomorphism \( F^\perp_{\rho} \approx T^\perp_{\rho}(P^n(C)) \). Then the map \( \phi \) is well defined and we have

**Proposition 7.6.**

1. \( \phi(\sum_{k+p=q} P^{k,q}_o) \) is dense in \( \Lambda^{p,q}(P^n(C)) \).
2. Let \( \Omega \) be the fundamental form on \( P^n(C) \). Then we have \( \phi(\Omega_o) = \Omega \).
3. \( \phi(W^p_o) = \phi(\bar{W}^p_o) = 0 \).
4. \( \phi(\tau^i) = 1 \).
5. \( \phi: \text{Ker } \partial^q \cap \text{Ker } \bar{\partial}^q \cap P^{q}; \rightarrow P^n(C) \) is injective.
6. For \( \alpha \in H^p; \), \( \phi(\alpha) \) is a primitive form on \( P^n(C) \) if and only if \( i(W_o)i(W_o)\alpha = 0 \).

**Proof.** We shall show here (5) and (6), the other statements being easily proved. Let \( \alpha \in \text{Ker } \partial^q \cap \text{Ker } \bar{\partial}^q \cap P^{q}; \). Assume \( \phi(\alpha) = 0 \). Then we have \( W_{\partial}^q \cap \bar{W}_{\partial}^q \cap \alpha = 0 \), i.e., \( \bar{W}_{\partial}^q \cap \alpha \in \text{Ker } \bar{e}(W_{\partial}^q) \). On the other hand we have \( \bar{\partial}^q \) \( (\bar{W}_{\partial}^q \cap \alpha) = -\bar{W}_{\partial}^q \cap \partial^q \alpha = 0 \). Therefore we have \( \bar{W}_{\partial}^q \cap \alpha \in \text{Ker } \partial^q \). Thus we have \( \bar{W}_{\partial}^q \cap \alpha = 0 \). In the same way as above, we see \( \alpha = 0 \). Thus we have proved (5). Let \( W_1, \ldots, W_n \) be a unitary base complementary to \( W_o \) at \( \rho \in S^{2n+1} \)
in $C^{n+1}$. For $\alpha \in H_k^p$, we have
\[
0 = \sum_{i=0}^n i(W_i)\bar{i}(W_i)\alpha = \sum_{i=1}^n i(W_i)\bar{i}(W_i)\alpha + i(W_0)\bar{i}(W_0)\alpha.
\]
It is clear that $\phi(\alpha)$ is primitive form if and only if the first term vanishes. Thus we have proved (6).

q.e.d.

According to (6) in Proposition 7.6, for an element of the last three direct summands in (7.2), its $\phi$-image is a primitive form. On the other hand for $\alpha \in \operatorname{Ker} \partial_6 \cap \operatorname{Ker} \partial_6 \cap H^p_k$, we put
\[
\alpha_1 = (n+2-p-q)\alpha + \sqrt{-1} \Omega \bar{i}(W_0)\bar{i}(W_0)\alpha,
\]
\[
\alpha_2 = i(W_0)\bar{i}(W_0)\alpha.
\]
Then we have

Lemma 7.7. $\phi(\alpha_1)$ and $\phi(\alpha_2)$ are primitive forms.

Proof. $\phi(\alpha_2)$ is a primitive form follows from (6) in Proposition 7.6. On the other hand we have
\[
\Lambda \phi(\alpha_1) = \phi(\Lambda \alpha_1 + \sqrt{-1} i(W_0)\bar{i}(W_0)\alpha_1)
\]
\[
= \phi \left\{ (n+2-p-q)\Lambda \alpha - \sqrt{-1} \Omega \bar{i}(W_0)\bar{i}(W_0)\alpha \right\}
\]
\[
+ \sqrt{-1} (n+3-p-q) i(W_0)\bar{i}(W_0)\alpha - i(W_0)\Omega \bar{i}(W_0)\bar{i}(W_0)\alpha
\]
\[
= \phi \left\{ -\sqrt{-1} (n+3-p-q) i(W_0)\bar{i}(W_0)\alpha \right\}
\]
\[
+ \sqrt{-1} (n+2-p-q) i(W_0)\bar{i}(W_0)\alpha + \sqrt{-1} i(W_0)\bar{i}(W_0)\alpha
\]
\[
= 0.
\]
q.e.d.

Lemma 7.8. $\alpha \neq 0$ if and only if $\alpha_2 \neq 0$.

Proof. Assume $\alpha_2 = 0$. Then $i(W_0)\bar{i}(W_0)\alpha = 0$. Thus we have $i(W_0)\alpha \in \operatorname{Ker} i(W_0)$. Since $\partial_6 i(W_0)\alpha = -i(W_0)\partial_6 \alpha = 0$, we have $i(W_0)\alpha = 0$. In the same way, we see $\alpha = 0$.

q.e.d.

Lemma 7.9. If $\alpha \neq 0$ and $k+p \neq 0$, then $\alpha_1 = 0$.

Proof. Assume $\alpha \neq 0$. We shall show $\partial_6 \alpha_1 = 0$.
\[
\sqrt{-1} \partial_6 \alpha_1 = \partial_6 \Omega \bar{i}(W_0)\bar{i}(W_0)\alpha
\]
\[
= \Omega \partial_6 \bar{i}(W_0)\bar{i}(W_0)\alpha
\]
\[
= -\Omega \bar{i}(W_0)\partial_6 \bar{i}(W_0)\alpha + (k+p) \Omega \bar{i}(W_0)\bar{i}(W_0)\alpha
\]
\[
= (k+p) \Omega \bar{i}(W_0)\alpha.
\]
By our assumption, we have $i(W_0)\alpha \neq 0$. Since $i(W_0)\alpha$ is a $(p, q-1)$-form and
\[ p + (q - 1) \leq n, \] we have \( \Omega_{\phi} (\mathcal{W}_0) \alpha \neq 0. \] Thus we have shown \( \partial \alpha \neq 0. \) q.e.d.

**Corollary 7.10.** We have a direct sum decomposition, 

\[ \phi(P^k_i \varphi) = \sum_{j \geq 0} \Omega^j \phi(H^k_j \varphi) \quad (k + p = l + q). \]

**Corollary 7.11.** \( \phi: \sum_{k+b=p+q} \Omega^i H^k_{p,q} \rightarrow \Lambda^p \cdot q(P^s(C)) \) is injective and its image is dense.

Combining this with Theorem 5.2, we have

**Proposition 7.12.** Assume \( k + p = l + q. \) Then we see that \( H^k_{p,q} \) contains only irreducible \( SU(n+1) \)-modules appearing in Proposition 7.5 with multiplicity one and no other submodules occur in \( H^k_{p,q}. \)

Now, put

\[ H^k_{p,0} = \{ \alpha \in H^k_{p,0} : \partial \alpha X = 0, \partial \alpha Y = 0 \} , \]
\[ H^k_{p,1} = \{ \alpha \in H^k_{p,1} : \partial \alpha X = 0, i(\mathcal{W}_0) \alpha = 0 \} , \]
\[ H^k_{p,-1} = \{ \alpha \in H^k_{p,-1} : i(\mathcal{W}_0) \alpha = 0, \partial \alpha X = 0 \} , \]
and
\[ H^k_{p,1} = \{ \alpha \in H^k_{p,1} : i(\mathcal{W}_0) \alpha = 0, i(\mathcal{W}_0) \alpha = 0 \} . \]

Then by (7.2), we have

\[ H^k_{p,q} = \phi^k_{p,0} \oplus \phi^k_{p,0} \oplus H^k_{p,q} \oplus H^k_{p,-q} \quad (k + p = 0, q + l = 0). \]

**Theorem 7.13.** Assume \( k + p = l + q \) and \( 0 \leq p + q \leq n. \) Then we have (1) \( \phi^k_{p,0} = \phi^k_{p,0} = \phi^k_{p,0} = \phi^k_{p,0} = \phi^k_{p,0} = \phi^k_{p,0} = \phi^k_{p,0}, \) etc., except for the modules listed in (1), are irreducible modules with the following highest weights;

<table>
<thead>
<tr>
<th>Module</th>
<th>Highest Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi^k_{p,0} )</td>
<td>( (\ell-1)(\Lambda_{\alpha} + \Lambda_{\beta}) + (p-q-1)\Lambda_{\alpha} + \Lambda_{\gamma} + \Lambda_{\gamma+1} )</td>
</tr>
<tr>
<td>( \phi^k_{p,1} )</td>
<td>( (\ell-1)(\Lambda_{\alpha} + \Lambda_{\beta}) + (p-q)\Lambda_{\alpha} + \Lambda_{\gamma} + \Lambda_{\gamma+1} )</td>
</tr>
<tr>
<td>( \phi^k_{p,-1} )</td>
<td>( (\ell-1)(\Lambda_{\alpha} + \Lambda_{\beta}) + (p-q)\Lambda_{\alpha} + \Lambda_{\gamma} + \Lambda_{\gamma+1} )</td>
</tr>
<tr>
<td>( \phi^k_{p,-q} )</td>
<td>( (\ell-1)(\Lambda_{\alpha} + \Lambda_{\beta}) + (p-q)\Lambda_{\alpha} + \Lambda_{\gamma} + \Lambda_{\gamma+1} )</td>
</tr>
<tr>
<td>( \phi^k_{p,q} )</td>
<td>( (\ell-1)(\Lambda_{\alpha} + \Lambda_{\beta}) + (p-q)\Lambda_{\alpha} + \Lambda_{\gamma} + \Lambda_{\gamma+1} )</td>
</tr>
<tr>
<td>( \phi^k_{p,-q} )</td>
<td>( (\ell-1)(\Lambda_{\alpha} + \Lambda_{\beta}) + (p-q)\Lambda_{\alpha} + \Lambda_{\gamma} + \Lambda_{\gamma+1} )</td>
</tr>
<tr>
<td>$p \neq 0$, $n$</td>
<td>$\mathcal{H}_{k,l}^{q=0}$</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>$q = 0$</td>
<td>$\mathcal{H}_{k,l}^{p=0}$</td>
</tr>
<tr>
<td>$p = 0$, $q = n$</td>
<td>$\mathcal{H}_{k,l}^{p=0}$</td>
</tr>
<tr>
<td>$p = n$, $q = 0$</td>
<td>$\mathcal{H}_{k,l}^{p=0}$</td>
</tr>
<tr>
<td>$p = q = 0$</td>
<td>$\mathcal{H}_{k,l}^{p=0}$</td>
</tr>
</tbody>
</table>

(3) $\phi(\mathcal{H}_{k,l}^{p=0}) \subset \text{Ker } \delta \cap \Lambda^{k+q}(P^*(C))$

$\phi(\mathcal{H}_{k,l}^{p=0}) \subset \text{Ker } \theta \cap \text{Ker } \delta^{k+q}(P^*(C))$ ($p = 0$)

$\phi(\mathcal{H}_{k,l}^{p=0}) \subset \text{Ker } \delta \cap \text{Ker } \delta^{k+q}(P^*(C))$ ($q = 0$).

Proof. It is clear that $E_{k+1}^{l} \subset \mathcal{H}_{k,l}^{p=0}$, $i(W_0)E_{k+1}^{l} \subset \mathcal{H}_{k,l}^{p=0}$, $i(W_0)E_{k+1}^{l} \subset \mathcal{H}_{k,l}^{p=0}$ and $i(W_0)E_{k+1}^{l} \subset \mathcal{H}_{k,l}^{p=0}$. Combining this fact with (7.5) and Proposition 7.12, we get (1) and (2). Moreover, we get (3) by comparing the irreducible $SU(n+1)$-modules appearing in $H_{k,l}^{p=0}$.

q.e.d.

## References


Added in proof. After submitting this paper, we knew the following two papers: