AN EXAMPLE OF THE COMPLETION OF RANK FUNCTIONS OVER SIMPLE UNIT REGULAR RINGS

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In this note, we are concerned with Von Neumann regular rings having (pseudo-) rank functions. Let \( R \) be a regular ring and \( N \) a pseudo-rank function on \( R \). Then \( N \) induces a pseudo-metric topology on \( R \), and \( \bar{R} \), the completion of \( R \) at this pseudo-metric, is a right and left self-injective regular ring. If \( N \) is an extremal pseudo-rank function, \( \bar{R} \) is simple moreover. It is known that there exist uncountable nonisomorphic simple right and left self-injective regular rings [1, Cor. 2.9].

From this observation, K.R. Goodearl asked for two different extremal rank functions \( P, Q \) on a given simple unit-regular ring \( R \), whether the \( P \)-completion of \( R \) is isomorphic to the \( Q \)-completion of \( R \) or not. ([3, Open problem 38]). Now we answer that this problem is negative. Let \( F \) be any field and \( K_1 (=1, 2) \) any quadratic extensions of \( F \). We give an example of a simple regular \( F \)-algebra \( R \) with two extremal rank functions \( P_i \) such that the center of the \( P_i \)-completion of \( R \) is \( K_i \) \((i=1, 2)\). In particular, put \( F=\mathbb{Q}, \ K_1=\mathbb{Q}(i), \) and \( K_2=\mathbb{Q}(\sqrt{2}) \). Then, since \( \mathbb{Q}(i) \) is not isomorphic to \( \mathbb{Q}(\sqrt{2}) \) over \( \mathbb{Q} \), the \( P_1 \)-completion of \( R \) is not isomorphic to the \( P_2 \)-completion of \( R \).

We use most of our terminologies and notations from Goodearl's book [3].

1. A construction an example

Let \( K_1, K_2 \) be quadratic extension fields of a field \( F \) and \( g_i: K_i \rightarrow M_2(F) \) \((i=1, 2)\) matrix representations of \( K_i \) over \( F \) with respect to regular representation of \( K_i \). We shall construct an \( F \)-algebra \( R \), as a direct limit of a sequence \( R_1 \rightarrow R_2 \rightarrow \cdots \) of semisimple \( F \)-algebras. We shall refer to K.R. Goodearl's example [2, Scheme I] and D. Handelman's one [4, p.114]. Let \( p_1, p_2, \ldots \) be integers \((p_n > 2)\). Define positive integers \( \omega(1), \omega(2), \ldots \) by setting \( \omega_1=1 \) and \( \omega_n= (p_{n-1}+2)(p_{n-2}+2)\cdots (p_1+2) \) and put

\[
R_n = M_{\omega(n)}(F) \otimes F K_1 \oplus M_{\omega(n)}(F) \otimes F K_2
\]

Next we shall define \( F \)-algebra maps from \( R_n \) to \( R_{n+1} \). Let \( \{1, v_i\} \) be \( F \)-basis of \( K_1 \). Then any element of \( M_{\omega(n)}(F) \otimes F K_1 \) is written by the following form; \( x \otimes 1 + y \otimes v_i \), where \( x, y \in M_{\omega(n)}(F) \). We use \( x \bigotimes y \) to denote the Kronecker product of matrices \( x, y \in M_{\omega(n)}(F) \). Let \( I_n \) be the identity matrix in \( M_n(F) \).
Define maps $G_i: M_{w(n)}(F) \otimes F K_i \to M_{w(n)}(F)$ by the rule; $z_i = x \otimes 1 + y \otimes v_i \to x \oplus I_2 + y \oplus g_i(v_i)$. Define maps $\phi_n: R_n \to R_{n+1}$ by the rule;

$$
\begin{pmatrix}
G_n(z_1) & \cdots & G_n(z_2)
\end{pmatrix}
\begin{pmatrix}
[ z_1, \ z_2 ]
\end{pmatrix}
\in R_n
$$

$$
\begin{pmatrix}
G_n(z_1) & \cdots & G_n(z_2)
\end{pmatrix}
\begin{pmatrix}
[ z_1, \ z_2 ]
\end{pmatrix}
\in R_{n+1}
$$

where $x, x', y$ and $y' \in M_{w(n)}(F)$.

Now define $R$ to be the limit of $\{R_n, \phi_n\}$ and let $\theta_n: R_n \to R$ natural embeddings. Obviously $R$ is a simple unit-regular $F$-algebra with the center $F$.

Next we shall determine all (extremal) rank functions on $R$. We use $P(R)$ to denote the set of all rank functions on $R$. Put $R'_n = M_{w(n)}(F) \oplus M_{w(n)}(F)$ for each $n$. We consider $R'_n$ as a sub-$F$-algebra of $R_n$ by the embedding $[x, y] \to [x \otimes 1, y \otimes 1]$ where $x, y \in M_{w(n)}(F)$. Put $\phi'_n = \phi_n| R'_n$, then $\phi'_n$ is as follows:

$$
\begin{pmatrix}
[x, \ y]
\end{pmatrix}
\in R'_n
$$

$$
\begin{pmatrix}
x & \cdots & x
\end{pmatrix}
\begin{pmatrix}
[ x \ y ]
\end{pmatrix}
\in R'_{n+1}
$$

Define $R'$ to be the limit of $\{R', \phi'_n\}$.

**Lemma 1.** $P(R)$ is affinely homeomorphic to $P(R')$ by the restriction map.

**Proof.** For any $N \in P(R_n)$ (resp. $P(R'_n)$),

$$N([A, B]) = \frac{\alpha \text{ rank } (A) + \beta \text{ rank } (B)}{w(n)}$$

, where $A \in M_{w(n)}(F) \otimes K_1$, $B \in M_{w(n)}(F) \otimes K_2$ (resp. $A, B \in M_{w(n)}(F)$), $\alpha = \beta = N([I_n, O])$ by [3, Cor. 16.6]. Then $P(R_n)$ is affinely homeomorphic to $P(R'_n)$ by the restriction map for all $n$. Since $P(R)$ (resp. $P(R')$) is the inverse limit of $\{P(R_n), \phi'_n\}$ (resp. $\{P(R'_n), \phi'_n\}$), by [3, Prop. 16.21], $P(R)$ is affinely homeomorphic to $P(R')$.

The structure of $P(R')$ has been determined by K.R. Goodearl [2, pp. 277-280]. For the sake of completeness, we shall again explain it.

Put $u(1) = 1$ and $u(n+1) = (p_n - 2) \cdots (p_1 - 2)$ for all $n \geq 1$. For $N \in P(R')$, there exist positive real numbers $\alpha_n(t)$ ($n = 1, 2, \cdots; t = 1, 2$) such that

1. $\alpha_n(1) + \alpha_n(2) = 1$ for all $n$
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\( (2) \quad \alpha_{n+1}(i) = \frac{(p_n+2)\alpha_n(i) - 2}{p_n - 2} \quad \text{for all } n, i \)

\( (3) \quad N([A, B]) = \frac{\alpha_1(1)\rank(x) + \alpha_2(2)\rank(y)}{w(n)} \quad \text{for all } [A, B] \in R', \)

where \([A, B] = \theta_n([x, y])\) for some \(n\) and \([x, y] \in R'. \) Conversely, if \(\{\alpha_n(i)\}\) are any positive real numbers satisfying (1) and (2), then (3) defines a rank function \(N\) on \(R'.\)

Now we assume that \(\lim_{n \to \infty} \frac{u(n)}{w(n)} > 0.\) Put \(\lambda = \frac{1}{2} \lim_{n \to \infty} \frac{u(n)}{w(n)}\)

We define

\[ \alpha_n(1) = \frac{1}{2} + \lambda \frac{u(n)}{w(n)} \quad \beta_n(1) = \frac{1}{2} - \lambda \frac{w(n)}{u(n)} \]

\[ \alpha_n(2) = \frac{1}{2} - \lambda \frac{w(n)}{u(n)} \quad \beta_n(2) = \frac{1}{2} + \lambda \frac{w(n)}{u(n)} \]

for all \(n \geq 1.\) Then \(\{\alpha_n(i)\}\) and \(\{\beta_n(i)\}\) satisfy the above conditions (1) and (2).

Let \(N_1\) (resp. \(N_2\)) be the rank function determined by \(\{\alpha_n(i)\}\) (resp. \(\{\beta_n(i)\}\)). by [2, Lemma 27], \(N_1\) and \(N_2\) are all extremal rank functions on \(R'.\) Therefore, by Lemma 1, \(N_i\) and \(N_2\) can be extended to extremal rank functions on \(R.\) \(N_i\) \((i=1, 2)\) induce metrics on \(R\) given by the rule; \(d_i(x, y) = N_i(x - y)\) for \(x, y \in R,\) which we call the \(N_i\)-metric \([3, \S 19].\) Let \(T_i\) be the completion of \(R\) with respect to \(N_i\)-metric \((i=1, 2).\) Then \(T_i\) are simple regular, right and left self-injective \(F\)-algebras by [3, Th. 19. 14].

2. Calculation of the centers of \(T_i\)

In this note, we shall calculate the center \(Z(T_i)\) of \(T_i.\) Let \(I_n\) be the identity matrix for \(M_n(F)\) and \(\theta_n\) the natural embedding: \(R_n \to R.\)

**Lemma 2.** If \(\sum \frac{1}{(P_n+2)} < \infty,\) then \(\{\theta_n([I_{w(n)} \otimes \alpha, 0])\}\) (resp. \(\theta_n([0, I_{w(n)} \otimes \beta])\)) is a Cauchy sequence with respect to \(N_1\)-metric (resp. \(N_2\)-metric) for each \(\alpha \in K_1\) (resp. \(\beta \in K_2).\)

**Proof.** Put \(K = K_1\) and \(N = N_1.\) For \(\alpha \in K\) and each \(n,\) we see that

\[
\phi_n([I_{w(n)} \otimes \alpha, 0]) = \begin{pmatrix}
g_1(\alpha) & \omega(n) \\
g_1(\alpha) & I_{w(n)} \otimes \alpha \end{pmatrix}
\begin{pmatrix}
p_n - 2 & 0 \\
0 & I_{w(n)} \otimes \alpha
\end{pmatrix}
\begin{pmatrix}
g_1(\alpha) & \omega(n) \\
0 & 0
\end{pmatrix},
\]

\[
\begin{pmatrix}
g_1(\alpha) & \omega(n) \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
p_n - 2 & 0 \\
0 & I_{w(n)} \otimes \alpha
\end{pmatrix}
\begin{pmatrix}
g_1(\alpha) & \omega(n) \\
0 & 0
\end{pmatrix}.
\]
Therefore, we have

\[
[I_{\omega(n+1)} \otimes \alpha, 0] - \phi_n([I_{\omega(n)} \otimes \alpha, 0]) \\
= \begin{pmatrix}
\alpha I_2 - g_1(\alpha) & \omega(n) \\
\alpha I_2 - g_1(\alpha) & 0 \\
\alpha & 2\omega(n) \\
\end{pmatrix}
\begin{pmatrix}
-g_1(\alpha) \\
-\omega(n) \\
0 \\
\end{pmatrix}.
\]

We can calculate that

\[
N(\theta_n([I_{\omega(n+1)} \otimes \alpha, 0]) - \theta_n([I_{\omega(n)} \otimes \alpha, 0])) \\
= N([I_{\omega(n+1)} \otimes \alpha, 0] - \phi_n([I_{\omega(n)} \otimes \alpha, 0]) \\
= \frac{1}{\omega(n+1)} \{ \alpha_{n+1}(1) \omega(n) \text{rank}(\alpha I_2 - g_1(\alpha)) + 2\omega(n) + \alpha_{n+1}(2) \omega(n) \text{rank}(g_1(\alpha)) \} \\
< \frac{1}{\omega(n+1)} \left[ \left( \frac{1}{2} + \frac{\omega(n+1)}{u(n+1)} \right) 4\omega(n) + \left( \frac{1}{2} - \frac{\omega(n+1)}{u(n+1)} \right) 4\omega(n) \right] \\
< \frac{4}{(p_n + 2)}.
\]

Then \{\theta_n([I_{\omega(n)} \otimes \alpha, 0])\} is a Cauchy sequence.

By Lemma 2, we define \(\tau_i(\alpha) = \text{lim} \theta_n([I_{\omega(n)} \otimes \alpha, 0])\) (resp. \(\tau_2(\beta) = \theta_n([0, I_{\omega(n)} \otimes \beta])\)) for each \(\alpha \in K_1\) (resp. \(\beta \in K_2\)). Then \(\tau_i: K_i \to T_i\) is a map as \(F\)-algebra for \(i = 1, 2\).

**Lemma 3.**

(1) \(\tau_i(K_i) \subseteq Z(T_i)\) for \(i = 1, 2\).

(2) \(\tau_i(a) = a\) for all \(a \in F\).

**Proof.** (1) For any \(r \in R\) and \(\alpha \in K_1\), we shall show that \(\tau_i(\alpha r) = r\tau_i(\alpha)\). Let \(r = \theta_n([x, y])\) for some \(n\) and \([x, y] \in R_n\). Since \([I_{\omega(b)} \otimes \alpha, 0] [x, y] = [x, y] [I_{\omega(b)} \otimes \alpha, 0]\) for all \(k > n\), we have that \(\tau_i(\alpha r) = r\tau_i(\alpha)\). Since \(T_1\) is the completion of \(R\) with respect to \(N_1\)-metric, we have that \(\tau_i(\alpha x) = x\tau_i(\alpha)\) for all \(x \in T_1\).

(2) Since \(a = \theta_n([I_{\omega(n)} \otimes a, I_{\omega(n)} \otimes a])\) for all \(a \in F\) and all \(n\), we see that

\[
N_1(a - \theta_n([I_{\omega(n)} \otimes a, 0])) \\
= N_1([0, I_{\omega(n)} \otimes a]) \\
= \alpha_n(2)
\]

Therefore we have that \(a = \text{lim} \theta_n([I_{\omega(n)} \otimes a, 0])\), because \(\text{lim} \alpha_n(2) = 0\).
Lemma 4. Let $p_1, p_2, \ldots$ be integers such that $\lim_{n \to \infty} \frac{u(n)}{w(n)} = \frac{1}{2}$. Then $\tau_i: K_i \rightarrow Z(T_i)$ is an isomorphism over $F$.

Proof. Put $T = T_1$ and $N = N_1$. We shall show that $\tau_1(K_1) = Z(T)$. First for any $x \in Z(T)$ and any real number $\varepsilon > 0$, there exists $r \in R$ such that $N(x - r) < \varepsilon / 4$. And there exists $r_{k(1)} \in R_{k(1)}$ such that $r = \theta_{k(1)}(r_{k(1)})$. We note that $r = \theta_m(r_m)$ for all $m \geq k(1)$, where some $r_m \in R_m$. For any $r_m$, there exist $z_m \in Z(R_m)$ and $y_m \in R_m$ such that $N(r_m - z_m) \leq N(r_m, y_m - y_m^2)$ by [1, Cor. 2.4].

Since $N(r_m - z_m) \leq N(r_m, y_m - y_m^2)$
\[ \leq N((r - x)\theta_m(y_m^2)) + N(\theta_m(y_m)(x - r)) \]
\[ \leq 2N(x - r) < \varepsilon / 2, \]
we see that for any $m \geq k(1)$,

\[(*) \quad N(x - \theta_m(z_m)) \leq N(x - r) + N(r - \theta_m(z_m)) \]
\[ \leq N(x - r) + N(r_m - z_m) < \varepsilon / 2. \]

Put $z_m = [I_{w(n)} \otimes \alpha_m, I_{w(n)} \otimes \beta_m]$ for some $\alpha_m \in K_1$ and $\beta_m \in K_2$. Since $\lim_{n \to \infty} \alpha_n(2) = 0$, there exists $k(2)$ such that $\alpha_n(2) < \varepsilon / 4$ for all $m \geq k(2)$. We see that for all $m \geq \max(k(1), k(2))$, \[ N(x - \theta_m([I_{w(n)} \otimes \alpha_m, 0])) \]
\[ \leq N(x - \theta_m(z_m)) + N([0, I_{w(n)} \otimes \beta_m]) \]
\[ \leq N(x - \theta_m(z_m)) + \alpha_m(2) < \varepsilon. \] (by \((*)\))

Since $\sum_{n=1}^{\infty} \frac{4}{(p_n + 2)} < \infty$, for $\varepsilon$, there exists a natural number $k(3)$ such that

\[(**) \quad \sum_{n=1}^{l} \frac{4}{(p_n + 2)} < \varepsilon \quad \text{for all} \quad l > k(3). \]

Select some $k \geq \max(k(i); i = 1, 2, 3)$. Then we have already seen that $\alpha_k \in K_1$ and $N(x - \theta_k([I_{w(k)} \otimes \alpha_k, 0])) < \varepsilon$. Put $\gamma = \alpha_k$. We shall show that $N(x - \tau_i(\gamma)) < \varepsilon$. There exists a positive integer $k(4) > k$ such that for any $m \geq k(4)$,

\[(***) \quad N(\theta_m([I_{w(n)} \otimes \gamma, 0] - \tau_i(\gamma))) < \varepsilon. \]

We see that for some $m \geq \max\{k(i); i = 1, 2, 3, 4\}$,
\[ N(x-\tau_1(\gamma)) \leq N(x-\theta_0([I_{u(0)} \otimes \gamma], 0]) + N(\theta_0([I_{u(0)} \otimes \gamma], 0]) - \theta_{k+1}([I_{u(k+1)} \otimes \gamma], 0]) \]
\[ + N(\theta_{m-1}([I_{u(m-1)} \otimes \gamma], 0]) - \theta_{m}([I_{u(m)} \otimes \gamma], 0]) + N(\theta_{m}([I_{u(m)} \otimes \gamma], 0]) - \tau_1(\gamma)) \]

, using the inequality in the proof of Lemma 2, (**) and (***)

\[ < \epsilon + \sum_{n=1}^{r-1} 4/(p_{i}+2) + \epsilon \]
\[ < 6\epsilon . \]

Since \( T \) is a simple ring, \( Z(T) \) is a field, so if \( \epsilon \) is less than 1/6, \( N(x-\tau_1(\gamma)) = 0 \). Therefore \( x \) belongs to \( \tau_1(K_1) \).

Now we shall give a negative answer for the Goodearl's problem No. 38 [3, p. 348].

**Example** There exists a simple unit-regular ring \( R \) such that

1. \( R \) has two extremal rank functions \( N_1, N_2 \).
2. The \( N_1 \)-completion of \( R \) is not isomorphic to the \( N_2 \)-completion of \( R \).

Proof. Set \( F=\mathbb{Q}, K_1=\mathbb{Q}(i) \) and \( K_2=\mathbb{Q}(\sqrt{2}). \) Put \( p_n=n^2+4n+2 \) for all \( n \), and construct \( R \) according to the previous method. Since \( \frac{w(n)}{u(n)} = \frac{2(n+2)(n+3)}{9n(n+1)} \),

we have \( \lim_{n \to \infty} \frac{u(n)}{w(n)} = \frac{2}{9} \). And we see that \( \sum_{n=1}^{\infty} \frac{1}{p_{n}+2} < \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)} = \frac{5}{12} \). By Lemma 4, the \( N_1 \)-completion \( T_1 \) of \( R \) is not isomorphic to the \( N_2 \)-completion \( T_2 \) of \( R \), because \( Z(T_1)=\mathbb{Q}(i) \) is not isomorphic to \( Z(T_2)=\mathbb{Q}(\sqrt{2}) \) over \( \mathbb{Q} \).

**References**


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