ON PROJECTIVE MODULES OVER DIRECTLY FINITE REGULAR RINGS SATISFYING THE COMPARABILITY AXIOM

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(Received November 6, 1984)

In [2], J. Kado has studied simple directly finite regular rings satisfying the comparability axiom, and completely determined the directly finiteness of projective modules over these rings. In this paper, without the assumption of simplicity in [2], we shall study projective modules over directly finite regular rings satisfying the comparability axiom. In Theorem 6, we shall give a criterion of the directly finiteness over these rings. Using this criterion, in Theorem 7, we shall show the following result: Let $R$ be a directly finite regular ring satisfying the comparability axiom. If $P$ and $Q$ are directly finite projective $R$-modules, then so is $P \oplus Q$.

Throughout this paper, $R$ is a ring with identity and $R$-modules are unitary right $R$-modules. If $M$ and $N$ are $R$-modules, then the notation $N \leq M$ (resp. $N \leq \oplus M$) means that $N$ is isomorphic to a submodule of $M$ (resp. $N$ is isomorphic to a direct summand of $M$). For a cardinal number $\alpha$ and an $R$-module $M$, $\alpha M$ denotes a direct sum of $\alpha$-copies of $M$.

First we recall some definitions and well-known results (cf. [1]).

**Definition.** A ring $R$ is **directly finite** if $xy=1$ implies $yx=1$, for all $x, y \in R$. An $R$-module $M$ is **directly finite** if $\text{End}_R(M)$ is directly finite. A ring $R$ (a module $M$) is **directly infinite** if it is not directly finite. It is well-known that $M$ is directly finite if and only if $M$ is not isomorphic to a proper direct summand of $M$ itself. A regular ring $R$ is said to **satisfy the comparability axiom** provided that, for any $x, y \in R$, either $xR \leq yR$ or $yR \leq xR$, or equivalently, for any finitely generated projective $R$-modules $P$ and $Q$, either $P \leq Q$ or $Q \leq P$. A ring $R$ is said to be **unit-regular** if, for each $x \in R$, there is a unit (i.e. an invertible element) $u$ of $R$ such that $uxx = x$.

**Lemma 1.** (a) Every directly finite regular ring satisfying the comparability axiom is unit-regular (cf. [1, Theorem 8.12]).

(b) Let $R$ be a unit-regular ring. Then,

(1) Every finitely generated projective $R$-module is directly finite ([1,}
Proposition 5.2]).

(2) Let $B, A_1, A_2, \ldots$ be projective $R$-modules. If each $A_n$ is finitely
generated and $A_1 \oplus \cdots \oplus A_n \leq B$ for all $n$, then $\oplus A_n \leq B$ ([1, Proposition 4.8]).

(3) Let $A$ be a finitely generated projective $R$-module. If $B$ and $C$ are any $R$-modules such that $A \oplus B \cong A \oplus C$, then $B \cong C$ ([1, Theorem 4.14]).

(4) Let $A$, $B$ and $C$ be projective $R$-modules such that $A = B \oplus C$. If $C$ is finitely generated, then $A$ is directly finite if and only if $B$ is directly finite.

An $R$-module $M$ is said to have the exchange property if, for any direct
decomposition $G = M' \oplus C = \oplus_{i \in I} D_i$ with $M' \cong M$ and the index set $I$, there
are submodules $D'_i \leq D_i$ ($i \in I$) such that $G = M' \oplus (\oplus_{i \in I} D'_i)$.

**Lemma 2.** Every projective module over a regular ring has the exchange
property.

Proof. This follows from [4, Theorem 3], [5, Proposition 3] and the
proof of [3, Lemma 1].

Let $R$ be a regular ring, and let $P$ be a countably generated, but not finitely
generated, projective $R$-module which has a cyclic decomposition $P = \oplus_{i \in I} P_i$
satisfying the condition

(*) $P_i \geq P_{i+1}$ for all $i$, and there exists no nonzero $R$-module $X$ such that $X \leq P_i$ for all $i$.

Consider the following conditions on \{$P_i$\}:

(A) There exists a positive integer $m$ such that

1. for each $i \geq m$, $P_i \leq t_i P_{i+1}$ for some positive integer $t_i$, and
2. $\oplus_{i=m}^\infty P_i \leq t P_m$ for some positive integer $t$.

(B) There exists an increasing sequence $1 = i_1 < i_2 < \cdots$, of positive integers
such that $P_{i_n} \geq \mathfrak{m}^n P_{i_{n+1}}$ for $n = 1, 2, \ldots$.

(C) There exists a positive integer $m$ for which the condition (1) of (A)
holds.

**Lemma 3.** Let $R$ be a directly finite regular ring satisfying the comparability
axiom. Then, for a countably generated, but not finitely generated, projective $R$-module $P$ with a cyclic decomposition $P = \oplus_{i \in I} P_i$ satisfying (*), either
(B) or (C) hold, but not both.

Proof. It is clear from Lemma 1 that if $P = \oplus_{i \in I} P_i$ does not satisfy (B),
then $P = \oplus_{i \in I} P_i$ satisfies (C). Assume that $P = \oplus_{i \in I} P_i$ does not satisfy (C). Then there exists an increasing sequence $j_1 < j_2 < \cdots$, of positive integers such
that, for each $n = 1, 2, \ldots$, $P_{i_n} \leq t P_{j_{n+1}}$ for all positive integers $t$, and so $P_{i_n} \geq \mathfrak{m}^{j_{n+1}} P_{i_{n+1}}$ by Lemma 1. Therefore, taking $i_1 = 1$ and $i_n = j_n$ for $n = 2, 3, \ldots$, we see that (B) holds for $P = \oplus_{i \in I} P_i$. 
Proposition 4. Let \( R \) be a directly finite regular ring satisfying the comparability axiom. For a countably generated, but not finitely generated, projective \( R \)-module \( P \) with a cyclic decomposition \( P = \bigoplus i_n P_i \) satisfying (*) and (C), the following are equivalent:

(a) \( P \) is directly finite.

(b) There exists a positive integer \( n > m \) such that \( \bigoplus i_n P_i \leq P_m \).

(c) There exists a positive integer \( t > 1 \) such that \( \bigoplus i_m P_i \leq tP_m \).

Proof. (a) \( \rightarrow \) (b). Assume that (b) does not hold. Then there exists an increasing sequence \( m < m_1 < m_2 < \cdots \), of positive integers such that \( P_{m_i+1} \oplus \cdots \oplus P_{m_i} \geq P_m \) for \( i = 1, 2, \cdots \) by Lemma 1. Thus \( \bigoplus \Gamma P_m \leq \bigoplus P \) and so \( P \) is directly infinite. (b) \( \rightarrow \) (c) is clear. (c) \( \rightarrow \) (a). Assume, to the contrary, that \( P \) is directly infinite. Putting \( Q = \bigoplus i_n P_i \), in view of Lemma 1, we see that \( Q \) is directly infinite; so there exists a nonzero \( R \)-module \( X \) such that \( \mathfrak{n}_0 X \leq Q \). Using (*), we can choose a positive integer \( i > m \) such that \( P_i \leq X \), and hence \( tP_m \leq \mathfrak{n}_0 P_m \leq \mathfrak{n}_0 tP_m \) \( \cdots \) \( t_{i-1} P_i \cong \mathfrak{n}_0 P_i \leq \mathfrak{n}_0 X \leq Q \leq tP_m \). This contradicts the directly finiteness of \( tP_m \).

Remark. Let \( R \) and \( P \) be as in Proposition 4. This \( P \) is not always directly finite in general. Take, for example, a field \( F \) and set \( R_n = M_2(n) \) for all \( n = 1, 2, \cdots \). Map each \( R_n \rightarrow R_{n+1} \) along the diagonal, i.e., map \( x \rightarrow (x \ 0), \) and set \( R = \lim R_n \) (see [1, Example 21.20]). Then \( R \) is a directly finite regular ring satisfying the comparability axiom and we can choose a cyclic projective \( R \)-module \( Q_n = x_n R \) such that \( N(x_n) = 1/2 \) for \( n = 1, 2, \cdots \), where \( N \) is a unique rank function on \( R \). Put \( Q_i = P_i^0 = \cdots = P_{i-1}^0 \) for \( i = 1, 2, \cdots \) and \( P = \bigoplus i_n P_i \). Then, using [2, Lemma 2.5], we see that \( P \cong \mathfrak{n}_0 R \) is a countably generated, but not finitely generated, directly infinite projective \( R \)-module with a cyclic decomposition \( P = \bigoplus i_n P_i \) satisfying (*) and (C).

Proposition 5. Let \( R \) be a directly finite regular ring satisfying the comparability axiom. Then, a countably generated, but not finitely generated, projective \( R \)-module \( P \) with a cyclic decomposition \( P = \bigoplus i_n P_i \), satisfying (*) and (B) is directly finite.

Proof. Putting \( \bar{P}_n = P_n^0 \oplus \cdots \oplus P_{n+1}^0 \) for each \( n \), we have \( P = (\bigoplus i_n P_i^0) \oplus (\bigoplus i_n \bar{P}_n) \). Let \( P = S \oplus T \) with \( P \cong S \) and assume \( T \neq 0 \). We have decompositions

\[ P_n = P_n^0 \oplus P_n^' \] and \[ \bar{P}_n = \bar{P}_n^0 \oplus \bar{P}_n^' \]

such that

\[ S \cong (\bigoplus i_n P_n^0) \oplus (\bigoplus i_n \bar{P}_n^0) \] and \[ T \cong (\bigoplus i_n P_n^0) \oplus (\bigoplus i_n \bar{P}_n^0) \]
for each \( n = 1, 2, \ldots \) (Lemma 2). Noting \( T \neq 0 \), we see by (*) that there exist a finitely generated nonzero submodule \( T' \) of \( T \) and a positive integer \( i_n' \) such that \( P_{i_n'} \leq T' \leq T \), and so that \( P_{i_n'} \leq U \), where \( U = (\bigoplus_{i=1}^{m} P_{i_i}) \oplus (\bigoplus_{i=n+1}^{m} P_{i_i}) \) for some positive integer \( m > n \). Then \( P \cong S \) shows that \( U \leq (\bigoplus_{i=n+1}^{m} P_{i_i}) \oplus (\bigoplus_{i=n+1}^{m} P_{i_i}) \leq P_{i_n'} \leq P \leq U \) by Lemma 1, which contradicts the directly finiteness of \( U \). Therefore \( P \) is directly finite.

We are now in a position to prove the main theorem.

**Theorem 6.** Let \( R \) be a directly finite regular ring satisfying the comparability axiom, and let \( P \) be a projective \( R \)-module. Then, \( P \) is directly finite if and only if

1. \( P \) is finitely generated or
2. \( P \) is a countably generated \( R \)-module with a cyclic decomposition \( P = \bigoplus_{i=1}^{\infty} P_{i_i} \) satisfying (*) and (A), or (*) and (B). If, in addition, \( R \) has the nonzero socle, then a projective \( R \)-module \( P \) is directly finite if and only if \( P \) is finitely generated.

**Proof.** Put \( P = \bigoplus_{i \in I} P_i \), where each \( P_i \) is a nonzero cyclic projective submodule of \( P \). Denote by \( |I| \) the cardinal number of \( I \). If \( |I| < \aleph_0 \), then \( P \) is directly finite by Lemma 1. Assume that \( |I| > \aleph_0 \). Given \( j \in I \), we set \( I_j = \{ i \in I | P_i \leq P_j \} \). Suppose that all \( I_j \) are finite sets. Then we can choose a countable sequence \( \{ i_1, i_2, \ldots \} \) of \( I \) such that \( I_{i_1} \leq I_{i_2} \leq \cdots \) and \( \bigcup_{i=1}^{\infty} I_i = I \). Hence there exists \( i_0 \in I - \bigcup_{i=1}^{\infty} I_i \) such that \( I_{i_0} \neq \bigcup_{i=1}^{\infty} I_i \), which contradicts to the fact that \( I_{i_0} \) is a finite set. Thus there must exist an infinite set \( I_j \) such that \( P_{i_0} \leq \bigoplus_{i \in I_j} P_i \leq P \); so \( P \) is directly infinite. Now suppose that \( |I| = \aleph_0 \). Then, in order to observe that \( P \) is directly finite, we can assume that \( P \) has a cyclic decomposition \( P = \bigoplus_{i=1}^{\infty} P_{i_i} \) with (*). Lemma 3, Propositions 4 and 5 show that \( P \) is directly finite if and only if \( P = \bigoplus_{i=1}^{\infty} P_{i_i} \) satisfies (A) or (B). Thus the first part of the theorem has proved. Next, assume that the socle of \( R \) is nonzero. Then there exists a simple right ideal \( S \) of \( R \) such that \( S \leq P_i \) for all \( i \in I \). This, together with (1) and (2), shows that \( P \) is directly finite if and only if \( P \) is finitely generated.

**Remark 1.** Let \( R \) be a nonzero simple directly finite regular ring satisfying the comparability axiom. Then, every non-finitely generated directly finite projective \( R \)-module \( P \) is a countably generated module with a cyclic decomposition \( P = \bigoplus_{i=1}^{\infty} P_{i_i} \) satisfying (*) and (A), because \( R \) has a strictly positive dimension function ([1, Corollary 16.15]).

**Remark 2.** A directly finite regular ring \( R \) with the comparability axiom is classified into three cases:

1. all non-finitely generated directly finite projective \( R \)-modules \( P \) have a cyclic decomposition \( P = \bigoplus_{i=1}^{\infty} P_{i_i} \) satisfying (*) and (A).
(2) all non-finitely generated directly finite projective $R$-modules $P$ have a cyclic decomposition $P = \bigoplus \mathbb{Z}P_i$ satisfying (\ast) and (B).

(3) all directly finite projective $R$-modules are finitely generated.

Proof. Assume that there exist non-finitely generated directly finite projective $R$-modules $P$ and $Q$ with cyclic decompositions $P = \bigoplus \mathbb{Z}P_i$ and $Q = \bigoplus \mathbb{Z}Q_i$ satisfying the following:

1. $P_i \geq P_{i+1}$ and $Q_i \geq Q_{i+1}$ for each $i$,
2. if $X \leq P_i$ (resp. $X \leq Q_i$) for all $i$, then $X = 0$ (resp. $X = 0$),
3. there exists a positive integer $m$ such that, for each $i = m, m+1, \ldots, P_i \leq t_i P_{i+1}$ for some positive integer $t_i$, and $\bigoplus \mathbb{Z}P_i \leq t P_m$ for some positive integer $t$, and
4. there exists an increasing sequence $1 = i_1 < i_2 < \ldots$, of positive integers such that $Q_{i_n} \geq Q_{i_{n+1}}$ for $n = 1, 2, \ldots$. Then we can choose positive integers $i_n$ and $m > m$ such that $P_m \geq Q_{i_n} \geq P_{m+1}$, and so $t_{m-1} \cdots t_m P_m \geq Q_{i_n} \geq Q_{i_{n+1}}$, which contradicts the directly finiteness of $t_{m-1} \cdots t_m P_m$.

From Lemma 1, Theorem 6 and Remark 2, we have the following result.

**Theorem 7.** Let $R$ be a directly finite regular ring satisfying the comparability axiom. If $P$ and $Q$ are directly finite projective $R$-modules, then so is $P \oplus Q$.

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**References**


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