ON FINITE POINT TRANSITIVE AFFINE PLANES
WITH TWO ORBITS ON $l_{\infty}$

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1. Introduction

Kallaher [3] proposed the following conjecture.

Conjecture. Let $\pi$ be a finite affine plane of order $n$ with a collineation
group $G$ which is transitive on the affine points of $\pi$. If $G$ has two orbits on the
line at infinity, then one of the following statements holds:

(i) The plane $\pi$ is a translation plane, and the group $G$ contains the group
of translations of $\pi$.

(ii) The plane $\pi$ is a dual translation plane, and the group $G$ contains the
group of dual translations of $\pi$.

The purpose of this paper is to study this conjecture. When
$G_A$ has two
orbits of length 1 and $n$ on the line at infinity, where $A$ is an affine point of $\pi$,
some work has been done on this conjecture. (See Johnson and Kallaher [2].)

Our notation is largely standard and taken from [3]. Let $P=P_{\infty}$ be the
projective extension of an affine plane $\pi$, and $G$ a collineation group of $P$.
If $P$ is a point of $P$ and $l$ is a line of $P$, then $G(P, l)$ is the subgroup of $G$ consisting
of all perspectivities in $G$ with center $P$ and axis $l$. If $m$ is a line of $P$, then
$G(m, m)$ is the subgroup consisting of all elations in $G$ with axis $m$.

In § 2 we prove the following theorem.

Theorem 1. Let $\pi$ be a finite affine plane of order $n$ with a collineation group
$G$ and let $\Delta$ be a subset of $l_{\infty}$ such that $|\Delta| = t \geq 2$, $(n, t) = 1$ and $(n, t - 1) = 1$. If
there is an integer $k_i > 1$ such that $|G(P, l_{\infty})| = k_i$ for all $P \in \Delta$ and there is an
integer $k_1 > 1$ such that $|G(Q, l_{\infty})| = k_1$ for all $Q \in l_{\infty} - \Delta$, then $\pi$ is a translation
plane, and $G$ contains the group $T$ of translations of $\pi$.

In § 3 and § 4, we prove the following theorem by using Theorem 1.

Theorem 2. Let $\pi$ be a finite affine plane of order $n$ with a collineation
group $G$ which is transitive on the affine points of $\pi$. If $G$ has two orbits of length
2 and $n - 1$ on $l_{\infty}$, then one of the following statements holds:
(i) The plane $\pi$ is a translation plane, and the group $G$ contains the group $T$ of translations of $\pi$.

(ii) $|G(\ell_\omega, \ell_\omega)| = n = 2^m$ for some $m \geq 1$, $G(P_1, \ell_\omega) = G(P_2, \ell_\omega) = 1$ and $|G(P, \ell_\omega)| = 2$ for all $P \in \ell_\omega - \{P_1, P_2\}$.

The planes which are not Andrê planes, satisfying the hypothesis of Theorem 2, include a class of translation planes of order $q^3$, where $q$ is an odd prime power. (See Suetake [4] and Hiramine [1].)

2. The proof of Theorem 1

In this section, we prove Theorem 1.

Let $\pi$ be a finite affine plane of order $n$ with a collineation group $G$, satisfying the hypothesis of Theorem 1. By Theorem 4.5 of [3], $G(\ell_\omega, \ell_\omega)$ is an elementary abelian $r$-group for some prime $r$ dividing $n$. Hence there exist positive integers $m$ and $s$ such that $k_1 = r^m$ and $k_2 = r^s$. Let $P$ be a point of $\pi$ such that $P \in \Delta$. Let $\ell$ be an affine line of $\pi$ such that $\ell \ni P$. Since $G(P, \ell_\omega)$ is semi-regular on $\ell - \{P\}$, $r^m | n$. Similarly, $r^s | n$. By definition, $G(\ell_\omega, \ell_\omega) = \bigcup_{P \in \ell_\omega} G(P, \ell_\omega)$ and $G(P, \ell_\omega) \cap G(Q, \ell_\omega) = 1$ for distinct points $P, Q \in \ell_\omega$. Thus

$$|G(\ell_\omega, \ell_\omega)| = 1 + \sum_{P \in \Delta} (|G(P, \ell_\omega)| - 1) + \sum_{Q \in \ell_\omega - \Delta} (|G(Q, \ell_\omega)| - 1)$$

$$= 1 + t(r^m - 1) + (n + 1 - t)(r^s - 1).$$

Since $r^m | |G(\ell_\omega, \ell_\omega)|$, it follows $0 \equiv 1 - t + (1 - t)r^s - 1 + t \pmod{r^m}$. Therefore $(t - 1)r^s \equiv 0 \pmod{r^m}$. Since $(t - 1, r) = 1$, this implies $r^s | r^m$. Thus $m \leq s$. On the other hand, since $r^s | |G(\ell_\omega, \ell_\omega)|$, it follows $0 \equiv 1 + t(r^m - 1) - 1 + t \pmod{r^m}$. Therefore $tr^m \equiv 0 \pmod{r^s}$. Since $(t, r) = 1$, this implies $r^s | r^m$. Thus $m \geq s$. Therefore $m = s$ and $k_1 = k_2$. By a result of Gleason (See Theorem 5.2 of [3].), the theorem holds.

3. The proof of Theorem 2 when $n$ is odd

In this section, we prove Theorem 2 when $n$ is odd.

Let $\pi$ be a finite affine plane of odd order $n$ with a collineation group $G$ which is transitive on the affine points of $\pi$, satisfying the hypothesis of Theorem 2. Then $G$ has an orbit $\Delta = \{P_1, P_2\}$ of length 2 on $\ell_\omega$. Let $A$ be an affine point of $\pi$. Let $\Phi$ be the set of the affine points of $\pi$, and let $\Omega = \Phi \cup \ell_\omega$. Then $G$ induces a permutation group on $\Omega$. $\Phi, \Delta$ and $\ell_\omega - \Delta$ are orbits of $G$. Since $(|\Phi|, |\Delta|) = (n^2, 2) = 1$ and $(|\Phi|, |\ell_\omega - \Delta|) = (n^2, n - 1) = 1$, by Theorem 3.3 of [3] $\Delta$ and $\ell_\omega - \Delta$ are orbits of $G_A$.

**Lemma 3.1.** $G_A$ includes an involutory homology of $\pi$. 
Proof. $G_A$ induces a permutation group on $\ell_\omega = \{P_1, P_2\}$. Since $n$ is odd, $|\ell_\omega - \{P_1, P_2\}| = n - 1$ is even. Let $S$ be a Sylow 2-subgroup of $G_A$. As $G_A$ is transitive on $\ell_\omega = \{P_1, P_2\}$, $n - 1 | |G_A|$. Hence $S \neq 1$. There exists an involution $\sigma$ in the center of $S$. Suppose that $\sigma$ is a Baer involution. If $P_1 \sigma = P_1$, then $P_2 \sigma = P_2$ and so $|\{P \in \ell_\omega - \Delta | P \sigma = P\}| = \sqrt{n} - 1$. This contradicts a result of Lüneburg. (See Corollary 3.6.1 of [3].) If $P_1 \sigma \neq P_1$, then $P_2 \sigma \neq P_2$ and so $|\{P \in \ell_\omega - \Delta | P \sigma = P\}| = \sqrt{n} + 1$. This is again a contradiction by Corollary 3.6.1 of [3]. Therefore $\sigma$ is an involution homology.

**Lemma 3.2.** Let $\sigma$ be an involutory homology of $\pi$ such that $\sigma \in G_A$. If $P_1 \sigma = P_1$, then $\pi$ is a translation plane, and $G$ contains the group $T$ of translations of $\pi$.

Proof. Since $P_1 \sigma = P_1$, $P_2 \sigma = P_2$. Assume that $\ell_\omega$ is the axis of $\sigma$. Then $\sigma \in G(A, \ell_\omega)$. By a result of André (See Corollary 10.1.3 of [3]), the lemma holds. Assume that $\ell_\omega$ is not the axis of $\sigma$. We may assume that $AP_1$ is the axis of $\sigma$. Then $\sigma \in G(P_2, AP_1)$. There exists $\tau \in G_A$ such that $P_1 \tau = P_2$. Clearly $P_2 \tau = P_1$. Since $P_2 \tau = P_1$ and $(AP_1) \tau = AP_2$, $\tau^{-1} \sigma \tau \in G(P_1, AP_2)$. Therefore $\sigma \tau^{-1} \sigma \tau \in G(A, \ell_\omega) = \{1\}$, by a result of Ostrom. (See Lemma 4.13 of [3].) Thus the lemma holds by Corollary 10.1.3 of [3].

**Lemma 3.3.** If $G_A$ includes an involutory homology of $\pi$ which does not fix $P_1$, then the following statements hold:

(i) If $P \in \ell_\omega - \{P_1, P_2\}$, then there exist $Q \in \ell_\omega - \{P_1, P_2, P\}$ and $\sigma \in G(Q, AP)$ such that $|\sigma| = 2$.
(ii) If $Q \in \ell_\omega - \{P_1, P_2\}$, then there exist $P \in \ell_\omega - \{P_1, P_2, Q\}$ and $\tau \in G(Q, AP)$ such that $|\tau| = 2$.

Proof. By assumption, there exists an involutory homology $\sigma$ of $\pi$ such that $\sigma \in G_A$ and $P_1 \sigma \neq P_1$. Clearly $P_2 \sigma \neq P_2$. There exists $P_0 \in \ell_\omega - \{P_1, P_2\}$ such that $AP_0$ is the axis of $\sigma$. Let $Q_0$ be the center of $\sigma$. Then $Q_0 \in \ell_\omega - \{P_1, P_2, P_0\}$. Let $P \in \ell_\omega - \{P_1, P_2\}$. Then there exists $\varphi \in G_A$ such that $P = P_0 \varphi$. Set $Q = Q_0 \varphi$. Clearly $Q \in \{P_1, P_2\}$. Since $\sigma \in G(Q_0, AP_0)$ and $(AP_0) \varphi = AP_0$, $\varphi^{-1} \sigma \varphi \in G(Q, AP)$. This yields the statement (i). Similarly, we have the statement (ii).

**Lemma 3.4.** If $G_A$ includes an involutory homology of $\pi$ which does not fix $P_1$, then one of the following statements holds:

(i) The plane $\pi$ is a translation plane and $G$ contains the group $T$ of translations of $\pi$.
(ii) If $P \in \ell_\omega - \{P_1, P_2\}$, then $G(P, AP) \neq 1$. 

Proof. Let \( P \in \ell_\omega - \{P_1, P_2\} \). By Lemma 3.3 (i), there exist \( Q \in \ell_\omega - \{P_1, P_2, P\} \) and \( \sigma \in G(Q, AP) \) such that \( |\sigma| = 2 \). On the other hand, by Lemma 3.3 (ii) there exist \( R \in \ell_\omega - \{P_1, P_2, Q\} \) and \( \tau \in G(R, AQ) \) such that \( |\tau| = 2 \). Assume that \( R = P \). Then \( \sigma \in G(Q, AP) \) and \( \tau \in G(P, AQ) \). By Lemma 4.13 of [3], \( \sigma \tau \in G(A, \ell_\omega) - \{1\} \). Thus the statement (i) holds by Corollary 10.1.3 of [3]. Assume that \( R \neq P \). Then since \( \tau \in G(R, AQ) \) and \( (AQ) \sigma = AQ \), \( \sigma^{-1} \tau \sigma \in G(R \sigma, AQ) \). As \( R = R \sigma \), \( \tau \sigma^{-1} \tau \sigma \in G(Q, AQ) - \{1\} \) by a result of Baer. (See Lemma 4.12 of [3].) Thus \( G(Q, AQ) \neq 1 \). On the other hand, since \( G_A \) acts transitively on \( \ell_\omega - \{P_1, P_2\} \), the statement (ii) holds.

Lemma 3.5. If \( G(P, AP) \neq 1 \) for all \( P \in \ell_\omega - \{P_1, P_2\} \), then there is an integer \( k > 1 \) such that \( |G(P, \ell_\omega)| = k \) for all \( P \in \ell_\omega - \{P_1, P_2\} \).

Proof. Let \( P \in \ell_\omega - \{P_1, P_2\} \). Let \( \ell \) be an affine line of \( \pi \) such that \( \ell \ni P \). By a result of Ostrom and Wagner (See Theorem 4.3 of [3]), there exists \( \tau \in G_\pi \) such that \( (AP) \tau = \ell \). Since \( G(P, AP) \neq 1 \), \( \tau^{-1} G(P, AP) \tau = G(P \tau, (AP) \tau) = G(P, \ell) \neq 1 \). Therefore by the dual of Corollary 4.6.1 of [3], \( G(P, \ell_\omega) \neq 1 \). On the other hand, since \( G_A \) acts transitively on \( \ell_\omega - \{P_1, P_2\} \), the lemma holds.

Lemma 3.6. If \( G(P, AP) \neq 1 \) for all \( P \in \ell_\omega - \{P_1, P_2\} \), then \( |G(P_1, \ell_\omega)| = |G(P_2, \ell_\omega)| > 1 \).

Proof. Since the order \( n \) of \( \pi \) is odd, by Lemma 3.5 \( |G(P, \ell_\omega)| \geq 3 \) for all \( P \in \ell_\omega - \{P_1, P_2\} \). Therefore

\[
\begin{align*}
|G(P, \ell_\omega)| &= \bigcup_{P \in \ell_\omega - \{P_1, P_2\}} |G(P, \ell_\omega)| \\
&= 1 + \sum_{P \in \ell_\omega - \{P_1, P_2\}} (|G(P, \ell_\omega)| - 1) \\
&\geq 1 + 2(n-1) \\
&= 2n - 1 \\
&> n.
\end{align*}
\]

Thus \( |G(\ell_\omega, \ell_\omega)| > n \). Hence by a result of Ostrom (See Theorem 4.6 of [3]), \( G(P_1, \ell_\omega) \neq 1 \) for all \( P \in \ell_\omega \). In particular \( G(P_1, \ell_\omega) \neq 1 \). There exists \( \tau \in G_A \) such that \( P_2 \tau = P_1 \). Thus \( |G(P_2, \ell_\omega)| = |\tau^{-1} G(P_2, \ell_\omega) \tau| = |G(P_1, \ell_\omega)| > 1 \). Hence the lemma holds.

Proof of Theorem 2 when \( n \) is odd: By Lemmas 3.2, 3.4, 3.5, 3.6 and Theorem 1, the theorem holds.

4. The proof of Theorem 2 when \( n \) is even

In this section, we prove Theorem 2 when \( n \) is even.

Let \( \pi \) be a finite affine plane of even order \( n \) with a collineation group \( G \).
which is transitive on the affine points of \( \pi \) satisfying the hypothesis of Theorem 2. Then \( G \) has an orbit \( \Delta = \{ P_1, P_2 \} \) of length 2 on \( \ell_\omega \).

**Lemma 4.1.** \( G \) includes a translation of order 2 of \( \pi \).

Proof. Since \( n^2 | |G|, 2 | |G| \). Let \( S \) be a Sylow 2-subgroup of \( G \). Then there exists an involution \( \sigma \) in the center of \( S \). By Corollary 3.6.1 of [3] the involution \( \sigma \) is neither a Baer involution, nor an affine elation. It follows that \( \sigma \) is a translation of \( \pi \).

**Lemma 4.2.** \( G(\ell_\omega, \ell_\omega) \) is an elementary abelian 2-group and \( |G(\ell_\omega, \ell_\omega)| \geq 2 \).

Proof. If \( n=2 \), then the lemma holds. Let \( n \neq 2 \). Considering the action of \( G \) on \( \ell_\omega \), by Lemma 4.1 there exist distinct points \( Q_1, Q_2 \in \ell_\omega \) such that \( G(Q_1, \ell_\omega) \neq 1 \) and \( G(Q_2, \ell_\omega) \neq 1 \). By Theorem 4.5 of [3], the lemma holds.

**Lemma 4.3.** If \( G(P, \ell_\omega) \neq 1 \), then the plane \( \pi \) is a translation plane, and the group \( G \) contains the group \( T \) of translations of \( \pi \).

Proof. There exists an involution \( \sigma_i \) such that \( \sigma_i \in G(P_i, \ell_\omega) \) for \( i \in \{ 1, 2 \} \). Then \( \sigma_1 \sigma_2 \in G(\ell_\omega, \ell_\omega) \) and \( |\sigma_1 \sigma_2| = 2 \). Let \( Q \) be the center of \( \sigma_1 \sigma_2 \). Then \( Q \in \ell_\omega - \{ P_1, P_2 \} \). Since \( G \) acts transitively on \( \ell_\omega - \{ P_1, P_2 \} \), there exists \( r \geq 1 \) such that \( |G(P, \ell_\omega)| = 2^r \) for all \( P \in \ell_\omega - \{ P_1, P_2 \} \). There exists \( s \geq 1 \) such that \( |G(Q, \ell_\omega)| = 2^s \). Let \( |G(\ell_\omega, \ell_\omega)| = 2^t \). Then \( t \geq r + s \). Since

\[
\sum_{P \in \ell_\omega - \{ P_1, P_2 \}} (|G(P, \ell_\omega)| - 1) + \sum_{Q \in \{ P_1, P_2 \}} (|G(Q, \ell_\omega)| - 1) \geq 2^r + (n-1)(2^s - 1) = 2^t. \tag{*}
\]

By the same argument as in the proof of Theorem 1, \( 2^r \equiv 0 \) (mod \( 2^t \)) and \( 2^s+1 \equiv 0 \) (mod \( 2^t \)). Thus \( s \leq r \leq s+1 \).

Suppose that \( r = s+1 \). From \( (*) \), \( 2^t = 1 + (n-1)(2^{s+1} - 1) = 2^t(2^{s+1}-1) \) follows. Therefore \( n = 2^{t}(2^{s+1}-1)^{-1} \). As \( n \) is an integer, this is a contradiction. Hence \( r = s \). By Theorem 5.2 of [3], the lemma holds.

**Lemma 4.4.** If \( G(P, \ell_\omega) = 1 \), then \( |G(\ell_\omega, \ell_\omega)| = n = 2^m \) for some \( m \geq 1 \), \( G(P, \ell_\omega) = 1 \) and \( |G(P, \ell_\omega)| = 2 \) for all \( P \in \ell_\omega - \{ P_1, P_2 \} \).

Proof. By assumption, \( G(P_1, \ell_\omega) = 1 \) follows. If \( P \in \ell_\omega - \{ P_1, P_2 \} \), then \( G(P, \ell_\omega) = 1 \). Therefore there exists an integer \( r \geq 1 \) such that \( |G(Q, \ell_\omega)| = 2^r \) for all \( Q \in \ell_\omega - \{ P_1, P_2 \} \). Suppose that \( r \geq 2 \). Then

\[
|G(\ell_\omega, \ell_\omega)| = \sum_{Q \in \ell_\omega - \{ P_1, P_2 \}} (|G(Q, \ell_\omega)| - 1) + 1 = (2^r - 1)(n-1) + 1
\]
By Theorem 4.6 of [3], it follows that $G(Q, \ell_w) \neq 1$ for all $Q \in \ell_w$. In particular $G(P, \ell_w) \neq 1$, a contradiction. Hence $r=1$. Therefore $|G(\ell_w, \ell_w)| = (2-1) \cdot (n-1)+1 = n$. Therefore there exists an integer $m \geq 1$ such that $n = 2^m$. Thus the lemma holds.

Proof of Theorem 2 when $n$ is even: By Lemmas 4.3 and 4.4, the theorem holds.

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References