THE MODULI SPACE OF YANG-MILLS CONNECTIONS OVER A KÄHLER SURFACE IS A COMPLEX MANIFOLD

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1. Introduction

Let $M$ be a compact, connected, oriented Riemannian 4-manifold. Let $P$ be a smooth principal $G$-bundle over $M$. For simplicity we assume that the Lie group $G=SU(n)$, $n^2$. An $SU(n)$-connection $A$ on $P$ is called self-dual (anti-self-dual) if curvature form $F(A)=dA-A\wedge A$ satisfies $*F(A)=\pm F(A)$. Each self-dual (anti-self-dual) connection is characterized as a connection minimizing the Yang-Mills functional $\int_M |F|^2 \ dv$ and then gives a solution to the Yang-Mills equation. That the second Chern class $c_2(P)<0(>0)$ for the adjoint bundle $g$ of $P$ is a topological restriction to $P$ in order to admit a self-dual (anti-self-dual) connection. The moduli space $\mathcal{M}$ of self-dual (anti-self-dual) connections, namely, the orbit space of self-dual (anti-self-dual) connections with respect to the group $G$ of gauge transformations has a structure of smooth manifold ([3], [7]).

A Kahler surface $M$ with a Kahler metric $g$, which is certainly a Riemannian 4-manifold, carries the canonical orientation induced from the complex structure. Relative to this orientation a connection $A$ is anti-self-dual if and only if its curvature is a 2-form of type $(1,1)$ which is primitive (that is, orthogonal to the Kahler form $\omega$). Therefore, by the integrability condition ([3]) each anti-self-dual connection induces a holomorphic structure on the complex adjoint bundle $g^c$. Since gauge-equivalent anti-self-dual connections give holomorphic structures which are isomorphic with respect to automorphisms of $g^c$, we have the canonical mapping from $\mathcal{M}$ to the moduli space of holomorphic structures on $g^c$. Furthermore an anti-self-dual $SU(n)$-connection $A$ naturally defines an Einstein-Hermitian structure on the associated holomorphic vector bundle $E=P\times_{SU(n)}\mathbb{C}^n$. We have also the fact that $E$ is $\omega$-semi-stable in the sense of Mumford and Takemoto ([9]). If $A$ is moreover irreducible, then $E$ is $\omega$-stable. On the other hand, over a nonsingular projective surface the moduli space of holomorphic, rank two vector bundles of fixed Chern classes is a quasi-projective variety ([12]). From these reasons together with an easy observation that the moduli space $\mathcal{M}$
has even dimension (Proposition 2.4), it is natural that $\mathcal{M}$ may possibly be a complex manifold ([1]). The aim of this paper is to show that $\mathcal{M}$ is indeed a complex manifold with singularities by using notion of holomorphic $(0,1)$-connections.

The singularities of $\mathcal{M}$ are described as gauge-equivalent classes $[A]$ of $\mathcal{M}$ either with non-zero 0-th cohomology $H^0$ or with non-zero second cohomology $H^2$ for a certain complex associated to the connection $A$. Denote by $\mathcal{K}$ the subset of $\mathcal{M}$ $\{[A] \in \mathcal{M}; H^0 \neq 0\}$. Then we obtain the following

**Theorem 1.** Let $M$ be a compact Kähler surface with a Kähler metric of positive total scalar curvature or with trivial canonical line bundle $K_M$. Let $P$ be a smooth principal $SU(n)$-bundle with second Chern class $c_2(Q_c) > 0$. If $\mathcal{M} \setminus \mathcal{K}$ is non-empty, then it is a complex manifold of dimension $c_2(Q_c) - (n^2 - 1)p_a(M)$, where $p_a(M)$ is arithmetic genus of $M$.

We denote by $H$ the space $H^0(M; \mathcal{O}(Q_c \otimes K_M))$ relative to the holomorphic structure on $Q_c$ induced from an anti-self-dual connection $A$. Theorem 1 is a direct consequence of the following theorem.

**Theorem 2.** Let $M$ be a compact Kähler surface, $P$ a smooth principal $SU(n)$-bundle with $c_2(Q_c) > 0$. If $(\mathcal{M} \setminus \mathcal{K})_0 = \{[A] \in \mathcal{M} \setminus \mathcal{K}; H = 0\}$ is non-empty, then it is a complex manifold of dimension $c_2(Q_c) - (n^2 - 1)p_a(M)$.

These theorems are obtained as follows. We first show in §2 that each $[A] \in (\mathcal{M} \setminus \mathcal{K})_0$ has a neighborhood in the first cohomology $H^1$ defining a local coordinate of $\mathcal{M}$. But such coordinate neighborhoods are not necessarily each other related holomorphically. Therefore we should verify by an indirect method that $(\mathcal{M} \setminus \mathcal{K})_0$ is in fact a complex manifold. For this purpose we define in §3 a holomorphic $(0,1)$-connection on the complexification $P_c$ of $P$. A holomorphic $(0,1)$-connection is a system of local $\mathfrak{sl}(n; \mathbb{C})$-valued $(0,1)$-forms satisfying a transition condition whose curvature form vanishes. In a manner analogous to the case of anti-self-dual $SU(n)$-connections we can define complex gauge transformations, moduli space of holomorphic $(0,1)$-connections and an elliptic complex which is a gauge field version of the Dolbeault complex. We obtain at §4 a canonical mapping $f$ from $\mathcal{M}$ to the moduli space of holomorphic $(0,1)$-connections which is injective and open over $(\mathcal{M} \setminus \mathcal{K})_0$ and then use the Atiyah-Singer index theorem and Kuranishi's integrating method together with the moment map due to Donaldson ([6]) to verify that the open subspace $f((\mathcal{M} \setminus \mathcal{K})_0)$ in the moduli is definitely a complex manifold of dimension $c_2(Q_c) - (n^2 - 1)p_a(M)$ (Proposition 5.1).

Holomorphic $(0,1)$-connections over a complex manifold are inseparably related to holomorphic structures on $g_c$. Then the moduli space of holomorphic connections reflects aspects and properties of the moduli of holomorphic struc-
tures on \( g^C \). See Ch. 2 of [13] and [2] as references for theory of holomorphic structures on a vector bundle over a compact complex manifold.

An announcement of this article is appeared in [8]. With respect to basical references we refer to [3] and [7].

2. Moduli space of anti-self-dual connections

Let \( M \) be a compact Kähler surface with a Kähler metric \( g \). We denote by \( \Lambda^k \) and \( \Lambda^{(p,q)} \) the vector bundles of real \( k \)-forms and of complex \( (p,q) \)-forms on \( M \), respectively. For a real vector bundle \( E \) and a complex vector bundle \( F \) we denote by \( \Omega^k(E) \) and \( \Omega^{(p,q)}(F) \) the space of all smooth \( k \)-forms with values in \( E \) and the space of all smooth \( (p,q) \)-forms with values in \( F \). Let \( P \) be a smooth principal bundle over \( M \) with gauge group \( SU(n) \). We denote by \( G \) and \( g \) the associated bundles \( P \times \text{Ad} SU(n) \) and \( P \times \text{Ad} \mathfrak{su}(n) \), respectively. We call \( g \) the adjoint bundle of \( P \).

Let \( \{ W_a \} \) be an open covering of \( M \) consisting of local trivializing neighborhoods of \( P \).

**Definition 2.1.** A system \( A = \{ A_a \} \) of local smooth \( SU(n) \)-valued 1-forms \( A_a \) defined over \( W_a \) is called an \( SU(n) \)-connection on \( P \), if \( A \) satisfies the cocycle condition;

\[
A_\beta = d g \cdot g^{-1} + g \cdot A_a \cdot g^{-1}
\]

(2.1)
on \( W_a \cap W_\beta \), where \( g = g_{a \beta} \) is a transition transition function of \( P \) over \( W_a \cap W_\beta \).

The set \( \mathcal{A} \) of all \( SU(n) \)-connections on \( P \) has an affine structure. That is, \( \mathcal{A} \) is given by \( \{ A + \alpha; \alpha \in \Omega^1(g) \} \) for a fixed \( SU(n) \)-connection \( A \). We call \( SU(n) \)-connection \( A \) irreducible when the covariant derivative \( d_A; \Omega^k(g) \rightarrow \Omega^{k+1}(g) \), \( \psi \mapsto d \psi + [\varphi, A] \) has trivial kernel. An \( SU(n) \)-connection is called reducible if it is not irreducible.

The complex surface \( M \) has the canonical orientation induced from the complex structure. The Hodge star operator \( * \) gives an endomorphism of \( \Lambda^2 \) with property \( ** = id \). We denote by \( \Lambda_+^2 \) and \( \Lambda_-^2 \) the eigenspaces of \( +1 \) and \( -1 \), respectively. The projection from \( \Lambda^2 \) onto \( \Lambda_+^2 \) is denoted by \( \rho_+ \). Over Kähler surface \( M \) we have the following ([7]). A real 2-form \( \alpha \) belongs to \( \Lambda_+^2 \) if and only if \( (1,1) \)-part of \( \alpha \) is proportional to the Kähler form \( \omega \), and a real 2-form \( \beta \) is in \( \Lambda_-^2 \) if and only if \( \beta \) is of type \( (1,1) \) and orthogonal to \( \omega \). A 2-form in \( \Lambda_+^2 \) (or in \( \Lambda_-^2 \)) is called self-dual (or anti-self-dual).

**Definition 2.2.** An \( SU(n) \)-connection \( A \) is called anti-self-dual if the curvature form \( F(A) = dA - A \wedge A \) which belongs to \( \Omega^2(g) \) satisfies \( *F(A) = -F(A) \), namely \( \rho_+ F(A) = 0 \).

The group \( G = \Gamma(M; G) \) of all smooth gauge transformations of \( P \) acts on \( \mathcal{A} \).
as \( g(A)=dg\cdot g^{-1}+g\cdot A\cdot g^{-1}, g\in G, A\in \mathcal{A} \). Let \( Z \) be the center of \( SU(n) \). Each element of \( Z \) defines a gauge transformation which commutes with all \( g \)'s of \( G \). It is easily seen that the center \( Z(G) \) of \( G \) coincides with \( Z \). The center \( Z=Z(G) \) acts trivially on \( \mathcal{A} \). Let \( A \) be an irreducible connection on \( P \). Then the isotropy subgroup \( \Gamma_A=\{g\in G; g(A)=A\} \) is just \( Z \). This fact is observed by the following. The endomorphism bundle \( \text{End}(E) \) of the associated vector bundle \( E=P\times \mathfrak{g} \), which is written as \( \text{End}(E)=P\times \mathfrak{su}(n;\mathbb{C}) \), decomposes into \( \text{End}(E)=1\oplus \mathfrak{g} \) as a \( \mathfrak{su}(n) \)-vector bundle, where \( 1 \) is a one-dimensional trivial bundle. The bundle \( G=P\times \mathfrak{g} \) is considered as a subbundle of \( \text{End}(E) \) with fibers consisting of \( SU(n) \). Then a gauge transformation \( g \) is in \( \Gamma \) if and only if \( g(A)=A=dg+[g,A]\cdot g^{-1}=dAg\cdot g^{-1}=0 \), that is, \( g \) is a parallel section of \( \text{End}(E) \). By the irreducibility of \( A \) \( g \) must be a constant multiple of identity transformation \( 1_E \), hence \( g\in Z \) since \( g \) takes values in \( SU(n) \). As a consequence the quotient group \( \tilde{G}=\tilde{G}/Z \) acts effectively on \( \mathcal{A} \) and freely on the subset of irreducible connections.

Denote by \( \mathcal{B} \) the quotient space \( \mathcal{A}/\tilde{G} \) and by \( \pi \) the projection of \( \mathcal{A} \) onto \( \mathcal{B} \). The equivalence class \( \pi(A) \) is denoted by \([A] \). Since \( F(g(A))=g\cdot F(A)\cdot g^{-1} \), \( g\in \tilde{G} \), \( g(A) \) is anti-self-dual for every anti-self-dual connection \( A \). The subset \( \mathcal{M} \) in \( \mathcal{B} \) given by \{anti-self-dual connections on \( P \})/\tilde{G} \) is called the moduli space of anti-self-dual connections on \( P \).

In order to introduce a local coordinate neighborhood for each \([A] \) of \( \mathcal{M} \) we define suitable topologies on \( \mathcal{B} \). On the spaces \( \Omega^k(g) \) the inner product \( \langle \cdot, \cdot \rangle_M \) is defined by \( \langle \phi, \psi \rangle_M=\int_M \langle \phi, \psi \rangle(x)dv, \langle \phi, \psi \rangle(x)dv=\text{Tr}\{\phi(x)\wedge \ast\psi(x)\}, p\geq 0 \). By using a partition of unity we also define the Sobolev’s norm \( |\cdot|_k \) on \( \Omega^k(g) \) for a positive integer \( k \). In the completion \( L^2_0(\Omega^k(g)) \) of \( \Omega^k(g) \) relative to \( |\cdot|_k \) the subspace \( \Omega^k(g) \) of all smooth sections is dense. Note that norms \( |\cdot|_0 \) and \( |\cdot|_1=\langle \cdot, \cdot \rangle_M^{1/2} \) are equivalent. Now we complete the space \( \mathcal{A} \) and the group \( G \). Namely, let \( \mathcal{A} \) be the space \( \{A_0+\alpha; \alpha\in L^2_0(\Omega^k(g)) \} \) for a fixed smooth connection \( A_0 \) and \( G \) the subset \( \{g\in L^2_0(\Gamma(M; \text{End}(E)); g \) takes values in \( SU(n) \} \). Then \( G \) and hence \( \tilde{G} \) acts on \( \mathcal{A} \) and we get the quotient topology on the space \( \mathcal{B}=\mathcal{A}/\tilde{G} \). In the following we assume that \( k \) is sufficiently large relative to the dimension of the base space \( M \) in order to apply Sobolev’s imbedding theorem.

For a connection \( A \) a subset \( U_A \) of \( \mathcal{A}\{A+\alpha; \alpha\in L^2_0(\Omega^k(g)), d_A^*\alpha=0 \} \) is said to be a slice at \( A \). Here \( d_A^*; \Omega^k(g)\rightarrow \Omega^{k-1}(g) \) is the formal adjoint of \( d_A \) relative to the inner product \( \langle \cdot, \cdot \rangle_M \). The proposition below is a consequence of Proposition 2.1.

Proposition 2.1. Let \( A \) be an irreducible connection. Then there is a positive \( \varepsilon \) such that \( U_{A,\varepsilon}=\{A+\alpha; |\alpha|_k<\varepsilon, d_A^*\alpha=0 \} \subset \mathcal{A} \) is homeomorphic to its image \( \pi(U_{A,\varepsilon}) \) through the restriction of \( \pi \) to \( U_{A,\varepsilon} \) and \( \pi(U_{A,\varepsilon}) \) gives a neighborhood of \([A] \) in \( \mathcal{B} \).
Proof. This proposition is shown in the proof of Theorem 6 in [5]. Then we give here a sketch of the proof. We define a mapping \( S: U_{A,\ast} \times \mathcal{O}/Z \to \mathcal{A} \), \( S(A+\alpha, g) = g(A+\alpha) \). Then \( S \) is smooth relative to the \( L^2 \)-topologies and its derivative at \( \alpha = 0 \) and \( g = \) the identity is given by

\[
DS; \operatorname{Ker} d_A^+ \times \Omega^0(g) \to \Omega^1(g),
\]

\[ (\alpha, \phi) \mapsto \alpha + d_A \phi, \]

which is an isomorphism since \( \operatorname{Ker} d_A = 0 \) and \( \Omega^1(g) = \operatorname{Im} d_A \oplus \operatorname{Ker} d_A^+ \). Then \( S \) gives a local diffeomorphism. Thus for a sufficiently small \( \varepsilon \) there is a neighborhood \( Q \) of \( A \) in \( \mathcal{A} \) which is written as \( S(U_{A,\ast} \times W) \), where \( W \) is a neighborhood in \( \mathcal{G} \). Namely, each \( A_1 \) in \( Q \) has a unique form \( A_1 = g(A+\beta), \beta \in U_{A,\ast}, g \in W \). By the aid of the semi-continuity of \( \dim \operatorname{Ker} d_A \) we can assume here that each connection of \( Q \) is irreducible. The proof is completed if we use the argument given at p. 448, 449 of [3].

Let \( \mathcal{K} \) be the subset of \( \mathcal{B} \) given by \( \{[A] \in \mathcal{B}; A \text{ is reducible}\} \). Since \( F(A) + \alpha = F(A) + d_A \alpha - \alpha \wedge \alpha \), a slice neighborhood \( U_{A,\ast} \) of \( [A] \in \mathcal{M} \setminus \mathcal{K} \) in \( \mathcal{M} \) can be given by an \( \varepsilon \)-neighborhood of a slice

\[
\{[A+\alpha]; |\alpha|_k < \varepsilon, d_A^+ \alpha = 0, d_A^+ \alpha = \alpha \#^{\#} \alpha\},
\]

(2.2)

where \( d_A^+ = p_+ \circ d_A \) and \( \#; \Omega^1(g) \times \Omega^1(g) \to \Omega^2^+(g) = \Gamma(M; \Lambda^2 \otimes g) \) is defined by \( \alpha \#^{\#} \beta = (1/2)p_+ (\alpha \wedge \beta + \beta \wedge \alpha) \).

To analyze more exactly the structure of neighborhoods of the moduli space \( \mathcal{M} \) we need notion of an elliptic complex and also the integrating method due to Kuranishi ([11]).

For any anti-self-dual \( SU(n) \)-connection \( A \) the following sequence presents an elliptic complex ([3, p. 444], [7, Proposition 2.4])

\[
0 \to \Omega^0(g) \xrightarrow{d_A^+} \Omega^1(g) \xrightarrow{d_A^+} \Omega^2(g) \to 0.
\]

(2.3)

If the connection \( A \) is irreducible, then 0-th cohomology group \( H^0_A \) vanishes. With respect to the second cohomology group \( H^2_A \) we have the following two propositions.

**Proposition 2.2.** Let \( A \) be an anti-self-dual connection. Then for each \( \Phi = \Phi^{2,0} + \Phi^{0,2} + \Phi^0 \otimes \omega \in \Omega^2_+(g) \)

\[
|d_A^+ \Phi|_{\tilde{\mathcal{M}}} = (1/2) \left\{ |\nabla_A \Phi^{2,0}|_{\tilde{\mathcal{M}}}^2 + |\nabla_A \Phi^{0,2}|_{\tilde{\mathcal{M}}}^2 + |d_A \Phi^0|_{\tilde{\mathcal{M}}}^2 \right\}
\]

\[ + (1/4) \int_M \operatorname{Scal}(g) \left\{ |\Phi^{2,0}|^2 + |\Phi^{0,2}|^2 \right\} dv .
\]

(2.4)

Here \( \nabla_A \) denotes the covariant derivative with respect to \( A \) together with the
Levi-Civita connection of the metric $g$ and $\text{Scal}(g)$ is the scalar curvature of $g$.

Notice that since each $\Phi$ in $\Omega^1(g)$ takes values in $\mathfrak{su}(n)$, $\Phi$ satisfies the reality condition, that is, $\Phi^0 \in \Omega^0(g)$ and $\Phi^{0,2} = -i(\Phi^{2,0})$.

**Proposition 2.3.** If an $SU(n)$-connection $A$ is anti-self-dual, then the second cohomology $H^2_A$ is $\mathbb{R}$-isomorphic to $H^2_A \oplus H$, where $H$ denotes the space of global holomorphic sections $H^0(M; \mathcal{O}(g^c \otimes K_M))$ with respect to the holomorphic structure $g^c$ on canonically induced from the $A$.

Proof of Proposition 2.2. It suffices to show the following Bochner-Weitzenböck formula with respect to a general connection $A$;

$$
\left| d_A^* \Phi \right|^2 = (1/2) \left( |\nabla_A \Phi^{2,0}|^2 + |\nabla_A \Phi^{0,2}|^2 \right) + |d_A \Phi|^2
+ \frac{1}{4} \int_M \text{Scal}(g) \left( |\Phi^{2,0}|^2 + |\Phi^{0,2}|^2 \right) dv
+ 4 \int_M \text{Re} \left( [\Phi^0, \sqrt{-1} F^{2,0}], \Phi^{2,0} \right) dv
- 2 \int_M \text{Re} \left( [\Phi^0, \sqrt{-1} F^0], \Phi^{2,0} \right) dv
$$

(2.5)

for $\Phi \in \Omega^3(g)$ and $F_+(A) = p_+ F(A) = F^{2,0} + F^{0,2} + F^0 \otimes \omega$.

Since

$$d_A^*(\Phi^{1,0} + \Phi^{0,1}) = \partial_A \Phi^{1,0} + \bar{\partial}_A \Phi^{0,1}
+ (1/2) \left< \bar{\partial}_A \Phi^{1,0} + \partial_A \Phi^{0,1}, \omega \right> \otimes \omega$$

(2.6)

and we have

$$d_A^*(\Phi^{2,0} + \Phi^{0,2}) = \partial_A^* \Phi^{2,0} + \bar{\partial}_A^* \Phi^{0,2},$$

(2.7)

and

$$d_A^*(\Phi^0 \otimes \omega) = \sqrt{-1} (\partial_A \Phi^0 - \bar{\partial}_A \Phi^0),$$

(2.8)

we obtain the following

$$d_A^* d_A^*(\Phi^{2,0} + \Phi^{0,2}) = \partial_A \partial_A^* \Phi^{2,0} + \bar{\partial}_A \bar{\partial}_A^* \Phi^{0,2}
+ (1/2) \left< \partial_A \partial_A^* \Phi^{2,0} + \bar{\partial}_A \bar{\partial}_A^* \Phi^{0,2}, \omega \right> \otimes \omega$$

(2.9)

and

$$d_A^* d_A^*(\Phi^0 \otimes \omega) = \sqrt{-1} \{\partial_A \partial_A \Phi^0 - \bar{\partial}_A \bar{\partial}_A \Phi^0
+ (1/2) \left< \partial_A \partial_A \Phi^0 - \partial_A \bar{\partial}_A \Phi^0, \omega \right> \otimes \omega \}. \quad (2.10)$$

Since $d_A d_A \Phi^0 = [\Phi^0, F(A)]$, (2.10) reduces to

$$d_A^* d_A^*(\Phi^0 \otimes \omega) = \sqrt{-1} \{[\Phi^0, F^{2,0}] - [\Phi^0, F^{0,2}]\}
+ (1/2) \left< \square_A \Phi^0 \right> \otimes \omega. \quad (2.11)$$
Here we denote by $\Box_A$ the rough Laplacian $-\sum g^\sigma \nabla_\sigma \nabla_\tau$. Hence the inner product $\langle d_A^*d_A^*(\Phi^0 \otimes \omega), \Phi \rangle_M$ is given by

$$\langle d_A^*d_A^*(\Phi^0 \otimes \omega), \Phi \rangle_M = \int_M 2 Re \langle [\Phi^0, \sqrt{-1} F^2], \Phi^2, \omega \rangle dv + \langle \Box_A \Phi^0, \Phi^0 \rangle_M. \tag{2.12}$$

On the other hand we have by an argument similar to [7, Lemma 3.3]

$$\partial_A \partial_A^* \Phi^2 = (1/2) \Box_A \Phi^2 + (1/4) \text{Scal}(g) \Phi^2 - (1/2) [\Phi^2, 2\sqrt{-1} F^0]. \tag{2.13}$$

By using the Ricci formula we obtain further

$$\langle \partial_A \partial_A^* \Phi^2, \omega \rangle = \sqrt{-1} \sum g^\sigma (\partial_A \partial_A^* \Phi^2) \mu^\sigma + (\sqrt{-1}/2) \sum g^\sigma g^\tau [\Phi_{\sigma\tau}, F_{\tau \sigma}]. \tag{2.14}$$

Therefore (2.5) is derived from these formulas.

Proof of Proposition 2.3. Since the curvature form $F(A)$ is of type $(1,1)$, the connection $A$ induces a holomorphic structure on the complex adjoint bundle $g^C$. Namely a smooth section $\Phi$ of $g$ satisfies $d_A \Phi = 0$ if and only if $\Phi$ is holomorphic relative to the holomorphic structure. Then the space $\{ \Phi \in \Omega^2(g^C); \partial_A \partial_A^* \Phi = 0 \}$ is isomorphic with the second cohomology $H^2(M; \mathcal{O}(g^C))$ from Theorem 4.1, ch. 3 in [10].

Moreover it is isomorphic with the space $H$ by the aid of Serre's duality theorem and the self-duality of $g^C$ as a vector bundle. In the course of the proof of Proposition 2.2 we can also verify that

$$|\partial_A^* \Phi^0|_M^2 = (1/2) |\nabla_A \Phi^0|^2 + (1/4) \int_M \text{Scal}(g) |\Phi^0|^2 dv \tag{2.15}$$

for $\Phi^0 \in \Omega^0(g^C)$. Thus we have

$$|d_A^* \Phi|^2 = |\partial_A^* \Phi^0|^2 + |\partial_A^* \Phi^2|^2 + |\partial_A^* \Phi^0|^2 + |d_A \Phi|^2 \tag{2.16}$$

from which the proposition follows easily.

REMARK 2.1. If the canonical line bundle $K_M$ is trivial, then $H$ is $C$-isomorphic to $(H^2)^C$. On the other hand, if the metric $g$ is of positive total scalar curvature, i.e., $\int_M \text{Scal}(g) dv > 0$, then $H$ vanishes.

By applying the Atiyah-Singer index theorem to complex (2.4), we have $[\{7\}]h^0 - h^1 + h^2 = -2c_2(g^C) + 2 \text{dim } SU(n) \cdot p_s(M)$, where $p_s(M)$ denotes the arithmetic genus of $M$ and $h^i = \dim_R H^i_A, i = 0, 1, 2$. If both $H^0$ and $H^2$ vanish, then $H^1$ has even dimension.

**Proposition 2.4.** The first cohomology group $H^1_A$ is $R$-isomorphic to the com-
plex vector space $\mathcal{H}^l = \{ \alpha^{(0,1)} \in \Omega^{(0,1)}(g, \mathfrak{g}) \mid \overline{\partial}_A \alpha^{(0,1)} = 0, \overline{\partial}^*_A \alpha^{(0,1)} = 0 \}$.

Proof. Each $\mathfrak{g}$-valued 1-form $\alpha$ splits into

$$\alpha = \alpha^{(1,0)} + \alpha^{(0,1)}, \quad \alpha^{(1,0)} = \sum_\mu \alpha_\mu d\omega^\mu \in \Omega^{(1,0)}(g, \mathfrak{g}),$$

$$\alpha^{(0,1)} = \sum_\mu \alpha_\mu d\bar{\omega}^\mu \in \Omega^{(0,1)}(g, \mathfrak{g}) \quad \text{with} \quad i(\alpha^{(1,0)}) = -\alpha^{(0,1)}.$$

We define a mapping $h: \Omega^l(g) \to \Omega^{(0,1)}(g, \mathfrak{g})$ by assigning $\alpha^{(0,1)}$ to $\alpha$. We show that $h|_{\mathcal{H}}$ gives an isomorphism of $H^1$ to $\mathcal{H}^l$. By an argument given in [7] we see that $d_A^* \alpha = 0$ if and only if

$$\sum g^{\mu \bar{\nu}} \nabla_\mu \alpha_\nu + \sum g^{\mu \bar{\nu}} \nabla_\mu \bar{\alpha}_{\bar{\nu}} = 0$$

(2.17)

and that $d_A \alpha = 0$ if and only if

$$\left\{ \begin{array}{ll}
\partial_{\alpha} \alpha^{(1,0)} = 0, & \overline{\partial}_A \alpha^{(0,1)} = 0, \\
\sum g^{\mu \bar{\nu}} (\nabla_\mu \alpha_\nu - \nabla_\mu \bar{\alpha}_{\bar{\nu}}) = 0.
\end{array} \right.$$  

(2.18)

Hence, if $\alpha$ is in $H^1$, then $\overline{\partial}_A \alpha^{(0,1)} = 0$ and $\overline{\partial}^*_A \alpha^{(0,1)} = -\sum g^{\mu \bar{\nu}} \nabla_\mu \bar{\alpha}_{\bar{\nu}} = 0$. Since $i(\alpha^{(1,0)}) = -\alpha^{(0,1)}$, the inverse implication is easily derived.

Remark 2.2. Proposition 2.4 is also established for a connection which is not necessarily anti-self-dual.

Now we define for each $[A]$ in the moduli space $\mathcal{M} \setminus \mathcal{K}$ a mapping $\Phi = \Phi_A; \Omega^l(g) \to \Omega^l(g)$ by $\Phi(\alpha) = -d_A^* (G_A (\alpha \# \alpha))$ ([2], [4]). Here $G_A$ is the Green operator of the Laplace operator $d_A^* \circ d_A^*$. Relative to the norms $| \cdot |_k$ we have

$$|d_A \alpha|_{k-1} \leq c_k |\alpha|_k,$$

(2.19)

$$|G_A \Psi|_{k+2} \leq c_k |\Psi|_k,$$

(2.20)

and

$$|\alpha \# \beta|_k \leq c_k |\alpha|_k |\beta|_k (2.21)$$

for $\alpha, \beta \in L^k_2(\Omega^l(g))$, $\Psi \in L^k_2(\Omega^l(\mathfrak{g}))$, where $c_k$ is a constant depending only on the manifold $M$ (Ch. 4 of [10], [11]). Therefore the mapping $\Phi_A; L^k_2(\Omega^l(g)) \to L^k_2(\Omega^l(g))$ is differentiable. Suppose that $H^1_A = 0$. Then we have on $\Omega^l(\mathfrak{g})$ $d_A^* \circ d_A^* \circ G_A = \text{id}$. Hence a slice neighborhood $U_A, \beta$, identified with $\mathcal{U}_{(A)}$ of $[A]$ is mapped by the $\Phi$ into $H^l_A$. Since the derivative of $\Phi$ at $\alpha = 0$ is identity, it has an inverse on a sufficiently small neighborhood $U = \{ \beta \in H^l_A \mid \beta|_M \leq \epsilon \}$.

Notice that by using a prior estimates of elliptic differential operators each $\beta$ in $L^k_2(\Omega^l(g))$ satisfying $(d_A d_A^* + d_A^* d_A^*) \beta = 0$ is a smooth section and norms $|\beta|_k$ and $|\beta|_M$ are equivalent.

As a consequence of these propositions we obtain
Proposition 2.5. Let \( M \) be a compact \( K\)ähler surface with a \( K\)ähler metric \( g \) and \( P \) a principal \( SU(n) \)-bundle with \( c_1(g^C) > 0 \). Suppose that either the canonical line bundle \( K_M \) is trivial or the metric is with positive total scalar curvature. Then, if the moduli space \( \mathcal{M} \setminus \mathcal{K} \) of irreducible anti-self-dual connections on \( P \) is not empty, it is a smooth manifold of dimension \( 2c_2(g^C) - 2(n^2 - 1) \cdot \rho_4(M) \).

Remark 2.3. On the subset \( \mathcal{B} \setminus \mathcal{K} = \{ [A] \in \mathcal{B}; A \) is irreducible\( \} \) we define a metric function \( \sigma \) (see for the precise discussion p. 448 in [3]); \( \sigma([A], [A_0]) = \inf_{g \in G} |A - g(A)|_M \). Since \( \sigma \) is continuous relative to the \( L^2 \)-topology, \( \mathcal{B} \setminus \mathcal{K} \) is a Hausdorff space. Therefore the moduli space \( \mathcal{M} \setminus \mathcal{K} \), a closed subset of \( \mathcal{B} \setminus \mathcal{K} \), is also Hausdorff with respect to the relative topology.

3. \((0,1)\)-connections and moduli space of holomorphic \((0,1)\)-connections

We denote by \( P^C \) a smooth principal \( SL(n; \mathbb{C}) \)-bundle given by extending the transition functions of the bundle \( P \) to \( SL(n; \mathbb{C}) \). The complexification \( g^C \) of \( g \) clearly coincides with \( P^C \times Ad \mathfrak{sl}(n; \mathbb{C}) \). Now we define on \( P^C \) a \((0,1)\)-connection and a holomorphic \((0,1)\)-connection as follows.

Definition 3.1. Let \( \{W_a\} \) be the open covering of \( M \) consisting of local trivializing neighborhoods of \( P \). A system \( A = \{A_a\} \), where each \( A_a \) is a smooth \( \mathfrak{sl}(n; \mathbb{C}) \)-valued \((0,1)\)-form defined over \( W_a \), is called a \((0,1)\)-connection on \( P^C \), when it satisfies the cocycle condition

\[
A_\beta = g \cdot g^{-1} + g \cdot A_a \cdot g^{-1}
\] (3.1)

on \( W_a \cap W_\beta \), where \( g = g_{a\beta} \) is the transition function of \( P \).

The set \( \mathcal{A}^{(0,1)} \) of all \((0,1)\)-connections on \( P^C \) has a structure of affine space. The group of complex gauge transformations \( \mathcal{G}^C = \Gamma(M; P^C \times Ad \mathfrak{sl}(n; \mathbb{C})) \) acts on \( \mathcal{A}^{(0,1)} \) in the form

\[
g(A) = g \cdot g^{-1} + g \cdot A \cdot g^{-1},
\] (3.2)

\( g \in \mathcal{G}^C, A \in \mathcal{A}^{(0,1)} \). We denote by \( \mathcal{B}^{(0,1)} \) the quotient space \( \mathcal{A}^{(0,1)} / \mathcal{G}^C \).

Remark 3.1. By its definition, each \((0,1)\)-connection is not a connection by itself. But we have a mapping \( h; \mathcal{A} \to \mathcal{A}^{(0,1)}; A \mapsto A^{(0,1)} \), where \( A^{(0,1)} \) is the \((0,1)\)-component of \( A \). Then \( h \) is one-to-one and onto, because for every \((0,1)\)-connection \( A = \{A_a\} \) on \( P^C \) a system \( \bar{A} = \{\bar{A}_a\} \) given by \( \bar{A}_a = A_a - (\bar{A}_a) \) satisfies (2.1) from (3.1) and it takes values in \( \mathfrak{su}(n) \), and hence it gives an \( \mathfrak{su}(n) \)-connection on \( P \) and \( h(\bar{A}) = A \).

A \((0,1)\)-connection \( A \) is called irreducible, if \( \bar{A}_a; \Omega^C(g^C) \to \Omega^{(0,1)}(g^C); \Psi \mapsto \bar{\Psi} + [\Psi, A] \) has trivial kernel. We call a \((0,1)\)-connection reducible when it is not irreducible.
For each $A \in \mathcal{A}^{(0,1)}$ the curvature form $F(A) = \bar{\partial}A - A \wedge A$ is defined. The curvature form $F(A)$ belongs to $\Omega^{(0,2)}(g^c)$.

**Definition 3.2.** A $(0,1)$-connection $A$ is called holomorphic if $F(A) = 0$.

**Remark 3.2.** Since the curvature form of a $(0,1)$-connection $A$ coincides with the $(0,2)$-component of the curvature form of the $SU(n)$-connection $\bar{A}$ induced from $A$, there exists for each holomorphic $(0,1)$-connection $A$ a holomorphic structure $J = J_A$ on $g^c$ relative to which $\bar{A}$ gives a hermitian holomorphic connection on $g^c$ in the usual sense ([4]). Namely, there exist smooth mappings $h_a; W_a \to SL(n; C)$ with properties that (i) $h_{a\beta}^\gamma = h_a^\gamma g_{a\beta}^\lambda h_\beta^\lambda$; $W_a \cap W_\beta \to SL(n; C)$ is holomorphic for each $\alpha$ and $\beta$ and (ii) $\bar{A}_a$ is transformed into a $(1,0)$-form $h_a(\bar{A}_a) = dh_a \cdot h_a^{-1} + h_{a\alpha} \cdot \bar{A}_a \cdot h_a^{-1}$ by $h_a$.

**Proposition 3.1.** Let $A$ be a holomorphic connection. Then the following sequence gives an elliptic complex:

\[ 0 \to \Omega^p(g^c) \to \Omega^{(0,p)}(g^c) \to \Omega^{(0,2)}(g^c) \to 0 \]  

(3.3)

**Proof.** Since $\bar{\partial}_A \bar{\partial}_A \Psi = [\Psi, F(A)]$ for $\Psi \in \Omega^0(g^c)$, the above sequence gives a complex. It is easily verified that the symbol sequence of the above is exact.

On the spaces $\Omega^{(0,p)}(g^c)$ we define inner products $\langle \cdot, \cdot \rangle_M$ by $\langle \Phi, \Psi \rangle_M = \int_M Tr(\Phi \wedge \ast(\Psi))$, $p=0,1,2$. Notice that these products are not $g^c$-invariant.

We set the subspaces $\mathcal{H}_p = \text{Ker} \Delta^p$ of $\Omega^{(0,p)}(g^c)$ by the aid of the complex Laplacians $\Delta^p, p=0,1,2$ associated to the above complex. Then by using the Atiyah-Singer index theorem we have the index of the complex (3.3) as

\[ h^p - h^1 + h^2 = ch(g^c) \{ ch(\Lambda^{0c}) - ch(\Lambda^{(0,1)}) + ch(\Lambda^{(0,2)}) \} \times e(TM)^{-1} \cdot \mathcal{L}(TM^c) \ [M] \]  

(3.4)

where $h^p = \text{dim}_{g^c} \mathcal{H}_p$. By a simple computation the index equals to $-c_4(g^c) + (n^2 - 1) \cdot p_4(M)$.

Since the group $G^c$ leaves the set of holomorphic $(0,1)$-connections invariant, we obtain its quotient space $\mathcal{H}_h$, called the moduli space of holomorphic $(0,1)$-connections.

The center of $SL(n; C)$ which coincides with the center of $SU(n)$ gives complex gauge transformations commuting with each $g$ of $G^c$. In the same way as the case of $SU(n)$ the center $Z(G^c)$ of $G^c$ is just the center $Z$ and it acts trivially on $\mathcal{A}^{(0,1)}$. Since $G^c$ is a subset of $\Gamma(M; \text{End } E) = \Gamma(M; 1) \oplus \Gamma(M; g^c)$ the isotropy subgroup $\Gamma g^c$ of each irreducible $(0,1)$-connection $A$ reduces to $Z$. Thus the quotient group $G^c = G^c/Z$ acts effectively on $\mathcal{A}^{(0,1)}$ and its action is free on the subset $\{ A \in \mathcal{A}^{(0,1)}; A$ is irreducible$\}$. Besides the inner product $\langle \cdot, \cdot \rangle_M$
we define on $\Omega^{0,\theta}(\mathfrak{g})$ the Sobolev's norms $| \cdot |_k$ and let $\mathcal{A}^{(0,1)}$ be \{ $A_0 + \alpha; \alpha \in L^2_0(\Omega^{0,\theta}(\mathfrak{g}))$ \} for a fixed smooth (0,1)-connection $A_0$. In $L^2_{k+1}$-topology $\mathcal{L}^\mathfrak{g}$ and hence $\mathcal{L}^\mathfrak{g}$ acts smoothly on $\mathcal{A}^{(0,1)}$. The quotient space $\mathcal{B}^{(0,1)} = \mathcal{A}^{(0,1)}/\mathcal{L}^\mathfrak{g}$ gets the canonical quotient topology by the projection $\pi'; \mathcal{A}^{(0,1)} \to \mathcal{B}^{(0,1)}$. We denote by $\mathcal{K}^{(0,1)} \{ [A] \in \mathcal{B}^{(0,1)}; A$ is reducible $\}$, the subset of $\mathcal{B}^{(0,1)}$.

Like an $SU(n)$-connection we call a subset $\mathcal{V}_A$ of $\mathcal{A}^{(0,1)} \{ A + \alpha; \alpha \in L^2_0(\Omega^{0,\theta}(\mathfrak{g})) \}$, $\overline{\partial}_A \alpha = 0$ a slice at $A$.

**Lemma 3.2.** Let $A$ be an irreducible (0,1)-connection on $P^\mathfrak{c}$. Then there exists for a sufficiently small $\varepsilon > 0$ a slice neighborhood $V_{A,\varepsilon} = \{ A + \alpha \in \mathcal{V}_A; |\alpha|_k < \varepsilon \}$ whose image $\pi'(V_{A,\varepsilon})$ gives a neighborhood of $[A]$ in $\mathcal{B}^{(0,1)}$.

**Proof.** Define a mapping $T; V_{A,\varepsilon} \times \mathcal{L}^\mathfrak{g}/Z \to \mathcal{A}^{(0,1)}; T(A + \alpha, g) = g(A + \alpha)$. Then in a manner similar to the case of $SU(n)$-connections, $T$ is smooth relative to the $L^2_\varepsilon$-topologies and its derivative at $\alpha = 0$ and $g =$-identity is written by

$$DT; \ker \overline{\partial}_A \times \Omega^0(\mathfrak{g}) \to \Omega^{0,\theta}(\mathfrak{g})$$

$$(\alpha, \psi) \mapsto \alpha + \overline{\partial}_A \psi .$$

Since $\ker \overline{\partial}_A = 0$ and $\Omega^{0,\theta}(\mathfrak{g}) = \text{Im} \overline{\partial}_A \ker \overline{\partial}_A$ $T$ is a local diffeomorphism. Therefore by using the argument which was used at the proof of Proposition 2.1 we obtain the lemma.

**Proposition 3.3.** Each irreducible $[A] \in \mathcal{M}_k$ has a neighborhood $\mathcal{V}_{[A]}$ which is given by the image of $V_{A,\varepsilon} = \{ A + \alpha \in \mathcal{V}_A; |\alpha|_k < \varepsilon, \overline{\partial}_A \alpha = 0, \overline{\partial}_A \alpha = \alpha \wedge \alpha \}$.

**Proof.** Since $F(A + \alpha) = F(A) + \overline{\partial}_A \alpha - \alpha \wedge \alpha$, this is a direct consequence of the above lemma.

Let $\Psi = \Psi_A$ be a mapping from $L^2_0(\Omega^{0,\theta}(\mathfrak{g}))$ to itself defined by $\Psi(\alpha) = \alpha - (\overline{\partial}_A)(G_\Delta(\alpha \wedge \alpha))$. Here $G_\Delta$ denotes the Green operator of $\Delta^\mathfrak{g}$. Assume now that the second cohomology group $\mathcal{H}^2$ vanishes. Then we see that $\overline{\partial}_A \alpha = 0$ and the slice neighborhood $V_{A,\varepsilon}$ is mapped through $\Psi$ into $\mathcal{H}^1$. Thus the slice neighborhood $V_{A,\varepsilon}$ is mapped through $\Psi$ into $\mathcal{H}^1$. Because over $L^2_0(\Omega^{0,\theta}(\mathfrak{g}))$ the derivative $D\Psi$ at $\alpha = 0$ is identity, $\Psi|_{V_{A,\varepsilon}}$ has an inverse over a small $\varepsilon$-neighborhood $V_\varepsilon$ of $\mathcal{H}^1$. We remark that $\Psi^{-1}\varepsilon_{\mathcal{H}^1}$ is holomorphic as a mapping from an open subset of a Banach space to a Banach space, since $\Psi$ is quadratic over the completed Banach space $L^2_0(\Omega^{0,\theta}(\mathfrak{g}))$ ([11]).

4. Canonical imbedding of $\mathcal{M} \setminus \mathcal{K}$ into $\mathcal{M} \setminus \mathcal{K}^{(0,1)}$

Let $A$ be an $SU(n)$-connection on the bundle $P$. Then the (0,1)-component $A^{(0,1)}$ of $A$ certainly defines a (0,1)-connection on the complexified bundle $P^\mathfrak{c}$ and the curvature $F(A^{(0,1)})$ is given by the (0,2)-component of $F(A)$. If $A$
is anti-self-dual, then \( F(A) \) is of type \((1,1)\), and hence \( A^{(0,1)} \) is holomorphic. Because \( \mathcal{G} \subset \mathcal{G}_c \), to each \([A] \) of \( \mathcal{M} \) we can assign \([A^{(0,1)}] \) of \( \mathcal{M}_h \). We denote this assignment by \( f \).

**Proposition 4.1.** If an anti-self-dual connection \( A \) is irreducible, then \( A^{(0,1)} \) is also irreducible.

**Proof.** Since \( A \) is anti-self-dual we have the formula \( \sum g^{\mu \nu} F_{\mu \nu}(A) = 0 \) ([7, Proposition 2.2]). Then we obtain for a nonzero \( \psi \) of \( \Omega^0(g^0) \) satisfying \( \partial_A \psi = 0 \) that

\[
\sum g^{\mu \nu} \nabla_\mu \nabla_\nu Tr(\psi \cdot \psi) = \sum g^{\mu \nu} Tr(\nabla_\mu \psi \cdot \nabla_\nu \psi) = 0.
\]

We integrate this over \( M \) to get \( \partial_A \psi = 0 \), that is, \( d_A \psi = 0 \). The sections \( \phi \) and \( \phi' \) of the adjoint bundle \( g \) given by \( \phi = \psi - i\bar{\psi} \) and \( \phi' = (1/\sqrt{-1}) (\psi + i\bar{\psi}) \), respectively, are parallel with respect to \( d_A \).

From this proposition we have \( f(\mathcal{M} \setminus \mathcal{K}) \subset \mathcal{M}_h \setminus \mathcal{K}^{(0,1)} \).

Now we show the following

**Proposition 4.2.** The mapping \( f \) restricted to \( \mathcal{M} \setminus \mathcal{K} \) is injective.

**Proof.** It suffices to verify that if there is for irreducible anti-self-dual connections \( A \) and \( A_1 \) \( g \in \mathcal{G} \) satisfying \( (A_1)^{(0,1)} = g(A^{(0,1)}) \), then \( g \) must lie in \( \mathcal{G} \).

By the way \( SU(n; C) \) has the following decomposition; \( SU(n; C) = H^+_0(n) \cdot SU(n) \), where \( H^+_0(n) \) means the set of all positive definite Hermitian matrices with determinant 1. This decomposition is invariant under the adjoint representation of \( SU(n) \), namely, if \( X \in SU(n; C) \) splits into \( X = X^h \cdot X^* \), then \( \psi \in \Omega^0(g^0) \) satisfying \( \partial_A \psi = 0 \) that

\[
\sum g^{\mu \nu} \nabla_\mu \nabla_\nu Tr(\psi \cdot \psi) = \sum g^{\mu \nu} Tr(\nabla_\mu \psi \cdot \nabla_\nu \psi) = 0.
\]

We integrate this over \( M \) to get \( \partial_A \psi = 0 \), that is, \( d_A \psi = 0 \). The sections \( \phi \) and \( \phi' \) of the adjoint bundle \( g \) given by \( \phi = \psi - i\bar{\psi} \) and \( \phi' = (1/\sqrt{-1}) (\psi + i\bar{\psi}) \), respectively, are parallel with respect to \( d_A \).

From this proposition we have \( f(\mathcal{M} \setminus \mathcal{K}) \subset \mathcal{M}_h \setminus \mathcal{K}^{(0,1)} \).

Now we apply the method of moment map developed at [6, p. 11]. Define for \( \{A_i\} \) a function \( m; R \rightarrow R \) by
\[ m(t) = \int_M R_4(t) \wedge \omega, \] (4.2)

where \( R_4(t) \) is a 2-form of type \((1,1)\) over \( M \) modulo \( \text{Im} \partial + \text{Im} \bar{\partial} \) satisfying
\[ \sqrt{-1} \partial \bar{\partial} R_4(t) = -Tr F_i \wedge F_i - (-Tr F_0 \wedge F_0). \] (4.3)

Then we have the following facts (Proposition 8 of [6]). Since \( A_0 \) is anti-self-dual, \( d/dt|_{t=0} m(t) = 0 \) and
\[ d^2/dt^2 m(t) = |d_A \psi|^2 \geq 0. \] (4.4)

Because \( m(t) \) is critical at also \( t=1 \), \( d^2/dt^2 m(t) = 0 \) identically, hence \( d_A \psi = 0 \).

Using the irreducibility of \( A_0 \) we have \( \psi = 0 \) and hence \( g_1 = \text{identity} \), that is, \( g \in \mathcal{G} \).

We define open subsets \((\mathfrak{H} \setminus J^0)_0 \) and \((\mathfrak{P} \setminus L^0)_0 \) of \( \mathfrak{H} \setminus J^0 \) and \( \mathfrak{P} \setminus L^0 \), respectively, by \((\mathfrak{H} \setminus J^0)_0 = \{ [A] \in \mathfrak{H} \setminus J^0; H_A = 0 \} \) and \((\mathfrak{P} \setminus L^0)_0 = \{ [A] \in \mathfrak{P} \setminus L^0; \mathcal{A}_\ell = 0 \} \). Since from Proposition 2.3 \( \mathfrak{H}^2(0,1) \approx H_A \) for the \((0,1)\)-component \( A^{(0,1)} \) of an anti-self-dual connection \( A \) we have \( f((\mathfrak{H} \setminus J^0)_0) \subset (\mathfrak{H} \setminus J^0)_0 \).

**Proposition 4.3.** \( f((\mathfrak{H} \setminus J^0)_0; (\mathfrak{H} \setminus J^0)_0 \rightarrow (\mathfrak{H} \setminus J^0)_0 \) is an open mapping.

**Proof.** Let \( U_{[A]} \) be a neighborhood of \([A] \in (\mathfrak{H} \setminus J^0)_0 \) identified with a slice neighborhood \( U_{A,\alpha} = \{ A + \alpha; |\alpha| < \varepsilon, d_A^\ast \alpha = 0, d_A^\ast \alpha = \alpha \} \). We notice that if \( \alpha \) is such a one-form its \((0,1)\)-component \( \alpha^{(0,1)} \), denoted by \( h(\alpha) \) in §2, satisfies \( \partial_A \alpha^{(0,1)} = \alpha^{(0,1)} \wedge \alpha^{(0,1)} \) but does not necessarily satisfy \( \partial_A^{(0,1)} = 0 \) for \( A = A^{(0,1)} \in \mathfrak{H}^{(0,1)} \). Let \( \mathcal{V}_{[A]} \) be a neighborhood of \([A] \in (\mathfrak{H} \setminus J^0)_0 \), written in the form of the image of a slice neighborhood \( V_{A',\gamma} = \{ A' + \gamma; |\gamma| < \varepsilon', (\bar{\partial}_A) \gamma^{(0,1)} = 0, \bar{\partial}_A \gamma^{(0,1)} = \gamma^{(0,1)} \wedge \gamma^{(0,1)} \} \).

**Assertion.** If we choose a sufficiently small \( \varepsilon \), then for any \( A + \alpha \) in \( U_{A,\alpha} \) there is a unique \( g = g_A \) in \( \mathcal{G}^0 \) close to the identity so that \( g(A' + h(\alpha)) \) belongs to \( V_{A',\gamma} \).

This assertion is shown as follows. Since \( g(A' + h(\alpha)) = (\bar{\partial}_A g) \cdot g^{-1} + g \cdot h(\alpha) \cdot g^{-1} + A', \) the \((0,1)\)-form \( \gamma' \) defined by \( A' + \gamma' = g(A' + h(\alpha)) \) is represented by \( \gamma' = (\bar{\partial}_A g) \cdot g^{-1} + g \cdot h(\alpha) \cdot g^{-1} \). The \((0,1)\)-connection \( A' + \gamma' \) is indeed holomorphic and satisfies \( \bar{\partial}_A \gamma' - \gamma' \wedge \gamma' = 0 \). Then \( \gamma' \) lies in \( V_{A',\gamma} \) if and only if for \( \bar{\partial}_A = \bar{\partial}_A' \)
\[ (\bar{\partial}_A) \{ (\bar{\partial}_A g) \cdot g^{-1} + g \cdot h(\alpha) \cdot g^{-1} \} = 0 \] (4.5)
If we set \( g = \exp \psi, \psi \in \Omega^2(g^0) \), then we reduce (4.5) to
\[ (\bar{\partial}_A g) \bar{\partial}_A \psi + (\bar{\partial}_A h(\alpha) - \partial [\partial_A \psi, h(\alpha)] + [\psi, \bar{\partial}_A h(\alpha)] + \bar{\partial}_A R(\psi, h(\alpha)) = 0, \] (4.6)
here \( R(\psi, h(\alpha)) \) is the remainder term of order not less than two. We operate...
the Green operator \( G_{A'} \) of \( \Delta_0^\alpha \) to (4.6) to deduce
\[
\psi + G_{A'}(\delta_{h}\phi(h(\alpha))) - G_{A'}(\delta_{h}\phi(h(\alpha))) + G_{A'}[\psi, \delta_{h}\phi(h(\alpha))] + G_{A}(\delta_{h}^{\ast}R) = 0. \tag{4.7}
\]
We remark that since \( \alpha = \alpha^{(1,0)} + \alpha^{(0,1)} = \sum (\alpha_{\mu}dz^{\mu} + \alpha_{\bar{\mu}}d\bar{z}^{\bar{\mu}}) \) satisfies \( d_{h}^{\ast}\alpha = 0 \) and \( d_{h}^{\ast}\alpha = \alpha_{\#}\alpha \),
\[
\delta_{h}^{\ast}\phi(h(\alpha)) = -(\sqrt{-1/2})^{\ast} \sum g^{\mu\bar{\nu}}[\alpha_{\mu}, \alpha_{\bar{\nu}}] \tag{4.8}
\]
and hence the \(|.|_{\ast}\)-norm of \( \delta_{h}^{\ast}\phi(h(\alpha)) \) is estimated by \( |\alpha|_{\ast} \).

By using the arguments of Section 3 in Ch. 4 of [10] and also of [3], [11] we obtain for a sufficiently small \( |\alpha|_{\ast} \) a unique smooth solution \( \psi(\alpha) \) to (4.7) in a neighborhood of \( 0 \in \Omega^{0}(\bar{g}^{c}) \). We see easily that \( \psi \) depends smoothly on \( \alpha \) and \( g_{\ast}(A' + h(\alpha)) \in V_{A',\epsilon} \) for \( g_{\ast} = \exp \psi(\alpha) \).

We remark that \( \psi(0) = 0 \) and from an implicit function theorem we have \( (d\psi(\alpha)/d\alpha)|_{\alpha = 0} = 0 \) and hence \( (dg_{\ast}/d\alpha)|_{\alpha = 0} = \text{id} \).

From the above assertion the mapping \( \bar{f} ; U_{A,\ast} \to V_{A',\epsilon} \) defined by \( A + \alpha \mapsto g_{\ast}(A' + h(\alpha)) \) is smooth. We show now that the composition of the following mappings
\[
U_{\ast}((\mathcal{H}(\mathcal{K})_{0}) \to U_{A,\ast} \to V_{A',\epsilon} \to V_{\epsilon}'(\subset \mathcal{K})_{0})
\]
is of maximal rank at \( \beta = 0 \) in \( H_{\lambda}^{\ast} \). Since \( (d\Phi_{A}/d\beta)|_{\beta = 0} \) is the identity mapping of \( H_{\lambda}^{\ast} \) and also \( (d\Phi_{A}/d\beta)|_{\beta = 0} \) gives the identity mapping of \( \mathcal{K}_{A} \), and further \( (d\bar{f}/d\alpha)|_{\alpha = 0}(\gamma) = \lim_{t \to 0} \{ g_{\gamma}(A' + h(t\gamma) - A') \} / t = h(\gamma) \) for each \( \gamma \in H_{\lambda}^{\ast} \), the derivative of the mapping at \( \beta = 0 \) coincides from Proposition 2.4 with \( h; H_{\lambda}^{\ast} \to \mathcal{K}_{A}^{\ast} \).

Because \( h \) is \( \mathcal{R} \)-isomorphic, it gives a local diffeomorphism at \( \alpha = 0 \) and then \( \bar{f} ; U_{A,\ast} \to V_{A',\epsilon} \) is open. Since \( \bar{f} \) is a lift of \( f \) on \( \mathcal{U}(U_{A}) \);

\[
\begin{array}{ccc}
U_{A,\ast} & \xrightarrow{f} & V_{A',\epsilon} \\
\downarrow \pi & & \downarrow \pi' \\
\mathcal{U}(U_{A}) & \xrightarrow{f} & \mathcal{C}U_{A'}(\subset (\mathcal{H}(\mathcal{K})_{0}) \to \mathcal{C}U_{A'}^{\prime}(\subset (\mathcal{H}(\mathcal{K})^{(0,1)})_{0}) \to \mathcal{C}U_{A'}(\subset (\mathcal{H}(\mathcal{K})^{(0,1)})_{0}) \to \mathcal{C}U_{A'}^{\prime}(\subset (\mathcal{H}(\mathcal{K})^{(0,1)})_{0}) \to \mathcal{C}U_{A'}(\subset (\mathcal{H}(\mathcal{K})^{(0,1)})_{0}) \to \mathcal{C}U_{A'}^{\prime}(\subset (\mathcal{H}(\mathcal{K})^{(0,1)})_{0})
\end{array}
\]

\( f \) is also open from the fact that \( \pi ; U_{A,\ast} \to \mathcal{U}(U_{A}) \) is a homeomorphism and \( \pi' ; V_{A',\epsilon} \to \mathcal{C}U_{A'}^{\prime} \) is open.

**Remark 4.1.** (1) The image \( f((\mathcal{H}(\mathcal{K})_{0}) \) is an open subspace in \( (\mathcal{H}(\mathcal{K})_{0}) \), identified with \( (\mathcal{H}(\mathcal{K})_{0}) \). (2) Although \( (\mathcal{H}(\mathcal{K})^{(0,1)})_{0} \) may not necessarily be Hausdorff, \( f((\mathcal{H}(\mathcal{K})^{(0,1)})_{0} \) is surely a Hausdorff space because \( (\mathcal{H}(\mathcal{K})_{0}) \) is Hausdorff from Remark 2.3. (3) Since the mapping \( \bar{f} ; U_{A,\ast} \to V_{A',\epsilon} \) provided in the above proof is locally diffeomorphic, we can choose sufficiently small \( \epsilon' \), if necessary, so that \( \pi'|_{V_{A',\epsilon}} \) gives a homeomorphism of \( V_{A',\epsilon} \) onto a neighborhood \( \mathcal{C}U_{A'}(\subset (\mathcal{H}(\mathcal{K})^{(0,1)})_{0}) \)
5. Complex structure of the moduli space

The aim of this section is to prove the following.

**Proposition 5.1.** The moduli space $f((\mathcal{M} \setminus \mathcal{K})_0)$ is a complex manifold of dimension $c_2(g^c) - (n^2 - 1)p_c(M)$, if it is not empty.

Proof. By Propositions 4.2 and 4.3 and also from (3) of Remark 4.1 we can assume that for each $[A] \in f((\mathcal{M} \setminus \mathcal{K})_0)$ and for a sufficiently small $V_A = V_{A,s}$ that the mapping $\Psi_A; V_A \rightarrow V_s = \{ \beta \in \mathcal{H}_1; |\beta|_M < \varepsilon \}$ defines a coordinate system for $f((\mathcal{M} \setminus \mathcal{K})_0)$.

Fix points $[A]$ and $[A']$ in $f((\mathcal{M} \setminus \mathcal{K})_0)$ with $\pi'(V_A) \cap \pi'(V_{A'}) = \phi$. We define subsets $B \subset V_A$ and $B' \subset V_{A'}$ by $B = \{ A + \alpha \in V_A; \pi'(A + \alpha) \in \pi'(V_{A'}) \}$ and $B' = \{ A' + \alpha' \in V_{A'}; \pi'(A' + \alpha') \in \pi'(V_A) \}$, respectively. Then for each $A + \alpha$ in $B$ there is a $g$ in $\mathcal{G}^c$ with $g(A + \alpha) \in B'$. Since the isotropy subgroup $\Gamma_1^c$ is finite, we can choose such a $g = g_0$ uniquely in $\mathcal{G}^c$ for $A + \alpha$.

Let $\{ \beta_1, \ldots, \beta_m \}$ and $\{ \beta'_1, \ldots, \beta'_m \}$ be orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_1'$, respectively, where $m$ is the dimension of $\mathcal{H}_1$, which is by assumption independent of $A$. Because $\Psi_A; V_s \rightarrow V_A$ is holomorphic, for $\beta(t) = \sum \beta_v t_v \in \mathcal{H}_1$, $t = (t_1, \ldots, t_m) \in \mathcal{C}^m$ with $||t|| < \varepsilon$ we have $\alpha(t) = \Psi^1_x(\beta(t))$ is holomorphic in $t$. Therefore, if we can show that $g_t = g_{s(t)}$ is holomorphic in $t$, then the composition of the mappings

$$
\Psi_A(B) (\subset V_s) \xrightarrow{\Psi_A^{-1}} B(\subset V_A) \xrightarrow{\text{the action of } g_t} B'(\subset V_{A'}) \xrightarrow{\Psi_A'} \Psi_A'(B')(\subset V_{s'})
$$

is also holomorphic in $t$, since $\Psi_{A'}(\alpha')$ is the harmonic part of $\alpha', \sum_{\beta'_v} <\alpha', \beta'_v>_M$.

We now verify the following assertion.

**Assertion.** The complex gauge transformations $g_t$ depend holomorphically on $t$.

It suffices for this purpose to prove that for any fixed $A + \alpha(t_0) \in B$ $g_t$ is holomorphic with respect to $A + \alpha(t)$ close to $A + \alpha(t_0)$. We set $\gamma(z) = \alpha(t_0 + z) - \alpha(t_0)$ and $h_z = g_{t_0 + z} \cdot (g_{t_0})^{-1}$. Then $\gamma(0) = 0$ and $h_0 = \text{id}$. If we define $\alpha'_t$ and $\sigma(z)$ in $\Omega^{0,1}(g^c)$ respectively by $A' + \alpha'_t = g_{t_0}(A + \alpha(t_0))$ and $\sigma(z) = g_z \cdot \gamma(z) \cdot (g_{t_0})^{-1}$, then for $t = t_0 \in \mathcal{G}^c$ $g_t(A + \alpha(t)) = (h_z \cdot g_{t_0})(A + \alpha(t_0) + \gamma(t))$ is written by

$$
g_t(A + \alpha(t)) = A' + \alpha'_t + (\bar{\partial}(\alpha'_t + \sigma), h_z) \cdot (h_z)^{-1} + h_z \cdot \sigma(z) \cdot (h_z)^{-1} . \quad (5.1)
$$

Since $h_z$ is close to $\text{id}$ in $\mathcal{G}^c$, there exists a unique $\psi(\zeta) \in \Omega^0(g^c)$ with $\psi(0) = 0$.
and $h_z = \exp \psi(z)$. Then (5.1) reduces to

$$g_t(A+\alpha(t)) = \bar{\partial}_{A''} \psi + A'' + \sigma(x) + R(\psi, \sigma)$$

(5.2)

for $A'' = A' + \alpha'$, where the remainder term $R(\psi, \sigma)$ is given by

$$R(\psi, \sigma) = (\bar{\partial}_{A''} \exp \psi) \cdot \exp(-\psi) - \bar{\partial}_{A''} \psi + \exp \psi \cdot \exp(-\psi) - \sigma.$$  

(5.3)

Notice that the remainder term indeed including $\bar{\partial}_{A''} \psi$ and $\sigma$ as linear terms can be represented more exactly by

$$R(\psi, \sigma) = (1/2) [\psi, \bar{\partial}_{A'} \psi] + [\psi, \sigma] + R_1(\psi, \bar{\partial}_{A'} \psi) + R_2(\psi, \sigma),$$

(5.4)

where $R_1$ and $R_2$ are written as matrix-power series of order not less than 3 with respect to $\psi$ and $\sigma$.

Since $\bar{\partial}_{A'} \alpha' = 0$, we see that $(\bar{\partial}_{A'})(g_t(A+\alpha(t)) - A') = 0$, namely $g_t(A+\alpha(t)) - A'$ belongs to the slice, if and only if from (5.2)

$$(\bar{\partial}_{A'}^k \bar{\partial}_{A''} \psi + (\bar{\partial}_{A'}^k \sigma + (\bar{\partial}_{A'}^k) R(\psi, \sigma) = 0.$$  

(5.5)

Because $G_{A''}. \Delta_{A''} \sigma = 0$ on $\Omega^2(g^c)$, the above reduces to

$$\psi + G_{A'}. \langle [\bar{\partial}_{A''} \psi, \alpha'] \rangle + G_{A'}(\bar{\partial}_{A'}^k \sigma) + G_{A''}(\bar{\partial}_{A'}^k) R(\psi, \sigma) = 0,$$

(5.6)

here $\bar{\partial}_{A''} \psi$ is the $(1,0)$-component of $d_{A''} \psi$ with respect to the $SU(n)$-connection $A''$ induced canonically from $A''$. Then by using the way quite similar to one to solve (4.7) we have a solution $\psi=\psi(z)$ to (5.6) depending smoothly on $z$. We operate on (5.6) $\bar{\partial}_z$ relative to the parameter $z$ to obtain

$$\bar{\partial}_z \psi + G_{A'}. \langle [\bar{\partial}_{A''} \psi, \alpha'] \rangle + G_{A'}(\bar{\partial}_{A'}^k \sigma) + G_{A''}(\bar{\partial}_{A'}^k) \bar{\partial}_z R(\psi, \sigma) = 0,$$

(5.7)

since $\bar{\partial}_z \sigma(z) = 0$ and $\bar{\partial}_z$ commutes with $G_{A''}$ and with $d_{A''}$. The term $\bar{\partial}_z R(\psi, \sigma)$ is obviously linear with respect to $\bar{\partial}_z \psi$. Define a linear operator $L = L_{\alpha_0}$ by $L(\Theta) = \Theta + G_{A'}. \langle [\bar{\partial}_{A''} \psi, \alpha'] \rangle$, $\Theta \in L_{A'+\alpha}^2(\Omega^2(g^c))$. Then $L$ satisfies

$$(1-c|\alpha'_{k+2}|^2) |\Theta|_{k+2} \leq |L(\Theta)|_{k+2} \leq (1+c|\alpha'_{k+1}|^2) |\Theta|_{k+2}$$

(5.8)

for a constant $c > 0$, independent of $\alpha'$. For each $\alpha'_{k+1}$ in a sufficiently small slice $V_{A'}$, $L_{\alpha_0}$ gives a bounded linear operator from (5.8). On the other hand by the remark on $R(\psi, \sigma)$ the norm $|\bar{\partial}_z R(\psi, \sigma)|_{k+1}$ is estimated by

$$|\bar{\partial}_z R(\psi, \sigma)|_{k+1} \leq c_1 |\bar{\partial}_z \psi|_{k+1} \{ |\sigma|_{k+1} T_1(|\psi|_{k+1}) + |\psi|_{k+2} T_2(|\psi|_{k+1}) \}$$

(5.9)

for some constant $c_1$, where $T_1(s)$ and $T_2(s)$ are power series of $s$ with convergence radius $\infty$.

Since $|\sigma(z)|_{k+1}$ is sufficiently small for small $|z|$, we can let $|\psi(z)|_{k+2}$ be also sufficiently small from (5.5). Thus by the aid of the lower estimate of $L |\bar{\partial}_z \psi|_{k+2} \leq c_2 |\bar{\partial}_z \psi|_{k+1} \leq c_2 |\bar{\partial}_z \psi|_{k+2}$, where $c_2 < 1$ for sufficiently small $|z|$,
therefore (5.7) admits only a trivial solution $\xi \psi = 0$, that is, $\psi = \psi(z)$ and consequently $g_t = (\exp \psi(z)) \cdot g_{t_0}$, $t = t_0 + z$, is holomorphic.

Proposition 5.1 follows from this assertion since $\dim_c \mathcal{H}^1 = c_2(G) - (n^2 - 1) \cdot p_a(M)$.

The proof of Theorem 2 is now completed if we pull back to $f(H \setminus K)$ the complex structure of $f(H \setminus K)_0$ through the $f$. Theorem 1 is a direct consequence of Theorem 2 from Remark 2.1 because $H^2_\mathcal{A} \cong H^2_\mathcal{B} \oplus H$ vanishes for every irreducible anti-self-dual connection $A$ over a Kähler surface $M$ which either admits a Kähler metric of positive total scalar curvature or is endowed with trivial canonical line bundle.

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References


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