ON THE SCHUR INDICES OF CERTAIN IRREDUCIBLE CHARACTERS OF REDUCTIVE GROUPS OVER FINITE FIELDS

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Introduction. Let $F_q$ be a finite field with $q$ elements, of characteristic $p$. Let $G$ be a connected, reductive linear algebraic group defined over $F_q$, with Frobenius endomorphism $F$, and let $G^F$ denote the group of $F$-fixed points of $G$. In [13], we investigated, under the assumption that the centre $Z$ of $G$ is connected, the rationality-properties of the characters $\chi_{\rho^F}$ of $G^F$ induced by certain linear characters $\lambda$ of a Sylow $p$-subgroup of $G^F$ and, using the results obtained there, proved some propositions concerning the Schur indices of the semisimple or regular irreducible characters of $G^F$. In this paper, we shall treat the general case, that is, the case that $Z$ is not necessarily connected. The main results are stated and proved in § 2. In particular, we get the following (see Corollary 1 to Proposition 1, § 2):

Theorem. Any irreducible Deligne-Lusztig character $\pm R_\chi$ of $G^F$ ([4]) has the Schur index at most two over the field $Q$ of rational numbers.

I wish to thank Professor N. Iwahori who kindly taught me properties of the Cartan matrices. I also thank Professor S. Endo for his kind advices during the preparation of the paper. The referee gave me valuable comments for the old version of the paper. Finally, I wish to dedicate this paper to the late Professor T. Miyata.

1. Some lemmas. Let $G$ and $F$ be as above. Let $B$ be an $F$-stable Borel subgroup of $G$ with the unipotent radical $U$ and $T$ an $F$-stable maximal torus of $B$. For a root $\alpha$ of $G$ (with respect to $T$), let $U_\alpha$ denote the root subgroup of $G$ associated with $\alpha$. Let $U$ be the subgroup of $U$ generated by the non-simple positive root subgroups $U_{\alpha}$ (the ordering on the roots is the one determined by $B$). Then $U/U_{\alpha}$ is commutative and can be regarded as the direct product $\prod_{\alpha \in \Delta} U_\alpha$, where $\Delta$ is the set of simple roots. As $FU_\alpha = U_\alpha$, $F$ acts on $U/U_{\alpha} = \prod_{\alpha \in \Delta} U_\alpha$ and this action is the one induced by the maps $F: U_\alpha \rightarrow FU_\alpha$, $\alpha \in \Delta$. Let $\rho$ be the permutation on the roots $\alpha$ given by $FU_\alpha = U_{\rho \alpha}$ and let $I$
be the set of orbits of \( \rho \) on \( \Delta \). For \( i \in I \), put \( U_i = \prod_{a \in I} U_a \). Then \( U/U = \prod_{i \in I} U_i \) and, as each \( U_i \) is \( F \)-stable, we have \( U_i/U_i = \prod_{i \in I} U_i \). For each \( i \in I \), put \( q_i = q^{n_i} \) and take one simple root \( \gamma_i \) in \( i \). Then, for each \( i \), there is an isomorphism \( \phi_i \) of \( U_i \) with the additive group of \( F_{q_i} \) such that \( \phi_i(tu^{-1}) = \gamma_i(t)\phi_i(u) \) for \( u \in U_i \) and \( t \in T \) (cf. Proof of 11.8 of Steinberg [17] and Carter [3], pp. 76-77). Thus the family \( \phi = (\phi_i)_{i \in I} \) defines an isomorphism

\[
\phi: U/U \cong \prod_{i \in I} U_i \cong \prod_{i \in I} F_{q_i}
\]

so that, for \( u = \prod_{i \in I} u_i \) with \( u_i \in U_i \) for \( i \in I \) and \( t \in T \), we have

\[
\phi(tu^{-1}) = \prod_{i \in I} \phi_i(t)(u_i).
\]

Now let \( \Lambda \) be the set of characters \( \lambda \) of \( U \) such that \( \lambda|U_1 = 1 \) and \( \Lambda_0 \) the set of characters \( \lambda \) in \( \Lambda \) such that \( \lambda|U_i \neq 1 \) for all \( i \in I \). Then we have

**Lemma 1.** Let \( \lambda \in \Lambda_0 \). Then \( \lambda_{G^F} \) is multiplicity-free (Gel'fand-Graev, Yokonuma, Steinberg) and any irreducible Deligne-Lusztig character \( \pm R^\lambda \) of \( G^F \) occurs in \( \lambda_{G^F} \) (Deligne-Lusztig).

By embedding \( G \) in the connected, reductive group \( G_{i} := (G \times T)/\{ (z, z^{-1}) \} \) \( z \in Z \) (\( Z \) is the centre of \( G \)) with connected centre and the same derived group ([4], 5.18) and (as to the second assertion) using properties of Green functions (cf. [3], 7.2.8 and 7.7), we are reduced to the case that \( Z \) is connected. In this case the lemma is proved in [4], Theorem 10.7 (or in [3], 8.1.3 and 8.4.5).

Our purpose is to study the rationality of the characters \( \lambda_{G^F}, \lambda \in \Lambda \). Suppose \( p = 2 \). Then, by (1), \( U/U \cong \prod_{i \in I} U_i \) is an elementary abelian 2-group, so that, for any \( \lambda \in \Lambda, \lambda \), hence \( \lambda_{G^F} \) is realizable in \( \mathbb{Q} \). Therefore, from now on, we shall assume that \( p \neq 2 \).

**Lemma 2.** Let \( \nu \) be a primitive element of \( F_p \) (i.e. \( F_p^\ast = \langle \nu \rangle \)). Then there exists an element \( t \) in \( T \) such that \( t^{p-1} = 1 \) (possibly \( t^{(p-1)/2} = 1 \)) and \( \alpha(t) = \nu^{\lambda} \) for all simple roots \( \alpha \).

It suffices to prove the lemma for the derived group \( G' \) of \( G \), hence for the simply-connected covering of \( G' \). If \( G \) is a simply-connected semisimple group, then we have \( G = G_1 \times \cdots \times G_m \), where, for \( 1 \leq i \leq m \), \( G_i \) is an \( F \)-stable simply-connected semisimple closed subgroup of \( G \) whose simple components are permuted by \( F \) cyclically, and the truth of the lemma for each \( G_i \) will imply that for \( G \). If \( G = G_1 \times F G_1 \times \cdots \times F^{n-1} G_1 \), where \( G_1 \) is an \( F^n \)-stable simply-connected simple closed subgroup of \( G \) for some \( n \geq 1 \), then \( T \) and \( B \), hence the set of simple roots has the corresponding decomposition, and it is easy to see that the truth of the lemma for \( G_1 \) with Frobenius map \( F^n \) implies that for
Thus we are reduced to the case that $G$ is a simply-connected simple group.

Suppose therefore that $G$ is such a group. Let $X(T) = \text{Hom}(T, G_m)$ and $Y(T) = \text{Hom}(G_m, T)$, and let $\langle , \rangle : X(T) \times Y(T) \rightarrow \mathbb{Z}$ be the natural pairing given by $\langle \chi, \chi' \rangle = \text{degree of } \chi \circ \chi'$ for $\chi \in X(T)$ and $\chi' \in Y(T)$. Let $\alpha_1, \ldots, \alpha_l$ be the simple roots (as to the numbering of the simple roots, we follow that of Bourbaki [2]) and let $\alpha_1^{\vee}, \ldots, \alpha_l^{\vee}$ be the corresponding simple coroots. Then, as $G$ is simply-connected, we have $Y(T) = \langle \alpha_1^{\vee}, \ldots, \alpha_l^{\vee} \rangle_{\mathbb{Z}}$, so that the mapping $h : (x_1, \ldots, x_l) \mapsto \prod_{i=1}^l \alpha_i^{\vee}(x_i)$ defines an isomorphism of $(G_m)^l$ with $T$. Then, for $1 \leq i \leq l$, we have

$$\alpha_i(h(x_1, \ldots, x_l)) = \prod_{j=1}^l x_j^{\langle \alpha_i, \alpha_j^{\vee} \rangle},$$

where $\langle \alpha_i, \alpha_j^{\vee} \rangle_{1 \leq i, j \leq l}$ is the Cartan matrix of $G$. We define an action of $F$ on $Y(T)$ by $F(\chi^{\vee}) = F \circ \chi'$ for $\chi' \in Y(T)$. Then we have

$$F(\alpha_i^{\vee}) = q(\rho \alpha_i)^{\vee}$$

for $1 \leq i \leq l$ (see [15], 11.4.7). It readily follows that, for $s \in T$, $s = h(x_1, \ldots, x_l)$, we have $F(s) = s$ if and only if $x_j = x_j^1$ if $\rho \alpha_i = \alpha_j$. Thus the proof of the lemma has been reduced to solving the following problem:

Find an element $t = h(x_1, \ldots, x_l)$ with $x_i \in F^\times$ for $1 \leq i \leq l$ such that $\prod_{j=1}^l x_j^{\langle \alpha_i, \alpha_j^{\vee} \rangle} = v^2$ for $1 \leq i \leq l$ and that $x_j = x_j^1$ (hence $x_j = x_i$) if $\rho \alpha_i = \alpha_j$.

When $G$ is adjoint, by the proof of Theorem 1 of [13], there is an element $s$ in $T^F$ of order $p-1$ such that $\alpha(s) = \nu$ for all simple roots $\alpha$. Hence it suffices to take $t = s^5$. Suppose therefore that $G$ is not adjoint. Then, as $p \neq 2$, $G$ is any one of the following types (Steinberg [17], 11.6; also see [3], 1.19): $A_l$ ($l \geq 1$), $B_l$ ($l \geq 2$), $C_l$ ($l \geq 2$), $D_l$ ($l \geq 3$), $E_6$, $E_7$, $^2A_l$ ($l \geq 1$), $^2D_l$ ($l \geq 3$), $^3D_4$, $^2E_6$. In each case, an element $t$ of $T^F$ having the property of the lemma (i.e. an solution $t$ of the problem above) can be given as follows (the Cartan matrices are listed up in the appendices of [2]):

<table>
<thead>
<tr>
<th>Type</th>
<th>$t$</th>
<th>$x_i = v^{l(l-1)/2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_l$</td>
<td>$h(x_1, \ldots, x_l)$</td>
<td>$x_i = v^{l(i-1)+l}$ (1 \leq i \leq l)</td>
</tr>
<tr>
<td>$B_l$</td>
<td>$h(x_1, \ldots, x_{l-1}, v^{l(l+1)/2})$</td>
<td>$x_i = v^{l(l-1)+l}$ (1 \leq i \leq l-1)</td>
</tr>
<tr>
<td>$C_l$</td>
<td>$h(x_1, \ldots, x_l)$</td>
<td>$x_i = v^{l(l-1)/2}$ (1 \leq i \leq l)</td>
</tr>
<tr>
<td>$D_l$</td>
<td>$h(x_1, \ldots, x_{l-1}, v^{l(l+1)/2})$</td>
<td>$x_i = v^{l(l-1)/2}$ (1 \leq i \leq l-2)</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$h(v^{16}, v^{22}, v^{10}, v^{12}, v^{16})$</td>
<td>$x_i = v^{l(l-1)/2}$ (1 \leq i \leq l)</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$h(v^{11}, v^{34}, v^{6}, v^{56}, v^{66}, v^{66}, v^{58}, v^{27})$</td>
<td>$x_i = v^{l(l-1)/2}$ (1 \leq i \leq l)</td>
</tr>
</tbody>
</table>

This completes the proof of Lemma 2.
Lemma 3. Assume that $q$ is an even power of $p$. Then there exists an element $t$ in $T^F$ such that $t^{q-1}=1$ (possibly $t^{-1}=1$) and $\alpha(t)=v$ for all simple roots $\alpha$.

As in the proof of Lemma 2, we can be reduced to the case that $G$ is a simply-connected simple group. When $G$ is adjoint Lemma 3 is proved in the proof of Theorem 1 of [13]. When $G$ is not adjoint $t$ can be given by replacing each $v$ in the above table with an element $e^Fq$ such that $\varepsilon z=v$. (We note that, when $G$ is a simply-connected simple group, an element $s=h(x_1, \ldots, x_l)$ of $T$ has the property of Lemma 3 if and only if the $x_i$ satisfy: (i) $x_i^{p-1}=1$ for $1\leq i \leq l$, (ii) $\prod_{j=1}^l x_j\langle e^{p-1} \rangle = v$ for $1\leq i \leq l$, and (iii) $x_j=x_i$ if $p \alpha_i=\alpha_j$.)

In the following, for an integer $m$ and a prime number $r$, $ord_m$ denotes the exponent of the $r$-part of $m$.

Lemma 4. Assume that $G$ is a (non-adjoint) simply-connected simple group of any one of the following types: $A_1$ with $2|l$ or $ord_2(l+1)>ord_2(p-1)$; $A_1$ with $2|l$; $B_l$ with $4|l(l-1)$; $D_l$ with either (a) $4|l(l-1)$ or (b) $ord_2(l-1)=1$ and $p \equiv -1 \pmod{4}$; $^2D_l$ with $4|l(l-1)$; $^2D_4$; $E_6$; $E_7$. Then there exists an element $t\in T^F$ such that $t^{q-1}=1$ and $\alpha(t)=v$ for all simple roots $\alpha$.

In fact, for an element $s=h(x_1, \ldots, x_l)$ of $T$, $s$ satisfies the property of Lemma 4 if and only if the $x_i$ satisfy: (i) $x_i\in F_q^+$, (ii) $\prod_{j=1}^l x_j\langle e^{p-1} \rangle = v$ for $1\leq i \leq l$, and (iii) $x_j=x_i$ if $p \alpha_i=\alpha_j$. By solving these equations, we find that an element $t$ having the property of the lemma can be given as follows:

<table>
<thead>
<tr>
<th>Type</th>
<th>$t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$ $^2A_1$ $2</td>
<td>l$</td>
</tr>
<tr>
<td>$A_1$ ord$_2(l+1)&gt;ord_2(p-1)$</td>
<td>$h(x_1, \ldots, x_l)$</td>
</tr>
<tr>
<td>$B_l$ $4</td>
<td>l(l-1)$</td>
</tr>
<tr>
<td>$D_l$ $^2D_l$ $4</td>
<td>l(l-1)$</td>
</tr>
<tr>
<td>$D_l$ ord$_2(l-1)=1$ $p \equiv -1 \pmod{4}$</td>
<td>$h(x_1, \ldots, x_{l-2}, x_{l-1}^{l(l+1)/4}, x_l^{(l^2-l+3p-3)/4}) = x_i = x_l^{l(l+1)/4}(l\leq i \leq l-2)$</td>
</tr>
<tr>
<td>$^2D_4$</td>
<td>$h(v^p, v^5, v^3, v^3)$</td>
</tr>
<tr>
<td>$E_6$ $^2E_6$</td>
<td>$h(v^p, v^{11}, v^{15}, v^{21}, v^{15}, v^p)$</td>
</tr>
</tbody>
</table>

Remark. If (at least) $G$ is split over $F_q$, then Lemmas 2, 4 above are implicit in Lehrer’s work [12] where he showed a method to calculate the image $a(T^F)$ of $T^F$ under the morphism $a: T\to (G_m)^l$ given by $a(s)=\prod_{i=1}^l \alpha_i(s)$ when $G$
is a simply-connected simple group (he has carried out the calculation when $G$ is a classical group). For our purpose, it is essential to know the order of $t$ (cf. § 2 below).

2. The main results. We recall that $p \neq 2$. Let $\zeta_p$ be a primitive $p$-th root of unity in the field $\mathbb{C}$ of complex numbers. Let $\tilde{F}_q = \text{Hom}(F_q, \mathbb{C}^*)$ (we consider $F_q$ as an additive group) and fix $\chi \in \tilde{F}_q$, $\chi \neq 1$. For $a \in F_q$, define $\chi_a \in \tilde{F}_q$ by $\chi_a(x) = \chi(ax)$ for $x \in F_q$. Then we have $\tilde{F}_q = \{ \chi_a | a \in F_q \}$ and $\{ \chi^* | \tau \in \text{Gal}(Q(\zeta_p)/Q) \} = \{ \chi_a | a \in F^*_q \}$.

In the following, if $\chi$ is a character of a finite group and $L$ is a field of characteristic zero, $L(\chi)$ is the field generated over $L$ by the values of $\chi$. If $\chi$ is irreducible, then $m_L(\chi)$ denotes the Schur index of $\chi$ with respect to $L$. If $L$ is an algebraic number field and $v$ is a place of $L$, then $L_v$ is the completion of $L$ at $v$. Now let $k$ be the quadratic subfield $Q(\sqrt{\varepsilon p})$, $\varepsilon = (-1)^{(p-1)/2}$, of $Q(\zeta_p)$.

**Proposition 1.** Let $G, F$ be as in Introduction. Let $\lambda \in \Lambda$, $\lambda \neq 1$. Then we have the following:

(i) $\lambda^{G_F}$ takes all its values in $k$; if $p \equiv -1 \pmod{4}$, $\lambda^{G_F}$ is realizable in $k$; if $p \equiv 1 \pmod{4}$, then, for any finite place $v$ of $k$, $\lambda^{G_F}$ is realizable in $k_v$.

(ii) Assume that $q$ is an even power of $p$. Then $\lambda^{G_F}$ takes all its values in $Q$ and, for any prime number $r \neq p$, $\lambda^{G_F}$ is realizable in $Q_r$.

(iii) If $G$ is an adjoint semisimple group or any one of the groups described in Lemma 4, then $\lambda^{G_F}$ is realizable in $Q_r$.

Proof of (i). Let $t$ be an element of $T^F$ having the property of Lemma 2. Then $z = t^{(p-1)/2}$ lies in the centre $Z^F$ of $G^F$ since $\alpha(z) = 1$ for all simple roots $\alpha$. Put $c = |<\zeta>|$ ($c = 1$ or 2). Let $M = <\ell>U^F$. Then $M$ acts on $\Lambda$ by $\lambda^m(u) = \lambda(mu^{-1})$ ($\lambda \in \Lambda$, $m \in M$, $u \in U^F$). Let $\lambda \in \Lambda$, $\lambda \neq 1$. Then, by (1), $\lambda$ can be expressed as $\lambda = (\lambda_i)_{i \in I}$ with $\lambda_i \in \tilde{F}_q$ for $i \in I$. And, by (2), we have

$$
\lambda' = ((\lambda_i)_{i \in I})_{i \in I} = ((\lambda_i)c)_{i \in I} = (\lambda^c)_{i \in I} = \lambda^{c_2},
$$

where $c$ is a suitable generator of $\text{Gal}(Q(\zeta_p)/Q)$. Thus, on $U^F$, we have

$$
\lambda^M = c \sum_{j=1}^{(p-1)/2} \lambda^{t_j} = c \sum_{j=1}^{(p-1)/2} \lambda^{c_2j},
$$

hence $Q(\lambda^M) = Q(\zeta_p)^{c_2} = k$. Therefore the values of $\lambda^{G_F} = (\lambda^M)^{G_F}$ lie in $k$.

Suppose $t^{(p-1)/2} = 1$. Then $\lambda^M$ is irreducible. By Gow’s argument [7], p. 104, we have $m_k(\lambda^M) = 1$: $\lambda^M|<\ell>$ is the character of the regular representation of $<\ell>$, hence $<\ell^M, 1_{Q^F}>_{Q^F} = 1$; hence, by Schur’s theorem (see e.g. Feit [5], 11.4), $m_k(\lambda_M) = 1$. Thus $\lambda^M$, hence $\lambda^{G_F} = (\lambda^M)^{G_F}$ is realizable in $k$.

Assume that $t^{(p-1)/2} \neq 1$. Then $\lambda^M$ is reducible and is equal to the sum $\mu_0 + \mu_1$ where, for $i = 0, 1$, $\mu_i$ is the irreducible character of $M$ induced by the
linear character of $\langle z \rangle U^F$ given by $z^j u \mapsto (-1)^j \lambda(u)$ ($j=0,1$). We have $Q(\mu_0)=Q(\mu_1)=k$. For $i=0,1$, the simple direct summand $A_i$ of the group algebra $k[M]$ of $M$ over $k$ corresponding to $\mu_i$ is isomorphic over $k$ to the cyclic algebra $((k(\zeta_p))/k, \sigma, (-1)^i)$ over $k$ (cf. Proof of Proposition 3.5 of Yamada [18]). $A_0$ clearly splits over $k$, hence $m_q(\mu_0)=1$ and $\mu_0$ is realizable in $k$. If $p \equiv -1 \pmod{4}$, then $\zeta_p$ is a norm in $k(\zeta_p)/k$, hence $A_1$ splits over $k$. Thus, in this case, $\mu_1$, hence $\lambda^M=\mu_0+\mu_1$ is realizable in $k$.

Proof of (ii). Let $t$ be an element of $T^F$ having the property of Lemma 3, and put $M=\langle t \rangle U^F$. Then, as $\lambda^t=\lambda^\sigma (\lambda \neq 1)$, on $U^F$, we have

$$\lambda^M = c \sum_{i=1}^{t^{-1}} \lambda^{t^i} = c \sum_{i=1}^{t^{-1}} \lambda^{\sigma^i} \quad (c = |\langle t^{t^{-1}} \rangle|).$$

Thus $Q(\lambda^M)=Q(\zeta_p)^{\langle \sigma \rangle}=Q$.

If $t^{t^{-1}}=1$, then $\lambda^M$ is irreducible and Gow's argument shows that $m_q(\lambda^M)=1$, hence $\lambda^{G^F}$ is realizable in $Q$. Suppose $t^{t^{-1}} \neq 1$. Then $\lambda^M$ is reducible and is equal to the sum $\mu_0+\mu_1$, where, for $i=0,1$, $\mu_i$ is the irreducible character of $M$ induced by the linear character of $\langle t^{t^{-1}} \rangle U^F$ given by $(t^{t^{-1}})^j u \mapsto (-1)^j \lambda(u)$. We have $Q(\mu_0)=Q(\mu_1)=Q$. For $i=0,1$, the simple direct summand $A_i$ of $Q[M]$ corresponding to $\mu_i$ is isomorphic over $Q$ to $Q(\zeta_p)/Q, \sigma, (-1)^i)$. $A_0$ splits, hence $\mu_0$ is realizable in $Q$. $A_1$ has the invariants $\frac{1}{2}$ mod 1 at $\infty$, $p$ and 0 mod 1 at any other place of $Q$. Thus, for any prime number $r \neq p, \mu_1$, hence $\lambda^M=\mu_0+\mu_1$ is realizable in $Q$.

Proof of (iii). When $G$ is adjoint the assertion is contained in Theorem 1 of [13]. Assume that $G$ is not adjoint. Let $t$ be an element of $T^F$ having the property of Lemma 4 and put $M=\langle t \rangle U^F$. Then $\lambda^M$ is irreducible and $Q(\lambda^M)=Q$. And, by Gow's argument, we have $m_q(\lambda^M)=1$. Thus $\lambda^M$, hence $\lambda^{G^F}=\lambda^M^{G^F}$ is realizable in $Q$.

We note that, for $G=SL_n$, $Sp_{2n}$, Proposition 1 is proved by Gow [7], [8].

**Corollary 1.** Let $G, F$ be as in Proposition 1. Recall that $p \neq 2$. Let be $\chi$ an irreducible character of $G^F$ such that $\langle \chi, \lambda^{G^F} \rangle^F=1$ for some $\lambda \in \Lambda$ (any irreducible component of $\lambda^{G^F}$ for $\lambda \in \Lambda_0$ has this property (see Lemma 1)). Then we have $m_q(\chi) \leq 2$. Thus, in particular, we have $m_q(\chi) \leq 2$ for any irreducible Deligne-Lusztig character $\chi=\pm R_\chi^\sigma$ of $G^F$. If $\lambda=1$, then $\chi^{G^F}$ is realizable in $Q$, hence we have $m_q(\chi)=1$. Assume that $\lambda \neq 1$. Let $r$ be any prime number and $\nu$ a place of $k$ lying above $r$. Then, by Proposition 1, we have $m_k(\chi)=1$, hence $m_q(\chi) \leq 2$ as $[k(\chi) \cap Q, (\chi)] \leq 2$. We also have $m_k(\chi) \leq 2$. Thus, $m_q(\chi)$, being the least
common multiple of the $m_q(\lambda)$ with $w$ running over all places of $Q$, is at most two. The last assertion follows from this fact and Lemma 1.

**Corollary 2.** Assume that $q$ is an even power of $p$. Let $\chi$ be an irreducible character of $G^F$ such that $\langle \chi, \chi^e \rangle_{G^F} = 1$ for some $\lambda \in \Lambda$. Then, for any prime number $r \neq p$, we have $m_q(\chi) = 1$.

This follows at once from Proposition 1, (ii).

**Corollary 3.** Assume that $G$ is an adjoint semisimple group or any one of the groups described in Lemma 4. Let $\chi$ be an irreducible character of $G^F$ such that $\langle \chi, \chi^e \rangle_{G^F} = 1$ for some $\lambda \in \Lambda$. Then we have $m_q(\chi) = 1$.

This follows from Proposition 1, (iii).

**Corollary 4.** Let $G, F$ be as in Proposition 1. Assume that $p$ is a good prime for $G$ ([16], I, 4.1). Let $\chi$ be an irreducible character of $G^F$ and let $u$ be a regular unipotent element in $G^F$. Then $\chi(u)$ is an algebraic integer in $k$, and if $p^r | \chi(1)$, we have $m_q(\chi) \leq 2$.

We first note that, as $p$ is good for $G$, $U^F$ is equal to the derived group of $U^F$, hence $\Lambda$ is the set of linear characters of $U^F$ (Howlett [9], Lehrer [11]), and that, if $u \in U^F$, then $\mu(u) = 0$ for any non-linear irreducible character $\mu$ of $U^F$ (Lehrer [11]).

Let $\mathfrak{o}_k$ be the ring of integers in $k$. We show that $\chi(u)$ belongs to $\mathfrak{o}_k$. We may assume that $u \in U^F$ as $u$ is conjugate to an element of $U^F$. Let $t$ be an element of $T^F$ having the property of Lemma 2, and let $\Lambda_1, \ldots, \Lambda_r$ be the orbits of $\langle t \rangle$ on $\Lambda$. Thus, as $\chi^e = \chi$, if we put $a_\lambda = \langle \chi, \chi^e \rangle_{U^F}$ for $\lambda \in \Lambda$, $a_\lambda$ is constant on each $\Lambda_i$. Hence we have

$$\chi(u) = \sum_{\lambda \in \Lambda} a_\lambda \lambda(u) = \sum_{i=1}^r a_i \left( \sum_{\lambda \in \Lambda_i} \lambda(u) \right),$$

where $a_i = a_\lambda$ on $\Lambda_i$. Each $\sum_{\lambda \in \Lambda_i} \lambda(u)$ is stable under the action of $\langle t \rangle$, hence under the action of $\langle \sigma^e \rangle$. Thus $\chi(u) \in \mathfrak{o}_k$.

To prove the second assertion, we embed $G$ in $G_1$ as in the proof of Lemma 1. Assume that $p \not| \chi(1)$ and take an irreducible character $\chi_1$ of $G_1^F$ such that $\langle \chi, \chi_1 | G^F \rangle_{G_1^F} \neq 0$. Then, by the Clifford theory, we have $\chi_1 | G^F = e(\chi^{(0)} + \chi^{(0)} + \cdots + \chi^{(0)})$, where $e$ is a positive integer dividing $(G^F : G_1^F)$ and $\chi^{(0)}, \chi^{(0)}, \ldots, \chi^{(0)}$ are the $G_1^F$-conjugates of $\chi = \chi^{(0)}(s | (G^F : G_1^F))$. Let $r$ be any prime number and $v$ a place of $k$ lying above $r$. Put $m_v = m_{h_1}(\chi^{(0)}) = \cdots = m_{h_s}(\chi^{(0)})$. For $1 \leq i \leq s$ and for $\lambda \in \Lambda$, put $a_i(\lambda) = \langle \chi^{(0)}, \lambda \rangle_{G_1^F}$. Then, by Proposition 1, (i), $m_v$ divides the $a_i(\lambda)$, $1 \leq i \leq s$, $\lambda \in \Lambda$. As $p \not| (G^F : G_1^F)$, $p \not| \chi_1(1)$, so that, by a theorem of Green-Lehrer-Lusztig (see [3], 8.3.6), we have $\chi_1(u) = \pm 1$. Therefore we have the expression
\[ \pm 1/m_\pi = \chi(u)/m_\pi = \{e \cdot \sum_{i=1}^{l} \chi^{(i)}(u)\}/m_\pi = e \cdot \sum_{i=1}^{l} \sum_{\alpha \in \Delta} (\alpha^{(i)})/m_\pi \cdot \lambda(u), \]

where the right-hand side is an algebraic integer and the left-hand side is a rational number. Hence \( m_\pi = 1, \) and \( m_q(\chi) \leq 2. \) As \( r \) is an arbitrary prime number, we hence have \( m_q(\chi) \leq 2. \) This completes the proof of Corollary 4.

**Corollary 5.** Assume that \( q \) is an even power of \( p \) and that \( p \) is good for \( G. \) Let \( u \) be a regular unipotent element in \( G^F. \) Then, for any irreducible character \( \chi \) of \( G^F, \) \( \chi(u) \) is a rational integer, and if \( p \nmid \chi(u), \) we have \( m_q(\chi) = 1 \) for any prime number \( r \neq p. \)

The proof is similar to the proof of Corollary 4 (we use Proposition 1, (iii)).

**Corollary 6.** Let \( G \) be an adjoint semisimple group or any one of the groups described in Lemma 4. Assume that \( p \) is good for \( G. \) Let \( u \) be a regular unipotent element in \( G^F \) and let \( \chi \) be an irreducible character of \( G^F. \) Then \( \chi(u) \) is a rational integer and if \( p \nmid \chi(u), \) we have \( m_q(\chi) = 1. \)

**Remark.** Lehrer [12] has calculated the values of the cuspidal irreducible characters of \( G^F \) at the regular unipotent elements of \( G^F \) when \( G \) is a semisimple group. As to the upper bound of the indices of the characters of related finite groups, we refer to Gow [8] for classical finite groups and Benard [1] and Feit [6] for the sporadic simple groups.

Let \( G \) be a connected, reductive algebraic group over an algebraically closed field \( K \) of characteristic \( p > 0 \) and \( F \) a surjective endomorphism of \( G \) such that \( G^F \) is finite. Then Lemma 2 still holds for such \( G^F, \) so that the statements in Proposition 1, (i) and in Corollary 1 (except for the comment for Lemma 1) hold for \( G^F. \) Assume that \( K \) is an algebraic closure of \( F \) and that some power of \( F \) is the Frobenius endomorphism relative to a rational structure on \( G \) over a finite subfield of \( K. \) Then Lemma 1 holds for \( G^F \) (cf. Carter [3], 8.1.3 and 8.4.5), so that all the statements in Corollary 1, hence the theorem in Introduction holds for \( G^F. \) If \( p \) is good for \( G, \) then the theorem of Green-Lehrer-Lusztig holds for \( G^F \) (if \( Z \) is connected: see [3], 8.3.6), so that Corollary 4 holds for \( G^F. \)

**3. Example.** We calculate all the local indices of the cuspidal irreducible Deligne-Lusztig characters \( \pm R_\pi^\pi \) of \( SL_n(F_q) \) when \( q \) is an even power of \( p \) \((\pm 2).\)

Let \( G \) be \( SL_n \) and \( F \) the endomorphism \((g_{ij}) \rightarrow (g_{ij}^q)\) \((q \text{ may be any power of any prime } p).\) Let \( T' \) be a minisotropic maximal torus of \( G \) and let \( W = N_G(T')^F/T'^F \) \((T' \text{ is unique up to } G^F\text{-conjugate}). \) Then, taking an element \( \gamma \) of order \((q^n-1)/(q-1)\) in \( F_q^\times, \) we have \( T'^F = \langle t_0 \rangle, \) where \( t_0 \) is \( G\)-conjugate to
diag(γ, γ*, ..., γr−1), and W=⟨w0⟩=Z|nZ, where w0 is defined by t00=wb0t0b0−1
=tw0∈N_G(T')F represents w0. (All these statements can be easily checked by
using [16], II, 1.3, 1.10 and 1.14.) W acts on T'F=Hom(T'F, C*) by
θw(s)=θ(se) for w∈W, θ∈T'F and s∈T'F. If θ is in general position, i.e., no
non-identity element of W fixes θ, then (−1)s−1R_θ^F is a cuspidal irreducible
character of G^F=SL_n(F_q) ([4], 7.4, 8.3).

Let θ∈T'F. Then, by [4], 4.2, for g∈G^F, if g=us=us (s semisimple, u
unipotent) is its Jordan decomposition, we have

(3) R_θ^F(g) = \frac{1}{|Z_G(s)^F|} \sum_{a∈G^F} Q_{aT'a^{-1}, \theta(a)(u)} \cdot \theta(h^{-1}s_h),

where the Q_{aT'a^{-1}, \theta(a)} are Green functions of Z_G(s) (which is connected since
G is simply-connected). It follows that, if s is not conjugate in G^F to any element
of T'F, we have R_θ^F(g)=0, and if s∈T'F, we have

(4) R_θ^F(g) = \frac{1}{|W(s)|} \sum_{w∈W} \theta^w(s),

where W(s)={w∈W|w^e=s} (we note that the minisotropic maximal tori of
Z_G(s) form a single Z_G(s)^F-conjugacy class (cf. [16], II, 1.3, 1.10 and 1.14)
and that any two elements of T' that are conjugate in G^F are conjugate under
the action of W). Thus, as the Green functions take integeral values, by put-
ing Θ(t_0)=ζ, we get from (4):

(5) Q(R_θ^F)=Q(∑_{w∈W} \theta^w) = Q(ζ^{e}+ζ^{e+1}+⋯+ζ^{e+r−1}).

Lemma 5. Assume that θ is in general position. Let q=p^m. We further
assume that n is even. Then we have

ord_2[Q_p(R_θ^F): Q_p]=ord_2 m.

Let φ be the automorphism of Q_p(ζ) defined by ξ^φ=ξ^q. Then φ has order
n (by assumption) and we have Q_p(ζ)^φ=Q_p(R_θ^F) (cf. (5)). Let f=[Q_p(ζ): Q_p]
and e=|⟨ξ⟩|. Then f is equal to the least integer h≥1 subject for the condi-
tion: p^h≡1 (mod e) (see Serre [14], p. 85). As φ^h≡1 and φ^h≡1 for 1≤i≤n−1,
we find that |f|mn but f^i/mi for 1≤i≤n−1 [in fact, if f/mi, then p^i−1|m_i−1, hence
|f|p^e−1, hence φ^i≡1]. This shows that ord_f=ord_m+ord_n for any prime divisor
r of n. Thus, in particular, we have ord_f=ord_m+ord_n. As

[Q_p(ζ): Q_p(R_θ^F)]=[Q_p(ζ): Q_p(ζ)^φ]=n, we hence have ord_2[Q_p(R_θ^F): Q_p]=ord_2 m,
as desired.

Remark. Professor K. Imura showed to the author (by an elementary
proof) that n=f/(m, f) and [Q_p(ζ)^φ: Q_p]=(m, f).
Proposition 2. Let \( \chi \) be any cuspidal irreducible Deligne-Lusztig character \((-1)^{n-1} R^F_\chi\) of \( G^F = SL_n(F_q) \), where we assume that \( q \) is an even power of \( p \neq 2 \). Then, if \( n \) is odd or \( \text{ord}_q n \geq 2 \), we have \( m_q(\chi) = 1 \). Assume that \( \text{ord}_q n = 1 \). Then we have \( m_q(\chi) = 1 \) for any prime number \( r \) and \( m_q(\chi) = m_R(\chi) \leq 2 \). And we have \( m_R(\chi) = 2 \) if an only if \( \chi \) is real and \( \chi(1) = \chi(1) \) (i.e., \( \theta(-1) = -1 \)).

Remark. Let \( \chi \) be as above. Assume that \( n \) is even and let \( n = 2m \). Fixing a generator \( \theta_0 \) of \( \hat{T}^F \), put \( \theta = \theta_0 \). Then the following can be shown:

(i) \( \chi \) is real if and only if \( \frac{q^m - 1}{q - 1} \mid i \).

(ii) Assume that \( \text{ord}_q n = 1 \) and let \( i = \frac{q^m - 1}{q - 1} \) \( i' \) with \( i' \in \mathbb{Z} \) (hence \( \chi \) is real). Then \( \theta(-1) = 1 \) if and only if \( i' \) is even, and the latter condition is equivalent to the condition that \( \theta \mid Z^F = 1 \).

Proof of Proposition 2. Let \( \chi \in \Lambda_0 \). Then, by Lemma 1, we have \( \langle \chi, \chi \rangle_{G^F} = 1 \). Thus, if \( n \) is odd or \( \text{ord}_q n > \text{ord}_q(p - 1) \), by Proposition 1, (iii), we have \( m_q(\chi) = 1 \). Assume that \( 1 \leq \text{ord}_q n \leq \text{ord}_q(p - 1) \). Let \( t \) be an element of \( T^F \) having the property of Lemma 3. Then, under our assumption, we have \( t^{p-1} = -1 \) (cf. Proof of Lemma 4 and Proof of Lemma 3.3 (a) of Gow [8]). Let us use the notation of the proof of Proposition 1, (ii). Then \( \chi^M = \mu + \mu_1 \). As \( \mu_i(-1) = (-1)^i \mu_i(1) \) for \( i = 0, 1 \), by Schur’s lemma, we have \( \langle \chi, \mu \rangle_M = 1 \) if \( \chi(-1) = \chi(1) \), and \( \langle \chi, \mu \rangle_M = 1 \) if \( \chi(-1) = -\chi(1) \). As \( \mu_0 \) is realizable in \( Q \), we have \( m_q(\chi) = 1 \) in the first case. Assume that \( \chi(-1) = -\chi(1) \). If \( r \) is any prime number \( \neq p \), then \( \mu_1 \) is realizable in \( Q_r \), hence we have \( m_q(\chi) = 1 \). As \( q \) is an even power of \( p \), by Lemma 5, we have \( 2 \mid \langle Q_\chi(Q) : Q_\chi(Q) \rangle \). Hence \( A_1 \otimes_{Q_\chi(Q)}(\chi) \) splits (see [14], Chap. XIII, § 3, Prop. 7), hence \( \mu_1 \) is realizable in \( Q_\chi(Q) \). Hence we have \( m_q(\chi) = m_q(\chi) = 1 \). Thus we have \( m_q(\chi) = m_R(\chi) \). If \( \chi \) is real, we must have \( m_R(\chi) = 2 \), since otherwise \( \chi \) will be realizable in \( R \), so that, by Schur’s theorem, we have \( (2 = m_R(\chi)) \langle \chi, \mu \rangle_M = 1 \), a contradiction. If \( \text{ord}_q n \geq 2 \), then \( \chi \) cannot be real since \( G^F \) contains a central element \( z \) of order 4 such that \( z^2 = -1 \) and \( \chi(z) = \pm \sqrt{-1} \chi(1) \) ([7], p. 107). Finally, we note that, by [4], 1.22, we have \( \chi(-1) = -\chi(1) \) if and only if \( \theta(-1) = -1 \). This completes the proof of Proposition 2.

References


