REGULARITY IN TIME OF THE SOLUTION OF PARABOLIC
INITIAL-BOUNDARY VALUE
PROBLEM IN L¹ SPACE

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1. Introduction

This paper is concerned with the regularity in t of the solution of the initial-boundary value problem of the linear parabolic partial differential equation

\begin{align*}
(1.1) & \quad \partial u(x, t)/\partial t + A(x, t, D)u(x, t) = f(x, t), \quad \Omega \times (0, T], \\
(1.2) & \quad B_j(x, t, D)u(x, t) = 0, \quad j = 1, \ldots, m/2, \quad \partial \Omega \times (0, T], \\
(1.3) & \quad u(x, 0) = u_0(x), \quad \Omega.
\end{align*}

Here \( \Omega \) is a not necessarily bounded domain in \( \mathbb{R}^N \) with boundary \( \partial \Omega \) satisfying a certain smoothness hypothesis. For each \( t \in [0, T] \) \( A(x, t, D) \) is a strongly elliptic linear differential operator of order \( m \), and \( \{B_j(x, t, D)\}_{j=1}^{m/2} \) is a normal set of linear differential operators of respective orders \( m_j < m \). It is assumed that the realization \( -A_p(t) \) of \( -A(x, t, D) \) in \( L^p(\Omega) \) under the boundary conditions \( B_j(x, t, D)u|_{\partial \Omega} = 0, \quad j = 1, \ldots, m/2 \), generates an analytic semigroup in \( L^p(\Omega) \) for any \( p \in (1, \infty) \). A sufficient condition for that, which is also necessary when \( p = 2 \), is given in S. Agmon [1]. Assuming moreover that the coefficients of \( A(x, t, D), \{B_j(x, t, D)\}_{j=1}^{m/2} \) and some of their derivatives in \( x \) belong to Gevrey’s class \( \{M_k\} \) \( [4], [6], [7] \) as functions of \( t \) and \( f \) also belongs to the same class as a function with values in \( L^p(\Omega) \), we show that the same is true of the solution of (1.1)–(1.3) considered as an evolution equation in \( L^p(\Omega) \) for any initial value \( u_0 \in L^p(\Omega) \). It should be noted here that if \( m_j = m - 1 \), the boundary condition \( B_j(x, t, D)u|_{\partial \Omega} = 0 \) is satisfied only in a variational sense.

In order to prove the result stated above we show that there exist positive constants \( K_0, K \) such that

\begin{equation}
(1.4) \quad ||(\partial/\partial t)^n(A(t) - \lambda)^{-1}|| \leq K_0 K^n M_n / |\lambda|,
\end{equation}

for any \( n = 0, 1, 2, \ldots, t \in [0, T] \) and \( \lambda \) in the sector \( \Sigma : |\arg \lambda| \geq \theta_0, 0 < \theta_0 < \pi/2 \), where \( A(t) \) is the realization of the operator \( A(x, t, D) \) in \( L^p(\Omega) \) under the boundary conditions \( B_j(x, t, D)u|_{\partial \Omega} = 0, \quad j = 1, \ldots, m/2 \). Once (1.4) is established, one
can apply the result of [9] to show that the estimates
\[ ||\frac{\partial}{\partial t} \left( \frac{\partial}{\partial t} + \gamma \frac{\partial}{\partial s} \right)^m U(t, s) || \leq L_0 L^{n+m+k} M_{n+m+k} (t-s)^{-n-k} \]
hold for \( n, m, k = 0, 1, 2, \cdots \) for the evolution operator \( U(t, s) \) to the equation
\[ du(t)/dt + A(t)u(t) = f(t), \quad 0 < t \leq T, \]
where \( L_0, L \) are some positive constants independent of \( n, m, k, t, s \). As for the solution \( u(t) \) of the inhomogeneous equation (1.6) satisfying the initial condition \( u(0) = u_0 \), if \( u_0 \) is an arbitrary element of \( L^p(\Omega) \) and \( f(t) \) is an infinitely differentiable function with values in \( L^q(\Omega) \) such that
\[ ||d^n f(t)/dt^n|| \leq F_0 F^n M_n, \quad 0 \leq t \leq T, \quad n = 0, 1, 2, \cdots, \]
for some constants \( F_0, F \), then we have
\[ ||d^n u(t)/dt^n|| \leq L_0 L^n M_n ||u_0|| t^{-n} + F_0 F^n M_n t^{-n}, \quad 0 < t \leq T \]
for \( n = 0, 1, 2, \cdots \), where \( F_0, F \) are constants depending only on \( d, F_0, F, L_0, L, T \).
Analogous results on the same equation in \( L^p(\Omega) \), \( 1 < p < \infty \), were proved in [9]. It was shown in [8] that the evolution operator \( U(t, s) \) of (1.6) exists if the coefficients of \( A(x, t, D) \), \( \{B_j(x, t, D)\}_{j=1}^m \) and some of their derivatives in \( x \) are once continuously differentiable in \( t \).
In [10] with the aid of the idea of R. Beals [2] and L. Hörmander [5] the estimates of the kernels \( G(x, y, \tau) \), \( K_\lambda(x, y) \) of operators \( \exp(-\tau A_\lambda), (A_\lambda - \lambda)^{-1} \) were established for \( 1 < p < \infty \), where \( A_\lambda = A_\lambda(t) \) for some fixed \( t \in [0, T] \). The operator \( \exp(-\tau A) \) in \( L^p(\Omega) \) was then defined as an integral operator with kernel \( G(x, y, \tau) \), and was shown to be an analytic semigroup with the infinitesimal generator \( -A = -A(t) \).
We use the same method to estimate the derivatives in \( t \) of the kernel of \( (A(t) - \lambda)^{-1} \). In order to make the paper self-contained we reproduce part of the argument of [10] which is relevant to the proof of our main result.

2. Assumptions and main theorem

Let \( \Omega \) be a not necessarily bounded domain of \( \mathbb{R}^n \) locally regular of class \( C^{2m} \) and uniformly regular of class \( C^m \) in the sense of F.E. Browder [3]. The boundary of \( \Omega \) is denoted by \( \partial \Omega \). We put \( D = (\partial/\partial x_1, \cdots, \partial/\partial x_N) \).

Let
\[ A(x, t, D) = \sum_{|\alpha| \leq m} a_\alpha(x, t) D^\alpha \]
be a linear differential operator of even order \( m \) with coefficients defined in \( \Omega \) for each fixed \( t \in [0, T] \), and let
\[ B_j(x, t, D) = \sum_{|\beta| \leq m_j} b_{j\beta}(x, t)D^\beta, \quad j = 1, \ldots, m/2 \]

be a set of linear differential operators of respective orders \( m_j < m \) with coefficients defined on \( \partial \Omega \) for each fixed \( t \in [0, T] \).

The principal parts of \( A(x, t, D) \) and \( B_j(x, t, D) \) are denoted by \( A^\dagger(x, t, D) \) and \( B_j^\dagger(x, t, D) \) respectively.

Let \( \{M_k, k = 0, 1, 2, \ldots \} \) be a sequence of positive numbers which satisfy the following conditions ([4], [6], [7]): for some positive constants \( d_0, d_1, d_2 \)

\begin{align*}
(2.1) & \quad M_{k+1} \leq d_k^2 M_k \quad \text{for all } k \geq 0, \\
(2.2) & \quad \binom{k}{j} M_{k-j} M_j \leq d_k M_k \quad \text{for all } k, j \text{ such that } 0 \leq j \leq k, \\
(2.3) & \quad M_k \leq M_{k+1} \quad \text{for all } k \geq 0, \\
(2.4) & \quad M_{j+k} \leq d_2^{j+k} M_j M_k \quad \text{for all } j, k \geq 0.
\end{align*}

We assume the following:

(A.1) For each \( t \in [0, T] \) \( A(x, t, D) \) is strongly elliptic, i.e. for all real vectors \( \xi \neq 0 \), all \( (x, t) \in \Omega \times [0, T] \)

\[ (-1)^m \Re \ A^\dagger(x, t, \xi) > 0. \]

(A.2) \( \{B_j(x, t, D)\}_{j=1}^m \) is a normal set of boundary operators, i.e. \( \partial \Omega \) is noncharacteristic for each \( B_j(x, t, D) \) and \( m_j = m_k \) for \( j \neq k \).

(A.3) For any \( (x, t) \in \partial \Omega \times [0, T] \) let \( \nu \) be the normal to \( \partial \Omega \) at \( x \) and \( \xi \neq 0 \) be parallel to \( \partial \Omega \) at \( x \). The polynomials in \( \tau \)

\[ B_j^\dagger(x, t, \xi + \tau \nu), \quad j = 1, \ldots, m/2, \]

are linearly independent modulo the polynomial in \( \tau \), \( \prod_{k=1}^{m/2} (\tau - \tau_k^\dagger(\xi, \lambda; x, t)) \) for any complex number \( \lambda \) with non-positive real part where \( \tau_k^\dagger(\xi, \lambda; x, t) \) are the roots with positive imaginary part of the polynomial in \( \tau \), \( (-1)^{m/2} A^\dagger(x, t, \xi + \tau \nu) - \lambda \).

(A.4) For each \( t \in [0, T] \) the formal adjoint

\[ A^\dagger(x, t, D) = \sum_{|\alpha| \leq m} a_\alpha^\dagger(x, t) D^\alpha \]

and the adjoint system of boundary operators

\[ B_j^\dagger(x, t, D) = \sum_{|\beta| \leq m_j} b_{j\beta}^\dagger(x, t) D^\beta, \quad j = 1, \ldots, m/2 \]

can be constructed.

(A.5) For \( |\alpha| = m \) \( a_\alpha(x, t) \) are uniformly continuous in \( \Omega \times [0, T] \). For \( |\alpha| \leq m a_\alpha(x, t), a_\alpha(x, t) \) have continuous derivatives in \( t \) of all orders in \( \Omega \times [0, T] \),
and there exist positive constants $B_q, B$ such that

\[(2.5)\quad |(\partial/\partial t)^k a(x, t)| \leq B_q B^k M_k \quad (x, t) \in \overline{\Omega} \times [0, T] \]

\[(2.6)\quad |(\partial/\partial t)^k a(x, t)| \leq B_0 B^k M_k \quad (x, t) \in \partial \Omega \times [0, T] \]

for $k = 0, 1, 2, \ldots$. For $j = 1, \ldots, m/2$ $D^q b_{j\beta}(x, t)$, \(|\gamma| \leq m - m_j, |\beta| \leq m_j\)$, and $D^q b'_{j\beta}(x, t)$, \(|\gamma| \leq m - m'_j, |\beta| \leq m'_j\)$, have continuous derivatives in $t$ of all orders on $\partial \Omega \times [0, T]$, and

\[(2.7)\quad |(\partial/\partial t)^k D^q b_{j\beta}(x, t)| \leq B_q B^k M_k \quad (x, t) \in \partial \Omega \times [0, T] \]

\[(2.8)\quad |(\partial/\partial t)^k D^q b'_{j\beta}(x, t)| \leq B_q B^k M_k \quad (x, t) \in \partial \Omega \times [0, T] \]

for $k = 0, 1, 2, \ldots$.

Let $W^{m,p}(\Omega)$ be the Banach space consisting of measurable functions defined in $\Omega$ whose distribution derivatives of order up to $m$ belong to $L^p(\Omega)$. The norm of $W^{m,p}(\Omega)$ is defined and denoted by

\[\|u\|_{m,p} = \left( \sum_{|\alpha| \leq m} \int_{\Omega} |D^\alpha u|^p dx \right)^{1/p}.\]

We simply write $\|\|_p$ instead of $\|\|_{0,p}$ to denote $L^p$-norm. We use the notation $\|\|$ to denote both the norm of $L^1(\Omega)$ and that of bounded linear operators from $L^1(\Omega)$ to itself.

For each $t \in [0, T]$ $A(t)$ is the operator defined as follows.

The domain $D(A(t))$ is the totality of functions $u$ satisfying the following three conditions:

(i) $u \in W^{m-1,q}(\Omega)$ for any $q$ with $1 \leq q < N/(N-1)$,

(ii) $A(x, t, D)u \in L^1(\Omega)$ in the sense of distributions,

(iii) for any $p$ with $0 < (N/m)(1-1/p) < 1$ and any $v \in W^{m,p'}(\Omega)$, $p' = p/(p-1)$ satisfying $B'_{j\beta}(x, t, D)v|_{\partial \Omega} = 0$, $j = 1, \ldots, m/2$,

\[(A(x, t, D)u, v) = (u, A'(x, t, D)v).\]

For $u \in D(A(t))$

\[(A(t)u)(x) = A(x, t, D)u(x).\]

We note that the boundary value of $B_j(x, t, D)u$ is defined and vanishes if $m_j < m-1$ for $u \in D(A(t))$.

It is known that $-A(t)$ generates an analytic semigroup in $L^1(\Omega)$. Hence there exist an angle $\theta_0 \in (0, \pi/2)$ and positive constants $C_1, C_2$ such that

\[(2.9)\quad \rho(-A(t)) \supset \Sigma \cap \{\lambda: |\lambda| \geq C_1\},\]

\[(2.10)\quad \|\lambda - A(t)\|^{-1} \leq C_2 / |\lambda| \quad \text{for} \quad \lambda \in \Sigma, |\lambda| \geq C_1,\]

where $\rho(A(t))$ stands for the resolvent set of $A(t)$ and $\Sigma$ is the closed sector.
\{ \lambda: \theta_n \leq \text{arg} \lambda \leq 2\pi - \theta_n \} \cup \{0\}.

We write (1.1)–(1.3) as an evolution equation in \( L^1(\Omega) \):

\begin{equation}
\frac{du(t)}{dt} + A(t)u(t) = f(t), \quad 0 < t \leq T,
\end{equation}

\begin{equation}
u(0) = u_0.
\end{equation}

Let \( U(t, s) \) be the evolution operator of (2.11) which is a bounded operator valued function defined in \( \Delta \) satisfying

\begin{equation}
\frac{\partial U(t, s)}{\partial t} + A(t)U(t, s) = 0,
\end{equation}

\begin{equation}\frac{\partial U(t, s)}{\partial s} - U(t, s)A(s) = 0,
\end{equation}

\begin{equation}U(s, s) = I, \quad 0 \leq s \leq T,
\end{equation}

where \( \Delta = \{(s, t): 0 \leq s < t \leq T\} \) and \( \bar{\Delta} = \{(s, t): 0 \leq s \leq t \leq T\} \). The existence of such an operator is known by [8].

Our main result is the following:

**Theorem.** Under the assumptions stated above the evolution operator \( U(t, s) \) of (2.11) is infinitely differentiable in \((s, t) \in \Delta\). There exist constants \( L_0, L \) such that

\[ ||(\partial/\partial t)^n(\partial/\partial s)^m(\partial/\partial s)^k U(t, s)|| \leq L_0L^nM_{n+m+k}^s(t-s)^{-n-k}, \]

for \( n, m, k = 0, 1, 2, \ldots \).

Let \( u(t) \) be the solution of the initial value problem (2.11), (2.12). If \( u_0 \) is an arbitrary element of \( L^1(\Omega) \) and \( f(t) \) is an infinitely differentiable function with values in \( L^1(\Omega) \) such that

\[ ||d^n f(t)/dt^n|| \leq F_0F^nM_n, \quad 0 \leq t \leq T, \quad n = 0, 1, 2, \ldots, \]

for some constants \( F_0, F \), then we have

\[ ||d^n u(t)/dt^n|| \leq L_0L^nM_n||u_0||t^{-n} + F_0F^nM_n t^{1-n}, \quad 0 \leq t \leq T, \]

for \( n = 0, 1, 2, \ldots \), where \( F_0, F \) are constants depending only on \( d_1, F_0, F, L_0, L, T \).

According to [9] it suffices to prove the following proposition in order to establish the above theorem.

**Proposition.** For any complex number \( \lambda \) such that \( \lambda \in \Sigma \) and \( |\lambda| \geq C_1 \), \((A(t) - \lambda)^{-1}\) is infinitely differentiable in \( t \in [0, T] \), and there exist positive constants \( K_0, K \) such that for \( n = 0, 1, 2, \ldots \)

\begin{equation}
||(\partial/\partial t)^n(A(t) - \lambda)^{-1}|| \leq K_0K^nM_n/|\lambda|.
\end{equation}
3. Preliminaries

For $1 < p < \infty$ the operator $A_p(t)$ is defined as follows:

$$D(A_p(t)) = \{ u \in W^{m,p}(\Omega) : B_j(x, t, D)u = 0 \text{ on } \partial \Omega \text{ for } j = 1, \ldots, m/2 \} ,$$

$$(A_p(t)u)(x) = A(x, t, D)u(x) \quad \text{for } u \in D(A_p(t)) .$$

Replacing $A(x, t, D)$ and $\{B_j(x, t, D)\}_{j=1}^{m/2}$ by $A'(x, t, D)$ and $\{B'_j(x, t, D)\}_{j=1}^{m/2}$ the operator $A'_p(t)$ is defined. According to S. Agmon [1] $A_p(t)$ generates an analytic semigroup in $L^p(\Omega)$, and with the aid of the argument of F.E. Browder [3] it is shown that the relation $A_p^*(t) = A'_p(t)$ holds where the left member stands for the adjoint operator of $A_p(t)$.

In what follows we assume that the coefficients of $B_j(x, t, D)$, $B'_j(x, t, D)$, $j = 1, \ldots, m/2$, are extended to the whole of $\Omega \times [0, T]$ so that (2.7), (2.8) hold there.

Slightly extending the argument of S. Agmon [1] it can be shown that there exist an angle $\theta_0 \in (0, \pi/2)$ and a constant $C_p > 0$ for each $p \in (1, \infty)$ such that for any $u \in W^{m,p}(\Omega)$, a complex number $\lambda$ satisfying $\theta_0 \leq \arg \lambda \leq 2\pi - \theta_0$, $|\lambda| > C_p$, and $t \in [0, T]$

$$\sum_{j=0}^{m/2} |\lambda|^{-m-j/m} |u_j|_{j,p} \leq C_p \left\{ \|(A(x, t, D) - \lambda)u\|_p \right. + \sum_{j=1}^{m/2} |\lambda|^{-m-j/m} \|g_j\|_p + \sum_{j=1}^{m/2} \|g_j\|_{m-j/m,p} \right\},$$

where $g_j$ is an arbitrary function in $W^{m-j/m,p}(\Omega)$ such that $B_j(x, t, D)u = g_j$ on $\partial \Omega$ for each $j = 1, \ldots, m/2$.

For a complex vector $\eta \in C^N$ put

$$A(x, t, D+\eta) = \sum_{|a| \leq m} a_w(x, t)(D+\eta)^a ,$$

$$B_j(x, t, D+\eta) = \sum_{|b| \leq m} b_{j\theta}(x, t)(D+\eta)^b ,$$

(cf. L. Hörmander [5]). As is easily seen the adjoint system of

$$(A(x, t, D+\eta), \{B_j(x, t, D+\eta)\} \) \quad (A'(x, t, D-\eta), \{B'_j(x, t, D-\eta)\}) .$$

For $1 < p < \infty$ $A'_p(t)$, $A''_p(t)$ are the operators defined by

$$D(A'_p(t)) = \{ u \in W^{m,p}(\Omega) : B_j(x, t, D+\eta)u = 0 \text{ on } \partial \Omega \text{ for } j = 1, \ldots, m/2 \} ,$$

$$(A'_p(t)u)(x) = A(x, t, D+\eta)u(x) \quad \text{for } u \in D(A'_p(t)) ,$$

$$D(A''_p(t)) = \{ u \in W^{m,p}(\Omega) : B'_j(x, t, D+\eta)u = 0 \text{ on } \partial \Omega \text{ for } j = 1, \ldots, m/2 \} ,$$

$$(A''_p(t)u)(x) = A'(x, t, D+\eta)u(x) \quad \text{for } u \in D(A''_p(t)) .$$

Lemma 3.1. For any $p \in (1, \infty)$ there exist positive constants $C_p$, $\delta_p$ such that for $\theta_0 \leq \arg \lambda \leq 2\pi - \theta_0$, $|\lambda| > C_p$, $t \in [0, T]$, $|\eta| \leq \delta_p |\lambda|^{1/m}$ the following ine-
qualities hold:

(i) for $u \in W^{m, \rho}(\Omega)$, $g_j \in W^{-m, \rho}(\Omega)$ such that $B_j(x, t, D+\eta)u = g_j$ on $\partial \Omega$, $j = 1, \ldots, m/2$,

$$\sum_{i=0}^{m} |\lambda|^{(m-i)/m} ||u||_{i, \rho} \leq C_{\rho} \{ ||A(x, t, D+\eta) - \lambda||u||_{\rho}$$
$$+ \sum_{j=1}^{m/2} |\lambda|^{(m-m_j)/m} ||g_j||_{\rho} + \sum_{j=1}^{m/2} ||g_j||_{m-m_j, \rho} \} ;$$

(ii) for $v \in W^{-m, \rho}(\Omega)$, $h_j \in W^{-m, \rho}(\Omega)$ such that $B_j(x, t, D+\eta)v = h_j$ on $\partial \Omega$, $j = 1, \ldots, m/2$,

$$\sum_{i=0}^{m} |\lambda|^{(m-i)/m} ||v||_{i, \rho} \leq C_{\rho} \{ ||A'(x, t, D+\eta) - \lambda||v||_{\rho}$$
$$+ \sum_{j=1}^{m/2} |\lambda|^{(m-m_j)/m} ||h_j||_{\rho} + \sum_{j=1}^{m/2} ||h_j||_{m-m_j, \rho} \} .$$

Proof. In the proof of (i) we denote by $C$ constants depending only on $N, m, B_\rho$, the upperbounds of the coefficients of $A(x, t, D)$ and the derivatives in $x$ of the coefficients of $B_j(x, t, D)$ of order up to $m - m_j$, $j = 1, \ldots, m/2$. As is easily seen

$$||A(x, t, D) - \lambda||u||_{\rho}$$
$$\leq ||(A(x, t, D+\eta) - \lambda)u||_{\rho} + ||(A(x, t, D+\eta) - A(x, t, D))u||_{\rho}$$
$$\leq ||(A(x, t, D+\eta) - \lambda)u||_{\rho} + C \sum_{i=0}^{m-1} \frac{\eta}{|\eta|} ||u||_{i, \rho} .$$

If we put

$$g'_j = (B_j(x, t, D) - B_j(x, t, D+\eta))u + g_j ,$$

then $g'_j \in W^{-m, \rho}(\Omega)$ and $g_j = B_j(x, t, D)u$ on $\partial \Omega$, and

$$||g'_j||_{\rho} \leq C \sum_{i=0}^{m_j-1} \frac{\eta}{|\eta|} ||u||_{i, \rho} + ||g_j||_{\rho} ;$$

$$||g'_j||_{m-m_j, \rho} \leq C \sum_{i=m-m_j}^{m_j-1} \frac{\eta}{|\eta|} ||u||_{i, \rho} + ||g_j||_{m-m_j, \rho} .$$

In view of (3.1) and the above inequalities

$$\sum_{i=0}^{m} |\lambda|^{(m-i)/m} ||u||_{i, \rho} \leq C_{\rho} \{ ||A(x, t, D+\eta) - \lambda||u||_{\rho}$$
$$+ C \sum_{i=0}^{m-1} \frac{\eta}{|\eta|} ||u||_{i, \rho} + C \sum_{j=1}^{m/2} |\lambda|^{(m-m_j)/m} \sum_{i=0}^{m_j-1} \frac{\eta}{|\eta|} ||u||_{i, \rho}$$
$$+ \sum_{j=1}^{m/2} |\lambda|^{(m-m_j)/m} ||g_j||_{\rho} + C \sum_{j=1}^{m/2} \sum_{i=m-m_j}^{m_j-1} \frac{\eta}{|\eta|} ||u||_{i, \rho}$$
$$+ \sum_{j=1}^{m/2} ||g_j||_{m-m_j, \rho} \} .$$
If $0 < \delta_p \leq 1$ and $|\eta| \leq \delta_p |\lambda|^{1/m}$ the right member of the above inequality does not exceed

\[
C_p \{ \|(A(x, t, D+\eta) - \lambda)u\|_p + C\delta_p \sum_{i=0}^{m-1} |\lambda|^{(m-i)/m} \|u\|_{i,p} \\
+ C\delta_p \sum_{j=1}^{m/2} |\lambda|^{(m-m_j)/m} \sum_{i=0}^{m_j-1} |\lambda|^{(m-j)/m} \|u\|_{i,p} \\
+ \sum_{j=1}^{m/2} \|g_j\|_{m-m_j,p} \}
\]

Choosing $\delta_p$ sufficiently small we easily complete the proof of (i). The proof of (ii) is similar.

Especially if $u \in D(A^p_\lambda(t))$, $v \in D(A^{*p}_\lambda(t))$ then we can choose $g_j = 0$, $h_j = 0$ in Lemma 3.1. Hence we obtain:

**Corollary.** If $\theta_0 \leq \arg \lambda \leq 2\pi - \theta_0$, $|\lambda| > C_p$, $t \in [0, T]$, $|\eta| \leq \delta_p |\lambda|^{1/m}$, then $\lambda \in \rho(A^p_\lambda(t))$, $\lambda \in \rho(A^{*p}_\lambda(t))$ and the following inequalities hold:

(3.2) $\|(A^p_\lambda(t) - \lambda)^{-1}\|_{B(L^p, L^p)} \leq C_p'/|\lambda|$, 

(3.3) $\|(A^{*p}_\lambda(t) - \lambda)^{-1}\|_{B(L^p, W^{m,p})} \leq C_p'$, 

(3.4) $\|(A^{*p}_\lambda(t) - \lambda)^{-1}\|_{B(L^p, W^{m,p})} \leq C_p'$, 

(3.5) $(A^p_\lambda(t))^* = A^{*p}_\lambda(-\bar{\bar{\mu}}(t))$.

Here and in what follows $B(L^p, L^p)$, $B(L^p, W^{m,p})$ stand for the sets of all bounded linear operators from $L^p(\Omega)$ to $L^p(\Omega)$, $W^{m,p}(\Omega)$ respectively.

**Lemma 3.2.** For any $p \in (1, \infty)$ there exist constants $C_{3,p}$, $C_{4,p}$ such that the following inequalities hold for any $n = 0, 1, 2, \ldots$, $\arg \lambda \in [\theta_0, 2\pi - \theta_0]$, $|\lambda| > C_p$, 

(3.6) $\|(\partial/\partial t)^n(A^p_\lambda(t) - \lambda)^{-1}\|_{B(L^p, L^p)} \leq C_{3,p} C_{4,p}^n M_n / |\lambda|$, 

(3.7) $\|(\partial/\partial t)^n(A^{*p}_\lambda(t) - \lambda)^{-1}\|_{B(L^p, W^{m,p})} \leq C_{3,p} C_{4,p}^n M_n$, 

(3.8) $\|(\partial/\partial t)^n(A^{*p}_\lambda(t) - \lambda)^{-1}\|_{B(L^p, W^{m,p})} \leq C_{3,p} C_{4,p}^n M_n$.

Proof. In the proof of this lemma we use the notation $C$ to denote constants depending only on $m$ and $N$. Letting $f$ be an arbitrary element of $L^p(\Omega)$, we put $u(t) = (A^p_\lambda(t) - \lambda)^{-1}f$. Then
Differentiating both sides of (3.9), (3.10) \( n \) times with respect to \( t \) we get

\[
(A(x, t, D+\eta) - \lambda)u^{(n)}(x, t) = \sum_{k=0}^{n-1} \binom{n}{k} A^{(n-k)}(x, t, D+\eta)u^{(k)}(x, t),
\]

\[
B_j(x, t, D+\eta)u^{(n)}(x, t) = -\sum_{k=0}^{n-1} \binom{n}{k} B_j^{(n-k)}(x, t, D+\eta)u^{(k)}(x, t),
\]

where \( A^{(n-k)} \) and \( B_j^{(n-k)} \) are differential operators obtained by differentiating \( n-k \) times the coefficients of \( A, B_j \) with respect to \( t \) and \( u^{(n)} = (\partial/\partial t)^nu \). Applying Lemma 3.1 we get

\[
\sum_{i=0}^{n} |\lambda|^{(m-i)/m}||u^{(i)}(t)||_{i,\rho} \leq C_\rho \left\{ \sum_{k=0}^{n-1} \binom{n}{k} A^{(n-k)}(x, t, D+\eta)u^{(k)}(t) ||_{\rho} + \sum_{j=1}^{\lfloor n/2 \rfloor} \sum_{k=0}^{n-1} \binom{n}{k} B_j^{(n-k)}(x, t, D+\eta)u^{(k)}(t) ||_{m-m_j, \rho} \right\}.
\]

The first term in the bracket of the right side of (3.11) does not exceed

\[
C \sum_{k=0}^{n-1} \binom{n}{k} B_k B^{n-k}M_{n-k} \sum_{i=0}^{n} |\lambda|^{(m-i)/m}||u^{(k)}(t)||_{i,\rho}
\]

\[
\leq C \sum_{k=0}^{n-1} \binom{n}{k} B_k B^{n-k}M_{n-k} \sum_{i=0}^{n} |\lambda|^{(m-i)/m}||u^{(k)}(t)||_{i,\rho}.
\]

It is easy to show that remaining terms in the bracket of the right side of (3.11) are not larger than the right side of (3.12). Hence

\[
\sum_{i=0}^{n} |\lambda|^{(m-i)/m}||u^{(i)}(t)||_{i,\rho} \leq C_\rho \sum_{k=0}^{n-1} \binom{n}{k} B_k B^{n-k}M_{n-k} \sum_{i=0}^{n} |\lambda|^{(m-i)/m}||u^{(k)}(t)||_{i,\rho}.
\]

Arguing as in [9: p. 542] we show the existence of constants \( C_{3,\rho}, C_{4,\rho} \) such that

\[
||u^{(n)}(t)||_{m,\rho} + |\lambda||u^{(n)}(t)||_{\rho} \leq C_{3,\rho}C_{4,\rho} M_n ||f||_{\rho}
\]

for \( n=0, 1, 2, \ldots \). Hence we have established (3.6), (3.7). The proof of (3.8) is similar.

We choose natural numbers \( l, s \) and exponents \( 2=q_1<q_2<\cdots<q_s<q_{s+1}=\infty \),
2 = r_1 < r_2 < \cdots < r_{l-1} < r_{l+1} = \infty as follows (R. Beals [2]):

(i) in case \( m > N/2 \), \( l = 2 \) and \( s = 1 \), hence \( 2 = q_1 < q_2 = \infty \) and \( 2 = r_1 < r_2 = \infty \);

(ii) in case \( m < N/2 \), \( s > N/2m \), \( s-1 > s > N/2m \), \( q_j-1 - q_{j+1} < m/N \) for \( j = 1, \ldots, s-1 \), \( q_{s-1}^{-1} > m/N > q_s^{-1} \), \( m - N/q_s \) is not a non-negative integer, \( r_j-1 - r_{j+1}^{-1} < m/N \) for \( j = 1, \ldots, l-1 \), \( r_{l-1}^{-1} > m/N > r_{l}^{-1} \), \( m - N/r_{l-1} \) is not a non-negative integer;

(iii) in case \( m = N/2 \), \( l = 4 \), \( s = 2 \), \( 2 = q_1 < q_2 = \infty \), \( 2 = r_1 < r_2 < r_3 = \infty \).

Adding some positive constant to \( A(x, t, D) \) if necessary, we may suppose in view of Lemma 3.2 that for any non-negative integer \( n \), complex number \( \lambda \in \Sigma = \{ \lambda : \arg \lambda \in [\theta_0, 2\pi - \theta_0] \} \cup \{ 0 \} \), complex vector \( \eta \in \mathbb{C}^N \) such that \( |\eta| \leq \delta |\lambda|^{1/n} \) and \( t \in [0, T] \)

\[
\tag{3.13}
\| (\partial/\partial t)^p (A_2^p(t) - \lambda)^{-1} \|_{B(L_t^p, L_t^p)} \leq C_3 C_4^p M_n |\lambda|,
\]

\[
\tag{3.14}
\| (\partial/\partial t)^p (A_2^p(t) - \lambda)^{-1} \|_{B(L_t^p, L_t^{m,n})} \leq C_3 C_4^p M_n
\]

for \( p = q_1, q_2, \ldots, q_s \), and

\[
\tag{3.15}
\| (\partial/\partial t)^p (A_3^p(t) - \lambda)^{-1} \|_{B(L_t^p, L_t^p)} \leq C_3 C_4^p M_n |\lambda|,
\]

\[
\tag{3.16}
\| (\partial/\partial t)^p (A_3^p(t) - \lambda)^{-1} \|_{B(L_t^p, L_t^{m,n})} \leq C_3 C_4^p M_n
\]

for \( p = r_1, r_2, \ldots, r_{l-1} \), where \( C_3, C_4 \) and \( \delta \) are some positive constants.

According to Sobolev's imbedding theorem there exists a positive constant \( \gamma \) such that for \( j = 1, \ldots, s \)

\[
\tag{3.17}
W^{m,s_j}(\Omega) \subset L^{q_j+1}(\Omega) \quad \text{and} \quad \|u\|_{q_j+1} \leq \gamma \|u\|_{m,s_j} \|u\|^{1-s_j}_{q_j}
\]

where \( 0 < a_j = (N/m)(q_j^{-1} - q_{j+1}^{-1}) < 1 \), and for \( j = 1, \ldots, l-1 \)

\[
\tag{3.18}
W^{m,r_j}(\Omega) \subset L^{r_j+1}(\Omega) \quad \text{and} \quad \|u\|_{r_j+1} \leq \gamma \|u\|_{m,r_j} \|u\|^{1-s_j}_{r_j}
\]

where \( 0 < a_{s+j} = (N/m)(r_j^{-1} - r_{j+1}^{-1}) < 1 \).

4. Estimates of the kernel of the derivatives of \( \exp(-\tau A(t)) \) \( (1) \)

In what follows we only consider the case (ii) of the previous section.

For complex numbers \( \lambda_1, \ldots, \lambda_s \in \Sigma \), a complex vector \( \eta \in \mathbb{C}^N \) such that

\[
\tag{4.1}
|\eta| \leq \delta \min \{ |\lambda_1|^{1/m}, \ldots, |\lambda_s|^{1/m} \},
\]

and \( t \in [0, T] \) we put

\[
\tag{4.2}
S(t) = (A_2^p(t) - \lambda_1)^{-1} \cdots (A_2^p(t) - \lambda_s)^{-1},
\]

\[
\tag{4.3}
T(t) = (A_2^p(t) - \lambda_{s+1})^{-1} \cdots (A_2^p(t) - \lambda_l)^{-1}.
\]

In view of (3.17)

\[
R((A_2^p(t) - \lambda_1)^{-1}) \subset W^{m,2}(\Omega) = W^{m,4}(\Omega) \subset L^2(\Omega).
\]
Hence, we may replace \((A_2^j(t) - \lambda_j)^{-1}\) in (4.2) by \((A_2^j(t) - \lambda_j)^{-1}\). Continuing this process we get

\[(4.4) \quad S(t) = (A_2^1(t) - \lambda_1)^{-1} \cdots (A_2^j(t) - \lambda_j)^{-1}(A_2(t) - \lambda_1)^{-1} .\]

By virtue of Sobolev's imbedding theorem we get

\[(4.5) \quad R(S(t)) \subset R((A_2^j(t) - \lambda_j)^{-1}) \subset B^{m-N/q_1} (\Omega) \]

where \(B^{m-N/q_1}(\Omega)\) is the set of all functions which have bounded, continuous derivatives of order up to \([m-N/q_1]\) in \(\Omega\) and have derivatives of order \([m-N/q_1]\) uniformly Holder continuous with exponent \(m-N/q_1-[m-N/q_1]\).

With the aid of (3.13), (3.14), (3.17) we get

\[(4.6) \quad \|\partial/\partial t\|^n (A_2^j(t) - \lambda_j)^{-1} f_{\xi_j} \leq \gamma \|\partial/\partial t\|^n (A_2^j(t) - \lambda_j)^{-1} f_{\xi_j} \leq C_2 \gamma \|\xi_j\|^n f_{\xi_j} ,\]

Using (4.4) and (4.6) for \(n=0\) we obtain

\[(4.7) \quad \|S(t)\|_{B^2 (\Omega)} \leq (\gamma C_2 M_0) \prod_{j=1}^\infty |\lambda_j|^{-s_j} .\]

Similarly we see that

\[(4.8) \quad R(T^\ast(t)) \subset B^{m-N/q_1} (\Omega) ,
\]

\[(4.9) \quad \|T^\ast(t)\|_{B^2 (\Omega)} = \|(A_2^j - \lambda_j)^{-1} \cdots (A_2 - \lambda_1)^{-1}\|_{B^2 (\Omega)} \leq (\gamma C_2 M_0)^{1-s} \prod_{j=1}^\infty |\lambda_j|^{-s_j} .\]

**Lemma 4.1** ([2]). *Let S and T be bounded linear operators in \(L^p(\Omega)\) such that \(R(S) \subset L^\infty (\Omega)\) and \(R(T^\ast) \subset L^\infty (\Omega)\). Then ST has a kernel \(k \in L^\infty (\Omega \times \Omega)\) satisfying

\[(4.10) \quad |K_{\lambda_1, \cdots, \lambda_i}(x, y; t)| \leq (\gamma C_2 M_0)^i \prod_{j=1}^i |\lambda_j|^{-s_j} .\]

If \(\eta\) is pure imaginary, \(e^{\eta} f \in L^p(\Omega)\) if and only if \(f \in L^p(\Omega)\), and hence

\[(A_2^j(t) - \lambda)^{-1} f = e^{-\eta}(A_2^j(t) - \lambda)^{-1}(e^{\eta} f) ,\]
which implies
\[ S(t)T(t)\psi = e^{-\eta}(A_2(t) - \lambda_i)^{-1} \cdots (A_2(t) - \lambda_j)^{-1}(e^{\eta}\psi) \, . \]

Hence, if we denote the kernel of
\[ (A_2(t) - \lambda_i)^{-1} \cdots (A_2(t) - \lambda_j)^{-1} \]
by \( K_{\lambda_i, \ldots, \lambda_j}(x, y; t) \), we have
\[ K_{\lambda_i, \ldots, \lambda_j}(x, y; t) = e^{(x-y)\eta} K_{\lambda_i, \ldots, \lambda_j}(x, y; t) \]
if \( \eta \) is pure imaginary. As is easily seen \( S(t)T(t) \) is a holomorphic function of \( \eta \) in \(|\eta| \leq \delta \min \{ |\lambda_1|^{1/m}, \ldots, |\lambda_v|^{1/m} \} \), and hence so is \( K_{\lambda_i, \ldots, \lambda_j}(x, y; t) \). Thus (4.11) also holds for complex vector \( \eta \). With the aid of (4.10), (4.11) we get when \( \eta \) is real
\[ |K_{\lambda_i, \ldots, \lambda_j}(x, y; t)| \leq (\gamma C_3 M_0) e^{(x-y)\eta} \prod_{j=1}^v |\lambda_j|^{\sigma_j - 1} \, . \]

Minimizing the right side of this inequality with respect to \( \eta \) we obtain
\[ |K_{\lambda_i, \ldots, \lambda_j}(x, y; t)| \leq (\gamma C_3 M_0) e^{(x-y)\eta} \prod_{j=1}^v |\lambda_j|^{\sigma_j - 1} \, . \]

Next, we estimate the derivatives of \( K_{\lambda_i, \ldots, \lambda_j}(x, y; t) \) in \( t \). For that purpose we first estimate the kernel of
\[ (\partial/\partial t)^{s-k} S(t)T(t) = \sum_{k=0}^s \binom{n}{k} (\partial/\partial t)^{s-k} S(t)(\partial/\partial t)^k T(t) \, . \]
Using (4.4), (4.6),
\[ ||(\partial/\partial t)^{s-k} S(t)||_{B(L^2, L^\infty)} \]
\[ = \sum_{k_1 + \cdots + k_s = s} (n-k)! \prod_{i=1}^s (\partial/\partial t)^{k_i} (A_2(t) - \lambda_i)^{-1} \leq (\gamma C_3 C_4)^{s-k} \prod_{j=1}^v |\lambda_j|^{\sigma_j - 1} \, . \]

Noting
\[ (\sigma_1 + \cdots + \sigma_s) M_{k_1} \cdots M_{k_s} \leq d_1^{-1} M_{k_1 + \cdots + k_s} \]
which can be easily shown by induction, we get

\begin{equation}
\|(\partial/\partial t)^{s-k} S(t)\|_B(t^2, L^m) \\
\leq (\gamma C_5)' d_1^{l-1} C_4^{s-k} M_{n-k} \prod_{j=1}^{l} |\lambda_j|^{s_j-1} \\
\leq (\gamma C_5)' d_1^{l-1} (C_6)^{s-k} M_{n-k} \prod_{j=1}^{l} |\lambda_j|^{s_j-1}.
\end{equation}

Similarly

\begin{equation}
\|(\partial/\partial t)^{s-k} T^*(t)\|_B(t^2, L^m) \\
\leq (\gamma C_5)' d_1^{l-1} (C_6(l-s))^k M_{n-k} \sum_{j=1}^{l} |\lambda_j|^{s_j-1}.
\end{equation}

With the aid of (4.10), (4.13), (4.14), (4.15) we get

\[ |(\partial/\partial t)^{n} K_{\lambda_1, \ldots, \lambda_n}(x, y; t)| \]
\[ \leq \sum_{k=0}^{n} \binom{n}{k} (\gamma C_5)' d_1^{l-2} C_4^{s-k} (l-s)^k M_{n-k} M_{n} \prod_{j=1}^{l} |\lambda_j|^{s_j-1} \]
\[ \leq (\gamma C_5)' d_1^{l-1} C_4^s \sum_{k=0}^{s} s^{s-k} (l-s)^k M_{n} \prod_{j=1}^{l} |\lambda_j|^{s_j-1} \]
\[ \leq (\gamma C_5)' d_1^{l-1} C_4^s (n+1) (\max \{ s, l-s \})^n M_{n} \prod_{j=1}^{l} |\lambda_j|^{s_j-1} \]
\[ \leq C_5 C_6 M_{n} \prod_{j=1}^{l} |\lambda_j|^{s_j-1}, \]

where \( C_5 = (\gamma C_5)' d_1^{l-1}, \ C_6 = e C_4 \max \{ s, l-s \}, \) where we used \( n+1 < e^n. \) By the argument through which we derived (4.12) we obtain

\begin{equation}
|((\partial/\partial t)^{n} K_{\lambda_1, \ldots, \lambda_n}(x, y; t)| \]
\[ \leq C_5 C_6^s M_{n} \exp \left\{- \delta \min \{ |\lambda_1|^{1/m}, \ldots, |\lambda_l|^{1/m} \} |x-y| \right\} \prod_{j=1}^{l} |\lambda_j|^{s_j-1} \]
\[ \leq C_5 C_6^s M_{n} \sum_{k=1}^{l} \exp \left\{- \delta \max \{ |\lambda_1|^{1/m}, |x-y| \} \right\} \prod_{j=1}^{l} |\lambda_j|^{s_j-1} .
\end{equation}

5. Estimates of the kernel of the derivatives of \( \exp (-\tau A(t)) \) (2)

We denote the kernel of \( \exp (-\tau A(t)) \) by \( G(x, y; \tau; t) \) which is also the kernel of \( \exp (-\tau A_\rho(t)), 1 < \rho < \infty. \)

Let \( \Gamma \) be a smooth contour running in \( \Sigma \) from \( \infty e^{-i\theta} \) to \( \infty e^{i\theta}. \) Then for \( |\arg \tau| < \pi/2 - \theta, \ t \in [0, T] \)

\[ \exp (-l\tau A_\rho(t)) = \left\{ \frac{1}{2\pi i} \int_{\Gamma} e^{-\lambda t} (A_\rho(t) - \lambda)^{-1} d\lambda \right\}^l \\
= \left\{ \frac{1}{2\pi i} \right\}^l \int_{\Gamma} \cdots \int_{\Gamma} e^{-\lambda_1 t - \cdots - \lambda_l t} (A_\rho(t) - \lambda_1)^{-1} \cdots (A_\rho(t) - \lambda_l)^{-1} d\lambda_1 \cdots d\lambda_l .
\]
Hence

\[(5.1) \quad G(x, y, l\tau; t) = \left(\frac{1}{2\pi i}\right) \int_{\Gamma} \cdots \int_{\Gamma} e^{-\lambda_{1} \tau - \cdots - \lambda_{l} \tau} K_{\lambda_{1}, \ldots, \lambda_{l}}(x, y; t) d\lambda_{1} \cdots d\lambda_{l}.\]

For any fixed \(x, y, \tau\), let \(\Gamma_{x,y,\tau}\) be the contour defined by

\[\{\lambda : |\arg \lambda| = \theta_0, \quad |\lambda| \geq a\} \cup \{\lambda : \lambda = ae^{i\theta}, \quad \theta_0 \leq \theta \leq 2\pi - \theta_0\}\]

where

\[(5.2) \quad a = \varepsilon (|x-y|/|\tau|)^{m/(m-1)} = \varepsilon \rho /|\tau|, \quad \rho = |x-y|^{m/(m-1)} / \tau^{1/(m-1)}\]

and \(\varepsilon\) is a positive constant which will be fixed later. If \(|\arg \lambda| = \theta_0\) and hence \(\lambda = re^{i\theta}, r > 0\), then

\[Re \lambda \tau = Re \lambda Re \tau - Im \lambda Im \tau = r Re \tau (cos \theta_0 - sin \theta_0 (Im \tau / Re \tau)) \geq r Re \tau (cos \theta_0 - sin \theta_0 (|Im \tau| / Re \tau)).\]

Thus if \(\tau\) is in the region

\[(5.3) \quad \frac{|Im \tau|}{Re \tau} \leq (1 - \varepsilon_0) \cos \theta_0 \sin \theta_0\]

for some \(\varepsilon_0, 0 < \varepsilon_0 < 1\), then

\[(5.4) \quad Re \lambda \tau \geq r Re \tau \cdot \varepsilon_0 \cos \theta_0 \geq c_i |\tau|\]

where \(c_i\) is some positive constant depending only on \(\varepsilon_0, \theta_0\). Differentiating both sides of (5.1) \(n\) times with respect to \(t\), deforming the contour \(\Gamma\) to \(\Gamma_{x,y,\tau}\) and using (4.16) we get

\[(5.5) \quad [(\partial / \partial t)^{n} G(x, y, l\tau; t)] \leq \frac{1}{1/2\pi i} \int_{\Gamma_{x,y,\tau}} \cdots \int_{\Gamma_{x,y,\tau}} e^{-\lambda_{1} \tau - \cdots - \lambda_{l} \tau} K_{\lambda_{1}, \ldots, \lambda_{l}}(x, y; t) d\lambda_{1} \cdots d\lambda_{l} | \leq \frac{1}{1/2\pi i} \int_{\Gamma_{x,y,\tau}} \cdots \int_{\Gamma_{x,y,\tau}} e^{-Re \lambda_{1} \tau - \cdots - Re \lambda_{l} \tau} \times C_2 C_3^l M_n \sum_{k=1}^{l} \exp (-\delta |\lambda_k|^{l/(m-1)} |x-y|) \prod_{j=1}^{l} |\lambda_j|^{l/(m-1)} d\lambda_1 \cdots d\lambda_l | \leq \frac{1}{1/2\pi i} \int_{\Gamma_{x,y,\tau}} \cdots \int_{\Gamma_{x,y,\tau}} e^{-Re \lambda_{1} \tau - \cdots - Re \lambda_{l} \tau} \times \exp (-\delta |\lambda_k|^{l/(m-1)} |x-y|) \prod_{j=1}^{l} |\lambda_j|^{l/(m-1)} d\lambda_1 \cdots d\lambda_l |.
The summand with \( k = 1 \) in the last member of (5.5) is

\[
(5.6) \quad \int_{\Gamma_{x, y, t}} e^{-Re \lambda_i \tau} \exp \left( -\delta |\lambda_1|^{1/m} |x-y| \right) |\lambda_1|^{s_i-1} |d\lambda_1|
\times \prod_{j=2}^l \int_{\Gamma_{x, y, t}} e^{-Re \lambda_j \tau} |\lambda_j|^{s_j-1} |d\lambda_j|.
\]

If we write

\[
(5.7) \quad \int_{\Gamma_{x, y, t}} e^{-Re \lambda_i \tau} \exp \left( -\delta |\lambda_1|^{1/m} |x-y| \right) |\lambda_1|^{s_i-1} |d\lambda_1|
= \int_{|\lambda_1|<a} + \int_{|\lambda_1|>a} = I + II,
\]

then

\[
(5.8) \quad I \leq 2\pi a^{s_i} \exp \left( a |\tau| - \delta a^{1/m} |x-y| \right)
= 2\pi (\varepsilon p/|\tau|)^{s_i} \exp \left( \varepsilon p - \delta \varepsilon^{1/m} \rho \right)
\leq C_i |\tau|^{-s_i} \exp \left( 2\varepsilon p - \delta \varepsilon^{1/m} \rho \right)
\]

where \( C_i \) is a positive constant such that

\[
2\pi \sigma^{s_{j}} \leq C_i \sigma^\sigma \quad \text{for} \quad \sigma > 0 , \quad j = 1, \ldots, l,
\]

and in view of (5.4)

\[
II \leq 2 \int_a^\infty \exp \left( -c_i r |\tau| - \delta r^{1/m} |x-y| \right) r^{s_i-1} dr.
\]

Suppose that \( x \neq y \). By the change of the variable \( r = a \sigma \) we get

\[
(5.9) \quad II = 2a^{s_i} \int_1^\infty \sigma^{s_i-1} \exp \left( -c_i \varepsilon p \sigma - \delta \varepsilon^{1/m} \rho \sigma^{1/m} \right) d\sigma
\leq 2a^{s_i} \exp \left( -\varepsilon p \sigma^{1/m} \right) \int_1^\infty \sigma^{s_i-1} \exp \left( -c_i \varepsilon p \sigma \right) d\sigma
= 2(\varepsilon p/|\tau|)^{s_i} \exp \left( -\varepsilon p \sigma^{1/m} \right) \Gamma(a_i) / (c_i \varepsilon p)^{s_i} = \frac{2 \Gamma(a_i)}{(c_i \varepsilon p)^{s_i}} \exp \left( -\varepsilon p \sigma^{1/m} \right).
\]

It is easy to show that (5.9) holds also in case \( x = y \). Combining (5.7), (5.8), (5.9) we get

\[
(5.10) \quad \int_{\Gamma_{x, y, t}} e^{-Re \lambda_i \tau} \exp \left( -\delta |\lambda_1|^{1/m} |x-y| \right) |\lambda_1|^{s_i-1} |d\lambda_1|
\leq C_8 |\tau|^{-s_i} \exp \left( 2\varepsilon p - \delta \varepsilon^{1/m} \rho \right),
\]

where \( C_8 = C_7 + 2 \max \{ \Gamma(a_j) c_i^{s_i}; j = 1, \ldots, l \} \).

For \( j = 2, \ldots, l \)

\[
(5.11) \quad \int_{\Gamma_{x, y, t}} e^{-Re \lambda_j \tau} |\lambda_j|^{s_j-1} |d\lambda_j| = \int_{|\lambda_j|<a} + \int_{|\lambda_j|>a}
\]
\[ e^{2\pi a \gamma^2} \frac{e^{2\pi \tau} + 2\Gamma(a_j)}{\Gamma(\kappa_j)\Gamma(\kappa_j)} \leq C \gamma^{-\zeta} \lambda^2 + 2\Gamma(a_j) \gamma^{\zeta} \lambda^2, \]

where \( C_0 = \max \{2\Gamma(a_j)\gamma^{-\zeta}; j = 1, \ldots, \ell \} \).

Combining (5.10) and (5.11), and noting \( \sum_{j=1}^\ell a_j = N/m \), we see that (5.6) is dominated by

\[ C_0 |\tau|^{-N/m} \exp \{(2\varepsilon - \delta \varepsilon \lambda)\rho\}. \]

Other summands in the last member of (5.5) is analogously estimated, and so we get

\[ |(\partial/\partial t)^*G(x, y, \tau; t)| \leq (2\pi)^{-1/4} C_0 C_5 C_6 M_s |\tau|^{-N/m} \exp \{(2\varepsilon - \delta \varepsilon \lambda)\rho\}. \]

Choosing \( \varepsilon \) so small that \( c_2 = \delta \varepsilon^{-m} - 2\varepsilon > 0 \), and replacing \( \tau \) by \( \tau/l \) we obtain

\[ |(\partial/\partial t)^*G(x, y, \tau; t)| \leq C_1 C_5 C_s M_s |\tau|^{-N/m} \exp \left(-c_2 \frac{|x-y|^m (m-1)}{\tau^{1/(m-1)}} \right) \]

for \( \tau \) in the region (5.3), where \( C_1 = (2\pi)^{-1/4 + \varepsilon \lambda^2} C_5 C_6 \).

6. Estimates of the derivatives of the kernel of \((A(t) - \lambda)^{-1}\)

If we denote the kernel of \((A(t) - \lambda)^{-1}\) by \( K_\lambda(x, y; t) \), then

\[ K_\lambda(x, y; t) = \int_0^\infty e^{\lambda t} G(x, y, \tau; t) d\tau. \]

First let \( \lambda \) be in the region

\[ \{ \lambda : \Re \lambda > 0, \Im \lambda > 0, \Re \lambda / \Im \lambda \leq (1 - \varepsilon_1 \tan \theta_1) \} \cup \{ \lambda : \Re \lambda \leq 0, \Im \lambda > 0 \} \]

where \( \varepsilon_1 \) and \( \theta_1 \) are arbitrary fixed constants such that \( 0 < \theta_1 < \pi/2 - \theta_0, 0 < \varepsilon_1 < 1 \).

Then the integral path in the right side of (6.1) may be altered to the ray \( \tau = re^{i\theta}, 0 < r < \infty \), since \( \Re \lambda \tau \leq -c_2 |\tau| |\lambda| \) for some positive constant \( c_2 \) on the ray. In view of (5.12)

\[ |(\partial/\partial t)^*K_\lambda(x, y; t)| \leq \int_0^\infty e^{\lambda t} (\partial/\partial t)^*G(x, y, \tau; t) d\tau \]

\[ \leq C_1 C_5 C_s M_s \int_0^\infty \exp \left(-c_2 |\tau| |\lambda| \right) |\tau|^{-N/m} \exp \left(-c_2 |x-y|^m (m-1) \tau^{-1/(m-1)} \right) d\tau. \]

First we consider the case \( N > m \). If \( x \neq y \), by the change of variable \( r = |x-y|^m s \)

\[ \int_0^\infty \exp \left(-c_2 |\tau| |\lambda| \right) |\tau|^{-N/m} \exp \left(-c_2 |x-y|^m (m-1) \tau^{-1/(m-1)} \right) d\tau \]

\[ = |x-y|^{m-N} \int_0^\infty s^{-N/m} \exp \left(-c_2 s^{-1/(m-1)} - c_2 |\lambda| \right) |x-y|^m s ds. \]
Putting $h=\left(\lambda \frac{1}{m} |x-y|\right)^{1-m}$ we see that the right side of the above equality does not exceed

$$|x-y|^m \int_0^h s^{-N/m} \exp \left(-c_4 h^{-\frac{1}{m}(m-1)} s \right) ds$$

$$+ |x-y|^m \int_h^\infty s^{-N/m} \exp \left(-c_4 h^{-\frac{m}{m}(m-1)} s \right) ds = I + II.$$  

If $|\lambda|^{1/m} |x-y| < 1$, then

$$\int_0^h s^{-N/m} \exp \left(-c_4 s^{-\frac{1}{m}(m-1)} \right) ds \leq \int_0^\infty s^{-N/m} \exp \left(-c_4 s^{-\frac{1}{m}(m-1)} \right) ds$$

$$= C_{12} \leq C_{15} \exp \left(-\lambda \frac{1}{m} |x-y| \right).$$

Note that $C_{12} < \infty$ since $N/m > 1$. If $|\lambda|^{1/m} |x-y| \geq 1$, then letting $C_{13}$ be a constant such that $\sigma (m-1) N/m \leq C_{13} \exp \left(-\frac{1}{m} \right)$ for any $\sigma > 0$ and noting that $h \leq 1$

$$\int_0^h s^{-N/m} \exp \left(-c_4 s^{-\frac{1}{m}(m-1)} \right) ds \leq C_{13} \int_0^\infty s^{-N/m} \exp \left(-2^{-1} c_4 s^{-\frac{1}{m}(m-1)} \right) ds$$

$$= C_{13} \exp \left(-2^{-1} c_4 |\lambda|^{1/m} |x-y| \right).$$

Thus

$$I \leq C_{14} |x-y|^m \exp \left(-2^{-1} c_4 |\lambda|^{1/m} |x-y| \right)$$

where $C_{14} = \max (C_{12}, C_{13})$. Next,

$$\int_h^\infty s^{-N/m} \left(-c_4 h^{-\frac{m}{m}(m-1)} s \right) ds \leq \exp \left(-c_4 h^{-\frac{1}{m}(m-1)} \right) \int_h^\infty s^{-N/m} ds$$

$$= (m/(N-m)) h^{-\frac{(N-m)}{m}} \exp \left(-c_4 h^{-\frac{1}{m}(m-1)} \right)$$

$$\leq C_{15} \exp \left(-2^{-1} c_4 h^{-\frac{1}{m}(m-1)} \right)$$

where $C_{15}$ is a constant such that

$$(m/(N-m)) \sigma (m-1) (N-m)/m \leq C_{15} \exp \left(2^{-1} c_4 \sigma \right)$$

for any $\sigma > 0$.

Hence

$$II \leq C_{15} |x-y|^m \exp \left(-2^{-1} c_4 |\lambda|^{1/m} |x-y| \right).$$

Combining (6.3), (6.4), (6.5) we obtain

$$|\partial /\partial t|^N K_\lambda (x, y; t) \leq C_{16} C_7 M_\alpha |x-y|^m \exp \left(-c_4 |\lambda|^{1/m} |x-y| \right),$$

in case $N > m$, where $C_{16} = C_{14} + C_{15}$, $c_4 = \min (c_2, c_3)/2$.

Next, we consider the case $N = m$. If $x \neq y$, by the change of variable $r = |x-y|^m$,

$$\int_0^\infty r^{-1} \exp \left(-c_4 r^m |\lambda| \right) \exp \left(-c_2 |x-y|^m r^{-1/2} r^m \right) dr$$
where $h=(|\lambda|^{1/m}|x-y|)^{1-m}$ as before.

If $|\lambda|^{1/m}|x-y|<1$, then putting $b=(|\lambda|^{1/m}|x-y|)^{1-m}=h^{m/(1-m)}>1$ we see that the right side of (6.7) does not exceed

$$
(6.8) \quad \int_0^{\lambda} s^{-1} \exp \left(-c_2 s^{-1/(m-1)}\right) ds + \int_{\lambda}^1 s^{-1} \exp \left(-c_2 b^{-1} s\right) ds
\leq \int_0^1 s^{-1} \exp \left(-c_2 s^{-1/(m-1)}\right) ds + \int_{\lambda}^1 s^{-1} \exp \left(-c_2 s\right) ds
= C_{17} + \log b = C_{17} + m \log \left(|\lambda|^{1/m}|x-y|\right)^{-1}
$$

where $C_{17}$ is a constant defined by the above relation. If $h^{1/(1-m)}=|\lambda|^{1/m}|x-y|\geq 1$, then the right side of (6.7) does not exceed

$$
(6.9) \quad \int_0^{\lambda} s^{-1} \exp \left(-c_2 s^{-1/(m-1)}\right) ds + \int_{\lambda}^1 s^{-1} \exp \left(-c_2 h^{-1/(m-1)} s - c_3 s^{-1/(m-1)}\right) ds
\leq \exp \left(-2^{-1} c_2 h^{-1/(m-1)}\right) \int_0^{\lambda} s^{-1} \exp \left(-2^{-1} c_2 s^{-1/(m-1)}\right) ds
+ \exp \left(-2^{-1} c_2 h^{-1/(m-1)}\right) \int_{\lambda}^1 s^{-1} \exp \left(-c_2 s^{-1/(m-1)}\right) ds
+ \int_{\lambda}^1 \int_0^{c_2 h^{-1/(m-1)}} \int_{\lambda}^1 \exp \left(-2^{-1} c_2 s\right) ds + \int_{\lambda}^1 s^{-1} \exp \left(-2^{-1} c_2 s\right) ds.
$$

In view of (6.8), (6.9) the right side of (6.7) is not greater than

$$
C_{19} \exp \left(-c_4 |\lambda|^{1/m}|x-y|\right) \left(1+\log^+ \left(|\lambda|^{1/m}|x-y|\right)^{-1}\right),
$$

where $C_{19} = \max(C_{17}, c_4^2, m c_4^2, C_{18})$ and $\log^+ \sigma = \log \sigma$ if $\sigma \geq 1$, $\log^+ \sigma = 0$ if $\sigma < 1$.

Thus in case $N=m$

$$
(6.10) \quad |(\partial/\partial t)^n K_4(x, y; t)|
\leq C_{19} C_{20} M_s \exp \left(-c_4 |\lambda|^{1/m}|x-y|\right) \left(1+\log^+ \left(|\lambda|^{1/m}|x-y|\right)^{-1}\right).
$$

Finally we consider the case $N<m$. Changing the variable by $r=s/|\lambda|$ and putting $\tilde{h}=|\lambda|^{1/m}|x-y|$

$$
\int_0^m \exp(-c_2 r |\lambda|) r^{-N/m} \exp \left(-c_2 s^{m/(m-1)-1/(m-1)}\right) dr
= |\lambda|^{N/m-1} \int_0^m s^{-N/m} \exp \left(-c_2 s\right) \exp \left(-c_2 \tilde{h}^{m/(m-1)} s^{-1/(m-1)}\right) ds
\leq |\lambda|^{N/m-1} \exp \left(-c_2 \tilde{h}\right) \int_0^{\tilde{h}} s^{-N/m} \exp \left(-c_2 s\right) ds
$$
\begin{align*}
+ |\lambda|^{N/m-1} \exp \left(-2^{-1} c_2 \bar{h} \right) \int_0^\infty s^{-N/m} \exp \left(-c_4 s/2\right) ds \\
\leq c_3^{N/m-1} \Gamma(1-N/m) |\lambda|^{N/m-1} \exp \left(-c_2 \bar{h}\right) \\
+ \left(c_3/2\right)^{N/m-1} \Gamma(1-N/m) |\lambda|^{N/m-1} \exp \left(-2^{-1} c_3 \bar{h}\right) \\
\leq C_20 |\lambda|^{N/m-1} \exp \left(-c_4 |\lambda|^{1/m} |x-y|\right),
\end{align*}

where \( C_20 = \{c_3^{N/m-1}+(c_3/2)^{N/m-1}\} \Gamma(1-N/m) \). Thus in case \( N < m \)

\begin{equation}
(6.11) \quad |(\partial/\partial t)^n K_0(x,y; t)| \leq C_{11} C_{20} C_2^n M_n |\lambda|^{N/m-1} \exp \left(-c_4 |\lambda|^{1/m} |x-y|\right).
\end{equation}

Summing up we see that the following estimate holds

\begin{equation}
(6.12) \quad |(\partial/\partial t)^n K_0(x,y; t)| \leq C_{21} C_2^n M_n \exp \left(-c_4 |\lambda|^{1/m} |x-y|\right) \times \left\{ \begin{array}{ll}
|x-y|^{m-N} & \text{if } N > m \\
1 + \log^+ \left(|\lambda|^{1/m} |x-y|\right)^{-1} & \text{if } N = m \\
|\lambda|^{N/m-1} & \text{if } N < m
\end{array} \right.
\end{equation}

for any \( n = 0, 1, 2, \ldots \), \( (x,y) \in \Omega \times \Omega \), \( t \in [0, T] \), \( \lambda \) in the region (6.2) where \( C_{21} = \max(C_{16}, C_{11} C_{19}, C_{11} C_{20}) \). It is clear that the same estimate holds for \( \lambda \) in the region

\[ \{ \lambda: \text{Re} \lambda > 0, \text{Im} \lambda < 0, \text{Re} \lambda/|\text{Im} \lambda| \leq (1-\varepsilon_i) \tan \theta_i \} \cup \{ \lambda: \text{Re} \lambda \leq 0, \text{Im} \lambda < 0 \}. \]

It follows readily from (6.12) that

\begin{equation}
(6.13) \quad \| (\partial/\partial t)^n (A(t) - \lambda)^{-1} \|_{B(L^1,L^\infty)} \leq C_{22} C_2^n M_n / |\lambda|
\end{equation}

for any \( n = 0, 1, 2, \ldots \), \( t \in [0, T] \), and \( \lambda \) in the region

\begin{equation}
(6.14) \quad \{ \lambda: \text{Re} \lambda > 0, \text{Re} \lambda/|\text{Im} \lambda| \leq (1-\varepsilon_i) \tan \theta_i \} \cup \{ \lambda: \text{Re} \lambda \leq 0 \}. \]
\end{equation}

Due to the closedness of \( \Sigma \) and the arbitrariness of \( \varepsilon_i \in (0, 1) \), \( \theta_i \in (0, \pi/2 - \theta_0) \) we see that there exist constants \( K_9, K \) such that (2.13) holds for any \( n = 0, 1, 2, \ldots \), \( \lambda \in \Sigma \), \( t \in [0, T] \), and the proof of the proposition and hence that of the main theorem is complete.

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