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# ON DOMINANT DIMENSION OF NOETHERIAN RINGS

Dedicated to Professor Hiroyuki Tachikawa on his 60th birthday

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Throughout this note, $R$ stands for a ring with identity and all modules are unital modules. In this note, for a given module $M$, we say that $M$ has *dominant dimension* at least $n$, written dom dim $M \geqq n$, if each of the first $n$ terms of the minimal injective resolution of $M$ is flat. Following Morita [5], we call $R$ left (resp. right) $QF$-3 if dom dim ${}_RR \geqq 1$ (resp. dom dim $R_R \geqq 1$). He showed that if $R$ is left noetherian and left $QF$-3 then it is also right $QF$-3. Thus, if $R$ is left and right noetherian, $R$ is left $QF$-3 if and only if it is right $QF$-3. Generalizing this, we will prove the following

**Theorem.** *Let $R$ be left and right noetherian. For any $n \geqq 1$,* dom dim ${}_RR \geqq n$ *if and only if* dom dim $R_R \geqq n$.

In case $R$ is artinian, our dominant dimension coincides with Tachikawa's one [8], and the above theorem has been established (see Tachikawa [9] for details).

In what follows, for a given left or right $R$-module $M$, we denote by $M^*$ the $R$-dual of $M$, by $\varepsilon_M: M \rightarrow M^{**}$ the usual evaluation map and by $E(M)$ the injective hull of $M$. We denote by mod $R$ (resp. mod $R^{op}$) the category of all finitely generated left (resp. right) $R$-modules, where $R^{op}$ stands for the opposite ring of $R$ and right $R$-modules are considered as left $R^{op}$-modules.

**1. Preliminaries.** In this section, we recall several known facts which we need in later sections.

**Lemma 1.1.** *Let $R$ be right noetherian. For any $N \in$* mod $R^{op}$ *and for any injective left $R$-module $E$,* $\mathrm{Hom}_R(\mathrm{Ext}_R^i(N, R), E) \simeq \mathrm{Tor}_i^R(N, E)$ *for $i \geqq 1$.*

Proof. See Cartan and Eilenberg [1, Chap. VI, Proposition 5.3].

**Lemma 1.2.** *Every finitely presented submodule of a flat module is torsionless.*

Proof. See Lazard [4, Théorème 1.2].

**Lemma 1.3.** *Let $R$ be right noetherian. Let $E$ be an injective left $R$-module* *and suppose that every finitely generated submodule of $E$ is torsionless. Then $E$ is flat.*

Proof. See Sato [6, Lemma 1.4]. His argument remains valid in our setting.

**Lemma 1.4.** *Let $R$ be left and right noetherian. Suppose that $R$ is left QF-3. An injective left $R$-module $E$ is flat if and only if it is cogenerated by $E({}_RR)$.*

Proof. Immediate by Lemmas 1.2 and 1.3.

**Lemma 1.5.** *Let $R$ be left noetherian. Suppose that* inj dim $R_R=n<\infty$. *For a minimal injective resolution $O\to{}_RR\to E_0\to E_1\to\cdots$, $E=\bigoplus_{i=0}^{n} E_i$ is an injective cogenerator.*

Proof. See Iwanaga [3, Theorem 2]. His argument remains valid in our setting.

2. **Proof of Theorem.** In order to prove the theorem, we need two more lemmas.

**Lemma 2.1.** *Let $R$ be left noetherian and $n\geqq 1$. For any $M\in\operatorname{mod} R$ with $\operatorname{Ext}_R^i(M, R)=0$ for $1\leqq i\leqq n$ and for any $L\in\operatorname{mod} R$ with* proj dim $L=m<n$, *$\operatorname{Ext}_R^i(M, L)=0$ for $1\leqq i\leqq n-m$.*

Proof. By induction on $m\geqq 0$. The case $m=0$ is clear. Let $m\geqq 1$ and let $0\to K\to P\to L\to 0$ be an exact sequence in mod $R$ with $P$ projective. Since proj dim $K=m-1$, by induction hypothesis $\operatorname{Ext}_R^i(M, K)=0$ for $1\leqq i\leqq n-m+1$. Applying the functor $\operatorname{Hom}_R(M, -)$ to the above exact sequence, we get $\operatorname{Ext}_R^i(M, L)\simeq\operatorname{Ext}_R^{i+1}(M, K)=0$ for $1\leqq i\leqq n-m$.

**Lemma 2.2.** *Let $R$ be left and right noetherian. Suppose that $R$ is left QF-3. For any $n\geqq 2$,* dom dim ${}_RR\geqq n$ *if and only if for an $M\in\operatorname{mod} R$, $M^*=0$ implies $\operatorname{Ext}_R^i(M, R)=0$ for $1\leqq i\leqq n-1$.*

Proof. Let $0\to{}_RR\xrightarrow{f_0}E_0\xrightarrow{f_1}E_1\to\cdots$ be a minimal injective resolution. For any $i\geqq 1$ we have an exact sequence of functors

$$\operatorname{Hom}_R(-, E_{i-1})\to\operatorname{Hom}_R(-, \operatorname{Im} f_i)\to\operatorname{Ext}_R^i(-, R)\to 0\,.$$

"Only if" part. For a given $M\in\operatorname{mod} R$ with $M^*=0$, by Lemma 1.2 $\operatorname{Hom}_R(M, E_i)=0$ for $1\leqq i\leqq n-1$. Thus $\operatorname{Hom}_R(M, \operatorname{Im} f_i)=0$, and by the above exact sequence $\operatorname{Ext}_R^i(M, R)=0$ for $1\leqq i\leqq n-1$.

"If" part. By induction on $i\geqq 0$, we show that $E_i$ is flat for $0\leqq i\leqq n-1$. By assumption, $E_0$ is flat. Let $1\leqq i\leqq n-1$ and suppose that $E_{i-1}$ is flat. For a given $M\in\operatorname{mod} R$ with $M^*=0$, we claim $\operatorname{Hom}_R(M, \operatorname{Im} f_i)=0$. We have

$\mathrm{Ext}_R^i(M, R)=0$. Also, by Lemma 1.2 $\mathrm{Hom}_R(M, E_{i-1})=0$. Thus by the above exact sequence $\mathrm{Hom}_R(M, \mathrm{Im} f_i)=0$. Hence $\mathrm{Im} f_i$ is cogenerated by $E({}_RR)$, and by Lemma 1.4 $E_i$ is flat.

We are now in a position to prove the theorem. It suffices to prove the "only if" part.

"Only if" part of Theorem. The case $n=1$ is due to Morita [5, Theorem 1]. Let $n\geqq 2$. Note that $R$ is left and right $QF$-3. Replacing $R$ with $R^{op}$ in Lemma 2.2, it suffices to show that for any $N\in \mathrm{mod}\, R^{op}$ with $N^*=0$ we have $\mathrm{Ext}_R^i(N, R)=0$ for $1\leqq i\leqq n-1$. For a given $N\in\mathrm{mod}\, R^{op}$ with $N^*=0$, we claim first that $\mathrm{Ext}_R^i(N, R)^*=0$ for $i\geqq 1$. For any $i\geqq 1$, by Lemma 1.1 $\mathrm{Hom}_R(\mathrm{Ext}_R^i(N, R), E({}_RR))\simeq\mathrm{Tor}_i^R(N, E({}_RR))=0$, thus $\mathrm{Ext}_R^i(N, R)^*=0$. Hence by Lemma 2.2 $\mathrm{Ext}_R^j(\mathrm{Ext}_R^i(N, R), R)=0$ for $i\geqq 1$ and $1\leqq j\leqq n-1$. Now, by induction on $i\geqq 1$, we show that $\mathrm{Ext}_R^i(N, R)=0$ for $1\leqq i\leqq n-1$. Let $\cdots\to P_1\xrightarrow{f_1}P_0\xrightarrow{f_0}N\to 0$ be an exact sequence in mod $R^{op}$ with the $P_i$ projective and put $N_i=\mathrm{Im} f_i$. Since $N^*=0$, we have an exact sequence

$$0\to P_0^*\xrightarrow{\beta_1}N_1^*\xrightarrow{\alpha_1}\mathrm{Ext}_R^1(N, R)\to 0\,.$$

Since $\mathrm{Ext}_R^1(\mathrm{Ext}_R^1(N, R), R)=0$, $\alpha_1$ splits. On the other hand, since $\mathrm{Ext}_R^1(N, R)^*=0$, $\mathrm{Hom}_R(\mathrm{Ext}_R^1(N, R), N_1^*)=0$. Thus $\mathrm{Ext}_R^1(N, R)=0$. Next, let $1<i\leqq n-1$ and suppose that $\mathrm{Ext}_R^j(N, R)=0$ for $1\leqq j\leqq i-1$. We have an exact swquence

$$0\to P_0^*\to\cdots\to P_{i-1}^*\xrightarrow{\beta_i}N_i^*\xrightarrow{\alpha_i}\mathrm{Ext}_R^i(N, R)\to 0\,.$$

Since $\mathrm{Ext}_R^j(\mathrm{Ext}_R^i(N, R), R)=0$ for $1\leqq j\leqq n-1$, and since proj dim $\mathrm{Im}\,\beta_i\leqq i-1<n-1$, by Lemma 2.1 $\mathrm{Ext}_R^1(\mathrm{Ext}_R^i(N, R), \mathrm{Im}\,\beta_i)=0$. Thus $\alpha_i$ splits. On the other hand, $\mathrm{Ext}_R^i(N, R)^*=0$ implies $\mathrm{Hom}_R(\mathrm{Ext}_R^i(N, R), N_i^*)=0$. Hence $\mathrm{Ext}_R^i(N, R)=0$.

**3. Left exactness of the double dual.** In this section, we establish the relation between the dominant dimension of a left and right noetherian ring $R$ and the behavior of the functor $(\ )^{**}$: mod $R\to$mod $R$. Compare our results with Colby and Fuller [2, Theorems 1 and 2].

**Proposition 3.1.** *Let $R$ be left and right noetherian. Then $R$ is left $QF$-3 if and only if the functor $(\ )^{**}$: mod $R\to$mod $R$ preserves monomorphisms.*

This is an immediate consequence of Morita [5, Theorem 1] and the following lemmas.

**Lemma 3.2.** *Let $R$ be left noetherian and right $QF$-3. For any monomorphism $\alpha\colon M\to L$ with $M, L\in\mathrm{mod}\, R$, $\alpha^{**}$ is monic.*

Proof. For a given exact sequence $0\to M\xrightarrow{\alpha}L\to K\to 0$ in mod $R$, we claim $(\text{Cok}\,\alpha^*)^*=0$. By Lemma 1.1 $\text{Hom}_R(\text{Ext}_R^1(K, R), E(R_R))\simeq\text{Tor}_1^R(E(R_R), K)=0$. Since Cok $\alpha^*$ is imbedded into $\text{Ext}_R^1(K, R)$, we get $\text{Hom}_R(\text{Cok}\ \alpha^*, E(R_R))=0$. Thus $(\text{Cok}\ \alpha^*)^*=0$, and $\alpha^{**}$ is monic.

**Lemma 3.3.** *Let $R$ be right noetherian. Suppose that for any monomorphism $\alpha$: $M\to L$ with $M$, $L\in$mod $R$ $\alpha^{**}$ is monic. Then $R$ is left QF-3.*

Proof. For a given $M\in\text{mod}\,R$ with $M\subset E({}_RR)$, we claim that $M$ is torsionless. Replacing $M$ with $M+R$ if necessary, we may assume $R\subset M$. Denote by $\iota$ the inclusion $R\hookrightarrow M$. Since $\iota^{**}$ is monic, so is $\iota^{**}\circ\varepsilon_R=\varepsilon_M\circ\iota$. Thus $R\cap\text{Ker}\,\varepsilon_M=0$, which implies Ker $\varepsilon_M=0$. Hence by Lemma 1.3 $E({}_RR)$ is flat.

Now we can prove the following

**Proposition 3.4.** *Let $R$ be left and right noetherian. Then* dom dim ${}_RR\geqq 2$ *if and only if the functor* ( )**: mod $R\to$mod $R$ *is left exact.*

Proof. "Only if" part. For a given exact sequence $0\to M\xrightarrow{\alpha}L\xrightarrow{\beta}K\to 0$ in mod $R$, we claim $(\text{Cok}\ \alpha^*)^*=0=\text{Ext}_R^1(\text{Cok}\ \alpha^*, R)$. Note that dom dim $R_R\geqq 2$. By Lemma 3.2, $\alpha^{**}$ is monic. Thus $(\text{Cok}\,\alpha^*)^*=0$, and by Lemma 2.2 $\text{Ext}_R^1(\text{Cok}\ \alpha^*, R)=0$. Hence the following sequence is exact:

$$0\to M^{**}\xrightarrow{\alpha^{**}}L^{**}\xrightarrow{\beta^{**}}K^{**}\,.$$

"If" part. By Lemma 3.3, $E({}_RR)$ is flat. For a given $M\in\text{mod}\,R$ with $M\subset E({}_RR)/R$, we claim that $M$ is torsionless. There is some $L\in\text{mod}\,R$ such that $L\subset E({}_RR)$ and $M=L/R$. By Lemma 1.2, $L$ is torsionless. We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0\to & R & \longrightarrow & L & \longrightarrow & M & \longrightarrow 0\\ & \downarrow\wr & & \downarrow\varepsilon_L & & \downarrow\varepsilon_M & \\ 0\to & R^{**} & \to & L^{**} & \to & M^{**}\,. & \end{array}$$

Since $\varepsilon_L$ is monic, so is $\varepsilon_M$. Thus by Lemma 1.4 $E(E({}_RR)/R)$ is flat.

## 4. Remarks.

In this final section, we make some remarks on noetherian rings of finite self-injective dimension.

The following proposition is essentially due to Iwanaga [3].

**Proposition 4.1.** *Let $R$ be left noetherian. Suppose that* inj dim ${}_RR<\infty$ *and that the last non-zero term of the minimal injective resolution of ${}_RR$ is flat. Then $R$ is quasi-Frobenius.*

Proof. Suppose to the contrary that ${}_RR$ is not injective. Put $n=$inj dim ${}_RR$

and let $0 \to {}_R R \to E_0 \to E_1 \to \cdots \to E_n \to 0$ be a minimal injective resolution. There is a torsion theory $(\mathcal{T}, \mathcal{F})$ in mod $R$ such that $\mathcal{F}$ consists of the modules $M \in \operatorname{mod} R$ with $\operatorname{Ext}_R^n(M, R)=0$. Note that $\mathcal{T}$ contains a simple module $L$. Since $E_n$ is flat, and since $\operatorname{Hom}_R(L, E_n) \simeq \operatorname{Ext}_R^n(L, R) \neq 0$, by Lemma 1.2 $L$ is torsionless, which implies $L \in \mathcal{F}$, a contradiction.

**Proposition 4.2.** *Let $R$ be left noetherian. Suppose that* inj dim $R_R <$ dom dim ${}_R R$. *Then $E({}_R R)$ is an injective cogenerator.*

Proof. Let $0 \to {}_R R \to E_0 \to E_1 \to \cdots$ be a minimal injective resolution and put $E = \bigoplus_{i=0}^{n} E_i$, where $n=$ inj dim $R_R$. By Lemma 1.5 $E$ is an injective cogenerator. Thus, since $E$ is flat, by Lemma 1.2 every $M \in \operatorname{mod} R$ is torsionless, namely $E({}_R R)$ is an injective cogenerator.

The next proposition generalizes Sumioka [7, Theorem 5].

**Proposition 4.3.** *Let $R$ be left and right noetherian and $n \geqq 1$. Suppose that* inj dim ${}_R R \leqq n \leqq$ dom dim ${}_R R$. *For a minimal injective resolution $0 \to {}_R R \to E_0 \to E_1 \to \cdots$, $E = \bigoplus_{i=0}^{n} E_i$ is an injective cogenerator if and only if* inj dim $R_R \leqq n$.

Proof. "Only if" part. Since $E_i$ is flat for $0 \leqq i \leqq n-1$, and since $E_i=0$ for $i>n$, $E_n$ and thus $E$ have weak dimension at most $n$. Thus by Lemma 1.1 $\operatorname{Hom}_R(\operatorname{Ext}_R^{n+1}(N, R), E) \simeq \operatorname{Tor}_{n+1}^R(N, E)=0$ for all $N \in \operatorname{mod} R^{op}$. Hence, since $E$ is an injective cogenerator, inj dim $R_R \leqq n$.

"If" part. By Lemma 1.5.

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