Introduction. Let $G$ be a connected semisimple Lie group with finite center. Let $\Gamma$ be a discrete subgroup of $G$ such that $\Gamma \backslash G$ is compact. Assume that $\Gamma$ has no elements with finite order other than the identity. Fix a Haar measure $dg$ on $G$. Then $dg$ induces the $G$-invariant measure on $\Gamma \backslash G$ and we can construct the right regular representation $\pi_\Gamma$ of $G$ on $L^2(\Gamma \backslash G)$. It is well-known that $\pi_\Gamma$ decomposes into the direct sum of irreducible unitary representations with finite multiplicity, up to unitarily equivalence. Let $\hat{G}$ be the set of all equivalence classes of irreducible unitary representations of $G$. For $U \in \hat{G}$, denote by $N_\Gamma(U)$ the multiplicity of $U$ in $\pi_\Gamma$.

Let $\hat{G}_d \subset \hat{G}$ be the discrete series of $G$. Assume that $\hat{G}_d$ is not empty. In this paper, we shall consider the multiplicity $N_\Gamma(U)$ of every class $U$ in $\hat{G}_d$. Langlands [8] showed that, if $U \in \hat{G}_d$ is integrable, we have the generic formula

$$N_\Gamma(U) = d(U) \cdot \operatorname{vol}(\Gamma \backslash G)$$

where $d(U)$ is the formal degree of $U$ and $\operatorname{vol}(\Gamma \backslash G)$ is the volume of $\Gamma \backslash G$. On the other hand, it has been known that there are examples of non-integrable classes in $\hat{G}_d$ for which the above Langlands’ formula breaks down. Also Hotta-Parthasarathy [6] obtained a sufficient condition for $U$ to satisfy Langlands’ formula.

Now, in [3], DeGeorge-Wallach proved a certain formula about $N_\Gamma(U)$ ($U \in \hat{G}_d$) which seems to explain the reason why Langlands’ formula breaks down for non-generic classes (c.f. Theorem 3.1.1). In fact, the formula of DeGeorge-Wallach shows that, in the case of non-generic classes, there can appear the terms of “trash” representations in the sense of Wallach [15].

The purpose of this paper is to apply the formula of DeGeorge-Wallach to the special cases of $G=\text{Spin}(1, 2m)$ ($m \geq 2$) and $G=\text{SU}(1, n)$ ($n \geq 2$) and provide the concrete formulas about $N_\Gamma(U)$ ($U \in \hat{G}_d$). The most part of our work is devoted to finding all classes in $\hat{G}$ with given infinitesimal character. In [1], Borel and Wallach make the same arguments for the special infinitesimal charac-
Their method depends on Langlands' classification of admissible representations of reductive groups. In this paper, we depend on detailed results by Thieleker (c.f. [12], [13], [14]) and Kraljevic (c.f. [7]).

Main results are Theorem 7.6.1 and Theorem 8.7.1. It will be interesting which classes in \( \hat{G} \) appear as "trash" representations. Wallach [15] and Ragozin-Warner [11] have obtained the similar formulas in the low dimensional cases.

1. The discrete series

In this section, we shall recall some known results about the discrete series of semisimple Lie groups.

1.1. Let \( G \) be a connected noncompact semisimple Lie group with finite center. Throughout this paper, we assume that \( G \) is a real form of a simply connected complex semisimple Lie group. We fix a Haar measure \( dg \) on \( G \), once and for all.

Let \( \hat{G} \) be the set of all equivalence classes of irreducible unitary representations of \( G \). We call \( U \in \hat{G} \) a discrete class if a representation in the class \( U \) is equivalent to a subrepresentation of the regular representation of \( G \) on \( L^2(G) \). Let us denote by \( \hat{G}_d \) the set of all discrete classes in \( \hat{G} \). We call \( \hat{G}_d \) the discrete series of \( G \).

Let \( K \) be a maximal compact subgroup of \( G \). It is known that \( G \) has a discrete class if and only if there exists a Cartan subgroup \( H \) of \( G \) which is contained in \( K \) (c.f. [17], II, p. 401). Hereafter we always assume that \( G \) has such a subgroup \( H \).

1.2. Let \( g_0, k_0 \) and \( h_0 \) be the Lie algebras of \( G, K \) and \( H \), respectively. Let \( p_0 \) be the orthogonal complement of \( k_0 \) in \( g_0 \) with respect to the Killing form of \( g_0 \). For any subalgebra \( u_0 \) of \( g_0 \), we shall denote by \( u \) the complexification of \( u_0 \).

Now \( h \) is a Cartan subalgebra of \( g \) which is contained in \( k \). Let \( h_R^* \) be the dual space of \( h_R = \sqrt{-1}h_0 \) and let \( h^* \) be the dual space of \( h \). Let \( \Delta \) be the root system of \( (g, h) \). For \( \alpha \in \Delta \), denote by \( g_{\alpha} \) the corresponding root space. A root \( \alpha \) is called to be compact (resp. noncompact) if \( g_{\alpha} \) is contained in \( k \) (resp. \( p \)). Denote by \( \Delta_c \) (resp. \( \Delta_n \)) the set of all compact (resp. noncompact) roots. The Killing form of \( g_0 \) induces the inner product \( \langle , \rangle \) on \( h_R^* \). Denote by \( \mathfrak{F} \) the set of all integral linear forms on \( h_R \); i.e.

\[
\mathfrak{F} = \left\{ \lambda \in h_R^* \middle| \frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \quad (\alpha \in \Delta) \right\}
\]

and define

\[
\mathfrak{F}_0 = \left\{ \lambda \in \mathfrak{F} \middle| \langle \lambda, \alpha \rangle \neq 0 \quad (\alpha \in \Delta) \right\}.
\]

An element \( \lambda \in \mathfrak{F}_0 \) is called a regular integral form on \( h_R \).
Let $W$ be the Weyl group of $(g, \mathfrak{h})$. Let $W_G$ be the Weyl group of $(G, H)$; i.e. the quotient group of the normalizer of $H$ in $G$ modulo $H$. The group $W$ and $W_G$ act on $\mathfrak{h}_R^*$ and we can identify $W_G$ with the subgroup of $W$ generated by reflections with respect to compact roots.

When we choose an ordering in $\mathfrak{h}_R^*$, put

\[
\Delta^+ = \{ \alpha \in \Delta | \alpha > 0 \}, \quad \Delta_+ = \Delta^+ \cap \Delta_c, \quad \Delta_+^* = \Delta^+ \cap \Delta_u,
\]

\[
\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha, \quad \rho_c = \frac{1}{2} \sum_{\alpha \in \Delta_c^+} \alpha, \quad \rho_u = \frac{1}{2} \sum_{\alpha \in \Delta_u^+} \alpha,
\]

\[
\mathfrak{S}^+ = \{ \lambda \in \mathfrak{S} | \langle \lambda, \alpha \rangle \geq 0 \quad (\alpha \in \Delta^+) \},
\]

\[
\mathfrak{S}_+^* = \mathfrak{S}_0 \cap \mathfrak{S}^+.
\]

If an integral form $\lambda$ belongs to $\mathfrak{S}^+$, we say that $\lambda$ is dominant with respect to $\Delta^+$. Also define the subset $W^1$ of $W$ by

\[
W^1 = \{ s \in W | s \Delta^+ \supseteq \Delta_+^* \}.
\]

Then the mapping $W_G \times W^1 \ni (s_1, s_2) \mapsto s_1 s_2 \in W$ is a bijection.

Harish-Chandra showed that there is a distinguished surjection $D: \mathfrak{S}_0 \to \hat{G}_d$ (c.f. [17], II, p. 407). For $\Lambda \in \mathfrak{S}_0$, denote by $D\Lambda$ the image of $\Lambda$ by $D$. Then, for $\Lambda, \Lambda' \in \mathfrak{S}_0$, we have $D\Lambda = D\Lambda'$ if and only if there exists an element $s \in W_G$ such that $\Lambda' = s\Lambda$ (c.f. [17], II, p. 407). Hence, if we choose a positive root system $\Delta^+$ of $\Delta$, $\hat{G}_d$ corresponds bijectively to the subset $\bigcup_{s \in W^1} s\mathfrak{S}_0^*$ of $\mathfrak{S}_0$.

1.3. Denote by $\hat{K}$ the set of all equivalence classes of irreducible $K$-modules. Let $\pi$ be a continuous representation of $G$ on a Hilbert space $E$. For convenience' sake, we shall often denote by the pair $(\pi, E)$ the representation $\pi$ on $E$. We say that $\pi$ is $K$-finite if it satisfies the following two conditions;

(1). The restriction $\pi|_K$ of $\pi$ to $K$ is a unitary representation of $K$.

(2). Each $\tau \in \hat{K}$ occurs with finite multiplicity in $\pi|_K$.

Let $(\pi, E)$ be a $K$-finite representation of $G$. Let $\tau \in \hat{K}$. Denote by $m(\pi; \tau)$ the multiplicity with which $\tau$ occurs in $\pi|_K$. Let $\Phi(\pi)$ be the set of all $\tau \in \hat{K}$ with $m(\pi; \tau) \neq 0$, which is repeated as many times as its multiplicity $m(\pi; \tau)$. We call $\Phi(\pi)$ the $K$-spectrum of $\pi$.

When we choose a positive root system $\Delta_+^*$ of $\Delta_u$, each $\tau \in \hat{K}$ is characterized by its highest weight. Put

\[
\mathfrak{S}_+^* = \left\{ \lambda \in \mathfrak{h}_R^* \left| \frac{2}{\langle \alpha, \alpha \rangle} \langle \lambda, \alpha \rangle \in \mathbb{Z} \quad \text{and} \quad \geq 0 \quad (\alpha \in \Delta_+^*) \right. \right\}.
\]

For $\gamma \in \mathfrak{S}_+^* \cap \mathfrak{S}_+^*$, denote by $\tau_\gamma$ the irreducible $K$-module with highest weight $\gamma$ and also denote by $\tau_\gamma$ the equivalence class of $\tau_\gamma$. Similarly, for $\gamma \in \mathfrak{S}_+^*$, we denote by $\tau_\gamma$ the irreducible $\mathfrak{g}$-module with highest weight $\gamma$. 
1.4. Now every irreducible unitary representation \( \pi \) of \( G \) is \( K \)-finite (c.f. [17], I, p. 318). In particular, if \( \pi \) belongs to a discrete class, the \( K \)-spectrum of \( \pi \) is known under the assumption that \( G \) is a linear group. We shall review it precisely.

Let \( \Lambda \in \mathfrak{F}_0 \) and \( \pi_\Lambda \) be a representation of \( G \) in the class \( D_\Lambda \). Choose a positive root system \( \Delta^+ \) with respect to which \( \Lambda \) is dominant. For \( \lambda \in \mathfrak{h}_R^* \), let \( Q(\lambda) \) be the number of distinct ways in which \( \lambda \) can be written as a sum of positive noncompact roots. Then we have the following theorem.

**Theorem 1.4.1** (Hecht-Schmid, [5]). Assume that \( G \) has a faithful finite dimensional representation. Let \( \gamma \in \mathfrak{F} \cap \mathfrak{F}_1 \). Then we have

\[
m(\pi_\Lambda; \gamma) = \sum_{\rho \in W} \varepsilon(s)Q(s(\gamma + \rho) - \Lambda - \rho),
\]

where \( \varepsilon(s) \) is the sign of \( s \in W_G \).

1.5. Finally, we refer to formal degrees of the discrete series. Let \( (\pi, E) \) be an irreducible unitary representation of \( G \). If \( \pi \) belongs to a discrete class, there is a positive real number \( d(\pi) \) such that

\[
\int_G |(\pi(g)u, v)|^2 dg = d(\pi)^{-1}(u, u)(v, v) \quad (u, v \in E)
\]

where \( (, ) \) is the inner product in \( E \) (c.f. [17], I, p. 351). We call \( d(\pi) \) the formal degree of \( \pi \).

For \( \Lambda \in \mathfrak{F}_0 \), denote by \( d(D_\Lambda) \) the formal degree of a representation in the class \( D_\Lambda \in \hat{G}_d \). If we choose a positive root system \( \Delta^+ \) as in 1.4., we have

\[
d(D_\Lambda) = c | \prod_{\alpha \in \Delta^+} <\Lambda, \alpha> | \tag{1.1}
\]

where \( c \) is the positive constant depending on the measure \( dg \) (c.f. [17], II, p. 407).

2. **Infinitesimal characters**

In this section, we shall review certain infinitesimal properties of \( K \)-finite representations of \( G \). Let \( G \) be as in \S 1.

2.1. Let \( (\pi, E) \) be a \( K \)-finite representation of \( G \). An element \( u \in E \) is called \( K \)-finite if \( \pi(K)u \) is contained in a finite dimensional subspace of \( E \). Denote by \( E_f \) the set of all \( K \)-finite elements of \( E \). For \( u \in E_f \) and \( X \in \mathfrak{g}_0 \), we can define

\[
\pi_f(X)(u) = \lim_{t \to 0} \frac{\pi(\exp tX)u - u}{t}
\]

and \( \pi_f(X)(u) \) belongs to \( E_f \) (c.f. [17], I, p. 326). In this way, we can induce the representation \( \pi_f \) of \( \mathfrak{g}_0 \) on \( E_f \). Denote by \( U(\mathfrak{g}) \) the universal enveloping
algebra of \( g \). The representation \( \pi_f \) of \( g_0 \) is canonically extended to the representation of \( U(g) \). This representation of \( U(g) \) is also denoted by \( \pi_f \) and is called the infinitesimal representation of \( \pi \).

Let \( \pi^1 \) and \( \pi^2 \) be \( K \)-finite representations of \( G \). If \( \pi^1 \) is equivalent to \( \pi^2 \) as representations of \( U(g) \), we say that \( \pi^1 \) is infinitesimally equivalent to \( \pi^2 \). In particular, let \( \pi^1 \) and \( \pi^2 \) be irreducible unitary representations. Then \( \pi^1 \) is unitarily equivalent to \( \pi^2 \) if and only if \( \pi^1 \) is infinitesimally equivalent to \( \pi^2 \) (c.f. [17], I, p. 329).

Let \( (\pi, E) \) be a \( K \)-finite representation of \( G \). We say that \( \pi \) is infinitesimally unitary if \( E \) has an inner product with respect to which \( \pi_f(X)u = X(u)u \) for all \( X \in Z(g) \) and \( u \in E_f \). Here \( \pi \) is called the infinitesimal character of \( \pi \). As it is well-known, an irreducible infinitesimally unitary representation of \( G \) is infinitesimally equivalent to a unique irreducible unitary representation of \( G \), up to unitarily equivalence (c.f. [17], I, p. 331).

2.2. Let \( Z(g) \) be the center of \( U(g) \). A \( K \)-finite representation \( (\pi, E) \) is said to be quasisimple if there exists a homomorphism \( \chi \) of \( Z(g) \) into \( C \) such that \( \pi_f(x)u = \chi(x)u \) for all \( x \in Z(g) \) and \( u \in E_f \). Here \( \chi \) is called the infinitesimal character of \( \pi \). As it is well-known, an irreducible infinitesimally unitary representation of \( G \) is quasisimple (c.f. [17], I, p. 318). For \( \pi \in \hat{G} \), denote by \( \chi_\pi \) the infinitesimal character of a representation in the class \( U \).

We shall state some preliminary facts about infinitesimal characters. Let \( \mathfrak{h} \) be a Cartan subalgebra of \( g \) in \( \S 1 \) and \( U(\mathfrak{h}) \) the universal enveloping algebra of \( \mathfrak{h} \). Since \( U(\mathfrak{h}) \) is the symmetric tensor algebra of \( \mathfrak{h} \), \( U(\mathfrak{h}) \) is naturally identified with the algebra of complex-valued polynomial functions on \( \mathfrak{h}^* \). Choose a positive root system \( \Delta^+ \) of \( \Delta \) and put \( \mathfrak{h}^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \). Then \( Z(g) \) is contained in the direct sum \( U(\mathfrak{h}) \oplus U(g)\mathfrak{h}^+ \). Let \( \varphi \) be the projection of \( Z(g) \) into \( U(\mathfrak{h}) \). For \( \lambda \in \mathfrak{h}^* \), define the homomorphism \( \chi_\lambda ; Z(g) \rightarrow C \) by
\[
\chi_\lambda(x) = \varphi(x)(\lambda - \rho) \quad (x \in Z(g)).
\]
Then it turns out that \( \chi_\lambda \) does not depend on the choice of \( \Delta^+ \) (c.f. [4], p. 231). By a well-known result of Harish-Chandra, every homomorphism \( \chi; Z(g) \rightarrow C \) is of the form \( \chi_\lambda \) for some \( \lambda \in \mathfrak{h}^* \). Moreover, for \( \lambda_1, \lambda_2 \in \mathfrak{h}^* \), we have \( \chi_{\lambda_1} = \chi_{\lambda_2} \) if and only if \( \lambda_2 = s_\lambda \lambda_1 \) for some \( s \in W \) (c.f. [4], p. 232). In particular, if \( \lambda \in \mathfrak{h}^* \) is a dominant regular integral form with respect to \( \Delta^+ \), \( \chi_\lambda \) is the central character of the irreducible \( g \)-module with highest weight \( \lambda - \rho \) (c.f. [4], p. 230).

2.3. Now let \( \mathfrak{h}' \) be another Cartan subalgebra of \( g \). For \( \lambda' \in (\mathfrak{h}')^* \), let \( \chi_{\lambda'}; Z(g) \rightarrow C \) be the homomorphism constructed from \( \lambda' \) in the same way as in 2.2. We shall study the relation between \( \chi_\lambda \) and \( \chi_{\lambda'} \). Let \( \theta \) be an automorphism of \( g \) such that \( \theta(\mathfrak{h}) = \mathfrak{h} \) and let \( \theta^*; \mathfrak{h}^* \rightarrow (\mathfrak{h}')^* \) be the linear isomorphism induced by \( \theta; \mathfrak{h} \rightarrow \mathfrak{h} \).
Lemma 2.3.1. If $\lambda \in \mathfrak{h}^*$ is a regular integral form, then we have $\chi_\lambda = \chi_{\theta^* \lambda}$ as homomorphisms of $Z(\mathfrak{g})$ into $C$.

Proof. We take a positive root system $\Delta^+$ with respect to which $\lambda$ is dominant. Let $V_{\lambda - \rho}$ be the irreducible $\mathfrak{g}$-module with highest weight $\lambda - \rho$. Then $\chi_\lambda$ is the central character of $V_{\lambda - \rho}$. On the other hand, if we take $\theta^*(\Delta^+)$ as a positive root system of $(\mathfrak{g}, \mathfrak{h})$, the highest weight of $V_{\lambda - \rho}$ relative to $\theta^*(\Delta^+)$ is $\theta^*(\lambda - \rho) = \theta^* \lambda - \theta^* \rho \in (\mathfrak{h})^*$. Hence $\chi_{\theta^* \lambda}$ is also the central character of $V_{\lambda - \rho}$. Thus the assertion follows.

3. Basic multiplicity formulas

In this section, we shall review multiplicity formulas obtained by DeGeorge and Wallach. It plays a basic role in this paper.

3.1. Let $G$ be as in § 1. Let $\Gamma$ be a discrete subgroup of $G$ such that $\Gamma \backslash G$ is compact. In this paper, we always assume that $\Gamma$ has no elements with finite order other than the identity. We fix the $G$-invariant measure $dg$ on $\Gamma \backslash G$. For $U \in \hat{G}$, we denote by $N_\Gamma(U)$ the multiplicity with which $U$ occurs in the right regular representation of $G$ on $L^2(\Gamma \backslash G)$.

Now we take the Cartan subalgebra $\mathfrak{h}$ as in § 1. Consider a special linear form $\Lambda \in \mathfrak{h}_0$ and the discrete class $D_A \subset \hat{G}_A$. Let $\chi_\Lambda$ be the homomorphism of $Z(\mathfrak{g})$ into $C$ constructed from $\Lambda$ as in 2.2. Denote by $\hat{G}_A$ the set of all $U \in \hat{G}$ such that its infinitesimal character is $\chi_\Lambda$; i.e.

$$\hat{G}_A = \{ U \in \hat{G} | \chi_0 = \chi_\Lambda \}.$$ 

Clearly we have $\hat{G}_A = \hat{G}_{sA}$ for all $s \in W$. Also it is known that $D_A \subset \hat{G}_A$ and the number of elements in $\hat{G}_A$ is finite (c.f. [3], p. 141).

In [3], DeGeorge and Wallach obtained a formula describing the relation among numbers in $\{ N_\Gamma(U) | U \in \hat{G}_A \}$. We shall precisely review it. Choose and fix a positive root system $\Delta^+$ with respect to which $\Lambda$ is dominant. Since rank $\mathfrak{g}_0 = \text{rank } \mathfrak{f}_0$, we can construct the half-spin representations of $\mathfrak{f}_0$ as in [6], p. 144. Denote by $L^+$ the half-spin representation of $\mathfrak{f}_0$ with highest weight $\rho_\Lambda$ and denote by $L^-$ the other one. Note that $L^\pm$ depends on the choice of $\Delta^+$. Let $\tau_{\Lambda - \rho_\Lambda}$ be the irreducible $\mathfrak{f}$-module with highest weight $\Lambda - \rho_\Lambda$. Then the tensor product $\tau_{\Lambda - \rho_\Lambda} \otimes L^\pm$ integrates to the representation of $K$ which is also denoted by $\tau_{\Lambda - \rho_\Lambda} \otimes L^\pm$ (c.f. [3], p. 139). For $K$-modules $\tau_1$ and $\tau_2$, denote by $\dim \text{Hom}_K(\tau_1, \tau_2)$ the intertwining number of $\tau_1$ and $\tau_2$. Let $U \in \hat{G}$ and let $\pi_U$ be a representation in the class $U$. Then define

$$\dim \text{Hom}_K(\tau_{\Lambda - \rho_\Lambda} \otimes L^\pm, U) = \dim \text{Hom}_K(\tau_{\Lambda - \rho_\Lambda} \otimes L^\pm, \pi_U \mid_K).$$

Theorem 3.1.1 (DeGeorge-Wallach, [3]). Let $\Lambda \in \mathfrak{h}_0$. Then we have
(3.1) \[ d(D_A) \text{vol}(\Gamma \backslash G) = N_\Gamma(D_A) + \sum_{\tau \in \Delta^+} N_\Gamma(U) \{ \dim \text{Hom}_K(\tau_{\Lambda-P} \otimes L^+, U) - \dim \text{Hom}_K(\tau_{\Lambda-P} \otimes L^-, U) \} \]

where \( d(D_A) \) is the formal degree of \( D_A \) and \( \text{vol}(\Gamma \backslash G) \) is the volume of \( \Gamma \backslash G \).

Our purpose in this paper is to obtain concrete formulas about \( N_\Gamma(D_A) \) by using Theorem 3.1.1. We divide our study into two steps as follows;

**Step 1:** To find out all elements in \( \hat{G}_\Lambda - \hat{G}_d \).

**Step 2:** To examine \( \dim \text{Hom}_K(\tau_{\Lambda-P} \otimes L^+, U) \) for every \( U \) in \( \hat{G}_\Lambda - \hat{G}_d \).

4. The nonunitary principal series

In order to find out irreducible unitary representations of \( G \) with given infinitesimal character, we need the family of representations of \( G \) which is called the nonunitary principal series. In this section, we shall make preparations about this family. Let \( G \) be as in § 1.

4.1. First we shall state some notations. Let \( G = KAN \) be an Iwasawa decomposition of \( G \). Let \( \alpha_0 \) and \( \eta_0 \) be the Lie algebras of \( A \) and \( N \). We write \( g = k(g) \exp((H(g)) n(g)) \) for \( g \in G \), where \( k(g) \in K \), \( H(g) \in \alpha_0 \) and \( n(g) \in N \).

Let \( \Sigma \) be the restricted root system of \( g_0 \) with respect to \( \alpha_0 \). Choose a positive root system \( \Sigma^+ \) of \( \Sigma \) which is associated with the subalgebra \( \eta_0 \) (c.f. [16], p. 164). Define \( \rho_\varphi = \frac{1}{2} \sum_{\alpha \in \Sigma^+} \alpha \).

Let \( M \) be the centralizer of \( A \) in \( K \) and let \( \eta_0 \) be the Lie algebra of \( M \). Let \( t_0 \) be a maximal abelian subalgebra of \( \eta_0 \) and let \( t_0^* \) be the dual space of \( t_0 = \sqrt{-1} t_0 \). Denote by \( \Delta_m \) the root system of \( (m, t) \). Choose a positive root system \( \Delta_m^+ \) of \( \Delta_m \) and put \( \rho_m = \frac{1}{2} \sum_{\alpha \in \Delta_m^+} \alpha \). Also define

\[ \Phi_m^+ = \left\{ \mu \in t_0^* \left| \frac{2\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \text{ and } \geq 0 \right. (\alpha \in \Delta_m^+) \right\} \]

Let \( \hat{M} \) be the set of all equivalence classes of irreducible \( M \)-modules. For \( \xi \in \hat{M} \), denote by \( \mu(\xi) \) the highest weight of \( \xi \). Then \( \mu(\xi) \) belongs to \( \Phi_m^+ \).

If we set \( \hat{g}_0^0 = t_0 \oplus \alpha_0 \), \( \hat{h}' \) is a Cartan subalgebra of \( \hat{g} \). In this section, we use this Cartan subalgebra \( \hat{h}' \). Let \( (\hat{h}_0)^* \) be the dual space of \( \hat{h}_0^0 = \sqrt{-1} t_0 \oplus \alpha_0 \). Canonically, \( t^* \) and \( \alpha^* \) can be considered as subsets of the dual space \( (\hat{h}_0)^* \) of \( \hat{h}' \). Denote by \( \Delta' \) the root system of \( (\hat{g}, \hat{h}') \). Then there exists a positive root system \( \Delta'^+ \) of \( \Delta' \) such that \( \Delta_m^+ \subset \Delta'^+ \) and \( \{ \alpha \in \Delta' \mid \alpha |_{\eta_0} \in \Sigma^+ \} \subset \Delta'^+ \) (c.f. [16], p. 169). This ordering of \( (\hat{h}_0)^* \) is called a compatible ordering with respect to \( \Sigma^+ \). Throughout this section, we fix this ordering in \( (\hat{h}_0)^* \). Set \( \rho' = \frac{1}{2} \sum_{\alpha \in \Delta'^+} \alpha \).

Then we have \( \rho' = \rho_m + \rho_\varphi \).
4.2. Let \( \xi \in \hat{M} \) and let \( \xi \) also be a representation of \( M \) on the vector space \( V_\xi \) which belongs to the class \( \xi \). Let \( C_{\xi} \) be the set of all continuous mappings \( f; K \rightarrow V_\xi \) such that \( f(km) = \xi(m)^{-1}f(k) \) for all \( k \in K \) and \( m \in M \). For \( f_1, f_2 \in C_{\xi} \), define
\[
(f_1, f_2) = \int_K (f_1(k), f_2(k))\,dk
\]
where \( dk \) is the normalized Haar measure on \( K \) such that \( \int_K dk = 1 \). Denote by \( E_\xi \) the completion of \( C_{\xi} \) with respect to this inner product \( (, ) \).

Let \( \nu \in \mathfrak{a}^\ast \). We define the representation \( \pi_{\xi, \nu} \) of \( G \) on the Hilbert space \( E_\xi \) as follows:
\[
(\pi_{\xi, \nu}(g)f)(k) = e^{-\nu + \rho_\nu(g^{-1}k)}f(k(g^{-1}k))
\]
for \( g \in G, k \in K \) and \( f \in C_{\xi} \). Then \( \pi_{\xi, \nu} \) is not necessarily unitary, but it is \( K \)-finite (c.f. [16], p. 232) and quasisimple (c.f. [9], p. 29). The family of representations \( \{\pi_{\xi, \nu}, \xi \in \hat{M}, \nu \in \mathfrak{a}^\ast\} \) is called the nonunitary principal series.

4.3. We shall review how the infinitesimal character of each representation in the nonunitary principal series is given. Let \( \pi_{\xi, \nu} \) be a representation in the nonunitary principal series. For \( \lambda' \in (\mathfrak{h'})^\ast \), denote by \( \chi_{\lambda'} \), the homomorphism of \( Z(\mathfrak{g}) \) into \( C \) constructed from \( \lambda' \) as in 2.2.. Then we have the following lemma.

**Lemma 4.3.1** (Lepowsky [9]). The infinitesimal character \( \chi_{\pi_{\xi, \nu}} \) of \( \pi_{\xi, \nu} \) is equal to \( \chi_{\mu(\xi) + \rho_m + \nu} \), where \( \mu(\xi), \rho_m \) and \( \nu \) are regarded as elements in \( (\mathfrak{h'})^\ast \).

**Proof.** In [9], \( \chi_{\pi_{\xi, \nu}} \) was given explicitly. We shall translate the result in [9] into our desirable form. In Proposition 8.9 of [9], take \( -\Delta^+ = \{ -\alpha | \alpha \in \Delta^+ \} \) as a positive root system of \( \Delta^m \). Note that \( \mu(\xi) \) is the lowest weight of \( \xi \) with respect to \( -\Delta^m \). Denote by \( \Delta' \) the positive root system of \( \Delta' \) determined by \( \Sigma^+ \) and \( -\Delta^+ \). Let \( \mathfrak{n}^+_i \) (resp. \( \mathfrak{n}^-_i \)) be the sum of positive (resp. negative) root spaces. Then we have
\[
Z(\mathfrak{g}) \subseteq U(\mathfrak{h'}) + \mathfrak{n}^+_i U(\mathfrak{g}) ,
Z(\mathfrak{g}) \subseteq U(\mathfrak{h'}) + U(\mathfrak{g}) \mathfrak{n}^-_i .
\]
In the above two direct sums, the projection mappings of \( Z(\mathfrak{g}) \) into \( U(\mathfrak{h'}) \) are the same (c.f. [4], p. 230). Denote by \( \varphi' \) this projection mapping. Then Proposition 8.9 of [9] implies that
\[
\chi_{\pi_{\xi, \nu}}(z) = \varphi'(z) (\mu(\xi) + \nu + \rho_\lambda)
\]
for \( z \in Z(\mathfrak{g}) \). On the other hand, use a positive root system \( -\Delta'_+ \) of \( \Delta' \) in the construction of \( \chi_{\lambda'} \) (\( \lambda' \in (\mathfrak{h'})^\ast \)). Then we have
\[\chi_{\mu(\xi)+\rho_m+\nu}(z) = \varphi'(z)((\mu(\xi)+\rho_m+\nu)-(\rho_m-\rho_a))\]

\[= \varphi'(z)(\mu(\xi)+\nu+\rho_a).\]

Hence we have \(\chi_{\pi,\xi,\nu}(z) = \chi'_{\mu(\xi)+\rho_m+\nu}(z)\).

4.4 Finally we shall prepare some facts about the \(K\)-spectrum of \(\pi,\nu\). Let \(\xi \in \hat{M}\) and \(\tau \in \hat{K}\). Denote by \(m(\tau; \xi)\) the multiplicity with which \(\xi\) occurs in the restriction of \(\tau\) to \(M\). If \(m(\tau; \xi) = 0\), \(\tau\) is called to be \(\xi\)-admissible. Denote by \(\hat{K}(\xi)\) the set of all \(\xi\)-admissible elements in \(\hat{K}\).

Let \(\nu \in \mathfrak{a}^*\). From the definition of \(\pi,\nu\), the restriction of \(\pi,\nu\) on \(K\) is the representation induced from the representation \(\xi\) of \(M\). Hence, by Frobenius' reciprocity theorem, we have \(m(\pi,\nu; \tau) = m(\tau; \xi)\) for \(\tau \in \hat{K}\). Thus the \(K\)-spectrum of \(\pi,\nu\) is the set of all \(\tau \in \hat{K}(\xi)\) repeated as many times as \(m(\tau; \xi)\).

For \(\tau \in \hat{K}\), let \(E_\xi(\tau) \subset E_\xi\) be the subspace of all vectors transformed by \(\pi(\xi,\nu)|_K\) according to \(\tau\). When \(\Phi\) is a subset of \(\hat{K}(\xi)\), define \(E_\xi(\Phi) = \bigoplus_{\tau \in \Phi} E_\xi(\tau)\), where \(\bigoplus\) means the orthogonal direct sum. Note that \(E_\xi(\Phi)\) is a \(K\)-invariant subspace of \(E_\xi\).

5. The subquotient theorem

In this section, for the sake of Step 1, we shall review the subquotient theorem of Harish-Chandra. It will give a method to find out the representations in question.

5.1. Let \((\pi, E)\) be a continuous representation of \(G\). Let \(E\) have a finite sequence of closed invariant subspaces

\[E = E_0 \supset E_1 \supset \cdots \supset E_{r-1} \supset E_r = \{0\}\]

such that, on each quotient space \(E_{i-1}/E_i\), \(\pi\) induces an irreducible representation of \(G\). Then \(\pi\) is said to have a finite composition series and each quotient representation \((\pi, E_{i-1}/E_i)\) is called an irreducible subquotient of \(\pi\).

It is known that every member of the nonunitary principal series has a finite composition series (c.f. [9], p. 33). The following theorem gives a method of realizing our exploring representations.

**Theorem 5.1.1** (Harish-Chandra; c.f. [9], p. 29). Every irreducible unitary representation of \(G\) is infinitesimally equivalent to an irreducible subquotient of some member of the nonunitary principal series.

5.2. Now, let \(\mathfrak{h}\) be the Cartan subalgebra in \(\S 1\). We consider a special linear form \(\Lambda \in \mathfrak{h}^*\). Let \(\Delta^+\) be the positive root system with respect to which \(\Lambda\) is dominant. On the other hand, let \(\mathfrak{h}'\) and \(\Delta'^+\) be as in 4.1. There is an automorphism \(\theta\) of \(\mathfrak{g}\) such that \(\theta(\mathfrak{h}') = \mathfrak{h}\) and \(\theta(\Delta^+) = \Delta'^+\). Identify \(\mathfrak{h}^*\) with \((\mathfrak{h}')^*\) by this isomorphism. From Lemma 2.3.1, if \(\lambda \in h^*\) is a regular integral
form, we have $\mathcal{X}_\Lambda = \mathcal{X}_\Lambda'$. Hence, for $\xi \in \hat{M}$ and $\nu \in \mathfrak{a}^*$, we have $\mathcal{X}_{\mu(\xi)+\rho_m+\nu} = \mathcal{X}_\Lambda$ if and only if there exists an element $s \in W$ such that $\mu(\xi)+\rho_m+\nu = s(\Lambda)$. Let $R(\Lambda)$ be the set of all pair $(\xi, \nu)$ ($\xi \in \hat{M}$, $\nu \in \mathfrak{a}^*$) such that $\mu(\xi)+\rho_m+\nu$ and $\Lambda$ are in the same $W$-orbit. From Lemma 4.4.1 and Theorem 5.1.1, we obtain the following proposition.

**Proposition 5.2.2.** Let $\Lambda \in \widehat{\mathfrak{g}}_0$ and $\pi$ be an irreducible unitary representation of $G$ such that $\mathcal{X}_\pi = \mathcal{X}_\Lambda$. Then $\pi$ is infinitesimally equivalent to an irreducible subquotient of $\pi_{\xi, \nu}$ for some $(\xi, \nu)$ in $R(\Lambda)$.

6. The decomposition of $\mathfrak{g}$-modules

In this section, we shall describe the direct sum decomposition of a tensor product of two $\mathfrak{g}$-modules. Also we shall review a known result about the $\mathfrak{g}$-module $L^\pm$. These facts will be used in studying Step 2.

6.1. Let $\mathfrak{g}_0$ and $\mathfrak{h}_0$ be as in §1. Let $\mathfrak{c}_0$ be the center of $\mathfrak{g}_0$ and let $\mathfrak{h}_0^\perp$ be the semisimple part $[\mathfrak{g}_0, \mathfrak{g}_0]$ of $\mathfrak{g}_0$. Then we have

$$\mathfrak{g}_0 = \mathfrak{c}_0 \oplus \mathfrak{h}_0^\perp$$

where $\mathfrak{h}_0^\perp$ is the maximal abelian subalgebra of $\mathfrak{g}_0$. We always regard $(\mathfrak{h}_0^\perp)^*$ and $\mathfrak{c}^*$ as subsets of $\mathfrak{g}_0^*$.  

First we shall examine the decomposition as $\mathfrak{f}$-modules. As the positive root system of $(\mathfrak{f}_0, \mathfrak{h}_0)$, we take $\Delta^+$ in §1. For $\gamma \in \widehat{\mathfrak{f}}_0^+ \cap (\mathfrak{h}_0)^*$, denote by $\tau^\gamma$ the irreducible $\mathfrak{f}_0$-module with highest weight $\gamma$. The following lemma is due to [2] (c.f. [2], Chapter VIII, §9, Proposition 2).

**Lemma 6.1.1.** Let $\beta$, $\gamma$ and $\delta$ be in $\widehat{\mathfrak{f}}_0^+ \cap (\mathfrak{h}_0)^*$. Then the multiplicity of $\tau^\gamma$ in the tensor product $\tau^\gamma \otimes \tau^\delta$ is given by

$$\sum_{s \in W} \varepsilon(s)M_{\mathfrak{g}_0}(\beta+\rho_k-s(\gamma+\rho_k))$$

where $M_{\mathfrak{g}_0}(\lambda)$ ($\lambda \in (\mathfrak{h}_0)^*$) is the multiplicity of a weight $\lambda$ in $\tau^\gamma$.

Once we know the decomposition as $\mathfrak{f}$-modules, we can easily obtain the decomposition as $\mathfrak{g}$-modules.

**Proposition 6.1.2.** Let $\tau$ be a $\mathfrak{g}$-module. Let $\beta$ and $\gamma$ be in $\widehat{\mathfrak{f}}_0^+$ and let $\tau_\beta$ and $\tau_\gamma$ be the irreducible $\mathfrak{f}$-modules with highest weight $\beta$ and $\gamma$ respectively. Then the multiplicity of $\tau_\beta$ in $\tau \otimes \tau_\gamma$ is given by

$$\sum_{s \in W} \varepsilon(s)M_{\mathfrak{g}}(\beta+\rho_k-s(\gamma+\rho_k))$$

where $M_{\mathfrak{g}}(\lambda)$ ($\lambda \in \mathfrak{h}^*$) is the multiplicity of a weight $\lambda$ in $\tau$. 

Proof. It is sufficient to prove the statement in the case that \( \tau \) is an irreducible \( \mathfrak{f} \)-module \( \tau_\delta (\delta \in H^\mathfrak{f}_\delta) \). Let \( \beta \) be in \( H^\mathfrak{f}_\delta \). If \( \tau_\beta \) occurs in \( \tau_\gamma \otimes \tau_\delta \) with nonzero multiplicity, the restriction of \( \beta|_c \) to \( c \) is equal to \( (\gamma + \delta)|_c \). We put

\[
H^\mathfrak{f}_\delta(\gamma, \delta) = \{ \beta \in H^\mathfrak{f}_\delta \mid \beta|_c = (\gamma + \delta)|_c \}.
\]

Assume that \( \beta \in H^\mathfrak{f}_\delta(\gamma, \delta) \). Then \( \tau_\beta \) occurs in \( \tau_\gamma \otimes \tau_\delta \) with multiplicity \( m \) if and only if \( \tau_\beta|_\mathfrak{g} \) occurs in \( \tau_\gamma|_\mathfrak{g} \otimes \tau_\delta|_\mathfrak{g} \) with multiplicity \( m \). Hence, by Lemma 6.1.1, the multiplicity of \( \tau_\beta \) in \( \tau_\gamma \otimes \tau_\delta \) is given by

\[
\sum_{s \in W_\mathfrak{g}} \varepsilon(s) M_{\tau_\beta}(\beta + \rho_k - s(\gamma + \rho_k))
\]

where \( \beta^1, \gamma^1 \) and \( \delta^1 \) are the restrictions of \( \beta, \gamma \) and \( \delta \) to \( \mathfrak{h}^1 \), respectively. Since, for \( \lambda \in \mathfrak{h}^* \) and \( s \in W_\mathfrak{g} \), we have \( s(\lambda)|_{\mathfrak{h}^1} = s(\lambda)|_{\mathfrak{h}^1} \) and \( s(\lambda)|_\mathfrak{e} = s(\lambda)|_\mathfrak{e} \), it follows that

\[
\{ \beta + \rho_k - s(\gamma + \rho_k) \}|_{\mathfrak{h}^1} = \beta^1 + \rho_k - s(\gamma^1 + \rho_k)
\]

\[
\{ \beta + \rho_k - s(\gamma + \rho_k) \}|_c = (\beta - \gamma)|_c
\]

\[
= \delta|_c.
\]

Hence, \( \beta + \rho_k - s(\gamma + \rho_k) \) is a weight of \( \tau_\beta \) if and only if \( \beta^1 + \rho_k - s(\gamma^1 + \rho_k) \) is a weight of \( \tau^1_\beta \). Therefore we have

\[
M_{\tau_\beta}(\beta + \rho_k - s(\gamma + \rho_k)) = M_{\tau^1_\beta}(\beta^1 + \rho_k - s(\gamma^1 + \rho_k)),
\]

and the statement is proved for \( \tau_\beta \) with \( \beta \in H^\mathfrak{f}_\delta(\gamma, \delta) \).

Next, assume that \( \beta \in H^\mathfrak{f}_\delta(\gamma, \delta) \). Clearly the multiplicity of \( \tau_\beta \) in \( \tau_\gamma \otimes \tau_\delta \) is equal to 0. On the other hand, since we have \( \{ \beta + \rho_k - s(\gamma + \rho_k) \}|_c = \delta|_c \), we obtain that

\[
M_{\tau_\beta}(\beta + \rho_k - s(\gamma + \rho_k)) = 0 \quad (s \in W_\mathfrak{g}).
\]

The proposition is proved.

6.2. Finally we shall state how the set of weights of \( L^\pm \) is given. Choose an ordering in \( H^\mathfrak{f}_\delta \) and appoint \( L^\pm \) with respect to this ordering. Let \( Q \) be a subset of \( \Delta^+ \). Denote by \( \langle Q \rangle \) the sum of all elements in \( Q \) and denote by \( |Q| \) the number of elements in \( Q \). The following lemma is known (c.f. [6], p. 144).

**Lemma 6.2.1.** The set of all weights of the \( \mathfrak{f} \)-module \( L^+ \) (resp. \( L^- \)) is given by

\[
\{ \rho_\circ - \langle Q \rangle \mid Q \subset \Delta^+, (-1)^{|Q|} = +1 \text{ (resp. } -1) \}
\]

and the multiplicity of each weight is the number of ways in which it can be expressed in the above form.
7. The case of $G=\text{Spin}(1, 2m)$

In this section, we shall study Step 1 and Step 2 in the case of $G=\text{Spin}(1, 2m)$ and we shall obtain concrete multiplicity formulas. In [12], [13] and [14], Thieleker classified all irreducible unitary representations of $\text{Spin}(1, 2m)$. We shall often use his results.

7.1. Let $m$ be an integer such that $m \geq 2$. Let $G$ be the group $\text{Spin}(1, 2m)$; i.e. the universal covering group of the identity component of $\text{SO}(1, 2m)$. The subgroup $K$ is isomorphic to the universal covering group $\text{Spin}(2m)$ of $\text{SO}(2m)$. As usual, we realize $\mathfrak{g}_0$, $\mathfrak{t}_0$ and $\mathfrak{p}_0$ as Lie algebras of matrices;

\[
\mathfrak{g}_0 = \left\{ \begin{pmatrix} 0 & u \\ -u^T & X \end{pmatrix} \middle| u \in \mathbb{R}^{2m}, X \in \mathfrak{so}(2m, \mathbb{R}) \right\}
\]

\[
\mathfrak{t}_0 = \left\{ K(X) = \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix} \middle| X \in \mathfrak{so}(2m, \mathbb{R}) \right\}
\]

\[
\mathfrak{p}_0 = \left\{ P(u) = \begin{pmatrix} 0 & u \\ -u^T & 0 \end{pmatrix} \middle| u = (u_1, u_2, \ldots, u_{2m}) \in \mathbb{R}^{2m} \right\}.
\]

Take $\mathfrak{h}_0$ as follows;

\[
\mathfrak{h}_0 = \left\{ H(h) = \begin{pmatrix} 0 & 0 \\ 0 & -h_1 & 0 \\ 0 & 0 & -h_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -h_m \end{pmatrix} \middle| h = (h_1, h_2, \ldots, h_m) \in \mathbb{R}^m \right\}
\]

For $1 \leq i \leq m$, define a linear form $e_i$ on $\mathfrak{h}$ by

\[
e_i(H(h_1, \ldots, h_m)) = \sqrt{-1} h_i.
\]

Then $\mathfrak{h}_R^*$ is spanned by $\{e_1, \ldots, e_m\}$ over $\mathbb{R}$ and the inner product in $\mathfrak{h}_R^*$ is given by

\[
\langle e_i, e_j \rangle = \frac{1}{2(2m-1)} \delta_{ij} \quad (1 \leq i, j \leq m).
\]

Choose and fix a lexicographic ordering in $\mathfrak{h}_R^*$ with respect to the basis $\{e_1, \ldots, e_m\}$. Under these situations, the following facts are easily seen;

\[
\Delta = \{ \pm e_i \ (1 \leq i \leq m), \pm (e_i+e_j) \ (1 \leq i < j \leq m) \}
\]

\[
\Delta^* = \{ e_i \ (1 \leq i \leq m), \ e_i \pm e_j \ (1 \leq i < j \leq m) \}.
\]
\[\Delta_t = \{e_i \pm e_j \mid 1 \leq i < j \leq m\}\]
\[\Delta_0 = \{e_i \mid 1 \leq i \leq m\}\]
\[\rho = \frac{1}{2} \sum_{i=1}^{m} (2m-2i+1)e_i\]
\[\rho_s = \sum_{i=1}^{m} (m-i)e_i\]
\[\rho_n = \frac{1}{2} \sum_{i=1}^{m} e_i\]
\[\mathcal{F} = \{\lambda = \sum_{i=1}^{m} \lambda_i e_i \mid \lambda_i - \lambda_{i+1} \in \mathbb{Z} \mid 1 \leq i \leq m-1, \ 2\lambda_m \in \mathbb{Z}\}\]
\[\mathcal{F}_0 = \{\lambda \in \mathcal{F} \mid \lambda_i \neq 0, \ \lambda_i \neq \lambda_j \ (i \neq j)\}\]
\[\mathcal{F}_+ = \{\lambda \in \mathcal{F} \mid \lambda_1 > \lambda_2 > \cdots > \lambda_m > 0\}\]
\[\mathcal{F}^- = \{\lambda \in \mathcal{F} \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{m-1} \geq |\lambda_m|\}\].

Note that, if \(\lambda = \sum_{i=1}^{m} \lambda_i e_i \in \mathcal{F}\), \(\{\lambda_1, \cdots, \lambda_m\}\) are either all integers or all half odd integers.

Let \(\Theta_m\) be the symmetric group of degree \(m\) and let \(\sigma \in \Theta_m\). Let \(\varepsilon_i = \{\varepsilon_i\}\) be a set of \(m\) signs; i.e. \(\varepsilon_i = 1\) or \(\varepsilon_i = -1\) for \(1 \leq i \leq m\). Define the orthogonal transformation \(s_{\sigma, \varepsilon}\) of \(\mathfrak{k}\) by
\[s_{\sigma, \varepsilon} = \varepsilon_i e_{\sigma(i)} \quad (1 \leq i \leq m)\].

The Weyl groups \(W\) and \(W_\sigma\) are given as follows;
\[W = \{s_{\sigma, \varepsilon} \mid \sigma \in \Theta_m, \ \varepsilon_i = 1 \text{ or } -1 \ (1 \leq i \leq m)\}\]
\[W_\sigma = \{s_{\sigma, \varepsilon} \mid \sigma \in \Theta_m, \ \varepsilon_i = 1 \text{ or } -1 \ (1 \leq i \leq m), \ \prod_{i=1}^{m} \varepsilon_i = 1\}\]

Define \(s_0 \in W\) by \(s_0(e_i) = e_i\) for \(1 \leq i \leq m-1\) and \(s_0(e_m) = -e_m\). Then we have \(W' = \{1, s_0\}\), where \(1\) denotes the identity element of \(W\). Hence the mapping \(D; \mathcal{F}_0 \cup \mathcal{F}_0 \rightarrow \hat{G}_d\) is bijective.

Take a maximal abelian subalgebra \(a_0\) of \(\mathfrak{p}_0\) defined by
\[a_0 = \{P(a, 0, \cdots, 0) \mid a \in \mathbb{R}\}\].

Then \(M\) is isomorphic to \(\text{Spin}(2m-1)\) and we have
\[m_0 = \left\{K(X) \mid X = \begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix} \ Y \in \mathfrak{s}(2m-1, \mathbb{R}) \right\}\].

Also take a maximal abelian subalgebra \(t_0\) of \(m_0\) defined by
\[t_0 = \{H(0, h_2, \cdots, h_m) \mid h_i \in \mathbb{R} \quad (2 \leq i \leq m)\}\].
Two Cartan subalgebras \( h' = t \oplus \alpha \) and \( h \) can be identified by the following isomorphism \( \theta \):

\[
\theta(P(a,0,\ldots,0) + H(0, h_2, \ldots, h_m)) = H(a, h_2, \ldots, h_m).
\]

Moreover, by this identification, the above ordering in \( h^*_R \) corresponds to a compatible ordering of \( (h'_R)^* \). In this way, we always identify \( (h'_R)^* \) with \( h^*_R \) together with their orderings.

As for \((m, t)_R\), \((t^R)^*_R\) is spanned by \( \{e_2, \ldots, e_m\} \) and, in \((t^R)^*_R\), the lexicographic ordering with respect to this basis is given. Hence \( \Delta^*_m, \rho_m \) and \( (t^R)^*_R \) are given as follows:

\[
\Delta^*_m = \{ e_i \ (2 \leq i \leq m) \}, \quad e_i + e_j \ (2 \leq i < j \leq m)
\]

\[
\rho_m = \sum_{i=2}^{m} \frac{2m-2i+1}{2} e_i
\]

\[
(\tau_m) = \left\{ \mu = \sum_{i=2}^{m} \mu_i e_i \ \middle| \ \mu_i - \mu_{i+1} \in \mathbb{Z} \text{ and } \mu_i \geq \mu_{i+1} \ (2 \leq i \leq m-1) \right\}
\]

Since \( M \) is simply connected and semisimple, the mapping \( \mu; \hat{M} \in \mathfrak{g} \rightarrow \mu(\xi) \in \mathfrak{g}_m^* \) is bijective. Also we have \( \alpha^* = CE_1 \subset \mathfrak{h}^* \). We shall identify \( \nu e_1 \in \mathfrak{a}^* \) with the complex number \( \nu \in \mathbb{C} \).

7.2. Now we shall set about Step 1. Consider and fix a special linear form \( \Lambda = \sum_{i=1}^{m} \Lambda_i e_i \in \mathfrak{g}_m^* \). Then the numbers \( \{\Lambda_1, \ldots, \Lambda_m\} \) are either all integers or all half odd integers and satisfy the inequalities

\[
(7.1) \quad \Lambda_1 > \Lambda_2 > \cdots > \Lambda_m > 0.
\]

Depending on Proposition 5.2.2, we shall first determine the set \( R(\Lambda) \subset \hat{M} \times \mathfrak{a}^* \). For \( \lambda \in \mathfrak{h}^* \), denote by \( \lambda|_t \) and \( \lambda|_{\mathfrak{a}} \) the restrictions of \( \lambda \) to \( t \) and to \( \mathfrak{a} \), respectively. If \((\xi, \nu) \in R(\Lambda)\), we have \( \mu(\xi) = s(\Lambda)|_t - \rho_m \) and \( \nu = s(\Lambda)|_{\mathfrak{a}} \) for some \( s \in W \). Hence it is necessary to find out all \( s \in W \) such that \( s(\Lambda)|_t - \rho_m \) belongs to \( \mathfrak{g}_m^* \). Let \( j \) be an integer such that \( 1 \leq j \leq m \). Define \( \sigma_j \in \mathfrak{S}_m \) and \( \epsilon_j \) by

\[
\sigma_j = (1, 2)(2, 3) \cdots (j-1, j)
\]

\[
\epsilon_j = \{1, 2, \ldots, j, \ldots, \}
\]

and put \( s_j = s_{\sigma_j} e_j \in W \).

**Lemma 7.2.1.** Let \( s \in W \). Then we have \( s(\Lambda)|_t - \rho_m \in \mathfrak{g}_m^* \) if and only if \( s \) is either \( s_j \) or \( s_{\sigma_j} \) for some \( 1 \leq j \leq m \).

**Proof.** Let \( \sigma \in \mathfrak{S}_m \) and \( \epsilon = \{\epsilon_i\} \). We have

\[
\epsilon_{\sigma, \epsilon}(\Lambda)|_t - \rho_m = \sum_{i=2}^{m} \left( \epsilon_{\sigma^{-1}(i)} \Lambda_{\sigma^{-1}(i)} - \frac{2m-2i+1}{2} \right) e_i.
\]
Hence \( s_{\sigma, \varepsilon}(\Lambda)|_t - \rho_m \in \mathfrak{g}^*_m \) if and only if

\[
\begin{cases}
\varepsilon_{\sigma^{-1}(i)} \Lambda_{\sigma^{-1}(i)} \geq \varepsilon_{\sigma^{-1}(i+1)} \Lambda_{\sigma^{-1}(i+1)} + 1 \\ \varepsilon_{\sigma^{-1}(m)} \Lambda_{\sigma^{-1}(m)} \geq \frac{1}{2}
\end{cases} \quad (2 \leq i \leq m-1)
\]

Clearly, \( \sigma_j \) and \( \varepsilon_j \) satisfy (7.2). Conversely assume that \( s_{\sigma, \varepsilon}(\Lambda)|_t - \rho_m \in \mathfrak{g}^*_m \). Then the left hand sides of (7.2) are positive. Since \( \Lambda_{\sigma^{-1}(i)} \) is positive for all \( i \), \( \varepsilon_{\sigma^{-1}(i)} \) must be 1 for \( 2 \leq i \leq m \). Moreover, from (7.2), we have \( \Lambda_{\sigma^{-1}(i+1)} > \Lambda_{\sigma^{-1}(i)} \) for \( 2 \leq i \leq m-1 \). Hence, from (7.1), we obtain \( \sigma^{-1}(2) < \sigma^{-1}(3) < \cdots < \sigma^{-1}(m) \). If we put \( \sigma^{-1}(1) = j \), we have necessarily \( \sigma = (1, 2)(2, 3) \cdots (j-1, j) \). The lemma is proved.

Denote by \( \xi_j \) the element of \( \tilde{M} \) with highest weight \( s^+_j(\Lambda)|_t - \rho_m \); i.e. the highest weight \( \mu(\xi_j) \) of \( \xi_j \) is given by

\[
\begin{align*}
\mu(\xi_1) &= \sum_{i=1}^m \left( \Lambda_i - \frac{2m - 2i + 1}{2} \right) e_i \\
\mu(\xi_j) &= \sum_{i=1}^m \left( \Lambda_i - \frac{2m - 2i + 1}{2} \right) e_i + \sum_{i=j+1}^m \left( \Lambda_i - \frac{2m - 2i + 1}{2} \right) e_i \\
\mu(\xi_m) &= \sum_{i=1}^m \left( \Lambda_i - \frac{2m - 2i + 1}{2} \right) e_i
\end{align*}
\]

Also note that \( s^+_j(\Lambda)|_t = \pm \Lambda_j \). After all, we obtain that

\[
R(\Lambda) = \{ (\xi_j, \Lambda_j), (\xi_j, -\Lambda_j) \mid 1 \leq j \leq m \}.
\]

Now consider the representations \( \pi_{\xi_j, \Lambda_j} \) and \( \pi_{\xi_j, -\Lambda_j} \) (\( 1 \leq j \leq m \)). From Proposition 5.2.2, an irreducible unitary representation \( \pi \) such that \( \chi_\pi = \chi_\Lambda \) is infinitesimally equivalent to an irreducible subquotient of \( \pi_{\xi_j, \Lambda_j} \) or \( \pi_{\xi_j, -\Lambda_j} \) for some \( 1 \leq j \leq m \).

For convenience' sake, we put \( \Lambda - \rho = \sum_{i=1}^m \Lambda_i e_i \); i.e.

\[
\Lambda_i = \Lambda_i - \frac{2m - 2i + 1}{2} \quad (1 \leq i \leq m).
\]

Hence we have

(7.3) \( \Lambda_1 \geq \Lambda_2 \geq \cdots \geq \Lambda_m \geq 0 \),

and we can write

\[
\begin{align*}
\mu(\xi_1) &= \sum_{i=1}^m \Lambda_i e_i \\
\mu(\xi_j) &= \sum_{i=1}^j (\Lambda_i - 1) e_i + \sum_{i=j+1}^m \Lambda_i e_i \\
\mu(\xi_m) &= \sum_{i=1}^m (\Lambda_i - 1) e_i
\end{align*}
\]
7.3. Next we shall find out all irreducible subquotients of $\pi_{\xi, \pm A_j} (1 \leq j \leq m)$, up to infinitesimally equivalence. In [13], Thieleker determined all irreducible subquotients of every representation in the nonunitary principal series. Let us rely on his results.

Let $\xi$ be in $\mathcal{M}$ and $\tau$ in $\hat{K}$. The highest weight $\mu(\xi)$ of $\xi$ and the highest weight $\gamma(\tau)$ of $\tau$ can be written as follows;

$$\mu(\xi) = \sum_{i=2}^{n} \mu(\xi)_i e_i$$

$$\gamma(\tau) = \sum_{j=2}^{n} \gamma(\tau)_j e_j .$$

By Lemma 4 in § 5 of [12], $\tau$ is $\xi$-admissible if and only if $\gamma(\tau)$ satisfies the following conditions;

$$\begin{align*}
\mu(\xi)_i - \gamma(\tau)_i &\in \mathbb{Z} \\
\gamma(\tau)_j &\geq \mu(\xi)_j & (2 \leq i \leq m) \\
\gamma(\tau)_j &\geq \gamma(\tau)_j & j \geq 2, j \neq i, j \neq m .
\end{align*}$$

Moreover, for every $\xi$-admissible $\tau$, we have $m(\tau; \xi) = 1$. Hence, by the statement in 4.4, we have

$$\Phi(\pi_{\xi, \gamma}) = \hat{K}(\xi)$$

$$= \{ \tau \in \hat{K} | \gamma(\tau) \text{ satisfies (7.5)} \} .$$

Also, note that every $K$-invariant subspace of $E_{\xi}$, and hence every $G$-invariant subspace of $E_{\xi}$, is of the form $E_{\xi}(\Phi)$ for a certain subset $\Phi$ of $\hat{K}(\xi)$.

Following Thieleker, we shall define some $K$-invariant subspaces of $E_{\xi}$, $(1 \leq j \leq m)$. In the case that $1 \leq j \leq m-1$, we have

$$\hat{K}(\xi_j) = \left\{ \tau_{\gamma} \in \hat{K} \left| \begin{array}{c}
\gamma_i - \overline{\gamma}_i \in \mathbb{Z} \\
\gamma_j \geq \overline{\gamma}_{j+1} \geq \cdots \geq \gamma_{j-1} \geq \overline{\gamma}_{j-1} + 1 \\
\gamma_j \geq \overline{\gamma}_m \geq |\gamma_m| 
\end{array} \right. \right\} ,$$

where $\overline{\gamma} = \sum_{i=1}^{m} \gamma_i e_i \in \mathbb{F} \cap \mathbb{F}^{*}$. Let $\Phi^{+}_j$ and $\Phi^{-}_j$ be subsets of $\hat{K}(\xi_j)$ defined as follows;

$$\Phi^{+}_j = \{ \tau_{\gamma} \in \hat{K}(\xi_j) | \gamma_j \geq \overline{\gamma}_j + 1 \}$$

$$\Phi^{-}_j = \{ \tau_{\gamma} \in \hat{K}(\xi_j) | \gamma_j < \overline{\gamma}_j + 1 \} ,$$

and put $E^{+}_j = E_{\xi_j}(\Phi^{+}_j)$ and $E^{-}_j = E_{\xi_j}(\Phi^{-}_j)$. In the case that $j = m$, we have

$$\hat{K}(\xi_m) = \left\{ \tau_{\gamma} \in \hat{K} \left| \begin{array}{c}
\gamma_i - \overline{\gamma}_i \in \mathbb{Z} \\
\gamma_i \geq \overline{\gamma}_{i+1} \geq \cdots \geq \gamma_{m-1} \geq \overline{\gamma}_{m-1} + 1 \geq |\gamma_m| 
\end{array} \right. \right\} .$$

Let $\Phi^{+}_m$, $\Phi^{-}_m$ and $\Phi^{\circ}_m$ be subsets of $\hat{K}(\xi_m)$ defined as follows;
\[ \Phi^+_m = \{ \gamma \in \hat{K}(\xi_m) \mid \gamma_m \geq \lambda_m + 1 \} \]

\[ \Phi^-_m = \{ \gamma \in \hat{K}(\xi_m) \mid \gamma_m \leq -(\lambda_m + 1) \} \]

\[ \Phi^*_m = \{ \gamma \in \hat{K}(\xi_m) \mid -(\lambda_m + 1) < \gamma_m < \lambda_m + 1 \} \]

and put \( E^+ = E^+_{\xi_m}(\Phi^+_m) \), \( E^- = E^-_{\xi_m}(\Phi^-_m) \) and \( E^* = E^*_{\xi_m}(\Phi^*_m) \).

**Remark 7.3.1.** In [14], our representation \((\pi^\xi, E_\xi)\) is denoted by \((\pi^\xi, L^\xi(K))\). Hence \( E^+_j \) (1 ≤ \( j \) ≤ \( m \)) corresponds to \( D^+_j(\Lambda_j, \xi_j) \) in [14] and \( E^* \) corresponds to \( D^*_m(\Lambda_m, \xi_m) \).

Under these notations, apply Theorem 2 and Theorem 3 in [14] to our representations \( \pi_{\xi, \pm \Lambda_1} \) (1 ≤ \( j \) ≤ \( m \)). For 1 ≤ \( j \) ≤ \( m - 1 \), \( E^+_j \) (resp. \( E^*_j \)) is an irreducible \( G \)-invariant subspace of \( E^+_{\xi, \pm \Lambda_1} \) under the action of \( \pi_{\xi_1, \pm \Lambda_1} \) (resp. \( \pi_{\xi_1, -\Lambda_1} \)). Also \( E^-_m \) and \( E^*_m \) are irreducible \( G \)-invariant subspaces of \( E^-_m \) under the action of \( \pi_{\xi_m, -\Lambda_m} \) and \( E^*_m \) is an irreducible \( G \)-invariant subspace of \( E^*_{\xi_m} \) under the action of \( \pi_{\xi_m, -\Lambda_m} \). There are no more irreducible \( G \)-invariant subspaces of \( E^+_{\xi_j} \) (1 ≤ \( j \) ≤ \( m \)) than the above.

Among these subrepresentations and quotient representations by them, there are following infinitesimal equivalences;

\[
(\pi_{\xi_j, \pm \Lambda_1}, E^+_j) \sim (\pi_{\xi_{j+1}, -\Lambda_{j+1}}, E^+_{j+1}) \quad (1 \leq j \leq m - 2)
\]

\[
(\pi_{\xi_m, -\Lambda_m}, E^+_m) \sim (\pi_{\xi_m, -\Lambda_m}, E^+_m)
\]

\[
(\pi_{\xi_j, -\Lambda_1}, E^+_j) \sim (\pi_{\xi_{j-1}, +\Lambda_1}, E^+_{j-1}) \quad (1 \leq j \leq m - 1)
\]

\[
(\pi_{\xi_m, +\Lambda_m}, E^+_{\xi_m}) \sim (\pi_{\xi_m, -\Lambda_m}, E^+_{\xi_m}) \quad (1 \leq j \leq m - 1)
\]

\[
(\pi_{\xi_m, -\Lambda_m}, E^+_m) \sim (\pi_{\xi_m, -\Lambda_m}, E^+_{m}) \quad (1 \leq j \leq m - 1)
\]

where the symbol \( \sim \) denotes the infinitesimal equivalence. Moreover, there is no more infinitesimal equivalences than the above. For the simplification, denote by \( \pi^+_j(\Lambda) \) (1 ≤ \( j \) ≤ \( m - 1 \)) the representation \( (\pi_{\xi_j, \pm \Lambda_1}, E^+_j) \) and denote by \( \pi^-_{\xi_m}(\Lambda) \) the representation \( (\pi_{\xi_m, -\Lambda_m}, E^+_{\xi_m}) \). After all, we have the following proposition.

**Proposition 7.3.2.** Up to infinitesimal equivalence, all irreducible subquotients of the nonunitary principal series whose infinitesimal characters are equal to \( \chi_\Lambda \) are given by

\[
\{ \pi^-_{\Lambda}(\Lambda), \pi^+_1(\Lambda), \pi^+_2(\Lambda), \cdots, \pi^+_m(\Lambda), \pi^-_m(\Lambda), \pi^-_m(\Lambda) \}.
\]

Moreover these representations are not infinitesimally equivalent to one another.

7.4. We shall examine which subquotients in Proposition 7.3.2 are infinitesimally unitary. Using Theorem 4 in [14], we have the following proposition.
Proposition 7.4.1.

(1). The representation \( \pi_\Lambda(\Lambda) \) is infinitesimally unitary if and only if \( \Lambda = \rho \). Moreover, \( \pi_\Lambda(\rho) \) is the trivial representation \( 1_G \) of \( G \).

(2). For \( 1 \leq j \leq m-1 \), \( \pi_j(\Lambda) \) is infinitesimally unitary if and only if \( \Lambda_{j+1} = \cdots = \Lambda_m = 0 \).

(3). The representation \( \pi_m(\Lambda) \) is always infinitesimally unitary.

Proof. By (3) of Theorem 4 in [14], \( \pi_\Lambda(\Lambda) \) is infinitesimally unitary if and only if \( \Lambda_2 = 0 \) and \( \Lambda = \frac{2m-1}{2} \). If \( \Lambda_1 = \frac{2m-1}{2} \), we have \( \Lambda_1 = 0 \) and hence, by (7.3), \( \Lambda_1 = \Lambda_2 = \cdots = \Lambda_m = 0 \). This implies \( \Lambda = 0 \). Also, by the definition of \( E_\Lambda \), \( \pi_\Lambda(\rho) \) is clearly the one-dimensional trivial representation of \( G \). Hence the assertion (1) follows.

The assertion (2) follows immediately by (3) of Theorem 4 in [1]. In the case of \( 1 = m \), \( \pi_m(\Lambda) \) is infinitesimally unitary if and only if \( \Lambda_{m-1} > 0 \). This inequality is always satisfied. Thus the assertion (3) is proved. The proof is completed.

Now, if \( \pi \) is an irreducible infinitesimally unitary representation of \( G \), \( \pi \) is infinitesimally equivalent to a unique irreducible unitary representation of \( G \), up to unitarily equivalence. Hence \( \pi \) determines a class in \( \hat{G} \). When \( \Lambda \) satisfies the condition that \( \Lambda_{j+1} = \cdots = \Lambda_m = 0 \), denote by \( U_j(\Lambda) \in \hat{G} \) the class which is determined by \( \pi_j(\Lambda) \). Also denote by \( U_m(\Lambda) \) the class which is determined by \( \pi_m(\Lambda) \).

By Theorem 5 in [14], \( U_m(\Lambda) \) is a discrete class. Note that, by Theorem 6 in [14], \( U_j(\Lambda) \) (\( 1 \leq j \leq m-1 \)) is not a discrete class. In effect, we have the following corollary and complete Step 1 for \( \Lambda \in \mathbb{R}_+^d \).

Corollary 7.4.2. Let \( \Lambda \in \mathbb{R}_+^d \). The subset \( \hat{G}_A - \hat{G}_d \) of \( \hat{G} \) is given as follows:

(1). If \( \Lambda_m \neq 0 \), we have \( \hat{G}_A - \hat{G}_d = \phi \).

(2). If \( \Lambda_{m-1} = \cdots = \Lambda_m = 0 \) and \( \Lambda_j \neq 0 \) for some \( 1 \leq j \leq m-1 \), we have \( \hat{G}_A - \hat{G}_d = \{ U_j(\Lambda), U_{j+1}(\Lambda), \cdots, U_{m-1}(\Lambda) \} \).

(3). If \( \Lambda_1 = \Lambda_2 = \cdots = \Lambda_m = 0 \) i.e. \( \Lambda = \rho \), we have \( \hat{G}_A - \hat{G}_d = \{ 1_G, U_1(\Lambda), U_2(\Lambda), \cdots, U_{m-1}(\Lambda) \} \), where \( 1_G \) is the class of the trivial representation of \( G \).

Remark 7.4.3. In fact, we have \( U_m(\Lambda) = D_\Lambda \in \hat{G}_d \) and \( U_m(\Lambda) = D_{\tau_\Lambda} \in \hat{G}_d \). This is shown by comparing \( K \)-spectra of these representations.

7.5. We shall go forward Step 2. As before, let \( \Lambda \) be in \( \mathbb{R}_+^d \). Our purpose is to compute \( \dim \text{Hom}_K(\tau_{\Lambda - \rho_k} \otimes L^z, U_j(\Lambda)) \) (\( 1 \leq j \leq m-1 \)) and \( \dim \text{Hom}_K(\tau_{\Lambda - \rho_k} \otimes L^z, 1_G) \).

Let \( j \) be an integer such that \( 1 \leq j \leq m-1 \). When \( \Lambda \) satisfies that \( \Lambda_{j+1} = \Lambda_{j+2} = \cdots = \Lambda_m = 0 \), a representation in the class \( U_j(\Lambda) \) has the same \( K \)-spectrum as \( \pi_j(\Lambda) \). Hence we shall examine \( \dim \text{Hom}_K(\tau_{\Lambda - \rho_k} \otimes L^z, \pi_j(\Lambda)|_K) \) and
Recall that the $K$-spectrum of $\pi_j(\Lambda)$ is given by

$$\Phi(\pi_j(\Lambda)) = \Phi_j = \{ \gamma_i - \Lambda_i \in \mathbb{Z} \mid (1 \leq i \leq m) \} \cup \{ \gamma_j - \Lambda_j + 1 \}$$.  

Also the $K$-spectrum of $1_K$ is given by $\Phi(1_K) = \{ \lambda_K \}$, where $\lambda_K$ is the trivial representation of $K$.  

Thus it is sufficient to compute the multiplicities of $\pi_j(\Lambda)$ in $\tau_{\Lambda \cdot \rho_s \otimes L^\pm}$ and the multiplicity of $1_K$ in $\pi_{\rho_s \otimes L^\pm}$.

In order to use Proposition 6.1.2, we shall prepare the following lemma.

Put $L = L^+ \oplus L^-$. 

**Lemma 7.5.1.** Let $\beta$ and $\gamma$ be in $\mathfrak{g}_K^+$.

1. Let $s \in W_G$. If $\beta + \rho_k - s(\gamma + \rho_k)$ is a weight of $L$, $s$ is the identity.
2. The multiplicity of $\tau_\beta$ in $\tau_{\gamma \otimes L}$ is equal to $M_L(\beta - \gamma)$, where $M_L(\lambda)$ $(\lambda \in \mathfrak{g}_K^+)$ is the multiplicity of the weight $\lambda$ in $L$.

**Proof.** Note that, by Lemma 6.2.1, the set of weights of $L^+$ (resp. $L^-$) is given by

$$\{ (\sum_{i=1}^m \varepsilon_i \xi_i) \mid \varepsilon_i = \pm 1 \}$$

and the multiplicity of each weight is equal to 1. If $\beta + \rho_k - s(\gamma + \rho_k)$ is a weight of $L$, we have

$$\beta + \rho_k - s(\gamma + \rho_k) = \frac{1}{2} \sum_{i=1}^m \varepsilon_i \xi_i$$

where $\varepsilon_i = 1$ or $-1$ for $1 \leq i \leq m$. Assume that $s$ is not the identity. Then there is an element $\alpha$ of $\Delta^+_k$ such that $s^{-1}(\alpha) \in -\Delta^+_k$. Since $\beta + \rho_k$ and $\gamma + \rho_k$ are dominant regular integral forms with respect to $\Delta^+_k$, we have

$$\frac{2 \langle \beta + \rho_k - s(\gamma + \rho_k), \alpha \rangle}{\langle \alpha, \alpha \rangle} = \frac{2 \langle \beta + \rho_k, \alpha \rangle + 2 \langle \gamma + \rho_k, -s^{-1}(\alpha) \rangle}{\langle \alpha, \alpha \rangle} \geq 2.$$  

On the other hand, if we set $\alpha = \xi_i \pm \xi_j$ for some $1 \leq i < j \leq m$, we have

$$\frac{2 \left( \frac{1}{2} \sum_{i=1}^m \varepsilon_i \xi_i, \alpha \right)}{\langle \alpha, \alpha \rangle} = \frac{\varepsilon_i \pm \varepsilon_j}{2} \leq 1.$$  

This is the contradiction. Thus the assertion (1) is proved. The assertion (2) follows immediately from Proposition 6.1.2 and the assertion (1). The lemma is proved.
The following proposition completes Step 2 for $\Lambda \in \mathfrak{B}$.

**Proposition 7.5.2.**

(1). Let $j$ be an integer such that $1 \leq j \leq m - 1$. Then we have

$$
\dim \text{Hom}_K(\tau_{\Lambda - \rho_n} \otimes L^+, \pi_j(\Lambda)|_K) = \begin{cases} 
1 & \text{if } m-j \text{ is even}, \\
0 & \text{if } m-j \text{ is odd},
\end{cases}
$$

$$
\dim \text{Hom}_K(\tau_{\Lambda - \rho_n} \otimes L^-, \pi_j(\Lambda)|_K) = \begin{cases} 
0 & \text{if } m-j \text{ is even}, \\
1 & \text{if } m-j \text{ is odd}.
\end{cases}
$$

(2). We have

$$
\dim \text{Hom}_K(\tau_{\rho_n} \otimes L^+, 1_g|_K) = \begin{cases} 
1 & \text{if } m \text{ is even}, \\
0 & \text{if } m \text{ is odd},
\end{cases}
$$

$$
\dim \text{Hom}_K(\tau_{\rho_n} \otimes L^-, 1_g|_K) = \begin{cases} 
0 & \text{if } m \text{ is even}, \\
1 & \text{if } m \text{ is odd}.
\end{cases}
$$

**Proof.** Let $\tau_\gamma \in \hat{K}$. Assume that $\tau_\gamma \in \Phi(\pi_j(\Lambda))$ and the multiplicity of $\tau_\gamma$ in $\tau_{\Lambda - \rho_n} \otimes L$ is not equal to 0. By (2) of Lemma 7.5.1, we have $M_L(\gamma - \Lambda + \rho_n) \neq 0$. Hence, by (7.10), we have $\gamma - \Lambda + \rho_n = \frac{1}{2} \sum_{i=1}^{m} \varepsilon_i e_i$ for some $\{\varepsilon_1, \ldots, \varepsilon_m\}$ such that $\varepsilon_i = 1$ or $-1$ ($1 \leq i \leq m$); i.e.

$$
\gamma_i = \Lambda_i - m + i + \frac{\varepsilon_i}{2} = \Lambda_i + \frac{\varepsilon_i + 1}{2} \quad (1 \leq i \leq m).
$$

On the other hand, $\{\gamma_1, \ldots, \gamma_m\}$ satisfy the inequalities (7.9). Therefore we have necessarily

$$
\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_j = 1 \quad \text{and} \quad \varepsilon_{j+1} = \varepsilon_{j+2} = \cdots = \varepsilon_m = -1,
$$

and hence we have

$$
(7.11) \quad \gamma = \sum_{i=1}^{j} (\Lambda_i + 1) e_i + \sum_{i=j+1}^{m} \Lambda_i e_i.
$$

Conversely, if $\gamma$ is given by (7.11), then we have clearly that $\tau_\gamma \in \Phi(\pi_j(\Lambda))$ and $\gamma - \Lambda + \rho_n = \frac{1}{2} (e_1 + \cdots + e_j - e_{j+1} - \cdots - e_m)$. Then, by (2) of Lemma 7.5.1, the multiplicity of $\tau_\gamma$ in $\tau_{\Lambda - \rho_n} \otimes L$ is equal to 1. Whether $\tau_\gamma$ occurs in $\tau_{\Lambda - \rho_n} \otimes L^+$ or $\tau_{\Lambda - \rho_n} \otimes L^-$ is determined by the sign of $(-1)^{m-j}$. Since each $\tau_\gamma \in \Phi(\pi_j(\Lambda))$ occurs with multiplicity 1 in $\pi_j(\Lambda)|_K$, the assertion (1) is proved.

Next, the highest weight of $1_K$ is 0. Since $-\rho_n$ is a weight of $L$, by (2) of
Lemma 7.5.1, the multiplicity of $1_\kappa$ in $\tau_{p_\kappa} \otimes L$ is equal to 1. Whether $1_\kappa$ occurs in $\tau_{p_\kappa} \otimes L^+$ or $\tau_{p_\kappa} \otimes L^-$ depends on the sign of $(-1)^n$. The assertion (2) immediately follows. The proof is completed.

7.6. Now, we have completed Step 1 and Step 2. From the basic formula in Theorem 3.1.1, we shall obtain the main theorem in the case of $G = \text{Spin}(1, 2m)$ $(m \geq 2)$.

Let $\Gamma$ be a torsion free discrete subgroup of $G$ such that $\Gamma \backslash G$ is compact. Then we have the following theorem.

**Theorem 7.6.1.** Let $\Lambda$ be in $\mathfrak{F}_0^+$ and set $\Lambda - \rho = \sum_{i=1}^m \alpha_i e_i$. Then we have the following formulas about the multiplicities $N_{\Gamma}(D_\Lambda)$ and $N_{\Gamma}(D_{s_0\Lambda})$;

(I). $N_{\Gamma}(D_\Lambda) = N_{\Gamma}(D_{s_0\Lambda})$;

(II). (1). If $\overline{\Lambda}_m = 0$,

$$N_{\Gamma}(D_\Lambda) = d(D_\Lambda) \text{vol}(\Gamma \backslash G).$$

(2). If $\overline{\Lambda}_j = \cdots = \overline{\Lambda}_m = 0$ and $\overline{\Lambda}_j \neq 0$ for some $1 \leq j \leq m-1$,

$$N_{\Gamma}(D_\Lambda) = d(D_\Lambda) \text{vol}(\Gamma \backslash G) + N_{\Gamma}(U_{m-1}(\Lambda)) - N_{\Gamma}(U_{m-2}(\Lambda)) + \cdots + (-1)^{m-j} N_{\Gamma}(U_j(\Lambda)).$$

(3). If $\overline{\Lambda}_1 = \cdots = \overline{\Lambda}_m = 0$, i.e. $\Lambda = \rho$,

$$N_{\Gamma}(D_\Lambda) = d(D_\Lambda) \text{vol}(\Gamma \backslash G) + N_{\Gamma}(U_{m-1}(\rho)) - N_{\Gamma}(U_{m-2}(\rho)) + \cdots + (-1)^m N_{\Gamma}(U_1(\rho)) + (-1)^{m+1}. $$

Proof. The assertion (II) follows directly from Theorem 3.1.1, Corollary 7.4.2 and Proposition 7.5.2.

We shall prove the assertion (I). To begin with, note that the positive root system of $\Delta$ with respect to which $s_0 \Delta$ is dominant is given by $s_0 \Delta^+$ and we have $s_0 \Delta^+ \cap \Delta = \Delta^+$. Compare the multiplicity formula (3.1) for $D_\Lambda$ with that for $D_{s_0\Lambda}$. From the formula (1.1), we have $d(D_\Lambda) = d(D_{s_0\Lambda})$. Since $\hat{G}_\Lambda = \hat{G}_{s_0\Lambda}$, it is sufficient to prove that

$$\dim \text{Hom}_K(\tau_{\Lambda - p_\kappa} \otimes L^\pm, U) = \dim \text{Hom}_K(\tau_{s_0\Lambda - p_\kappa} \otimes L^\pm, U)$$

for $U \in \hat{G}_\Lambda - \hat{G}_d$, where $L^\pm$ in the right hand side is appointed with respect to the positive root system $s_0 \Delta^+$.

Since $s_0 \Delta^+ \cap \Delta = s_0 \Delta^+$, the set of weights of $L^\pm$ associated with $s_0 \Delta^+$ is given by

$$\{s_0 \langle Q \rangle - s_0 \rho_\kappa | Q \subset \Delta^+, (-1)^{m-1 | Q |} = \pm 1 \}.$$  

Let $\gamma$ be in $\mathfrak{F}_d^+$. Then $s_0 \gamma$ is also in $\mathfrak{F}_d^+$. For $s \in W_\rho$ and $Q \subset \Delta^+$, $\gamma + \rho_\kappa - s \Lambda = \langle Q \rangle - \rho_\kappa$ if and only if $s_0 \gamma + \rho_\kappa - (s_0 s \gamma) (s_0 \Lambda) = s_0 \langle Q \rangle - s_0 \rho_\kappa$. Also the mapping;
$W_G \cong s \to s_0 s^{-1} \in W$ is a bijection of $W_G$ onto $W_G$ and we have $\varepsilon(s) = \varepsilon(s_0 s_0^{-1})$.

Hence, from Proposition 6.1.2, the multiplicity of $\tau_0$ in $\tau_{s_0} \otimes L^\pm$ is equal to the multiplicity of $\tau_{s_0}$. On the other hand, from (7.6), $\tau_0$ belongs to the $K$-spectrum of $\pi_j^0(\Lambda)$ ($1 \leq j \leq m-1$) if and only if $\tau_{s_0}$ belongs to it. Also, we have clearly that $\tau_0 = 1_K$ if and only if $\tau_{s_0} = 1_K$. After all, we obtain the following equalities;

\[
\dim \text{Hom}_K(\tau_{s_0} \otimes L^\pm, \pi_j^0(\Lambda)) = \dim \text{Hom}_K(\tau_{s_0} \otimes L^\pm, \pi_j^0(\Lambda)) \quad (1 \leq j \leq m-1),
\]
\[
\dim \text{Hom}_K(\tau_{s_0} \otimes L^\pm, 1_G) = \dim \text{Hom}_K(\tau_{s_0} \otimes L^\pm, 1_G).
\]

Hence the assertion (I) follows. Thus the theorem is proved.

**Remark 7.7.2.**

(1). In the above theorem, the condition on $\Lambda$ in (1) of (II) agree with the condition (i) of Theorem in the introduction of [6] (c.f. [6], p. 176).

(2). In the case that $m=2$, the above theorem gives the formula in Theorem 4.5 of [11] (c.f. [11], p. 306).

(3). The set of classes $\{1_G, U_1(\rho), U_2(\rho), \ldots, U_{m-1}(\rho), D_1, D_2\}$ corresponds with $\Pi'(G) = \{J_0, J_1, \ldots, J_{m-1}, D_1, D_2\}$ in [1], Chapter VI, § 4, 4.5. To be more precise, $U_j(\rho)$ ($1 \leq j \leq m-1$) corresponds with $J_j$ and $1_G$ with $J_0$. Also, these classes contribute to Mutsushima's formula about Betti numbers of the manifold $\Gamma \setminus G/K$ (c.f. [15], p. 174).

**8. The case of $G=SU(1, n)$**

In this section, we shall pursue the same argument as in §7 in the case of $G=SU(1, n)$ ($n \geq 2$). It is more complicated than that in §7, but the methods are the same. We will depend largely on the classification of all irreducible unitary representations of $SU(1, n)$ obtained by Kraljevic (c.f. [7]).

8.1. Let $n$ be an integer such that $n \geq 2$. Let $G$ be the group $SU(1, n)$ and $K$ the group $U(n)$ imbedded in $SU(1, n)$ as follows;

\[
K = \left\{ \begin{pmatrix} b & 0 \\ 0 & X \end{pmatrix} \middle| X \in U(n), \quad b = \det X \right\}.
\]

Then the Lie algebras $\mathfrak{k}_0$ and $\mathfrak{p}_0$ are given by

\[
\mathfrak{k}_0 = \left\{ \begin{pmatrix} b & 0 \\ 0 & X \end{pmatrix} \middle| X \in \mathfrak{u}(n), \quad b = -\text{tr} X \right\},
\]
Take a Cartan subalgebra $h_0$ as follows;

$$h_0 = \left\{ H(h_0, \ldots, h_n) = \begin{pmatrix} h_0 & 0 \\ 0 & h_n \end{pmatrix} \mid h_0 \in \sqrt{-1}R, \sum_{i=0}^n h_i = 0 \right\}.$$ 

The semisimple part $t_1 = [t, t]$ of $t$ is isomorphic to $\mathfrak{sl}(n, C)$. If we set

$$c_0 = \left\{ c \begin{pmatrix} n & -1 \\ 0 & -1 \end{pmatrix} \right\} \subset C,$$

$c_0$ is the center of $t$ and we have

$$t = c_0 \oplus t_1, \quad h = c_0 \oplus h_1$$

where $h_1$ is the Cartan subalgebra of $t$, which is given by

$$h_1 = \{ H(h_0, \ldots, h_n) \in h \mid h_0 = 0 \}.$$ 

Define a linear form $e_i (0 \leq i \leq n)$ on $h$ by

$$e_i(H(h_0, \ldots, h_n)) = h_i.$$ 

Then $h^*_R$ is the hyperplane of the $(n+1)$-dimensional real vector space $V$ spanned by $\{e_0, \ldots, e_n\}$ which is given by

$$h^*_R = \{ \sum_{i=0}^n \lambda_i e_i \mid \lambda_i \in R, \sum_{i=0}^n \lambda_i = 0 \}.$$ 

Also $h^*$ and $h^*_t$ are given as follows;

$$h^* = \{ \sum_{i=0}^n \lambda_i e_i \mid \lambda_i \in C, \sum_{i=0}^n \lambda_i = 0 \}$$

$$h^*_t = \{ \sum_{i=0}^n \lambda_i e_i \in h^* \mid \lambda_0 = 0 \}.$$ 

The inner product in $h^*_R$ is given by

$$\langle e_i, e_j \rangle = \frac{1}{2(n+1)} \delta_{ij} \quad (0 \leq i, j \leq n).$$

Take a lexicographic ordering in $V$ relative to the basis $\{e_0, \ldots, e_n\}$. This ordering in $V$ induces an ordering in $h^*_R$. Fix this ordering in $h^*_R$. Then the following facts are easily seen;
\[ \Delta = \{ e_i - e_j | 0 \leq i, j \leq n, \quad i \neq j \} \]
\[ \Delta^+ = \{ e_i - e_j | 0 \leq i < j \leq n \} \]
\[ \Delta'_i = \{ e_i - e_j | 1 \leq i < j \leq n \} \]
\[ \Delta'_+ = \{ e_0 - e_i | 1 \leq i \leq n \} \]
\[ \rho = \frac{1}{2} \sum_{i=0}^{n} (n-2i)e_i \]
\[ \rho_+ = \frac{1}{2} \sum_{i=1}^{n} (n-2i+1)e_i \]
\[ \rho_\Lambda = \frac{1}{2} (ne_0 - \sum_{i=1}^{n} e_i) \]
\[ \Phi = \{ \lambda = \sum_{i=0}^{n} \lambda_i e_i \in h^*_\mathfrak{g}_0 | \lambda_i - \lambda_j \in \mathbb{Z} \quad (0 \leq i, j \leq n) \} \]
\[ \Phi_0 = \{ \lambda \in \Phi | \lambda_i - \lambda_j = 0 \quad (0 \leq i, j \leq n, \quad i \neq j) \} \]
\[ \Phi^*_0 = \{ \lambda \in \Phi | \lambda_0 > \lambda_1 > \cdots > \lambda_{n-1} > \lambda_n \} \]
\[ \Phi_i = \{ \lambda \in h^*_\mathfrak{g}_0 | \lambda_i - \lambda_j \in \mathbb{Z} \quad (1 \leq i, j \leq n), \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \} . \]

Note that, if \( \lambda = \sum_{i=0}^{n} \lambda_i e_i \in \Phi \), we have
\[ (n+1) \lambda_i \in \mathbb{Z}; \quad \text{i.e.} \quad \lambda_i \in \frac{1}{n+1} \mathbb{Z} \quad (0 \leq i \leq n) . \]

Now we need the Weyl group \( W \) and \( W_\mathfrak{g} \). Let \( \Sigma_{n+1} \) be the group of permutations of the set \{0, 1, 2, \cdots, n\}. For \( \sigma \in \Sigma_{n+1} \), define an orthogonal transformation \( s_\sigma \) of \( h^*_\mathfrak{g}_0 \) by
\[ s_\sigma(e_i) = e_{\sigma(i)} \quad (0 \leq i \leq n) . \]

Then \( W \) and \( W_\mathfrak{g} \) are given as follows;
\[ W = \{ s_\sigma | \sigma \in \Sigma_{n+1} \} \]
\[ W_\mathfrak{g} = \{ s_\sigma | \sigma \in \Sigma_{n+1}, \quad \sigma(0) = 0 \} . \]

In the following, we shall always identify \( s_\sigma \) with \( \sigma \). Then we have
\[ W^1 = \{ 1 = \sigma_0, \quad \sigma_1, \cdots, \sigma_n \} , \]
where \( \sigma_l (0 \leq l \leq n) \) is the permutation \( (0, 1, \cdots, l) \). Therefore, there is the bijection between \( \mathcal{G}_2 \) and \( \bigcup_{l=0}^{n-1} \Phi_l^* \).

Next, we take a maximal abelian subalgebra \( \mathfrak{a}_0 \) of \( \mathfrak{p}_0 \) as follows;
\[ \mathfrak{a}_0 = \{ P(0, \cdots, 0, a) | a \in \mathbb{R} \} . \]

Then \( M \) and \( m_0 \) are given by
The semisimple part \( m_1 = [m, m] \) of \( m \) is isomorphic to \( \mathfrak{sl}(n-1, \mathbb{C}) \). If we put

\[
c_m = \left\{ c \cdot H\left(\frac{n-1}{2}, -1, \cdots, -1, \frac{n-1}{2}\right) \mid c \in \mathbb{C} \right\},
\]

\( c_m \) is the center of \( m \) and we have \( m = c_m + m_i \). Take a maximal abelian abelian subalgebra \( t_0 \) of \( m_0 \) as follows;

\[
t_0 = \{ H(b, h_1, \cdots, h_{n-1}, b) \mid h_i, b \in \sqrt{-1} \mathbb{R}, \sum_{i=1}^{n-1} h_i + 2b = 0 \}.
\]

Then we have \( t = c_m + t_i \), where \( t_i \) is a Cartan subalgebra of \( m_i \) defined by

\[
t_i = \{ H(0, h_1, \cdots, h_{n-1}, 0) \mid h_i \in \mathbb{C}, \sum_{i=1}^{n-1} h_i = 0 \}.
\]

Two Cartan subalgebras \( \mathfrak{h} \) and \( \mathfrak{h}' = \mathfrak{a} + \mathfrak{t} \) are identified by the following isomorphism \( \theta \);

\[
\theta(P(0, \cdots, 0, a) + H(b, h_1, \cdots, h_{n-1}, b)) = H(b+a, h_1, \cdots, h_{n-1}, b-a).
\]

Also, by \( \theta \), our ordering in \( \mathfrak{h}_R^* \) corresponds to a compatible ordering of \( (\mathfrak{h}_R^*)^* \) and induces an ordering in \( t_R^* \). In this way, we always identify \( (\mathfrak{h}')^* \) and \( \mathfrak{h}_R^* \). Regarding \( t^* \), \( t_R^* \) and \( \mathfrak{a}^* \) as subsets of \( \mathfrak{h}_R^* \), we have

\[
t^* = \{ \mu = \sum_{i=0}^{n} \mu_i e_i \in \mathfrak{h}_R^* \mid \mu_0 = \mu_n \}
\]

\[
t_R^* = \{ \mu \in t^* \mid \mu_0 = \mu_n = 0 \}
\]

\[
\mathfrak{a}^* = \left\{ \nu \frac{e_0 - e_n}{2} \mid \nu \in \mathbb{C} \right\}.
\]

We shall identify \( \nu \cdot \frac{e_0 - e_n}{2} \in \mathfrak{a}^* \) with the complex number \( \nu \) and hence identify \( \mathfrak{a}^* \) with \( \mathbb{C} \).

As for the Lie algebra \( m \), we have

\[
\Delta_m = \{ e_i - e_j \mid 1 \leq i, j \leq n-1, \quad i \neq j \}
\]

\[
\Delta_m^* = \{ e_i - e_j \mid 1 \leq i < j \leq n-1 \}.
\]
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(8.1) \( \mathcal{B}_m^+ = \left\{ \mu = \sum_{i=0}^{n} \mu_i e_i \in \mathbb{B}_R \left| \begin{array}{c} \mu_0 = \mu_n, \mu_i - \mu_j \in \mathbb{Z} \quad (0 \leq i, j \leq n) \\ \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \end{array} \right. \right\} \).

8.2. Following Kraljevic [7], we shall introduce some notations. Let \( r \) be a positive integer. We define the subset \( R_r^+ \) of \( R_r^+ \) by

\[
R_r^+ = \left\{ p = (p_0, \cdots, p_r) \in R_r^+ \left| \begin{array}{c} p_{i+1} - p_i \in \mathbb{Z} \quad (1 \leq i \leq r) \\ p_1 \geq p_2 \geq \cdots \geq p_r \end{array} \right. \right\}.
\]

For \( p = (p_0, \cdots, p_r) \in R_r^+ \) and \( q = (q_0, \cdots, q_{r+1}) \in R_{r+1}^+ \), we write \( p < q \) if \( p \) and \( q \) satisfy that

\[
\begin{align*}
q_{i+1} - p_i & \in \mathbb{Z} \quad (1 \leq i \leq r) \\
p_{i+1} & \geq q_0 \geq p_1 \geq \cdots \geq q_r \geq p_r \geq q_{r+1}.
\end{align*}
\]

Also, for \( p, q \in R_r^+ \), we write \( (p_0, \cdots, p_r) < (\infty, q_0, \cdots, q_r) \) (resp. \( (p_0, \cdots, p_r) \leq (q_0, \cdots, q_r) \)) if \( (p_0, \cdots, p_r) < (q_0, \cdots, q_r) \) (resp. \( (p_0, \cdots, p_r) \leq (q_0, \cdots, q_r, p_r) \)).

Let \( \gamma = \sum_{i=0}^{n} \gamma_i e_i \in \mathbb{B}_R^+ \). Then \( \gamma \) belongs to \( \mathcal{B} \cap \mathcal{B}_r^+ \) if and only if \( \{ \gamma_0, \cdots, \gamma_r \} \) satisfy the conditions

\[
\begin{align*}
\gamma_i & \in \left( \frac{1}{n+1} \mathbb{Z} \right) \quad (1 \leq i \leq n) \\
\gamma_i - \gamma_j & \in \mathbb{Z} \quad (1 \leq i, j \leq n) \\
\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n
\end{align*}
\]

If we put

\[
\left( \frac{1}{n+1} \mathbb{Z} \right)^n = \left\{ p \in R_r^+ \left| p_i \in \left( \frac{1}{n+1} \mathbb{Z} \right) \quad (1 \leq i \leq n) \right. \right\},
\]

the mapping; \( \mathcal{B} \cap \mathcal{B}_r^+ \in \gamma \rightarrow (\gamma_0, \cdots, \gamma_r) \in \left( \frac{1}{n+1} \mathbb{Z} \right)^n \) is a bijection. Thus we shall always identify an element \( \gamma \) of \( \mathcal{B} \cap \mathcal{B}_r^+ \) with an element \( (\gamma_0, \cdots, \gamma_r) \) of \( \left( \frac{1}{n+1} \mathbb{Z} \right)^n \). Similarly, \( \mu = \sum_{i=0}^{n} \mu_i e_i \in \mathcal{B} \cap \mathcal{B}_m^+ \) can be identified with an element \( \mu = (\mu_0, \cdots, \mu_m) \) of \( \left( \frac{1}{n+1} \mathbb{Z} \right)^m \). Hence \( R \) is isomorphic to the set \( \left( \frac{1}{n+1} \mathbb{Z} \right)^n \) and \( M \) is isomorphic to the set \( \left( \frac{1}{n+1} \mathbb{Z} \right)^{n-1} \) (c.f. [7], p. 35).

8.3. Now, we shall start off with Step 1. Once and for all, take a special element \( \Lambda = \sum_{i=0}^{r} \Lambda_i e_i \in \mathcal{B}_r^+ \) and fix. Then we have
Lemma 8.3.1. Let $h$ and $k$ be integers such that $0 \leq h < k \leq n$. Define 
\[ \sigma_{\pm, h, k} \in \mathfrak{S}_{n+1} \] 
by 
\[ \sigma_{\pm, h, k} = \begin{cases} 
0, 1, \ldots, h-1, h, h+1, \ldots, k-1, k, k+1, \ldots, n \\
1, 2, \ldots, h, 0, h+1, \ldots, k-1, n, k, \ldots, n-1 
\end{cases} \]

Then, an element $\sigma \in W$ satisfies that $\sigma \Lambda|_t - \rho_m \in \mathfrak{S}_m^*$ if and only if $\sigma$ is either $\sigma_{\pm, h, k}$ or $\sigma_{\mp, h, k}$ for some $0 \leq h < k \leq n$.

Proof. For $\sigma \in \mathfrak{S}_{n+1}$, we have 
\[ \sigma \Lambda|_t - \rho_m = \sum_{i=1}^{n-1} \left( \Lambda_{\sigma^{-1}(i)} - \frac{n-2i}{2} \right) e_i + b(e_0 + e_n) \]
where $b \in \mathcal{R}$ is given by $2b + \sum_{i=1}^{n-1} \Lambda_{\sigma^{-1}(i)} = 0$. Assume that $\sigma \Lambda|_t - \rho_m$ belongs to $\mathfrak{S}_m^*$. From (8.1), we have 
\[ \Lambda_{\sigma^{-1}(i)} - \frac{n-2i}{2} \leq \Lambda_{\sigma^{-1}(i+1)} - \frac{n-2(i+1)}{2} \quad (1 \leq i \leq n-2), \]
and hence we have 
\[ \Lambda_{\sigma^{-1}(i)} - \Lambda_{\sigma^{-1}(i+1)} \geq 1 \quad (1 \leq i \leq n-2). \]
Since $\Lambda$ satisfies (8.3), it follows that 
\[ \sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(n-1). \]
Therefore, if we put $k=\text{the minimum of} \{\sigma^{-1}(0), \sigma^{-1}(n)\}$ and $h=\text{the maximum of} \{\sigma^{-1}(0), \sigma^{-1}(n)\}$, $\sigma$ is necessarily either $\sigma_{\pm, h, k}$ or $\sigma_{\mp, h, k}$. The lemma is proved.

Let $h$ and $k$ be integers such that $0 \leq h < k \leq n$. We denote by $\xi_{h, k}$ the element of $\bar{M}$ with highest weight $\sigma_{\pm, h, k} \Lambda|_t - \rho_m$ and put $\nu_{h, k} = \sigma_{\pm, h, k} \Lambda|_t$; i.e. the highest weight of $\xi_{h, k}$ is given by 
\[ \mu(\xi_{h, k}) = \left( \Lambda_0 - \frac{n-2}{2}, \Lambda_1 - \frac{n-4}{2}, \ldots, \Lambda_{k-1} - \frac{n-2h}{2}, \right. \]
\[ \left. \Lambda_{k+1} - \frac{n-2(k+1)}{2}, \ldots, \Lambda_{k-1} - \frac{n-2(k-1)}{2}, \right. \]
\[ \left. \Lambda_{k+1} - \frac{n-2k}{2}, \ldots, \Lambda_n - \frac{n+2}{2} \right). \]
and we have

$$\nu_{h,k} = \pm (\Lambda_k - \Lambda_h).$$

In the above formulas, if $h=0$, (8.4) means

$$\mu(\xi_{0,h}) = \left( \Lambda_0 - \frac{n-2}{2}, \Lambda_1 - \frac{n-4}{2}, \ldots, \Lambda_{k-1} - \frac{n-2(k-1)}{2}, \right.$$

$$\left. \Lambda_{k+1} - \frac{n-2k}{2}, \ldots, \Lambda_n - \frac{n+2}{2} \right).$$

and if $k=n$, (8.4) means

$$\mu(\xi_{h,n}) = \left( \Lambda_0 - \frac{n-2}{2}, \Lambda_1 - \frac{n-4}{2}, \ldots, \Lambda_{h-1} - \frac{n-2h}{2}, \right.$$\n
$$\left. \Lambda_{h+1} - \frac{n-2(h+1)}{2}, \ldots, \Lambda_{n-1} - \frac{n+2}{2} \right).$$

Throughout this section, the case of $h=0$ or $k=n$ should be suitably appreciated, even if it is not specially offered. Moreover, for convenience' sake, define $\Lambda_0, \ldots, \Lambda_n$ by

$$\Lambda - \rho = \sum_{i=0}^{n} \bar{\Lambda}_i e_i.$$

Then we have

$$\bar{\Lambda}_i = \Lambda_i - \frac{n-2i}{2} \quad (0 \leq i \leq n)$$

$$\bar{\Lambda}_0 \geq \bar{\Lambda}_1 \geq \ldots \geq \bar{\Lambda}_n,$$

and we can write

(8.5) \quad $\mu(\xi_{h,k}) = (\bar{\Lambda}_0 + 1, \ldots, \bar{\Lambda}_{h-1} + 1, \bar{\Lambda}_{h+1}, \ldots, \bar{\Lambda}_{k-1}, \bar{\Lambda}_{k+1} - 1, \ldots, \bar{\Lambda}_n)$

Now consider the representations $\pi_{\xi_{h,k}, \nu_{h,k}^+}$ ($0 \leq h < k \leq n$) in the nonunitary principal series. From Proposition 5.2.2, an irreducible unitary representation $\pi$ such that $\mathcal{X}_\pi = \mathcal{X}_\lambda$ is infinitesimally equivalent to an irreducible subquotient of $\pi_{\xi_{h,k}, \nu_{h,k}^+}$ or $\pi_{\xi_{h,k}, \nu_{h,k}^-}$ for some $0 \leq h < k \leq n$.

8.4. Using results of [7], we shall determine all irreducible subquotients of $\pi_{\xi_{h,k}, \nu_{h,k}^\pm}$ ($0 \leq h < k \leq n$), up to infinitesimally equivalence.

Let $h$ and $k$ be integers such that $0 \leq h < k \leq n$. By Proposition 6.1 in [7] and (8.5), the $K$-spectrum of $\pi_{\xi_{h,k}, \nu_{h,k}^\pm}$ is given by

$$\Phi(\pi_{\xi_{h,k}, \nu_{h,k}^\pm}) = \hat{K}(\xi_{h,k})$$

$$= \{ \gamma \in \hat{K} | (\gamma_1, \ldots, \gamma_n) > (\bar{\Lambda}_0 + 1, \ldots, \bar{\Lambda}_{h-1} + 1, \bar{\Lambda}_{h+1}, \ldots, \bar{\Lambda}_{k-1}, \bar{\Lambda}_{k+1} - 1, \ldots, \bar{\Lambda}_n - 1) \}$$
where we use the notation $\succ$ in the sense of (8.2).

We shall apply Theorem 7.5 and Theorem 8.7 in [7] to our representation $\pi_{\xi,h,k}^\varepsilon$. Following Kraljevic, put

$$s_j(\xi) = 2\mu(\xi)_j + n - 2j + \sum_{i=1}^{j} \mu(\xi)_i \quad (1 \leq j \leq n-1)$$

for $\xi \in \hat{M}$, where $\mu(\xi) = (\mu(\xi)_1, \ldots, \mu(\xi)_n-1) \in \mathfrak{g}^*_m \cap \mathfrak{g}$ is the highest weight of $\xi$ (c.f. [7], p. 48). In the case of $\xi = \xi_{h,k}$, we have

$$s_j(\xi_{h,k}) = \begin{cases} 2\Lambda_j - \Lambda_h - \Lambda_k & (1 \leq j \leq h) \\ 2\Lambda_j - \Lambda_h - \Lambda_k & (h+1 \leq j \leq k-1) \\ 2\Lambda_j - \Lambda_h - \Lambda_k & (k \leq j \leq n-1) \end{cases}.$$

Since $\nu_{h,k}^\varepsilon \equiv \pm (\Lambda_h - \Lambda_k)$, it follows that

$$s_j(\xi_{h,k}) - \nu_{h,k}^\varepsilon \in \mathbb{Z} \quad (1 \leq j \leq n-1)$$

$$s_{h+1}(\xi_{h,k}) > \nu_{h,k}^\varepsilon > s_{h+2}(\xi_{h,k})$$

$$s_{h+1}(\xi_{h,k}) > \nu_{h,k}^\varepsilon > \nu_{h+1}^\varepsilon > s_{h+2}(\xi_{h,k}).$$

Hence, if $h+1 = k$ (resp. $h+1 < k$), $\xi_{h,k} \in \hat{M}$ and $\nu_{h,k}^\varepsilon \in \mathcal{C}$ satisfy the condition of the case (c) (resp. (d)) in Theorem 7.5 in [7]. By Theorem 7.5 and Theorem 8.7 in [7], $K$-spectra of all irreducible subquotients of $\pi_{\xi,h,k}^\varepsilon$ are given as follows;

(1). If $h+1 = k$, there can occur only the following three $K$-spectra;

$$\Phi_+^{(\xi_{h,k+1})} = \begin{cases} \tau_{\gamma} \in \hat{K} & (\gamma_1, \ldots, \gamma_{h+1}) < (\infty, \Lambda_0 + 1, \ldots, \Lambda_{h+1}) \\ & (\gamma_{h+2}, \ldots, \gamma_m) < (\Lambda_{h+2} - 1, \ldots, \Lambda_n - 1, -\infty) \end{cases}$$

$$\Phi_0^{(\xi_{h,k+1})} = \begin{cases} \tau_{\gamma} \in \hat{K} & (\gamma_1, \ldots, \gamma_{h+1}) < (\infty, \Lambda_0 + 1, \ldots, \Lambda_{h+1} + 1) \\ & (\gamma_{h+2}, \ldots, \gamma_m) < (\Lambda_{h+2} - 1, \ldots, \Lambda_n - 1, -\infty) \end{cases}$$

$$\Phi_-^{(\xi_{h,k+1})} = \begin{cases} \tau_{\gamma} \in \hat{K} & (\gamma_1, \ldots, \gamma_{h+1}) < (\infty, \Lambda_0 + 1, \ldots, \Lambda_{h+1} + 1) \\ & (\gamma_{h+2}, \ldots, \gamma_m) < (\Lambda_{h+2} - 1, \ldots, \Lambda_n - 1, -\infty) \end{cases}$$

(2). If $h+1 < k$, there can occur only the following four $K$-spectra;

$$\Phi_{++}^{(\xi_{h,k})} = \begin{cases} \tau_{\gamma} \in \hat{K} & (\gamma_1, \ldots, \gamma_{h+1}) < (\infty, \Lambda_0 + 1, \ldots, \Lambda_{h+1}) \\ & (\gamma_{h+2}, \ldots, \gamma_m) < (\Lambda_{h+2} - 1, \ldots, \Lambda_n - 1, -\infty) \end{cases}$$

$$\Phi_{+-}^{(\xi_{h,k})} = \begin{cases} \tau_{\gamma} \in \hat{K} & (\gamma_1, \ldots, \gamma_{h+1}) < (\infty, \Lambda_0 + 1, \ldots, \Lambda_{h+1}) \\ & (\gamma_{h+2}, \ldots, \gamma_m) < (\Lambda_{h+2} - 1, \ldots, \Lambda_n - 1, -\infty) \end{cases}$$

$$\Phi_{-+}^{(\xi_{h,k})} = \begin{cases} \tau_{\gamma} \in \hat{K} & (\gamma_1, \ldots, \gamma_{h+1}) < (\infty, \Lambda_0 + 1, \ldots, \Lambda_{h+1}) \\ & (\gamma_{h+2}, \ldots, \gamma_m) < (\Lambda_{h+2} - 1, \ldots, \Lambda_n - 1, -\infty) \end{cases}$$

$$\Phi_{--}^{(\xi_{h,k})} = \begin{cases} \tau_{\gamma} \in \hat{K} & (\gamma_1, \ldots, \gamma_{h+1}) < (\infty, \Lambda_0 + 1, \ldots, \Lambda_{h+1}) \\ & (\gamma_{h+2}, \ldots, \gamma_m) < (\Lambda_{h+2} - 1, \ldots, \Lambda_n - 1, -\infty) \end{cases}$$
On the other hand, by Theorem 9.2 in [7], two irreducible subquotients of representations \( \{\pi_h \mid 0 \leq h < k \leq n \} \) are infinitesimally equivalent to each other if and only if they have the same \( K \)-spectrum. By (1) and (2) above, among \( K \)-spectra of all irreducible subquotients of \( \{\pi_h \mid 0 \leq h < k \leq n \} \), there are \( \frac{(n+1)(n+2)}{2} \) kinds of \( K \)-spectra which are different from one another. They are given as follows;

\[
\Phi_{\pm}(\xi_{h,k}) = \left\{ \begin{array}{l}
\tau_\gamma \in \hat{K} \\
\left( (\gamma_1, \ldots, \gamma_k) < (\alpha_0 + 1, \ldots, \alpha_{k-1} + 1) \right) \\
(\gamma_{k+1}, \ldots, \gamma_n) < (\alpha_k, \ldots, \alpha_{k-1}) \\
(\gamma_{k+1}, \ldots, \gamma_n) < (\alpha_k - 1, \ldots, \alpha_{n-1} - 1, -\infty)
\end{array} \right. 
\]

where \( l \) runs over \( \{0, 1, \ldots, n\} \) and \( (p, q) \) runs over the set \( \{(p, q) \mid 0 \leq p < q \leq n\} \).

Denote by \( \pi_i(\Lambda) \) the irreducible subquotient with \( K \)-spectrum \( \Phi_i(\Lambda) \) and denote by \( \pi_{p,q}(\Lambda) \) the one with \( K \)-spectrum \( \Phi_{p,q}(\Lambda) \). After all we have the following proposition.

**Proposition 8.4.1.** Up to infinitesimal equivalence, all irreducible subquotients of the nonunitary principal series whose infinitesimal characters are equal to \( \chi_\Lambda \) are given as follows;

\[
\pi_i(\Lambda) \quad (0 \leq i \leq n), \quad \pi_{p,q}(\Lambda) \quad (0 \leq p < q \leq n).
\]

Moreover these representations are not infinitesimally equivalent to one another.

8.5. We shall examine which subquotients in Proposition 8.4.1 are infinitesimally unitary. Using Proposition 11.4 in [7], we have the following proposition.

**Proposition 8.5.1.**

1. For \( 0 \leq i \leq n \), \( \pi_i(\Lambda) \) is always infinitesimally unitary.

2. For \( 0 \leq p < q \leq n \), \( \pi_{p,q}(\Lambda) \) is infinitesimally unitary if and only if \( \Lambda_i - \Lambda_{i+1} = 1 \) for all \( p \leq i < q - 1 \).
Proof. By easy computations, we can prove that $\Phi/(\Lambda)$ satisfies the inequalities (6) in (ii) of Proposition 11.4 in [7]. As for $\pi_{p,\delta}(\Lambda)$, the condition in (iii) of Proposition 11.4 in [7] is clearly equivalent to our condition.

Now, let $\bar{G}$ be the universal covering group of $G$. Canonically, we can regard $\pi/(\Lambda)$ as a representation of $\bar{G}$. Since $\pi/(\Lambda)$ is infinitesimally unitary, there exists an irreducible unitary representation $\pi/(\Lambda)'$ of $\bar{G}$ which is infinitesimally equivalent to $\pi/(\Lambda)$. Then $\pi/(\Lambda)'$ turns out to be a representation of $G$ (c.f. Remark 8.5.2). Denote by $U/(\Lambda)\in\bar{G}$ the class which contains $\pi/(\Lambda)'$. Similarly, when $\pi_{p,\delta}(\Lambda)$ is infinitesimally unitary, denote by $U_{p,\delta}(\Lambda)\in\bar{G}$ the class which is determined by $\pi_{p,\delta}(\Lambda)$ in the same way.

**Remark 8.5.2.** Let $\psi: \bar{G} \to G$ be the covering epimorphism and put $\bar{K} = \psi^{-1}(K)$. Let $(\pi^1, E^1)$ and $(\pi^2, E^2)$ be $\bar{K}$-finite representations of $\bar{G}$. Assume that $\pi^1$ is infinitesimally equivalent to $\pi^2$. Then $\pi^1$ is a representation of $G$ if and only if $\pi^2$ is so. In fact, $\pi^1|_{\bar{K}}$ is equivalent to $\pi^2|_{\bar{K}}$ as representations of $\bar{K}$ on the spaces $E^1$ and $E^2$, respectively. Also the kernel of $\psi$ is contained in $\bar{K}$. Hence, for an element $k$ of the kernel of $\psi$, $\pi^1(k)$ is the identity operator of $E^1$ if and only if $\pi^2(k)$ is the identity operator of $E^2$.

**Proposition 8.5.3.** For $0 \leq l \leq n$, $U/(\Lambda)$ is a discrete class. More precisely, we have $U/(\Lambda) = D_{\sigma,\Lambda} \in \bar{G}_d$.

Proof. It is sufficient to prove that the $K$-spectrum $\Phi/(\Lambda)$ of $U/(\Lambda)$ is the same as the $K$-spectrum of $D_{\sigma,\Lambda}$. To begin with, we shall show that the $K$-spectrum of $D_{\sigma,\Lambda}$ contains $\Phi/(\Lambda)$.

In order to apply Theorem 1.4.1 to $D_{\sigma,\Lambda}$, take an ordering in $\bar{b}_K^+$ with respect to which $\sigma_{\Lambda}$ is dominant; i.e. take $\sigma_{\Lambda}(\Delta^+)$ as the positive root system of $\Delta$. Then the set of noncompact positive roots is given by

$$(8.8) \quad \{e_i - e_0 \quad (1 \leq i \leq l), \quad e_0 - e_j \quad (l+1 \leq j \leq n)\}$$

and we have

$$\rho_s = \frac{n-2l}{2} e_0 + \frac{1}{2} (e_1 + \cdots + e_i - e_{i+1} - \cdots - e_n).$$

Let $\lambda = \sum_{i=1}^n \lambda_i e_i$ be in $\bar{b}_K^+$. In our case, $Q(\lambda)$ in Theorem 1.4.1 is equal to either 1 or 0 and $Q(\lambda) = 1$ if and only if

$$\begin{cases} 
\lambda_i \in \mathbb{Z} & (0 \leq i \leq n) \\
\lambda_i \geq 0 & \text{if } 1 \leq i \text{ and } i \leq l \\
\lambda_i \leq 0 & \text{if } l+1 \leq i \text{ and } i \leq n.
\end{cases}$$

Let $\gamma_s$ be in $\Phi/(\Lambda)$. From (8.6), $\gamma = \sum_{i=0}^n \gamma^s_i e_i$ satisfies the following inequalities;
\[ \Lambda_{i-2} - 1 \geq \gamma_i + \frac{n-2i}{2} \geq \Lambda_{i-1} \quad \text{if } 1 \leq i \text{ and } i \leq l, \]
\[ \Lambda_{i-1} - 1 \geq \gamma_i + \frac{n-2i}{2} \geq \Lambda_{i+1} \quad \text{if } l+1 \leq i \text{ and } i \leq n \]

where \( \Lambda_{-1} \) should be considered \( \infty \) and \( \Lambda_{n+1} \) should be considered \( -\infty \). Let \( \sigma \) be an element of \( W_G \). Since \( \sigma(0) = 0 \), we have

\[
(8.10) \quad \sigma(\gamma + \rho_b) - \sigma_i \Lambda - \rho_a = \gamma_0 - \Lambda - \frac{n-2i}{2} e_0 + \sum_{i=1}^{l} \left( \gamma_{\sigma^{-1}(i)} + \frac{n-2\sigma^{-1}(i)}{2} - \Lambda_{i-1} \right) e_i + \sum_{i=l+1}^{n} \left( \gamma_{\sigma^{-1}(i)} + \frac{n-2\sigma^{-1}(i)}{2} - \Lambda_{i+1} \right) e_i.
\]

If there is an integer \( i \) such that \( 1 \leq i \), \( i \leq l \) and \( i < \sigma^{-1}(i) \), we have \( \Lambda_{i-1} \geq \Lambda_{\sigma^{-1}(i)-2} \) and hence, from (8.9), we have

\[
\gamma_{\sigma^{-1}(i)} + \frac{n-2\sigma^{-1}(i)}{2} - \Lambda_{i-1} \leq \Lambda_{\sigma^{-1}(i)-2} - 1 - \Lambda_{i-1} < 0.
\]

Also, if there is an integer \( i \) such that \( l+1 \leq i \), \( i \leq n \) and \( i > \sigma^{-1}(i) \), we have \( \Lambda_i \leq \Lambda_{\sigma^{-1}(i)+1} \) and hence, from (8.9), we have

\[
\gamma_{\sigma^{-1}(i)} + \frac{n-2\sigma^{-1}(i)}{2} - \Lambda_i + 1 \geq \Lambda_{\sigma^{-1}(i)+1} - \Lambda_i + 1 > 0.
\]

Therefore, in these two cases, we have

\[
Q(\sigma(\gamma + \rho_b) - \sigma_i \Lambda - \rho_a) = 0.
\]

If otherwise, \( \sigma \) satisfies the condition that

\[
\sigma^{-1}(i) \leq i \quad \text{if } 1 \leq i \text{ and } i \leq l,
\]
\[
\sigma^{-1}(i) \geq i \quad \text{if } l+1 \leq i \text{ and } i \leq n.
\]

Hence \( \sigma \) must be the identity. If we put \( \sigma^{-1}(i) = i \) \( (1 \leq i \leq n) \) in (8.10), we have clearly

\[
Q((\gamma + \rho_b) - \sigma_i \Lambda - \rho_a) = 1.
\]

After all, by Theorem 1.4.1, the multiplicity of \( \tau_\gamma \) in a representation in the class \( D_{\sigma_i \Lambda} \) is equal to 1. Therefore \( \Phi_i(\Lambda) \) is contained in the \( K \)-spectrum of \( D_{\sigma_i \Lambda} \).

Now, by Proposition 5.2.2, the \( K \)-spectrum of \( D_{\sigma_i \Lambda} \) is one of \( \{ \Phi_{l'}(\Lambda) \quad (0 \leq l' \leq \min(\Lambda, n)), \Phi_{p,q}(\Lambda) \quad (0 \leq p < q \leq n) \} \). Each \( \Phi_{p,q}(\Lambda) \) does not contain \( \Phi_l(\Lambda) \) and, if \( l' \neq l \), \( \Phi_{l'}(\Lambda) \) does not also contain \( \Phi_l(\Lambda) \). Hence, \( \Phi_i(\Lambda) \) is necessarily the \( K \)-spectrum of \( D_{\sigma_i \Lambda} \). The proposition is proved.
From Proposition 8.4.1, 8.5.1 and 8.5.3, we obtain the following corollary and complete Step 1 for \( \Lambda \in \mathfrak{B}_0^+ \).

**Corollary 8.5.4.** For \( \Lambda \in \mathfrak{B}_0^+ \), the set \( \hat{G}_{\Lambda} - \hat{G}_d \) is given as follows:

1. If \( \Lambda_i - \Lambda_{i+1} \geq 2 \) for all \( 0 \leq i \leq n-1 \), we have \( \hat{G}_{\Lambda} - \hat{G}_d = \emptyset \).
2. If \( \Lambda_i - \Lambda_{i+1} = 1 \) for some \( 0 \leq i \leq n-1 \), put
   
   \[
   \begin{align*}
   p_\Lambda &= \min \{ i | 0 \leq i \leq n-1, \Lambda_i - \Lambda_{i+1} = 1 \} \\
   q_\Lambda &= \max \{ i | 1 \leq i \leq n, \Lambda_{i-1} - \Lambda_i = 1 \}.
   \end{align*}
   \]

   The we have \( \hat{G}_{\Lambda} - \hat{G}_d = \{ U_{p,\Lambda}(\Lambda) | p_\Lambda \leq p < q_\Lambda \} \).
3. In particular, if \( \Lambda = \rho \), we have \( \hat{G}_{\Lambda} - \hat{G}_d = \{ U_{p,\rho}(\rho) | 0 \leq p < q \leq n \} \) where \( U_{p,\rho}(\rho) \) is the class 1-1.

8.6. Let us go forward Step 2. Once and for all, we fix \( \Lambda \in \mathfrak{B}_0^+ \) and an integer \( l \) such that \( 0 \leq l \leq n \). We shall study Step 2 for \( \sigma_\Lambda \in \mathfrak{B}_0^+ \). Our purpose is to compute

\[
\dim \text{Hom}_K(\tau_{\sigma_\Lambda - \rho_\Lambda} \otimes L^+, \pi_{p,\Lambda}(\Lambda) | k)
\]

for all \( 0 \leq p < q \leq n \).

Take an ordering in \( \mathfrak{h}_K^+ \) as in the proof of Proposition 8.5.3 and fix. From (8.8) and Lemma 6.2.1, the set of weights of \( L^+ \) (resp. \( L^- \)) is given by

\[
(8.11) \quad \left\{ \frac{1}{2} \sum_{i=1}^{n} \varepsilon_i (e_0 - e_i) \mid \|_L \right. \\
\left. \left( \prod_{i=1}^{n} \varepsilon_i \right)^\cdot(-1)^i = +1 \text{ (resp. } -1) \right\}.
\]

and the multiplicity of each weight is equal to 1.

The following lemma is proved by the same arguments as in Lemma 7.5.1.

**Lemma 8.6.1.** Let \( \beta \) and \( \gamma \) be in \( \mathfrak{b}_0^+ \).

1. Let \( \sigma \in W_\gamma \). If \( \beta + \rho_\Lambda - \sigma (\gamma + \rho_\Lambda) \) is a weight of \( L_\gamma \), \( \sigma \) is the identity.
2. The multiplicity of \( \tau_\beta \) in \( \tau_\gamma \otimes L \) is equal to \( M_L(\beta - \gamma) \), where \( M_L(\lambda) \) \((\lambda \in \mathfrak{h}_K^+)\) is the multiplicity of the weight \( \lambda \) in \( L \).

Using this lemma, we can obtain the following proposition and complete Step 2 for \( \sigma_\Lambda \) \((0 \leq l \leq n)\).

**Proposition 8.6.2.** Let \( l, p \) and \( q \) be integers such that \( 0 \leq l \leq n \) and \( 0 \leq p < q \leq n \).

1. If \( p \leq l \leq q \), we have
   
   \[
   \dim \text{Hom}_K(\tau_{\sigma_\Lambda - \rho_\Lambda} \otimes L^+, \pi_{p,\Lambda}(\Lambda) | k) = \begin{cases} 1 & \text{if } p+q \text{ is even,} \\ 0 & \text{if } p+q \text{ is odd,} \end{cases}
   \]
\[
\dim \text{Hom}_k(\tau_{\sigma_i \Lambda - \rho_k} \otimes L^-, \pi_{\rho, \varepsilon}(\Lambda) |_k) = \begin{cases} 
0 & \text{if } p+q \text{ is even}, \\
1 & \text{if } p+q \text{ is odd}, 
\end{cases}
\]

(2). If \( l < p \) or \( q < l \), we have
\[
\dim \text{Hom}_k(\tau_{\sigma_i \Lambda - \rho_k} \otimes L^\pm, \pi_{\rho, \varepsilon}(\Lambda) |_k) = 0.
\]

Proof. Let \( \tau_\gamma \in \hat{K} \). Assume that \( \tau_\gamma \in \Phi_{\rho, s}(\Lambda) \) and the multiplicity of \( \tau_\gamma \) in \( \tau_{\sigma_i \Lambda - \rho_k} \otimes L \) is not equal to 0. By (2) of Lemma 8.6.1, we have \( M_\delta(\gamma - \sigma_i \Lambda + \rho_k) \neq 0 \). Hence, by (8.11), we have
\[
\gamma - \sigma_i \Lambda + \rho_k = \frac{1}{2} \sum_{i=1}^n \varepsilon_i \varepsilon_i - e_i
\]
for some \( \{\varepsilon_1, \ldots, \varepsilon_n\} \) such that \( \varepsilon_i = 1 \) or \(-1\) (1 \( \leq i \leq n \)); i.e.
\[
\begin{align*}
\gamma_i &= \bar{\lambda}_{i-1} + \frac{1-\varepsilon_i}{2} & \text{if } 1 \leq i \text{ and } i \leq l, \\
\gamma_i &= \bar{\lambda}_i + \frac{1-\varepsilon_i}{2} & \text{if } l+1 \leq i \text{ and } i \leq n.
\end{align*}
\]

(8.12)

On the other hand, \( \{\gamma_1, \ldots, \gamma_n\} \) satisfy the condition (8.7). Hence we have
\[
\begin{align*}
\bar{\lambda}_{i-2} + 1 &\geq \gamma_i \geq \bar{\lambda}_{i-1} + 1 & \text{if } 1 \leq i \text{ and } i \leq p \\
\bar{\lambda}_{i-1} &\geq \gamma_i \geq \bar{\lambda}_i & \text{if } p+1 \leq i \text{ and } i \leq q \\
\bar{\lambda}_{i-1} &\geq \gamma_i \geq \bar{\lambda}_{i+1} - 1 & \text{if } q+1 \leq i \text{ and } i \leq n.
\end{align*}
\]

(8.13)

If \( p > l \), there is an integer \( i_0 \) such that \( l+1 \leq i_0 \leq p \). Then, by (8.12) and (8.13), we have
\[
\bar{\lambda}_{i_0} + \frac{1-\varepsilon_{i_0}}{2} \geq \bar{\lambda}_{i_0-1} + 1
\]
and hence we have \( \bar{\lambda}_{i_0} \leq \bar{\lambda}_{i_0-1} + 1 \). This is inconsistent with the fact that \( \bar{\lambda}_{i_0} \leq \bar{\lambda}_{i_0-1} \). Therefore we have \( p \leq l \). Similarly, if \( q < l \), we have a contradiction. Hence we have also \( q \geq l \). Thus the assertion (2) is proved.

Now, assume that \( p \leq l \leq q \). By (8.12) and (8.13), we have the following inequalities;
\[
\begin{align*}
\bar{\lambda}_{i-2} + 1 &\geq \bar{\lambda}_{i-1} + \frac{1-\varepsilon_i}{2} \geq \bar{\lambda}_{i-1} + 1 & \text{if } 1 \leq i \text{ and } i \leq p, \\
\bar{\lambda}_{i-1} &\geq \bar{\lambda}_{i-1} + \frac{1-\varepsilon_i}{2} \geq \bar{\lambda}_i & \text{if } p+1 \leq i \text{ and } i \leq l, \\
\bar{\lambda}_{i-1} &\geq \bar{\lambda}_i + \frac{1-\varepsilon_i}{2} \geq \bar{\lambda}_i & \text{if } l+1 \leq i \text{ and } i \leq q, \\
\bar{\lambda}_{i-1} - 1 &\geq \bar{\lambda}_i + \frac{1-\varepsilon_i}{2} \geq \bar{\lambda}_{i+1} - 1 & \text{if } q+1 \leq i \text{ and } i \leq n.
\end{align*}
\]
These inequalities imply that

\[
\begin{cases}
\varepsilon_i = -1 & \text{if } 1 \leq i \leq p, \\
\varepsilon_i = 1 & \text{if } p+1 \leq i \leq l, \\
\varepsilon_i = -1 & \text{if } l+1 \leq i \leq q, \\
\varepsilon_i = 1 & \text{if } q+1 \leq i \leq n.
\end{cases}
\]  
(8.14)

After all, by (8.12), we have

\[
\gamma = (\Lambda_0 + 1, \Lambda_1 + 1, \ldots, \Lambda_p + 1, \ldots, \Lambda_{l-1}, \Lambda_{l+1}, \ldots, \Lambda_q, \Lambda_{q+1}, -1, \ldots, \Lambda_{n-1})
\]  
(8.15)

\[
\prod_{i=1}^{n} \varepsilon_i = (-1)^{\varepsilon + q - l}.
\]  
(8.16)

Conversely, if \(\gamma\) is given by (8.15), it is clear that \(\tau_\gamma \in \Phi_{p,q}(\Lambda)\) and we have

\[
\gamma - \sigma l \Lambda + \rho = \frac{1}{2} \sum_{i=1}^{n} \varepsilon_i (e_0 - e_i)
\]

where \(\{\varepsilon_1, \ldots, \varepsilon_n\}\) are given by (8.14). Then, by (2) of Lemma 8.6.1, the multiplicity of \(\tau_\gamma\) in \(\tau_{\sigma l \Lambda - \rho} \otimes L\) is equal to 1. Whether \(\tau_\gamma\) occurs in \(\tau_{\sigma l \Lambda - \rho} \otimes L^+\) or \(\tau_{\sigma l \Lambda - \rho} \otimes L^-\) is determined by the sign of \(\left(\prod_{i=1}^{n} \varepsilon_i\right) \cdot (-1)^l\). By (8.16) we have \(\left(\prod_{i=1}^{n} \varepsilon_i\right) \cdot (-1)^l = (-1)^{\varepsilon + q - l}\). Since \(\tau_\gamma\) occurs in \(\pi_{p,q}(\Lambda)\) with multiplicity 1, the assertion (2) follows. The proposition is proved.

8.7. Now, we have completed Step 1 and Step 2. From the basic formula in Theorem 3.1.1, we shall obtain the main theorem in the case of \(G = SU(1, n)\) \((n \geq 2)\).

Let \(\Gamma\) be a torsion free discrete subgroup of \(G\) such that \(\Gamma \backslash G\) is compact. Then we have the following theorem.

**Theorem 8.7.1.** Let \(\Lambda\) be in \(S^+_0\) and set \(\Lambda = \sum_{i=0}^{n} \Lambda_i e_i\). Let \(l\) be an integer such that \(0 \leq l \leq n\). Then we have the following formulas for the multiplicity \(N_\Gamma(D_{\sigma l,\Lambda})\).

1. If \(\Lambda_l - \Lambda_{l+1} \geq 2\) and \(\Lambda_{l-1} - \Lambda_l \leq 2\), we have

\[
N_\Gamma(D_{\sigma l,\Lambda}) = d(D_{\sigma l,\Lambda}) \cdot \text{vol}(\Gamma \backslash G).
\]

2. If either \(\Lambda_l - \Lambda_{l+1} = 1\) or \(\Lambda_{l-1} - \Lambda_l = 1\), put

\[
p_\Lambda = \text{the minimum of } \{i \mid 0 \leq i \leq n-1, \Lambda_i - \Lambda_{i+1} = 1\}
\]

\[
q_\Lambda = \text{the maximum of } \{i \mid 1 \leq i \leq n, \Lambda_{i-1} - \Lambda_i = 1\},
\]

and define
Then we have
\[ N_T(D_{x,\Lambda}) = d(D_{x,\Lambda}) \text{ vol } (\Gamma \setminus G) + \sum_{(p, r) \in J_{l, \Lambda}} (-1)^{\rho + s - 1} N_T(U_{p, s}(\Lambda)) \cdot \]

(3). In particular, if \( \Lambda_i - \Lambda_{i+1} = 1 \) for all \( 0 \leq i \leq n - 1 \), that is, \( \Lambda = \rho \), we have
\[ N_T(D_{x,\rho}) = d(D_{x,\rho}) \text{ vol } (\Gamma \setminus G) + \sum_{0 \leq p \leq q \leq \rho} (-1)^{\rho + s - 1} N_T(U_{p, s}(\rho)) \]

where \( N_T(U_{0, n}(\rho)) = N_T(1_G) = 1 \).

Proof. All assertions immediately follow from Theorem 3.1.1, Corollary 8.5.4 and Proposition 8.6.2.

Remark 8.7.2.
(1). In the above theorem, the condition on \( \Lambda \) in (1) agree with the condition (i) of Theorem in the introduction of [6] (c.f. [6], p. 176).
(2). In the case that \( n=2 \), our theorem gives some formulas in [15] (c.f. [15], p. 192).
(3). In the same way as the case of \( G=\text{Spin}(1, 2m) \), the set of classes \( \{U_{p, q}(\rho) (0 \leq p < q \leq n), D_{x,\rho} (0 \leq l \leq n)\} \) corresponds with \( \Pi^0(G) = \{J_{l, j} | i, j \geq 0, i+j \leq n-1\} \cup \{D_0, D_1, \ldots, D_n\} \) in [1], Chapter VI, §4, 4.8; that is, \( U_{p, q}(\rho) \) corresponds with \( J_{p, n-q} \) and \( D_{x,\rho} \) with \( D_l \). Also, these classes contribute to Matsushima’s formula about Betti numbers of the manifold \( \Gamma \setminus G / K \) (c.f. [15], p. 175, p. 191).

References


Department of Mathematics
University of Osaka Prefecture
Sakai, Osaka 591
Japan