ON HYPERSURFACES OF A COMPLEX GRASSMANN MANIFOLD $G_{m+n,n}(\mathbb{C})$

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On Kahler submanifolds of a complex projective space, J-I Hano [5] has studied complete intersections of hypersurfaces in a complex projective space and proved that if a complete intersection $M$ of hypersurfaces is an Einstein manifold with respect to the induced metric then $M$ is a complex projective space or a complex quadric. The purpose of this note is to investigate hypersurfaces of a complex Grassmann manifold by using Hano's method. Let $G_{m+n,n}(\mathbb{C})$ denote the complex Grassmann manifold of $n$-planes in $\mathbb{C}^{m+n}$. Let $X$ be a compact complex hypersurface of $G_{m+n,n}(\mathbb{C})$. Then $X$ defines a positive divisor on $G_{m+n,n}(\mathbb{C})$ and hence a holomorphic line bundle $\{X\}$ on $G_{m+n,n}(\mathbb{C})$. We denote by $c(X)$ the Chern class of the line bundle $\{X\}$. Since the second cohomology group $H^2(G_{m+n,n}(\mathbb{C}), \mathbb{Z})$ is isomorphic to $\mathbb{Z}$, we can write $c(X) = a(X) \cdot \sigma$, where $a(X) \in \mathbb{N}$ and $\sigma$ is a generator of $H^2(G_{m+n,n}(\mathbb{C}), \mathbb{Z})$. We call $a(X)$ the degree of $X$. We equip an hermitian inner product on $\mathbb{C}^{m+n}$. The complex Grassmann manifold $G_{m+n,n}(\mathbb{C})$ has a Kahler metric invariant under the action of the unitary group $U(m+n)$. Moreover we may assume that $m \geq n$. Under these notations, we have the following Theorem.

Theorem. Let $X$ be a compact complex hypersurface of a complex Grassmann manifold $G_{m+n,n}(\mathbb{C})$ and $a(X)$ the degree of $X$. If $a(X) \geq r + 2$, where $r = (m+n) - mn - 1$ and $n \geq 2$, $X$ is not an Einstein manifold with respect to the induced metric.

1. Preliminaries

Let $G_{m+n,n}(\mathbb{C})$ be the complex Grassmann manifold of $n$-planes in $\mathbb{C}^{m+n}$. An element of $G_{m+n,n}(\mathbb{C})$ can be given by a non-zero decomposable $n$-vector $\Lambda = X_1 \wedge \cdots \wedge X_n \neq 0$ defined up to a constant factor. If $(e_1, \cdots, e_{m+n})$ denotes a fixed frame in $\mathbb{C}^{m+n}$, we can write

\begin{equation}
\Lambda = \sum p_{i_1 \cdots i_n} e_{i_1} \wedge \cdots \wedge e_{i_n} \quad (1 \leq i_1, \cdots, i_n \leq m+n)
\end{equation}

where the $p_{i_1 \cdots i_n}$'s are skew-symmetric in their indices. The $p_{i_1 \cdots i_n}$ are called the
Plücker coordinates in $G_{m+n,n}(C)$. By considering $p_{i_1 \cdots i_n}$ as the homogeneous coordinates of the complex projective space $P^n(C)$ of dimension $\mu = \binom{m+n}{n} - 1$, we get an imbedding $j: G_{m+n,n}(C) \rightarrow P^n(C)$.

We equip an hermitian inner product in $C^{m+n}$. Then we can define a Kähler metric on $G_{m+n,n}(C)$ which is invariant under the action of the unitary group $U(m+n)$. We also have the Fubini-Study metric on the complex projective space $P^n(C)$ induced from the hermitian inner product in the $n$-th exterior product $\Lambda^n C^{m+n}$ of $C^{m+n}$. Then the imbedding $j$ is isometric with respect to these Kähler metrics (cf. for example [3] §8).

From now on we identify $G_{m+n,n}(C)$ with the image of the imbedding $j$. Let $I(V)$ denote the ideal associated to a subvariety $V$ of $P^n(C)$. We recall the generators of the ideal $I(G_{m+n,n}(C))$. Let $i_1, \cdots, i_{n-1}$ be $n-1$ distinct numbers which are chosen from a set $\{1, \cdots, m+n\}$ and let $j_0, \cdots, j_n$ be $n+1$ distinct numbers chosen from the same set. We define homogeneous polynomials $Q(i_1 \cdots i_{n-1}, j_0 \cdots j_n)$ of degree 2 on $C^{n+1}$ by

\[
Q(i_1 \cdots i_{n-1}, j_0 \cdots j_n) = \sum_{k=0}^{n} (-1)^k p_{i_1 \cdots i_{n-1} j_k} j_0 \cdots \hat{j_k} \cdots j_n,
\]

Then it is known that $Q(i_1 \cdots i_{n-1}, j_0 \cdots j_n) = 0$ are the generators of the ideal $I(G_{m+n,n}(C))$ (See [7] Chapter 7 §6 Theorem 2 and §7 Theorem 1). The relations $Q(i_1 \cdots i_{n-1}, j_0 \cdots j_n) = 0$ are called the quadratic $p$-relations.

Let $\pi$ denote the canonical projection of $C^{n+1} - (0)$ onto the complex projective space $P^n(C)$. The triple $(C^{n+1} - (0), \pi, P^n(C))$ is a principal $\mathbb{C}^*$-bundle over $P^n(C)$. Let $E$ be the standard line bundle over $P^n(C)$ associated to the above principal bundle. We denote by $H^1(M, \theta^*)$ the group of all equivalent classes of holomorphic line bundles over a compact complex manifold $M$. On the line bundles over a Grassmann manifold $G_{m+n,n}(C)$, the following propositions are known.

**Proposition 1.1.** Let $H$ denote the dual bundle of $E$ over $P^n(C)$. Then, for any integer $k > 0$, the inclusion map $j: G_{m+n,n}(C) \rightarrow P^n(C)$ induces the surjective map $j^*: H^k(P^n(C), H^k) \rightarrow H^k(G_{m+n,n}(C), j^*H^k)$, that is, every holomorphic section of the line bundle $j^*H^k$ is given by the restriction of a section of the line bundle $H^k$ on $P^n(C)$.

**Proposition 1.2.** The inclusion map $j: G_{m+n,n}(C) \rightarrow P^n(C)$ induces the canonical isomorphism $j^*: H^1(P^n(C), \theta^*) \rightarrow H^1(G_{m+n,n}(C), \theta^*)$. Moreover each positive divisor $X$ of $G_{m+n,n}(C)$ is the complete intersection of $G_{m+n,n}(C)$ and a subvariety $Y$ of codimension 1 of $P^n(C)$. Furthermore, for an irreducible subvariety $X$ of codimension 1 in $G_{m+n,n}(C), I(X) = I(G_{m+n,n}(C)) + (F)$ where $F$ is an irreducible homogeneous polynomial on $C^{n+1}$.

For a compact connected complex submanifold $X$ of codimension 1 in $G_{m+n,n}(C)$, let $[X]$ denote the positive divisor defined by $X$ and $c(X)$ the Chern class of the line bundle $\{X\}$ defined by $[X]$. Since $H^2(G_{m+n,n}(C), Z) \cong Z$, $c(X) = a(X) \sigma$ where $a(X) \in N$ and $\sigma$ is a generator of $H^2(G_{m+n,n}(C), Z)$. We call $a(X)$ the degree of $X$. Note that the degree of an irreducible subvariety $Y$ of codimension 1 of $P^n(C)$ corresponding to $X$ is given by $a(X)$.

2. The canonical line bundle

With respect to the hermitian inner product on $C^{m+1}$ induced from the hermitian inner product on $C^{m+n}$, the square of the norm $||z||$ is given by

$$\sum_{i_1 < \cdots < i_n} |p_{i_1 \cdots i_n}(z)|^2$$

for an orthonormal frame $(e_{i_1}, \cdots, e_{m+n})$ of $C^{m+n}$. The function $||z||^2$ can be regarded as a hermitian fiber metric on the standard line bundle $E$ on $P^n(C)$. A unique connection of type $(1, 0)$ on $E$ is determined by the fiber metric $||z||^2$ on $E$ and gives rise to the curvature form $-\Omega$ on $P^n(C)$. The form $\Omega$ is the associated (1, 1)-form of the Fubini-Study metric on $P^n(C)$; $\pi^*\Omega = \frac{\sqrt{-1}}{2\pi} d'd'^* \log ||z||^2$.

Let $K, K(G_{m+n,n}(C))$ and $K(X)$ be the canonical line bundle of $P^n(C)$, $G_{m+n,n}(C)$ and $X$ respectively. The normal bundle of $X$ in $P^n(C)$ is a holomorphic vector bundle over $X$ whose fiber dimension is $r+1 = m+n+1$. We denote by $N$ the $(r+1)$-th exterior product of the dual bundle of the normal bundle of $X$ in $P^n(C)$. Denoting by $\pi$ the inclusion $X \subset P^n(C)$, we have

$$(2.1)$$

$${K} = K(X) \cdot N.$$
Now we shall consider the structure of the holomorphic line bundle $N$ on $X$. Let $Q(\beta_1, \ldots, \beta_n)$ be a homogeneous polynomial of degree 2 on $\mathbb{C}^{n+1}$ defined by (1.2). It is obvious that $Q(\beta_1, \ldots, \beta_n)$ has the following properties:

1. $Q(\beta_1, \ldots, \beta_n)$ is alternating with respect to $\beta_1, \ldots, \beta_{n-1}$.
2. $Q(\beta_1, \ldots, \beta_n)$ is alternating with respect to $\beta_n, i_1, \ldots, i_n$.
3. If $\{\beta_1, \ldots, \beta_n\} \subset \{\beta, i_1, \ldots, i_n\}$, then $Q(\beta_1, \ldots, \beta_n, i_1, \ldots, i_n) = 0$.

Furthermore we have a following lemma which gives the relations between these polynomials.

**Lemma 2.1.** On $\pi^{-1}(U_{i_1, \ldots, i_n})$,

(a) $Q(\beta_1, \ldots, \beta_n, k_i, \ldots, i_n) = -Q(\beta_1, \ldots, \beta_n, \beta_{n+1}, \ldots, i_n)$

(b) $Q(\beta_1, \ldots, \beta_n, i_1, \ldots, i_{n-1}, i_{n+1})

Proof. Straightforward computation.

Let $(i_1, \ldots, i_n)$ be an $n$-tuples such that

$1 \leq i_1 < i_2 < \cdots < i_n \leq m+n$

and let $(i_1, \ldots, i_n, s_1, \ldots, s_m)$ be the permutation of $(1, \ldots, m+n)$ such that

$1 \leq s_1 < \cdots < s_m \leq m+n$.

For a permutation $(i_1, \ldots, i_m)$ of $(1, \ldots, n)$, we introduce a linear order $\prec$ on $\{1, \ldots, m+n\}$ by $i_1 \prec i_2 \prec \cdots \prec i_m \prec s_{i_1} \prec \cdots \prec s_{i_m}$. We denote $\{\beta = (\beta_1, \ldots, \beta_n) \mid \beta \prec \}$
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— $Q_\beta$ — by $C(i_1, \ldots, i_n, -\Theta)$. The associated lexicographical order on $C(i_1, \ldots, i_n, -\Theta)$ is called an admissible order with respect to $(i_1, \ldots, i_n)$. If the linear order $\Theta$ on \{1, \ldots, m+n\} is given by $i_1 \Theta \cdots \Theta i_n \Theta s_1 \Theta \cdots \Theta s_m$, the admissible order is called principal with respect to \{1, \ldots, i_n\}. For an admissible order with respect to $(i_1, \ldots, i_n)$, we define a subset $I(i_1, \ldots, i_n, -\Theta)$ of $C(i_1, \ldots, i_n, -\Theta)$ by

$$
\left\{ \beta = (\beta_1, \ldots, \beta_n) \mid \beta = (i_1, \ldots, i_t, \ldots, i_n, s_t), \ l = 1, \ldots, n; \ t = 1, \ldots, m, \ or \ \beta = (i_1, \ldots, i_n) \right\}.
$$

Note that $I(i_1, \ldots, i_n, -\Theta) = I(i_1, \ldots, i_n, -\Theta')$ for $-\Theta, -\Theta'$ admissible orders, with respect to $(i_1, \ldots, i_n)$ and the number of elements in $I(i_1, \ldots, i_n, -\Theta)$ is $mn+1$. Moreover, $Q(\beta_1, \ldots, i_n) \equiv 0$ for $\beta \in I(i_1, \ldots, i_n, -\Theta)$ by (2.2) 3).

For an admissible order $-\Theta$ with respect to $(i_1, \ldots, i_n)$, we define a holomorphic $r$-form $\tilde{q}^g_{i_1, i_n}$ on $C^{i_1, i_n}$ by

$$
(2.3) \quad \tilde{q}^g_{i_1, i_n}(z) = \bigwedge_{\beta \in C(i_1, \ldots, i_n, -\Theta)} dQ(\beta_1, \ldots, i_n)
$$

where we take the exterior product of $dQ(\beta_1, \ldots, i_n)$ according to the admissible order $-\Theta$ on $C(i_1, \ldots, i_n, -\Theta)$. If the admissible order $-\Theta$ is principal, we denote $\tilde{q}^g_{i_1, i_n}$ by $q_{i_1, i_n}$.

**Lemma 2.2.** Let $-\Theta, -\Theta'$ be admissible orders with respect to $(i_1, \ldots, i_n)$. Then we have

$$
(2.4) \quad \tilde{q}^g_{i_1, i_n}(z) = \varepsilon(-\Theta, -\Theta') q^g_{i_1, i_n}(z)
$$

for $z \in \pi^{-1}(V_{i_1, i_n})$, where $\varepsilon(-\Theta, -\Theta') \in \{\pm 1\}$.

**Proof.** Let $-\Theta$ be a linear order on \{1, \ldots, m+n\} given by $i_1 \Theta \cdots \Theta i_n \Theta s_1 \cdots \Theta s_m$. Since the symmetric group of $m$ elements is generated by transpositions \{(k, k+1) \mid k=1, \ldots, m-1\}, we may assume that the admissible order $-\Theta'$ is given by a linear order

$$
i_1 \Theta' \cdots \Theta i_n \Theta' s_1 \Theta' \cdots \Theta s_{i_k+1} \Theta' s_{i_{k+1}} \Theta' s_{i_k} \Theta' \cdots \Theta' s_{i_m-2} \Theta' s_{i_m}.$$

Let $\beta$ be an element of $C(i_1, \ldots, i_n, -\Theta') - I(i_1, \ldots, i_n, -\Theta')$. Then $\beta$ is of the form either

1) $\beta = (\beta_1, \ldots, \beta_n), \ \beta_t \neq s_{i_k}, s_{i_{k+1}}$ for any $t = 1, \ldots, n$,
2) $\beta = (\beta_1, \ldots, \beta_n), \ \beta_t = s_{i_k}$ for some $t$ and $\beta_a \neq s_{i_k}$ for $a \neq t$,
3) $\beta = (\beta_1, \ldots, \beta_n), \ \beta_t = s_{i_{k+1}}$ for some $t$ and $\beta_a \neq s_{i_k}$ for $a \neq t$,
4) $\beta = (\beta_1, \ldots, \beta_n), \ \beta_t = s_{i_{k+1}}, \ \beta_{t+1} = s_{i_k}$ for some $t+1 < n$,
5) $\beta = (\beta_1, \ldots, \beta_{n-2}, s_{i_{k+1}}, s_{i_k})$.

In the cases of 1), 2) and 3), $\beta \in C(i_1, \ldots, i_n, -\Theta) - I(i_1, \ldots, i_n, -\Theta)$. In the case of 4),

$$
Q(\beta_1, \ldots, i_n) = Q(\beta_1, \ldots, i_{t-1}, s_{i_k}, s_{i_{k+1}}) \beta_{t+1} \cdots = -Q(\beta_1, \ldots, i_{t-1}, s_{i_k}, s_{i_{k+1}}, \beta_{t+2} \cdots, \beta_{n} i_n).$


(2.2) 1). Note that $(\beta_1 \cdots \beta_t s_{i_k} \beta_{t+1} \cdots \beta_n) \in C(I_i \cdots I_n, -\emptyset) - I(I_i \cdots I_n, -\emptyset)$. In the case of 5), we have
\[
Q(\beta_1 \cdots \beta_n - s_{i_k} i_t \cdots i_n) = -Q(\beta_1 \cdots \beta_n - s_{i_k} i_t \cdots i_n) + \sum_{\sigma=1}^{n} (-1)^{\sigma+n-1} \frac{P_{\sigma(t+1)}(i_1 \cdots i_n)}{\beta_{t+1}-i_n} Q(i_1 \beta_1 \cdots \beta_{t+1} s_{i_k} i_t \cdots i_n) + \sum_{\sigma=1}^{n} (-1)^{\sigma+n-1} \frac{P_{\sigma} i_t \cdots i_n}{\beta_{t+1}-i_n} Q(i_1 \beta_1 \cdots \beta_n - s_{i_k} i_t \cdots i_n)
\]
by Lemma 2.1 (a). Note that $(\beta_1, \ldots, \beta_{t+1}, s_{i_k}, s_{i_{k+1}}) \in C(I_i \cdots I_n, -\emptyset) - I(I_i \cdots I_n, -\emptyset)$, $i_t \in s_{i_k}$ and $i_n \in s_{i_{k+1}}$. By (2.2) 1), $Q(i_1 \beta_1 \cdots \beta_{t+1} s_{i_k} i_t \cdots i_n) = Q(\beta_1 \cdots \beta_{t+1} s_{i_k} i_t \cdots i_n)$ where $\beta_1, \ldots, \beta_{t+1}$ is a permutation of $i_t, i_{t+1}, \ldots, i_{n-2}$ such that $\beta_1 \cdots \beta_{t+1} = s_{i_k} s_{i_{k+1}}$. If $Q(i_1 \beta_1 \cdots \beta_{t+1} s_{i_k} i_t \cdots i_n) = 0$, then $(\beta_1, \ldots, \beta_{t+1}) \in C(I_i \cdots I_n, -\emptyset) - I(I_i \cdots I_n, -\emptyset)$ and $\beta=(\beta_1, \ldots, \beta_{t+1}, s_{i_k})$ is of the form of the case 2). Similarly, $Q(i_1 \beta_1 \cdots \beta_{t+1} s_{i_k} i_t \cdots i_n) = Q(\beta_1 \cdots \beta_{t+1} s_{i_k} i_t \cdots i_n)$ where $\beta_1, \ldots, \beta_{t+1}$ is a permutation of $i_t, i_{t+1}, \ldots, i_{n-2}$ such that $\beta_1 \cdots \beta_{t+1} = s_{i_k} s_{i_{k+1}}$. If $Q(i_1 \beta_1 \cdots \beta_{t+1} s_{i_k} i_t \cdots i_n) = 0$, then $(\beta_1, \ldots, \beta_{t+1}, s_{i_k}) \in C(I_i \cdots I_n, -\emptyset) - I(I_i \cdots I_n, -\emptyset)$ and $\beta=(\beta_1, \ldots, \beta_{t+1}, s_{i_k})$ is of the form of the case 3). Now we get our claim by taking differential.

q.e.d.

Let $(i_1 \cdots i_j \cdots i_n)$ be a permutation of $(1 \cdots m+n)$. We define a linear order $<$ on $(1, \ldots, m+n)$ by $i_1 < \cdots < i_j < \cdots < i_n < i_j < \cdots < i_m$. We define a set $C(i_1 \cdots i_j \cdots i_n, -\emptyset)$ by $\beta=(\beta_1 \cdots \beta_n) | \beta_1 < \cdots < \beta_n$ and a subset $I(i_1 \cdots i_j \cdots i_n, -\emptyset) \subset C(i_1 \cdots i_j \cdots i_n, -\emptyset)$ of $C(i_1 \cdots i_j \cdots i_n, -\emptyset)$ by

$$\beta \in C(i_1 \cdots i_j \cdots i_n, -\emptyset) \implies \begin{cases} \beta=(i_1 \cdots i_j \cdots i_n, -\emptyset) \\ \text{or} \\ \beta=(i_1 \cdots i_j \cdots i_n, -\emptyset) \end{cases} \implies \begin{cases} t=1, \ldots, m; l=1, \ldots, j, \ldots, n \\ \text{or} \\ \beta=(i_1 \cdots i_j \cdots i_n, -\emptyset) \end{cases}$$

Lemma 2.3.

$$dQ(\gamma_i \cdots i_j \cdots i_n, -\emptyset) = \epsilon(i_1 \cdots i_n, -\emptyset) dQ(\gamma_i \cdots i_j \cdots i_n, -\emptyset)$$

where $\epsilon(i_1 \cdots i_n, -\emptyset) \in \{\pm 1\}$ and the exterior product is taken according to the lexicographical order induced from the linear order $<$. Proof. Note that there is a natural bijection between $C(i_1 \cdots i_n, -\emptyset) - I(i_1 \cdots i_n, -\emptyset)$ and $C(i_1 \cdots i_j \cdots i_n, -\emptyset) - I(i_1 \cdots i_j \cdots i_n, -\emptyset)$. We denote this map by

$$f: C(i_1 \cdots i_n, -\emptyset) - I(i_1 \cdots i_n, -\emptyset) \rightarrow C(i_1 \cdots i_j \cdots i_n, -\emptyset) - I(i_1 \cdots i_j \cdots i_n, -\emptyset)$$

Then, for $\beta \in C(i_1 \cdots i_n, -\emptyset) - I(i_1 \cdots i_n, -\emptyset)$, $Q(\beta i_1 \cdots i_n)$ and $Q(f(\beta) i_1 \cdots i_j \cdots i_n)$ coincide up to sign by (2.2) 1) and 2).

q.e.d.
Let \((i_1 \cdots i_j \cdots i_n, s_1 \cdots s_m)\) be a permutation of \((1, \ldots, m+n)\). We define a linear order \(\prec\) on \(\{1, \ldots, m+n\}\) by

\[
i_1 \prec \cdots \prec i_j \prec \cdots \prec i_n \prec s_k \prec i_j \prec \cdots \prec s_k \prec \cdots \prec s_m.
\]

We define a set \(C(i_1 \cdots i_j \cdots i_n, s_1 \cdots s_m, \prec)\) by \(\{\beta = (\beta_1 \cdots \beta_n) | \beta_1 \prec \cdots \prec \beta_n\}\) and a subset \(I(i_1 \cdots i_j \cdots i_n, s_1 \cdots s_m, \prec)\) by

\[
\begin{cases}
\beta \in C(i_1 \cdots i_j \cdots i_n, s_1 \cdots s_m, \prec) = \{ \beta = (i_1 \cdots i_j \cdots i_n, s_1 \cdots s_m, \beta_1 \cdots \beta_n) | \beta_1 < \cdots < \beta_n \} \\
\beta = (i_1 \cdots i_j \cdots i_n, s_1 \cdots s_m) \text{ or } \beta = (i_1 \cdots i_j \cdots i_n, s_1 \cdots s_m) \\
t = 1, \ldots, k, \ldots, m, \\
l = 1, \ldots, j, \ldots, m, \ldots.
\end{cases}
\]

**Lemma 2.4.** For \(l = 1, \ldots, j, \ldots, n, t = 1, \ldots, k, \ldots, m, Q(i_1 \cdots i_j \cdots i_n, s_1 \cdots s_m)\) is a permutation of \((1, \ldots, m+n)\). We define a linear order \(\prec'\) on \(\{1, \ldots, m+n\}\) by \(i_1 \prec' \cdots \prec' i_j \prec' \cdots \prec' i_n \prec' s_k \prec' \cdots \prec' s_m\).

We define a set \(C(i_1 \cdots i_j \cdots i_n, s_1 \cdots s_m, \prec')\) by \(\{\beta = (\beta_1 \cdots \beta_n) | \beta_1 \prec' \cdots \prec' \beta_n\}\) and a subset \(I(i_1 \cdots i_j \cdots i_n, s_1 \cdots s_m, \prec')\) of \(C(i_1 \cdots i_j \cdots i_n, s_1 \cdots s_m, \prec')\) by \(I(i_1 \cdots i_j \cdots i_n, s_1 \cdots s_m, \prec')\). We put

\[
V(i_1 \cdots i_j \cdots i_n, \prec) = C(i_1 \cdots i_j \cdots i_n, \prec) - I(i_1 \cdots i_j \cdots i_n, \prec)
\]

and

\[
V(i_1 \cdots i_j \cdots i_n, \prec') = C(i_1 \cdots i_j \cdots i_n, \prec') - I(i_1 \cdots i_j \cdots i_n, \prec').
\]

Let \(h = \{(1, \ldots, m+n), \prec\} \rightarrow \{(1, \ldots, m+n), \prec'\}\) be an order preserving bijection defined by

\[
\begin{cases}
\hat{h}(i) = i \text{ for } i \neq i_j, s_k \\
\hat{h}(i_j) = s_k \\
\hat{h}(s_k) = i_j.
\end{cases}
\]

Then \(h\) induces order preserving bijections

\[
h : C(i_1 \cdots i_j \cdots i_n, \prec) \rightarrow C(i_1 \cdots i_j \cdots i_n, \prec')
\]

and
h: \( I(i_1, \ldots, i_j \cdot i_s, \prec) \rightarrow I(i_1, \ldots, i_j, \prec') \).

Hence, we have an order preserving bijection

\[ h: V(i_1, \ldots, i_j \cdot i_s, \prec) \rightarrow V(i_1, \ldots, i_j, \prec') \].

**Proposition 2.5.** On \( \pi^{-1}(V_{i_1, \ldots, i_n}) \),

\[
\bigwedge_{\gamma \in V(i_1, \ldots, i_j \cdot i_s, \prec)} dQ(\beta i_1, \ldots, i_j \cdot i_s) \\
= \epsilon(i_1, \ldots, i_j \cdot i_s, i_1, \ldots, i_j) \left( \frac{p_{i_1, \ldots, i_j \cdot i_s}}{p_{i_1, \ldots, i_s}} \right)^t \\
\bigwedge_{\gamma \in V(i_1, \ldots, i_j \cdot i_s, \prec)} dQ(\gamma i_1, \ldots, i_j \cdot i_s)
\]

where \( \epsilon(i_1, \ldots, i_j \cdot i_s, i_1, \ldots, i_j) \) is constant and valued in \( \{\pm 1\} \), and \( t = r - (n-1)(m-1) \).

**Proof.** By Lemma 2.4, we have

\[
Q(i_1, \ldots, i_j \cdot i_s, i_1, \ldots, i_j, i_s) = \pm Q(i_1, \ldots, i_j \cdot i_s, i_1, \ldots, i_j, i_s)
\]

for \( l = 1, \ldots, j, \), \( n = 1, \ldots, k \), \( m = 1, \ldots, t \). In other words, for

\[
\beta = (i_1, \ldots, i_j \cdot i_s, i_1, \ldots, i_j, i_s) \quad (l = 1, \ldots, j, \), \( n = 1, \ldots, k \), \( m = 1, \ldots, t \)
\]

\[
Q(\beta i_1, \ldots, i_j \cdot i_s, i_1, \ldots, i_j) = \pm Q(h(\beta)i_1, \ldots, i_j \cdot i_s, i_1, \ldots, i_j)
\]

We put

\[
S(i_1, \ldots, i_j \cdot i_s)
\]

\[
= \left\{ \beta \in V(i_1, \ldots, i_j \cdot i_s, \prec) \mid \beta = (i_1, \ldots, i_j \cdot i_s, i_1, \ldots, i_s) \right\}
\]

and

\[
S(i_1, \ldots, i_j \cdot i_s)
\]

\[
= \left\{ \beta \in V(i_1, \ldots, i_j \cdot i_s, \prec) \mid \beta = (i_1, \ldots, i_j \cdot i_s, i_1, \ldots, i_s) \right\}
\]

Obviously \( h(S(i_1, \ldots, i_s)) = S(i_1, \ldots, i_s) \). Now we claim that on \( \pi^{-1}(U_{i_1, \ldots, i_n}) \)

\[
(2.5) \quad Q(\beta i_1, \ldots, i_j \cdot i_s) = \pm \frac{p_{i_1, \ldots, i_j \cdot i_s}}{p_{i_1, \ldots, i_s}} Q(h(\beta)i_1, \ldots, i_j \cdot i_s)
\]

\[
+ \sum_{\gamma \in S(\beta)} P_\gamma \left( \frac{p_{i_1, \ldots, i_s}}{p_{i_1, \ldots, i_s}} \right) Q(\gamma i_1, \ldots, i_j \cdot i_s)
\]

where \( P_\gamma \left( \frac{p_{i_1, \ldots, i_s}}{p_{i_1, \ldots, i_s}} \right) \) denotes a polynomial of \( \frac{p_{i_1, \ldots, i_s}}{p_{i_1, \ldots, i_s}} \), for each
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Case 1. \( \beta = (i_1 \cdots i_j \cdots i_n) \) where \( l = 1, \ldots, j, \ldots, n-1, l < \alpha_a (a=1, \ldots, t), \mu_b \neq k (b=1, \ldots, q) \).

By Lemma 2.1 (b) and (2.2) 1) 2),

\[
Q(i_1 \cdots i_j \cdots i_{n-1}) = (-1)^{n-j+1} \prod_{l=1}^{n-1} \frac{Q(i_1 \cdots i_j \cdots i_{n-1})}{Q(i_1 \cdots i_j \cdots i_{n-1})} + \sum_{s+j} (-1)^{s+n-j} \prod_{l=1}^{n-1} \frac{Q(s+1 \cdots i_j \cdots i_{n-1})}{Q(s+1 \cdots i_j \cdots i_{n-1})} + (-1)^{s+n-j} \prod_{l=1}^{n-1} \frac{Q(s+1 \cdots i_j \cdots i_{n-1})}{Q(s+1 \cdots i_j \cdots i_{n-1})}.
\]

Note that \( Q(i_1 \cdots i_j \cdots i_{n-1}) = 0 \) if and only if \( a \leq l \) and \( a \neq \alpha_1, \ldots, \alpha_t \).

By (2.2) 1) and Lemma 2.4, we also have

\[
Q(s_0 i_1 \cdots i_j \cdots i_{n-1}) = \pm Q(i_1 \cdots i_j \cdots i_{n-1}).
\]

Put

\[
\gamma = (i_1 \cdots i_j \cdots i_{n-1}).
\]

Then

\[
\gamma < (i_1 \cdots i_j \cdots i_{n-1})
\]

for \( a \leq l \).

By Lemma 2.1 (a) and (2.2) 2),

\[
Q(i_1 \cdots i_j \cdots i_{n-1}) = -Q(i_1 \cdots i_j \cdots i_{n-1}) + \sum_{s+j} (-1)^{s+n-j} \prod_{l=1}^{n-1} \frac{Q(s+1 \cdots i_j \cdots i_{n-1})}{Q(s+1 \cdots i_j \cdots i_{n-1})} + (-1)^{s+n-j} \prod_{l=1}^{n-1} \frac{Q(s+1 \cdots i_j \cdots i_{n-1})}{Q(s+1 \cdots i_j \cdots i_{n-1})}.
\]

Note that

\[
Q(i_1 \cdots i_j \cdots i_{n-1}) \equiv 0 \quad (t + l = n - 1),
\]

\[
Q(i_1 \cdots i_j \cdots i_{n-1}) \equiv 0 \quad (t + l = n - 1)
\]

and
Thus the first term in the right hand side of (2.7) is identically zero. Obviously

\[ Q(i_1, \ldots, i_j, \ldots, i_{a_1} s_{a_1} \ldots s_{\mu_k}, \ldots, t_1, \ldots, t_n) = 0 \]

if \( q \geq 2 \). Inductively we get

\[(2.8) \]

\[ Q(i_1, \ldots, i_j, \ldots, i_{a_1} s_{a_1} \ldots s_{\mu_k}, \ldots, t_1, \ldots, t_n) = \sum_{r \neq s \neq \beta} P_r \left( \ldots, \frac{p_{r \ldots, s \ldots}}{p_{t_1 \ldots, t_n}} \right) Q(\gamma i_1, \ldots, i_n) \]

for some polynomial functions \( P_r \). Hence we get our claim (2.5) in this case.

By the same way, we can show our claim in the following cases:

**Case 2.** \( \beta = (i_1, \ldots, i_j, \ldots, i_{a_1} s_{a_1} \ldots s_{\mu_k}) \)

\( l = 1, \ldots, j, n-1, t \geq 0, q \geq 2, l < \alpha_{s} = j \)

\( a = 1, \ldots, t \) \( \mu_{b} + k \) \( b = 1, \ldots, q \).

**Case 3.** \( \beta = (i_1, \ldots, i_j, \ldots, i_{a_1} s_{a_1} \ldots s_{\mu_k}) \)

\( l = 1, \ldots, j, n-1, t \geq 0, q \geq 2, l < \alpha_{s} = j \) \( a = 1, \ldots, t \) \( \mu_{b} + k \) \( b = 1, \ldots, q \).

**Case 4.** \( \beta = (i_1, \ldots, i_j, \ldots, i_{a_1} s_{a_1} \ldots s_{\mu_k}) \)

\( l = 1, \ldots, j, n, t \geq 0, q \geq 2, l < \alpha_{s} = j \) \( a = 1, \ldots, t \) \( \mu_{b} + k \) \( b = 1, \ldots, q \).

Hence, on \( \pi^{-1}(V_{i_1 \ldots, i_n}) \), we have

\[ dQ(\beta i_1, \ldots, i_j, i_{a_1} s_{a_1} \ldots s_{\mu_k}) = \pm dQ(h(\beta), i_1, \ldots, i_n) \]

for \( \beta \in S(i_1, \ldots, i_j, i_{a_1} s_{a_1} \ldots s_{\mu_k}) \) and

\[ dQ(\beta i_1, \ldots, i_j, i_{a_1} s_{a_1} \ldots s_{\mu_k}) = \pm \frac{p_{i_1 \ldots, i_j, i_{a_1} s_{a_1} \ldots s_{\mu_k}}}{p_{i_1 \ldots, i_n}} dQ(h(\beta), i_1, \ldots, i_n) \]

\[ + \sum_{r \neq s \neq \beta} P_r \left( \ldots, \frac{p_{r i_1 \ldots, i_{a_1} s_{a_1} \ldots s_{\mu_k}}}{p_{i_1 \ldots, i_n}} \right) dQ(\gamma i_1, \ldots, i_n) \]

for \( \beta \in V(i_1, \ldots, i_n) \), \( \langle \rangle \rightarrow - S(i_1, \ldots, i_n) \).

Since \( h \) is order preserving and the number of elements in \( S(i_1, \ldots, i_n) \) is \((n-1)(m-1)\), we get Proposition 2.5. q.e.d.

**Proposition 2.6.** For \( n \)-tuples \((i_1, \ldots, i_n), (j_1, \ldots, j_n) (1 \leq i_1 < \cdots < i_n \leq m + n, \)
on $\pi^{-1}(V_{i_1...i_n})$, where $\varepsilon(j_1...j_n, i_1...i_n)$ is constant and valued in $\{\pm 1\}$.

Proof. It is enough to see that for $n$-tuples $(i_1, ..., i_n)$ and $(i_1', ..., i_n')$

(1)$i_1<...<i_j<i_{j+1}<...<i_n\leq m+n$

(2.9) $q_{i_1...i_n} = \varepsilon(i_1...i_n, i_1...i_n) \left( \frac{p_{i_1...i_n}}{p_{i_1...i_n}} \right)^t q_{i_1...i_n}$

on $\pi^{-1}(V_{i_1...i_n})$.

By Lemma 2.2, 2.3 and Proposition 2.5, the equality (2.9) holds on $\pi^{-1}(V_{i_1...i_n}) \cap \pi^{-1}(V_{i_1...i_n})$. Since $q_{i_1...i_n}$ and $q_{i_1...i_n}$ are holomorphic forms on $C^{m+n}$, the equality (2.9) holds on $\pi^{-1}(V_{i_1...i_n})$. q.e.d.

Lemma 2.7. For $n$-tuples $(i_1, ..., i_n)$, $(j_1, ..., j_n)$, $(k_1, ..., k_n)$, $\varepsilon(i_1, ..., i_n, j_1, ..., j_n) \varepsilon(j_1, ..., j_n, k_1, ..., k_n) \varepsilon(k_1, ..., k_n, i_1, ..., i_n) = 1$ on $V_{i_1...i_n} \cap V_{j_1...j_n} \cap V_{k_1...k_n}$.

Proof. Since

\[ q_{i_1...i_n}(z) = (p_{i_1...i_n}(z))^t q_{i_1...i_n}(z) + \text{other terms}, \]

\[ q_{i_1...i_n}(z) = 0 \quad \text{for} \quad z \in \pi^{-1}(V_{i_1...i_n}). \]

By Proposition 2.6, we get

\[ \varepsilon(i_1, ..., i_n, j_1, ..., j_n) \varepsilon(j_1, ..., j_n, k_1, ..., k_n) \varepsilon(k_1, ..., k_n, i_1, ..., i_n) = 1 \]

on $\pi^{-1}(V_{i_1...i_n}) \cap \pi^{-1}(V_{j_1...j_n}) \cap \pi^{-1}(V_{k_1...k_n})$. Since $\varepsilon(i_1, ..., i_n, j_1, ..., j_n)$ is constant, we get our claim.

q.e.d.

Lemma 2.8 (Principle of monodromy). Let $G$ be an abelian group and $M$ a simply connected manifold. Let $U = \{U_a\}$ be an open covering of $M$ such that each $U_a$ is connected. Then $H^1(U, G) = (0)$.


Applying Lemma 2.8, for the complex Grassmann manifold $G_{m+n}(C)$ and the system of transition functions $\{\varepsilon(i_1, ..., i_n, j_1, ..., j_n)\}$, we get a system of constant functions $\{\delta(i_1, ..., i_n)\}$ such that $\varepsilon(i_1, ..., i_n, j_1, ..., j_n) = \delta(j_1, ..., j_n)^{-1} \delta(i_1, ..., i_n)$. We put $q_{i_1...i_n} = \delta(i_1, ..., i_n) q_{i_1...i_n}$. Then, by Proposition 2.6, we have
By Proposition 1.2, a compact complex hypersurface $X$ of $G_{m+n}(C)$ is the complete intersection of $G_{m+n}(C)$ and an irreducible subvariety $Y$ of codimension 1 in $P^m(C)$. Let $(F)$ denote the homogeneous ideal associated to $Y$. Note that the degree of homogeneous polynomial $F$ on $C^{m+1}$ is the degree of $X$ and $W_{r+1} = \{ x(\in V_{r+1}) | F(x) = 0 \}$.

**Lemma 2.9.** On $\pi^{-1}(W_{r+1})$, $q_{r+1}$ and $dF \neq 0$.

Proof. Suppose that there is a point $z_0 \in \pi^{-1}(W_{r+1})$ such that $(q_{r+1} \wedge dF)(z_0) = 0$. Since $\pi^{-1}(X)$ is a complex submanifold of $C^{m+1} - (0)$, there are open neighborhoods $U$ of $z_0$ in $C^{m+1} - (0)$ and holomorphic functions $f_j (j=1, \ldots, r+1)$ such that $U \cap \pi^{-1}(X) = \{ z \in U | f_j(z) = 0, j=1, \ldots, r+1 \}$ and $(df_j)(j=1, \ldots, r+1)$ are linearly independent for $z \in U \cap \pi^{-1}(X)$. By the Nullstellensatz for prime ideals ([4] chap. 2A Theorem 7),

$$f_j = \sum_a q_{ja} Q_a + h_j F$$

where $q_{ja}, h_j$ are holomorphic functions on $U$ and $Q_a$ are generators of the ideal $I(G_{m+n}(C))$. Thus we have

$$(df_j)(z_0) = \sum_a q_{ja}(z_0)(dQ_a)(z_0) + h_j(z_0)(dF)(z_0).$$

By Lemma 2.1 a) and b) and (2.2), we see that for each $Q_a$

$$(dQ_a)(z_0) = \sum_{\gamma \in \Gamma(G_{1^n}, 0)} C_a(\gamma)(dQ(\gamma_1 \cdots i_n))(z_0)$$

for some $C_a(\gamma) \in C$. Hence, $\bigwedge_{j=1}^{r+1}(df_j)(z_0) = c(q_{r+1} \wedge dF)(z_0)$ for some $c \in C$ and hence $\bigwedge_{j=1}^{r+1}(df_j)(z_0) = 0$. This is a contradiction. q.e.d.

We define a local holomorphic section $t_{r+1}$ of the line bundle $N$ on $W_{r+1}$ by

$$t_{r+1}(x) = (s_{r+1} q_{r+1}(dF))_x$$

for $x \in W_{r+1}$.

**Lemma 2.10.** The system of transition functions associated to the local trivialization $(W_{r+1}, t_{r+1})$ of the line bundle $N$ is $(t^* s_{r+1} 2r+1 \wedge dF)$, where $a$ is the degree of $X$. In particular, $N = t^* E^{2r+1 - a}$.

Proof. By Lemma 2.9, we have $t_{r+1}(x) \neq 0$ for any $x \in W_{r+1}$ since $Q(\beta_{r+1} \cdots i_n)$ are of degree 2 and $F$ is of degree $a$. 

(2.10) 

$$q_{j_1 \cdots j_n} = \left( \begin{array}{c} p_{j_1 \cdots j_n} \\ p_{j_1 \cdots j_n} \\ \end{array} \right)^t q_{j_1 \cdots j_n} \text{ on } \pi^{-1}(V_{j_1 \cdots j_n})$$
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The canonical line bundle $K$ of $P^\mu(C)$, the holomorphic line bundle of co-vectors of bi-degree $$(\mu,0)$$ on $P^\mu(C)$, is isomorphic to $E^{\mu+1}$. By (2.1) and Lemma 2.10,

$$K(X) = \epsilon^*E^{\mu+n-a},$$

since $t = r - (n-1)(m-1)$.

**Remark.** Let $j: G_{m+n,a}(C) \to P^\mu(C)$ be the inclusion. Then $K(G_{m+n,a}(C)) = j^*E^{m+n}$ ([1] §16). Let $X$ be a compact complex submanifold of codimension 1 in $G_{m+n,a}(C)$ and $\iota_0: X \to G_{m+n,a}(C)$ the inclusion. Then $K(X) = (j \circ \iota_0)^*E^{m+n-a}$, by considering the normal bundle $N(X,G_{m+n,a}(C))$ of $X$ in $G_{m+n,a}(C)$ and by Proposition 1.2.

The first Chern class of $X$, which is the Chern class of the dual bundle $K(X)^*$ of $K(X)$, is the cohomology class containing the form $(m+n-a)\omega$, where $\omega = \epsilon^*\Omega$ is the Kähler form on $X$ associated to the induced Kähler metric on $X$. We shall determine a local section $k_{i_1...i_n}$ of $K(X)^*$ on each $W_{i_1...i_n}$ so that the system of transition functions associated to the local trivialization $(W_{i_1...i_n}, k_{i_1...i_n})$ is $(\epsilon^*g_{i_1...i_n,j_1...j_n})^{-(m+n)}$. We put

$$l_{i_1...i_n} = (-1)^{(i_1...i_n)}\prod_{\sigma_1 < \sigma_2} \partial/\partial u_{i_1...i_n, \sigma_1...\sigma_n}$$
on $U_{i_1...i_n}$, where we take the exterior product of $\partial/\partial u_{i_1...i_n, \sigma_1...\sigma_n}$ according to the natural lexicographical order. Then $(U_{i_1...i_n}, l_{i_1...i_n})$ is the local trivialization of the holomorphic line bundle $K$ on $P^\mu(C)$ and the system of transition functions is $(g_{i_1...i_n,j_1...j_n}^{\mu+1})$.

**Lemma 2.11.** Let $k_{i_1...i_n}$ be a local holomorphic section of $K(X)^*$ on $W_{i_1...i_n}$ defined by

$$k_{i_1...i_n}(x) = l_{i_1...i_n}(x) \perp t_{i_1...i_n}(x)$$

for $x \in W_{i_1...i_n}$, where $\perp$ denotes the right interior multiplication. Then the system of transition functions associated to the local trivialization $(W_{i_1...i_n}, k_{i_1...i_n})$ of $K(X)^*$ is $(\epsilon^*g_{i_1...i_n,j_1...j_n}^{-(m+n)})$.

**Proof.** By (2.1) and Lemma 2.10, $(k_{i_1...i_n}, W_{i_1...i_n})$ is a local trivialization of $K(X)^*$ and the system of transition functions is $(\epsilon^*g_{i_1...i_n,j_1...j_n})^{-(\mu+1)+2r+a-t}$. Since $-(\mu+1)+2r+a-t = -a-(m+n)$, we get our claim. q.e.d.
3. The relation between volumes

Let \( C_n \) denote the set \( \{(i_1, \ldots, i_n) | 1 \leq i_1 < \cdots < i_n \leq m+n\} \). For an element \( i = (i_1, \ldots, i_n) \in C_n \), we put

\[
q_i = \sum H_{\lambda_1, \ldots, \lambda_r}^i \, dp_{\lambda_1} \wedge \cdots \wedge dp_{\lambda_r}
\]

where the summation runs over all \((\lambda_1, \ldots, \lambda_r) \in C_n \times \cdots \times C_n\) such that \( \lambda_1 < \cdots < \lambda_r \) with respect to the lexicographical order \(<\) on \( C_n \). Note that \( H_{\lambda_1, \ldots, \lambda_r}^i \) are homogeneous polynomials of degree \( r \).

\textbf{Proposition 3.1.} There exist homogeneous polynomials \( H_{\lambda_1, \ldots, \lambda_r}^i \) of degree \((n-1)(m-1)\) on \( C^{n+1} \) such that

\[
H_{\lambda_1, \ldots, \lambda_r}^i = p_i^l H_{\lambda_1, \ldots, \lambda_r}^i \quad \text{on} \quad \pi^{-1}(V_i) \quad \text{for each} \quad i \in C_n.
\]

\textbf{Proof.} By (2.10), we have

\[
H_{\lambda_1, \ldots, \lambda_r}^i = \left( \frac{p_j}{p_i} \right)^{(n-1)(m-1)} H_{\lambda_1, \ldots, \lambda_r}^j
\]

on \( \pi^{-1}(V_i \cap V_j) \) for each \((\lambda_1, \ldots, \lambda_r)\). Thus we get

\[
\frac{H_{\lambda_1, \ldots, \lambda_r}^i}{p_i^l} = \left( \frac{p_j}{p_i} \right)^{(n-1)(m-1)} \frac{H_{\lambda_1, \ldots, \lambda_r}^j}{p_j^l}
\]

On \( V_i \cap V_j \). Hence, \( \{H_{\lambda_1, \ldots, \lambda_r}^i/p_i^l\}_{i \in C_n} \) define a holomorphic section of the line bundle \( j^* H^{(n-1)(m-1)} \). Note that a holomorphic section of line bundle \( H^{(n-1)(m-1)} \) on \( P^u(C) \) is nothing but a homogeneous polynomial of degree \((n-1)(m-1)\) on \( C^{n+1} \). By Proposition 1.1, there is a homogeneous polynomial \( H_{\lambda_1, \ldots, \lambda_r} \) of degree \((n-1)(m-1)\) on \( C^{n+1} \) such that

\[
\frac{H_{\lambda_1, \ldots, \lambda_r}}{p_i^{(n-1)(m-1)}} = \frac{H_{\lambda_1, \ldots, \lambda_r}^i}{p_i^l} \quad \text{on} \quad V_i.
\]

Thus we get (3.2). q.e.d.

Now we have

\[
q_i = p_i^l \sum H_{\lambda_1, \ldots, \lambda_r} \, dp_{\lambda_1} \wedge \cdots \wedge dp_{\lambda_r}
\]

on \( \pi^{-1}(V_i) \) for each \( i \in C_n \), and hence

\[
q_i \wedge dF = p_i^l \sum G_{\lambda_1, \ldots, \lambda_{r+1}} \, dp_{\lambda_1} \wedge \cdots \wedge dp_{\lambda_{r+1}}
\]

on \( \pi^{-1}(W_i) \), where \( G_{\lambda_1, \ldots, \lambda_{r+1}} (\lambda_1 < \cdots < \lambda_{r+1}) \) are homogeneous polynomials of degree \((n-1)(m-1)+(a-1)\).

For homogeneous polynomials \( P_1, \ldots, P_s \) on \( C^{n+1} \), we put
where the summation runs over all \((\lambda_1, \ldots, \lambda_s) \in C_\mu \times \cdots \times C_s\) such that \(\lambda_1 < \cdots < \lambda_s\) with respect to the lexicographical order \(<\) on \(C_n\), and we define

\begin{equation}
\| dp_1 \wedge \cdots \wedge dp_s \|^2 (z) = \sum | P_{\lambda_1 \cdots \lambda_s} (z) |^2
\end{equation}

for \(z \in C^{n+1}\). Then we have

\begin{equation}
\| g_i \wedge dF \|^2 (z) = | p_i (z) |^2 \sum | G_{\lambda_1 \cdots \lambda_r} (z) |^2
\end{equation}

for \(z \in \pi^{-1}(W_i)\).

Now we can define a \(C^\infty\)-function \(\varphi: X \to \mathbb{R}\) by

\begin{equation}
\varphi (x) = \frac{\| g_i \wedge dF \|^2 (z)}{| p_i (z) |^2 | z |^{2((n-1)(m-1)+(a-1))}}
\end{equation}

where \(z \in \pi^{-1}(x)\).

Note that \(\varphi (x) = (\sum | G_{\lambda_1 \cdots \lambda_r} (z) |^2 | z |^{2((n-1)(m-1)+(a-1))}) / | z |^{2((n-1)(m-1)+(a-1))}\) for \(z \in \pi^{-1}(x), x \in X\).

Since the dual bundle \(K(X)^*\) of the canonical line bundle \(K(X)\) is the line bundle of \((mn-1)\) vectors of bi-degree \((mn-1, 0)\), the set of hermitian fiber metrics on \(K(X)^*\) and the set of positive volume elements on \(X\) are canonically in one to one correspondence. Let \(\vartheta\) denote the volume element on \(X\) corresponding to the fiber metric \(\vartheta^* | z |^{2(a-(m+n))}\) on \(K(X)^*\). Then the curvature form of the connection determined by the fiber metric \(\vartheta^* | z |^{2(a-(m+n))}\) is \((m+n-a)\omega\), where \(\omega = \vartheta^* \Omega\) is the Kahler form of the induced metric on \(X\).

Now the relation between two volume elements \(\omega_{mn-1}\) and \(\vartheta\) is given by the following Proposition.

**Proposition 3.2.** Let \(\varphi\) be a \(C^\infty\)-function on \(X\) defined by (3.9). Then

\begin{equation}
\omega_{mn-1} = \frac{(mn-1)!}{(2\pi)^{mn-1}} \vartheta \varphi \quad \text{on} \quad X.
\end{equation}

We need several lemmas to prove Proposition 3.2. Note that the norm defined by (3.7) does not depend on the choice of unitary cartesian coordinates on \(C^{n+1}\). That is, for a unitary matrix \(A \in U(\mu+1)\) and homogeneous polynomials \(P_j\), we put \(P_j (w) = P_j (A^{-1} w)\) for \(w \in C^{n+1}\). Then

\begin{equation}
\| dp_1 \wedge \cdots \wedge dp_s \|^2 (z) = \| dp_1' \wedge \cdots \wedge dp_s' \|^2 (w)
\end{equation}

for \(w = A z, z \in C^{n+1}\).

In order to prove Proposition 3.2, it suffices to verify (3.10) at an arbitrary point \(x_0 \in X\). Fix a point \(x_0 \in X\) and let \(z_0\) denote an element of \(C^{n+1}\) such that
$||z_0||=1$ and $\pi(z_0)=x_0$. For an element $A \in U(\mu+1)$, let $p'_i$ denote $p'_i = \sum_j A_j p_j$, where $A=(A_j)$, and put $w=(\cdots, p'_i, \cdots)$. For a homogeneous polynomial $P$ of degree $k$ on $C^{n+1}$, put $P'(w)=P(A^{-1}w)$, $P'_i(w)=P'(w)/(p'_i)^k$, where $i_0=(1, \cdots, n) \in C_n$ and put $u'_{i_0, \lambda}(x)=p'_i(z)/p'_i(z)(z \in \pi^{-1}(x))$. \(\lambda \in C_n, (\lambda \neq i_0)\).

**Lemma 3.3.** If $x_0 \in W, (i \in C_n)$, there is an element $A \in U(\mu+1)$ such that $p'_i(x_0)=1, p'_i(x_0)=0$ for $j \in C_n, j \neq i_0$ and $(dQ'(\beta, i)_{i_0}) (\beta \in C(i, -\infty) - I(i, -\infty)) (where the order is principal with respect to $i$), $(dF'_i)_{i_0}$ are linear combination of $u'_{i_0, \lambda}(x_0) (\lambda \in C(i_0, <)-I(i_0, <)\), $(dF'_{i_0,12-n-1+1})_{i_0}$.

**Proof.** By a routine computation of linear algebra.

Now we put $p_j=\sum_j B_j p'_j$ and $\psi=(\partial u'_{i_0, \lambda}/\partial u'_{i_0, \lambda})(x_0)$.

**Lemma 3.4.**

(3.12) $\psi=(B;B;B;B;B;B;B;B;B)$

for $\lambda \neq i, (i \neq i_0, \lambda), \lambda, \nu \in C_n$.

**Proof.** Straightforward computation.

Let $J(i_0, <)$ denote $I(i_0, <) - \{i_0, (12-n-1n+1)\}$. We put $J(i_0, <)=\{\nu_1, \cdots, \nu_{mn-1}\}$ with $\nu_k<\nu_{k+1}(k=1, \cdots, mn-2), C(i, -\infty) - I(i, -\infty)=\{\beta_1, \cdots, \beta_r\}$ with $\beta_i=\beta_i (l=1, \cdots, r-1)$ and $C(i_0, <)-I(i_0, <)=k_{l_1, \cdots, l_r}$ with $l_i<\lambda_{i+1}$ $(s=1, \cdots, r-1)$.

**Lemma 3.5.** Let $k, \rho$ be the holomorphic section of $K(x)^* \in W, \text{ defined in Lemma 2.12. Then, at } x_0 \in W_i$,

(3.13) $k_1(x_0)=(-1)^{\nu_{m-1} \cdot \delta(i) \cdot [\det (C;\psi)]^{-1}}$

$\times \left( p'_i(x_0) \right)^{2r+\delta} \frac{\partial (Q'(\beta_1, i)_{i_0}, \cdots, Q'(\beta_r, i)_{i_0}, F'_0) (x_0)}{\partial (u'_{i_0,12-n-1n+1}, u'_{i_0, \lambda_1}, \cdots, u'_{i_0, \lambda_r})} (x_0)$

$\times (\partial \partial u'_{i_0, \lambda_1} \wedge \cdots \wedge \partial \partial u'_{i_0, \lambda_{mn-1}})_{x_0}$.

**Proof.** For a homogeneous polynomial $P$ of degree $k$ on $C^{n+1}$, put $P_i=P/(p_i)^k$ on $U_i$. By the definition,

$t_i(x_0)=\delta(i) \delta(x_0) (dQ'(\beta_1, i) \wedge \cdots \wedge dQ'(\beta_r, i) \wedge dF'_i)_{x_0}$.

Thus

$t_i(x_0)=\delta(i) (dQ'(\beta_1, i, \cdots \wedge dQ'(\beta_r, i) \wedge dF'_i)_{x_0}$

$=\delta(i) (p'_i(x_0)^{2r+\delta} (dQ'(\beta_1, i)_{i_0}, \cdots \wedge dQ'(\beta_r, i)_{i_0} \wedge dF'_i)_{x_0}$

On the other hand, we have
By the definition of $k_i$,

$$k_i(x_0) = (-1)^{(q(i)-1)\delta(i)} \cdot [\det (C^i)]^{-1} (p'_{i_o}/p_i)(x_0)^{2r+\alpha}$$

$$\times \left( \prod_{\beta \in C(i)} \partial/\partial u'_{i_{0,\alpha}}(x_0) \right) \wedge (dQ'(\beta_{1, i_{0, \beta}}) \wedge \cdots \wedge dQ'(\beta_{r, i_{0, \beta}}) \wedge dF'_{i_{0}})_{x_0}$$

By Lemma 3.3, we get (3.13). q.e.d.

Now the local expression of the volume element $\nu$ at $x_0$ is given by the following Lemma.

**Lemma 3.6.**

(3.14) \[ \nu_{x_0} = \sqrt{-1}^{mn-1} |\det (C^i)|^{2.1} |(p'_{i_o}/p_i)(x_0)|^{-2(m+n+2r)} \]

$$\times \left( \prod_{\beta \in C(i)} \partial/\partial u'_{i_{0,\alpha}}(x_0) \right) \wedge (d\nu' \wedge d\eta')(x_0)$$

where \((d\eta')_{x_0} = (du'_{i_{0,\alpha}} \wedge \cdots \wedge du'_{i_{0,\gamma_{x_0-1}}}x_0)\)

Proof. By the definition, $\nu$ is the volume element on $X$ corresponding to the fiber metric $\iota^*||z||^{2(s-(m+n))}$ on $K(X)^*$. Note that

$$1 + \sum_{\alpha \in C(i)} |(p'_{i_o}/p_i)(x_0)|^2 = |(p'_{i_o}/p_i)(x_0)|^2$$

Put

$$T_i(x_0) = (-1)^{(q(i)-1)\delta(i)} \cdot [\det (C^i)]^{-1}$$

$$\times (p'_{i_o}/p_i)(x_0)^{2r+\alpha} \cdot \frac{\partial Q'(\beta_{1, i_{0, \beta}}) \wedge \cdots \wedge dQ'(\beta_{r, i_{0, \beta}}) \wedge dF'_{i_{0}}}{\partial (u'_{i_{0,12}, \alpha_1} \wedge \cdots \wedge u'_{i_{0,\gamma_{x_0-1}}})}_{(x_0)}$$

Then $\nu_{x_0}$ is given by

$$\frac{1}{|T_i(x_0)|^2} |(p'_{i_o}/p_i)(x_0)|^{2(s-(m+n))}(d\nu' \wedge d\eta')(x_0)$$

Hence

$$\nu_{x_0} = \sqrt{-1}^{mn-1} |\det (C^i)|^{2.1} |(p'_{i_o}/p_i)(x_0)|^{-2(m+n+2r)}$$

$$\times \left( \prod_{\beta \in C(i)} \partial/\partial u'_{i_{0,\alpha}}(x_0) \right) \wedge (d\nu' \wedge d\eta')(x_0)$$

q.e.d.

**Lemma 3.7.** At $x_0 \in W_i$,

(3.15) \[ \varphi(x_0) = |(p'_{i_o}/p_i)(x_0)|^{2r} \cdot \frac{\partial Q'(\beta_{1, i_{0, \beta}}) \wedge \cdots \wedge dQ'(\beta_{r, i_{0, \beta}}) \wedge dF'_{i_{0}}}{\partial (u'_{i_{0,12}, \alpha_1} \wedge \cdots \wedge u'_{i_{0,\gamma_{x_0-1}}})}_{(x_0)} \]
Proof. Fix $c \in C^*$ so that $|c_\infty(x_0)|^2 = 1$. Then $|c|^2 \cdot (1 + \sum_{a \in C_\infty} |(p_a)(x_0)|^2) = 1$ and $|c|^2 = \frac{1}{|\langle p_i, p_i \rangle(x_0)|^2}$. Note that

$$
\varphi(x_0) = \frac{||q_i \wedge dF||^2(c_\infty(x_0))}{|c|^2 ||c_\infty(x_0)||^{2(n-1)(m-1)+(d-1)}} = \frac{||q_i' \wedge dF'||^2(1, 0, \ldots, 0)}{|c|^2} \text{ by (3.11).}
$$

Since

$$
\frac{\partial Q'(\beta, i)}{\partial p_{i_0}'}(1, 0, \ldots, 0) = 0 \quad \text{for} \quad k = 1, \ldots, r, \frac{\partial F'}{\partial p_{i_0}'}(1, 0, \ldots, 0) = 0
$$

and

$$
\frac{\partial Q'(\beta, i)}{\partial p_{i_0}'}(1, 0, \ldots, 0) = \frac{\partial Q'(\beta, i)_{i_0}}{\partial u_{i_0,j}'}(x_0),
$$

$$
\frac{\partial F'}{\partial p_{i_0}'}(1, 0, \ldots, 0) = \frac{\partial F'_{i_0}}{\partial u_{i_0,j}'}(x_0) \quad \text{for} \quad j \in C_{\infty}, j \neq i_0,
$$

$$
||q_i' \wedge dF||^2(1, 0, \ldots, 0) = ||dQ'(\beta, i) \wedge \cdots \wedge dQ'(\beta, i) \wedge dF'||^2(1, 0, \ldots, 0)
$$

$$
= \left| \frac{\partial Q'(\beta, i)_{i_0}, \ldots, Q'(\beta, i)_{i_0}}{\partial (u_{i_0,12\cdots \gamma - 1\gamma + 1}, u_{i_0,1}, \ldots, u_{i_0,\lambda})} (x_0) \right|^2
$$

by Lemma 3.3.

By Lemma 3.3, the Kahler form $\omega$ of the induced metric on $X$ is given by

$$
\omega_{x_0} = \frac{\sqrt{-1}}{2\pi} \left( \sum_{\nu \in C_{\infty}, \nu \neq \gamma} du_{i_0,\nu} \wedge d\bar{u}_{i_0,\nu} \right) \text{ at } x_0 \in X.
$$

Hence,

$$
\omega_{x_0}^{mn-1} = \left( \frac{\sqrt{-1}}{2\pi} \right)^{(mn-1)^2} \left( mn-1 \right)! \left( d\eta' \wedge \bar{d}\eta' \right)_{x_0}.
$$

**Lemma 3.8.**

$$
|\det (C^\lambda_a)|^2 = |(p_{i_0})(x_0)|^{2(n+1)}
$$

Proof. Put $D^\lambda_a = B_{\lambda}^a B_{\lambda}^\nu - B_{\lambda}^\nu B_{\lambda}^a$ for $\lambda \neq \nu$, $i \neq i_0$, $\lambda, \nu \in C_{\infty}$. Note that

$$
|\det (D^\lambda_a)|^2 = \det (D^\lambda_a) \cdot \det ((\sum_{a \neq i_0} D^\lambda_a \bar{D}^\lambda_a)_{\lambda \neq i_0}),
$$

and that

$$
\sum_{a \neq i_0} D^\lambda_a \bar{D}^\lambda_a = \sum_{a \neq i_0} (B_{\lambda}^a B_{\lambda}^\nu - B_{\lambda}^\nu B_{\lambda}^a)(\bar{B}_{\lambda}^a B_{\lambda}^\nu - \bar{B}_{\lambda}^\nu B_{\lambda}^a)
$$

$$
= \sum_{a \in C_{\infty}} (B_{\lambda}^a B_{\lambda}^\nu - \bar{B}_{\lambda}^\nu B_{\lambda}^a)(\bar{B}_{\lambda}^a B_{\lambda}^\nu - \bar{B}_{\lambda}^\nu B_{\lambda}^a)
$$

$$
= \delta_{\lambda \gamma} |B_{\lambda}^\nu|^2 + B_{\lambda}^\nu B_{\lambda}^\nu,
$$

where $\delta_{\lambda \gamma}$ is the Kronecker delta.
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Thus
\[ |\det (D^a)|^2 = |\det (\delta_{\lambda_\alpha} B^i_{\alpha}|^2 + \sum_{\lambda
ot= \alpha} |B^i_{\lambda}|^2) \]
\[ = |B^i_{\alpha}|^{2\lambda} \sum_{X=0} \sum_{X=0} |B^i_{\alpha}|^{2\lambda} |B^i_{\alpha}|^2 \]
\[ = |B^i_{\alpha}|^{2\lambda} \sum_{X=0} \sum_{X=0} |B^i_{\alpha}|^{2\lambda} |B^i_{\alpha}|^2 \]
Now
\[ |\det (C^\alpha)|^2 = |B^i_{\alpha}|^{-2\times 2\lambda} |\det (D^a)|^2 \]
\[ = |B^i_{\alpha}|^{-2\times 2\lambda} \sum_{X=0} \sum_{X=0} |B^i_{\alpha}|^{2\lambda-1} \]
Since \( B^i_{\alpha} = (p^i_{\alpha}/p^i_{\alpha}) \), we get our claim. q.e.d.

Proof of Proposition 3.2.

By Lemma 3.6, Lemma 3.7 and Lemma 3.8, we have
\[ \phi(x_0)^{b_{x_0}} = (\sqrt{-1})^{(m-n-1)^2} \left( (p^i_{\alpha}/p^i_{\alpha}) \right)^{2\lambda} (d\eta' \wedge d\bar{\eta}')_{x_0} \]
Since
\[ r - t = (m-1)(n-1) = mn - (m+n)+1 \]
\[ \mu + t - 2r = \mu + 1 - r - (m+n)-mn + m+n - 1 = \mu + 1 - r - mn - 1 = 0 \]
Hence
\[ \phi(x_0)^{b_{x_0}} = (\sqrt{-1})^{(m-n-1)^2} (d\eta' \wedge d\bar{\eta}')_{x_0} \]
Now our claim follows from (3.16).


Let \( g_0 \) denote the Kähler metric on \( X \) induced from the Fubini-Study metric on \( P^n(C) \). Then \( (X, g_0) \) is an Einstein manifold if and only if \( \phi \) is a constant function on \( X \).

Proof. The Ricci form of the Kähler metric \( g_0 \) on \( X \) is the curvature form of the connection of type (1.0) on the holomorphic line bundle \( K(X)^* \) determined by the volume element \( \omega_{m+n-1} \). Suppose that \( g_0 \) is Einstein, that is, the Ricci form is a constant multiple of the Kähler form \( \omega \). Then the Ricci form is harmonic. On the other hand, the volume element \( \nu \) determines the curvature form \( \nu \omega \), which is also harmonic. Since the Ricci form and \( (m+n-a)\omega \) are both curvature form of the bundle \( K(X)^* \), they are cohomologous. Thus the Ricci form must be \( (m+n-a)\omega \). Since \( \omega^{m+n-1} \) and \( \nu \) define the same curvature form, \( d'd'' \log \phi = 0 \), and hence \( \log \phi \) is a harmonic function on \( X \). This implies that \( \phi \) is a constant function. Conversely, if \( \phi \) is a constant function, then the
metric $g_0$ is Einstein.

q.e.d.

4. The dual map and Veronese map

In this section we recall the dual map and Veronese map due to Hano [5].

Let $\wedge^{r+1}(\mathbb{C}^{m+1})^*$ denote the $(r+1)$-th exterior product of the dual space of the vector space $\mathbb{C}^{m+1}$. We identify the tangent space of $\mathbb{C}^{m+1}$ at a point with $\mathbb{C}^{m+1}$ itself. We regard $(q, \wedge dF)_z$ as an element in $\wedge^{r+1}(\mathbb{C}^{m+1})^*$. Let $(\xi_{\lambda_1 \ldots \lambda_{r+1}})$ be the standard base of $\wedge^{r+1}(\mathbb{C}^{m+1})^*$. Then

$$(q, \wedge dF)_z = (p(z))^t \sum G_{\lambda_1 \ldots \lambda_{r+1}}(z)\xi_{\lambda_1 \ldots \lambda_{r+1}} \quad \text{for} \quad z \in \pi^{-1}(W_i).$$

Now we define a map $G: \mathbb{C}^{m+1} \rightarrow \wedge^{r+1}(\mathbb{C}^{m+1})^*$ by

$$(4.1) \quad G(z) = \sum G_{\lambda_1 \ldots \lambda_{r+1}}(z)\xi_{\lambda_1 \ldots \lambda_{r+1}}.$$

We denote by $P'(\mathbb{C})$ the complex projective space associated to the complex vector space $\wedge^{r+1}(\mathbb{C}^{m+1})^*$, where $d+1=\dim S_k$. Since the map $G: \mathbb{C}^{m+1} \rightarrow \wedge^{r+1}(\mathbb{C}^{m+1})^*$ is a polynomial map of degree $(n-1)(m-1)+(a-1)$ and $G(z) \neq 0$ for $z \in \pi^{-1}(X)$, it induces a holomorphic map $g: X \rightarrow P'(\mathbb{C})$. We call $g$ the dual map of $X$ in $P'(\mathbb{C})$. Let $|m|$ be the norm of an element $w$ in $\wedge^{r+1}(\mathbb{C}^{m+1})^*$ induced from the hermitian inner product on $\mathbb{C}^{m+1}$. Let $\Omega'$ be the Fubini-Study form on $P'(\mathbb{C})$ determined from $|w|^2$.

**Proposition 4.1** (cf. [5] Proposition 3). The induced metric $g_0$ on $X$ is Einstein if and only if the reciprocal image of the Fubini-Study metric on $P'(\mathbb{C})$ under the dual map $g$ is $(n-1)(m-1)+(a-1)$ times of the induced metric; $g^*\Omega' = ((n-1)(m-1)+(a-1))\omega$.

Proof. Since the degree of $G$ is $(n-1)(m-1)+(a-1)$, the reciprocal image of the standard line bundle $E'$ over $P'(\mathbb{C})$ under the map $g$ is $E' \otimes E'(n-1)(m-1)+(a-1)$ where $E$ denotes the standard line bundle over $P'(\mathbb{C})$. We regard $|w|^2$ as the fiber metric on $E'$ over $P'(\mathbb{C})$. Its reciprocal image under $g$ is the restriction of $\sum |G_{\lambda_1 \ldots \lambda_{r+1}}(z)|^2$ to $\pi^{-1}(X)$ and is a fiber metric on $E' \otimes E'(n-1)(m-1)+(a-1)$. Then

$$(n-1)(m-1)+(a-1) \quad \text{log} \quad (\sum |G_{\lambda_1 \ldots \lambda_{r+1}}(z)|^2).$$

Now our claim follows from Corollary of Proposition 3.2. q.e.d.

Let $S_k$ be the vector space of homogeneous polynomials on $\mathbb{C}^{m+1}$ of degree $k$ and $S_k^*$ the dual space of $S_k$. We denote by $P^d(\mathbb{C})$ the complex projective space associated to $S_k^*$, where $d+1=\dim S_k$. Each point $z \in \mathbb{C}^{m+1}$ defines a linear function $\Psi(z)$ on $S_k$ given by $\Psi(z)(P) = P(z)$ for $P \in S_k$. We denote by $\psi$ the map $z \mapsto \Psi(z)$. The polynomial map $\Psi$ induces an injective holomorphic map...
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(4.2) \( \psi: P^m(C) \rightarrow P^d(C) \)

if \( k \geq 1 \). The map \( \psi \) is called the Veronese map of degree \( k \).

For simplicity we denote the Plücker coordinate \((\cdots, p_k, \cdots)\) by \((z_0, \cdots, z_\mu)\).

With respect to the hermitian inner product on \( S_k \) induced from the one on \( C^{n+1} \), the set of all monomials

\[
(e_0^\cdots e_\mu)(\nu_0! \cdots \nu_\mu)!^{1/2}, \nu_0 + \cdots + \nu_\mu = k
\]
is a unitary base of \( S_k \). Moreover

\[
|z_0^\cdots z_\mu!(\nu_0! \cdots \nu_\mu)!^{1/2}|^2 = |z|^2k!.
\]

Obviously the reciprocal image of the standard line bundle over \( P^d(C) \) under the map \( \psi \) is \( E^k \). By (4.4), if \( \Omega'' \) denotes the Fubini-Study form on \( P^d(C) \), then \( \psi^*\Omega'' = k\Omega \). That is, the Veronese map \( \psi \) is homothetic and the ratio of the metrics is the degree \( k \) of the map \( \psi \).

Now we specify \( k \) to be \((n-1)(m-1)+(a-1)\), and define a linear map

\[
L: S^{*}_{(n-1)(m-1)+(a-1)} \rightarrow \wedge^{r+1}(C^{a+1})^*\]

so that \( L \circ \psi = G \) on the cone \( \pi^{-1}(X) \). Let \((\xi_0, \cdots, \xi_\mu)\) be the dual base of the unitary base of \( S_{(n-1)(m-1)+(a-1)} \) chosen above.

Since \( G_{\lambda_1 \cdots \lambda_{r+1}} \) is of degree \((n-1)(m-1)+(a-1)\),

\[
G_{\lambda_1 \cdots \lambda_{r+1}} = \sum_{\nu_0 \cdots \nu_\mu} a(\lambda_1 \cdots \lambda_{r-1}; \nu_0 \cdots \nu_\mu)(z_0^\cdots z_\mu!(\nu_0! \cdots \nu_\mu)!^{1/2}).
\]

Using these coefficients, a linear map \( L \) is defined by

\[
L(\xi_0, \cdots, \xi_\mu) = \sum a(\lambda_1 \cdots \lambda_{r+1}; \nu_0 \cdots \nu_\mu)\xi_{\lambda_1 \cdots \lambda_{r+1}}.
\]

By the way \( L \) is defined, it is clear that

\[
(L \circ \psi)(z) = G(z) \quad \text{for} \quad z \in \pi^{-1}(X).
\]

Consider the rational map \( l: P^d(C) \rightarrow P^r(C) \) induced from the linear map

\[
L: S^{*}_{(n-1)(m-1)+(a-1)} \rightarrow \wedge^{r+1}(C^{a+1})^*\].

The map \( l \) is holomorphic at a point \( x \in P^d(C) \) if the image under \( L \) at a point of \( S^{*}_{(n-1)(m-1)+(a-1)} \) lying over \( x \) is not zero. Since \( ||g, \wedge dF||^2 \) vanishes nowhere on \( \pi^{-1}(W) \), \( L \) does not vanishes at each point on the image of \( \pi^{-1}(X) \) under \( \psi \). Therefore \( l \) is holomorphic on \( \psi(X) \).

**Proposition 4.2.** Let be \( \psi \) the Veronese map of degree \((n-1)(m-1)+(a-1)\) of \( P^m(C) \) into \( P^d(C) \) and let \( g \) be the dual map of \( X \) into \( P^r(C) \). Then there is a projective transformation \( \tilde{l} \) of \( P^d(C) \) into \( P^r(C) \) which is holomorphic on \( \psi(X) \) and satisfies the equality \((l \circ \psi)(x) = g(x) \) for \( x \in X \). Moreover the induced metric on \( X \) is Einstein if and only if the restriction of \( \tilde{l} \) to \( \psi(X) \) is everywhere locally isometric with respect to the induced metric on \( \psi(X) \) and the Fubini-Study metric on \( P^r(C) \).

Now we have the following Lemma due to Hano ([5] Lemma 7).

**Lemma 4.3.** Let $\Phi$ be a linear map of $\mathbb{C}^{a+1}$ into $\mathbb{C}^{i+1}$ and $\phi$ the induced projective transformation of $P^i(\mathbb{C})$ into $P^i(\mathbb{C})$. Let $U$ be a connected algebraic submanifold in $P^i(\mathbb{C})$ which is not contained in any hyperplane in $P^i(\mathbb{C})$. We equip on $U$ the metric induced from a Fubini-Study metric on $P^i(\mathbb{C})$, and on $P^i(\mathbb{C})$ a Fubini-Study metric. Suppose that the restriction of $\phi$ to $U$ is holomorphic and locally isometric everywhere, then $\Phi$ is a constant multiple of an isometry, and particularly $\Phi$ is injective.

Now we have the following necessary condition from Lemma 4.3.

**Proposition 4.4** (cf. [5] Hano §8). Let $X$ be a hypersurface of $G_{m+n,a}(\mathbb{C})$ of degree $a$. If the induced isometric metric on $X$ is Einstein, then

\[
\dim (S_{(n-1)(m-1)+(a-1)}^i I_{(n-1)(m-1)+(a-1)}) = \left(\begin{array}{c}
m+1 \\
r+1
\end{array}\right),
\]

where $I_{(n-1)(m-1)+(a-1)} = S_{(n-1)(m-1)+(a-1)} \cap I(X)$.

**Proof.** For $P \in S_{(n-1)(m-1)+(a-1)}$, the equation $\langle \xi, P \rangle = 0$, $\xi \in S_{(n-1)(m-1)+(a-1)}^i$, defines a hyperplane in $P^d(\mathbb{C})$. By the definition of the Veronese map $\psi$, a homogeneous polynomial $P$ in $S_{(n-1)(m-1)+(a-1)}^i$ defines a hyperplane containing $\psi(X)$ if and only if $P$ belongs to $I_{(n-1)(m-1)+(a-1)}$. Thus, the minimal linear variety $P^d(\mathbb{C})$ containing $\psi(X)$ is the intersection of those hyperplanes each of which is associated to a polynomial in $I_{(n-1)(m-1)+(a-1)}$. Its dimension $d'$ is given by $\dim (S_{(n-1)(m-1)+(a-1)} I_{(n-1)(m-1)+(a-1)}) - 1$. Let $C_{d' + 1}$ be the subspace in $S_{(n-1)(m-1)+(a-1)}^i$ perpendicular to the subspace $I_{(n-1)(m-1)+(a-1)}$. Let $L'$ be the restriction to $C_{d' + 1}$ of the linear map $L: S_{(n-1)(m-1)+(a-1)}^{*i} \rightarrow \bigwedge^{r+1} (\mathbb{C}^{a+1})^*$, and let $l'$ be the restriction to $P^d(\mathbb{C})$ of projective transformation $l$. Now the connected algebraic submanifold $\psi(X)$ in $P^d(\mathbb{C})$ is not contained in any hyperplane of $P^d(\mathbb{C})$. By Proposition 4.2, the restriction to $\psi(X)$ of $l'$ is everywhere locally isometric. Applying Lemma 4.3, to $\psi(X)$ in $P^d(\mathbb{C})$, we see that the linear map

\[
L': C_{d' + 1} \rightarrow \bigwedge^{r+1} (\mathbb{C}^{a+1})^*
\]

is injective, and hence we get (4.8). q.e.d.

5. **Proof of Theorem**

Let $J$ denote the ideal $I(G_{m+n,a}(\mathbb{C}))$ of homogeneous polynomials $S$ on $\mathbb{C}^{a+1}$.

**Lemma 5.1.** Let $J_k$ denote $J \cap S_k$. Then

\[
\dim (S_k/J_k) = \prod_{i=1}^{n} \prod_{j=a+1}^{m} \frac{k+i-j}{j-i}
\]
Proof. By Proposition 1.1, the inclusion \( j: G_{m+n,n}(C) \rightarrow P^n(C) \) induces a surjective linear map

\[
j^*: H^k(P^n(C), H^k) \rightarrow H^k(G_{m+n,n}(C), j^*H^k).\]

Noting that \( H^k(P^n(C), H^k) \) is the space of homogeneous polynomials \( S_k \) of degree \( k \),

\[
\text{Ker} j^* = \{ P \in S_k | P(z) = 0 \text{ for any } z \in \pi^{-1}(G_{m+n,n}(C)) \}
\]

\[
= J \cap S_k.
\]

Hence, \( \dim (S_k/J_k) = \dim H^k(G_{m+n,n}(C), j^*H^k) \).

On the other hand, by a Theorem of Borel-Weil [2] and the dimension formula of Weyl [10], we have

\[
\dim H^k(G_{m+n,n}(C), j^*H^k) = \prod_{i=1}^{m+n} \frac{k+j-i}{j-i}.
\]

q.e.d.

**Lemma 5.2.** Let \( I_k \) denote \( I(X) \cap S_k \). Then

\[
\dim (S_k/I_k) = \dim (S_k/J_k) - \dim (S_{k-a}/J_{k-a})
\]

if \( k \geq a \), where \( a \) is the degree of \( X \).

Proof. Let \( [X] \) denote the non-singular divisor defined by \( X \) and \( \{ X \} \) the holomorphic line bundle on \( G_{m+n,n}(C) \) defined by \( [X] \). Then there is an exact sequence

\[
0 \rightarrow j^*H^{k-s} \rightarrow j^*H^k \rightarrow \hat{\iota}^*H^k \rightarrow 0
\]

of holomorphic sheaves on \( G_{m+n,n}(C) \). (cf. [6])

Then (5.1) induces the following exact sequence of cohomologies

\[
0 \rightarrow H^0(G_{m+n,n}(C), j^*H^{k-s}) \rightarrow H^0(G_{m+n,n}(C), j^*H^k) \rightarrow H^0(X, \hat{\iota}^*H^k) \rightarrow H^0(G_{m+n,n}(C), j^*H^{k-s}) \rightarrow \ldots
\]

Since \( H^0(G_{m+n,n}(C), j^*H^{k-s}) = 0 \) if \( k \geq a \), by a theorem of Bott [2],

\[
\dim H^0(X, \hat{\iota}^*H^k) = \dim H^0(G_{m+n,n}(C), j^*H^k) - \dim H^0(G_{m+n,n}(C), j^*H^{k-s}).
\]

On the other hand, \( j^*: H^0(P^n(C), H^k) \rightarrow H^0(G_{m+n,n}(C), j^*H^k) \) is surjective, and hence \( \iota^*: H^0(P^n(C), H^k) \rightarrow H^0(X, \iota^*H^k) \) is surjective if \( k \geq a \). Noting that \( \text{Ker} \iota^* = I(X) \cap S_k \), we have

\[
\dim (S_k/I_k) = \dim H^0(X, \iota^*H^k)
\]

\[
= \dim H^0(G_{m+n,n}(C), j^*H^k) - \dim H^0(G_{m+n,n}(C), j^*H^{k-s})
\]

\[
= \dim (S_k/J_k) - \dim (S_{k-a}/J_{k-a}).
\]

q.e.d.
Proof of Theorem. Put $k = (n-1)(m-1) + (a-1)$. If $n \geq 2$ and $m \geq n$, then $k \geq a$. Thus, by Lemma 5.2,

$$\dim (S_{(n-1)(m-1)+(a-1)}/I_{(n-1)(m-1)+(a-1)})$$

$$= \dim (S_{(n-1)(m-1)+(a-1)}/J_{(n-1)(m-1)+(a-1)})$$

$$- \dim (S_{(n-1)(m-1)-1}/J_{(n-1)(m-1)-1}).$$

By Lemma 5.1, we see that $\dim (S_k/I_k)$ is increasing in $k$. Hence, it is enough to prove the following inequality (5.3) by Proposition 4.4;

$$(5.3) \dim (S_{\mu-(m+n)+2}/I_{\mu-(m+n)+2}) > \left(\frac{\mu+1}{mn}\right).$$

By Lemma 5.1, we have

$$\dim (S_k/J_k) = \frac{(k+1)(k+2)^2 \cdots (k+n)^n (k+m)^m (k+m-1)^{n-1} \cdots (k+m+n-1)}{1 \cdot 2 \cdot n^n m^m (m+1)^{n-1} \cdots (m+n-1)}.$$ 

Thus

$$\dim (S_{\mu-(m+n)+2}/I_{\mu-(m+n)+2}) - \left(\frac{\mu+1}{mn}\right)$$

$$= \frac{(\mu+1)\mu^2(\mu-1)^2 \cdots (\mu-n+2)^n (\mu-n+1)^{n-1} \cdots (\mu-m-n+3)}{1 \cdot 2 \cdot 3 \cdots (mn)}$$

$$\times (\mu-n-1) \cdots (\mu+2-mn)$$

$$> \frac{1}{(mn)!} (\frac{\mu+1)(\mu-1)^2 \cdots (\mu-n+2)^n (\mu-n+1)^{n-1} \cdots (\mu-m-n+3)}{\mu-m-n-3}$$

$$\times (\mu-1) \cdots (\mu+2-mn)$$

$$- (\mu-m-n+1)(\mu-m-n)^2 \cdots (\mu-n)^n (\mu-n-1)^{n-1} \cdots (\mu-n-2) 

$$

$$> (\mu+1)\mu(\mu-1) \cdots (\mu+2-mn)$$

$$= (\mu+1)\mu(\mu-1) \cdots (\mu-m-n+3)$$

$$\times (\mu-m-n+1) \cdots (\mu-m-n+4) - (\mu-m-n+2) \cdots (\mu-mn+2)$$

$$> (\mu+1)\mu(\mu-1) \cdots (\mu-mn+3)(\mu-mn+2).$$

Now we have

$$(\mu+1)\mu(\mu-1) \cdots (\mu-m-n+2)^n (\mu-m-n+1)^{n-1} \cdots (\mu-m-n+3)$$

$$- (\mu+1)\mu(\mu-1) \cdots (\mu+2-mn)$$

$$= (\mu+1)\mu(\mu-1) \cdots (\mu-m-n+3)$$

$$\times (\mu-m-n+1) \cdots (\mu-m-n+4) - (\mu-m-n+2) \cdots (\mu-mn+2)$$

$$> (\mu+1)\mu(\mu-1) \cdots (\mu-mn+3)(\mu-mn+2).$$

On the other hand,

$$(\mu-mn+3) - (mn-m-n+2) = \left(\frac{m+n}{n}\right) - 2mn + m + n > 0.$$
Thus we have
\[ (\mu+1)\mu(\mu-1)\cdots(\mu-mn+3)(mn-w-n+2) \]
\[ - (mn-1)\cdots(mn-m-n+1)^{n-1}(mn-n)^{\cdots}(mn-m)^{(mn-m-1)^{n-1}} \]
\[ \times (mn-m-n+1)>(\mu+1)\mu\cdots(\mu-mn+3)(mn-m-n+2)-(2mn-m-m) \]
\[ \times (mn-m-n+2)(mn-m-n+1)>0. \]

Hence, we get (5.3). q.e.d.

**Remark.** In the case of \( G_{5,2}(C) \), we can see that if the degree \( a(X) \) of \( X \) satisfies \( a(X)\geq 3 \) a hypersurface \( X \) is not an Einstein manifold with respect to the induced metric by the same way. But we do not know the cases when \( a(X)=1, 2 \).

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**References**


