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FLAT TORSIONFREE MODULES AND QF-3 RINGS

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Since Thrall [25] proposed the concept of QF-3 algebra as a generalization of QF algebras, several extensions of this concept have been proposed for general rings. Perhaps the most spread notion of QF-3 ring is the following: A ring R is called a left QF-3 ring if it has a minimal faithful left R-module (see [23]). However, other authors proposed alternative notions, mainly for noetherian rings. We will look at two of them that seem very intersting. The first one is due to Morita [16] and it has been investigated recently by Hoshino [9], [10]. A ring R is said to be left Morita-QF-3 ring (shortly, MQF-3) if $E({}_RR)$ is a flat left R-module. The second one was proposed by Sumioka [20], [21], [22]. The ring R is called a left Sumioka-QF-3 ring (shortly, left SQF-3) if every finitely generated submodule of $E({}_RR)$ is torsionless. Every commutative domain is MQF-3 and SQF-3. It is well-known that in the case of left Artinian rings, these three concepts are equivalent and, moreover, they are right-left symmetric.

The aim of Section 2 is to find relations between the different extensions of QF-3 rings mentioned above. As a consequence of the main result of Section 2 (Theorem 2.7) we will show that if R has D.C.C. on rationally closed left ideals then R is left or right MQF-3 if and only if R is left or right SQF-3 (Corollary 2.8). Moreover, these rings are precisely those that Masaike characterized [14, Theorem 2] as the rings with a semi-primary QF-3 two sided maximal quotient ring.

The unifying idea to prove these results comes from the problem of existence of flat covers [4], [5]. In connection with this problem we proposed in [7] to investigate the rings R for which the class $\mathcal{F}_0$ of the submodules of flat left R-modules is a torsionfree class. To be exact, we say that R is a left FTF ring if there is a hereditary torsion theory τ_0 on R-Mod such that $\mathcal{F}_0$ is the class of all τ_0-torsionfree left R-modules. The key result is that the left MQF-3 (or left SQF-3) rings with D.C.C. on rationally closed left ideals are precisely the τ_0-artinian left FTF rings.

A ring R is said to be a left IF ring if every injective left R-module is flat. In other words, $\mathcal{F}_0=R$-Mod. Thus, IF rings are the "trivial" FTF rings.

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The concept of left IF ring was proposed by Colby [1] and Jain [11] as a generalization of regular and QF rings. The class of left FTF rings includes these rings, and we enlarge by means of our concept the class including semiprime left and right Goldie rings (see Proposition 3.6), semiprimary QF-3 rings (Corollary 2.11) and a large collection of SQF-3 and MQF-3 rings (Theorem 2.7 and Corollary 2.8).

Section 3 is essentially devoted to prove two results about localization. The first one is that for a left FTF ring R, the localized $Q_{\tau_0}(R)$ with respect to τ_0 is a left FTF ring (Theorem 3.2). The second main theorem of this section (Theorem 3.7) characterizes the left SQF-3 rings that have a QF left classical ring of fractions.

1. Preliminaries and General Notation

We denote by R an associative ring with identity, by $E(_RR)$ the injective hull of R as left R-module and by λ the Lambek torsion theory on the category R-Mod of all unital left R-modlues. The λ-torsionfree left R-modules are precisely the $E(_RR)$-torsionless left R-modules. We call R left MQF-3 if $E(_RR)$ is a flat left R-module and left SQF-3 if every finitely generated submodule of $E(_RR)$ is torsionless.

Let τ be a hereditary torsion theory on R-Mod and M a left R-module. By $\tau(M)$ we denote the largest τ-torsion submodule of M. A submodule N of M is τ-closed in M if M/N is τ-torsionfree, and it is τ-dense in M if M/N is τ-torsion. When $M=R$ and N is a left ideal of R, we say simply that N is a τ-closed or τ-dense left ideal. We say that M is τ-finitely generated if M contains a τ-dense finitely generated submodule. If M is finitely generated then M is said to be τ-finitely presented whenever M has a finite free presentation with τ-finitely generated kernel. The ring R is τ-coherent [12] if every finitely generated left ideal is τ-finitely presented, and R is τ-noetherian if every finitely generated left R-module is τ-finitely presented. Equivalently, R is τ-noetherian if and only if R satisfies A.C.C. on τ-closed left ideals. R is said to be τ-artinian if it satisfies D.C.C. on τ-closed left ideals. Every τ-artinian ring is τ-noetherian [15, Theorem 1.4] and every τ-noetherian ring is clearly τ-coherent. For a submodule N of M we will use the notation $Cl_\tau^M(N)$ for the τ-closure of N in M, defined by the condition $Cl_\tau^M(N)/N=\tau(M/N)$. If any direct limit of τ-torsionfree modules is τ-torsionfree, then τ is said to be of finite type. If, in addition, the localization functor Q_τ: R-Mod$\rightarrow Q_\tau(R)$-Mod is exact, then τ is said to be perfect. It is known [19, exercise XI.6] that τ is perfect if and only if every left $Q_\tau(R)$-module is τ-torsionfree as left R-module.

Consider the class $\mathcal{F}_0^R$ of left R-modules defined by the following condition: $M\in\mathcal{F}_0^R$ if and only if there is a monomorphim of left R-modules from M to some flat left R-module. If there is not risk of confusion, we use the notation

$\mathcal{F}_0$. We will say that R is a left FTF ring ("flat is torsionfree") if $\mathcal{F}_0$ is the class of all τ_0-torsionfree left R-modules for some hereditary torsion theory τ_0 on R-Mod. Analogously we can define right FTF rings with notations $\mathcal{F}_0'$ and τ_0'. For left and right FTF ring R we say simply that R is FTF, and this convention is valid for any other one-sided concept (e.g., a QF-3 ring is a left and right QF-3 ring).

We refer to [19] for all torsion-theoretic notions used in this paper.

2. FTF AND QF-3 RINGS

We start with an easy characterization of left FTF rings in terms of the behavior of the flat modules under injective hulls and direct products.

Proposition 2.1. *A ring R is left FTF if and only if the following conditions are satisfied:*

(1) *If M is a flat left R-module then $E(M)$ is a flat left R-module.*

(2) *If $\{M_i: i\in I\}$ is a family of injective flat left R-modules then the direct product $\Pi\{M_i: i\in I\}$ is a flat left R-module.*

It is evident that a left FTF ring is left MQF-3. An objective of this section is to find conditions on R to have the converse of this fact. The following are partial results that will be basic tools to prove our main result. Recall from [2] that a (left or right) module M is said to be π-flat if every direct product of copies of M is a flat module. On the other hand, M is FP-injective whenever $\mathrm{Ext}_R^1(P, M)=0$ for every finitely presented module P. Finally, we use the notation $M^+=\mathrm{Hom}_{\boldsymbol{Z}}(M, \boldsymbol{Q}/\boldsymbol{Z})$.

Proposition 2.2. *Let R be a λ-coherent left MQF-3 ring.*

(*i*) *R is right FTF.*

(*ii*) *R is left FTF if and only if $E(_RR)$ is π-flat if and only if R is τ_0'-coherent.*

Proof. Since $E=E(_RR)$ is flat, we can consider the hereditary torsion theory on Mod-R $\kappa=\mathrm{Ker}(-\otimes_R E)$ whose κ-torsion right R-modules are the R-modules annihilated by $-\otimes_R E$. We will prove that the κ-torsionfree right R-modules are precisely the submodules of flat right R-modules and, so, R is right FTF with $\tau_0'=\kappa$. Let M be a flat right R-module and construct the commutative diagram of morphisms of abelian groups

$$\begin{array}{ccc} M\otimes_R R & \longrightarrow & M\otimes_R E \\ \uparrow & & \uparrow \\ \kappa(M)\otimes_R R & \rightarrow & \kappa(M)\otimes_R E \end{array}$$

It is evident that the morphism at the bottom row is monic. But $\kappa(M)\otimes_R E$

$=0$, so $\kappa(M)=0$. Therefore, $\mathcal{F}_0'\subseteq\mathcal{F}(\kappa)$. By definition, a right R-module M is κ-torsion if and only if $(M\otimes_R E)^+=0$. But $(M\otimes_R E)^+$ is canonically isomorphic to $\mathrm{Hom}_R(M, E^+)$. This means that M is κ-torsion if and only if $\mathrm{Hom}_R(M, E^+)=0$. Hence, E^+ is an injective cogenerator of κ. We claim that E^+ is a π-flat right R-module. If this happens, then every κ-torsionfree right R-module embeds in a flat right R-module and the equality $\mathcal{F}_0'=\mathcal{F}(\kappa)$ holds. To prove the claim observe that every direct product P of copies of E^+ can be obtained as $P=S^+$, where S is a direct sum of copies of E. Since S is an FP-injective left R-module and R is λ-coherent, [12, Theorem 3.3] assures that $P=S^+$ is a flat right R-module. Therefore, R is right FTF with $\tau_0'=\kappa$.

Now, if R is left FTF then E is π-flat obviously. Moreover, if E is π-flat then R is κ-coherent by [12, Corollary 3.5]. Finally, if R is τ_0'-coherent, a right-handed version of the foregoing proof runs to prove that R is left FTF. ∎

Proposition 2.3. *If R is a ring then R is left FTF and τ_0-coherent if and only if R is right FTF and τ_0'-coherent.*

Proof. This is an immediate consequence of Proposition 2.2 since a τ_0-coherent left FTF ring must be λ-coherent. ∎

Proposition 2.4. *Assume that R is a left MQF-3 ring.*

(1) *If M is a λ-finitely presented left R-module then $M/\lambda(M)$ is torsionless.*

(2) *If R satisfies A.C.C. on left annihilators, then R is λ-noetherian, left SQF-3 and right FTF.*

Proof. (1) Let M be a λ-finitely presented left R-module. Our objective is to show that $\lambda(M)=\cap\{\mathrm{Ker} f: f\in\mathrm{Hom}_R(M,R)\}$. From this equality, it will be clear that $M/\lambda(M)$ is torsionless. To check the equality $\lambda(M)=\cap\{\mathrm{Ker} f: f\in\mathrm{Hom}_R(M, R)\}$, assume in a first step that M is finitely presented. Observe that, since R is λ-torsionfree, the inclusion $\lambda(M)\subseteq\cap\{\mathrm{Ker} f: f\in\mathrm{Hom}_R(M, R)\}$ holds immediately. For the other inclusion, assume that $x\in M$ but $x\notin\lambda(M)$. There is $f\in\mathrm{Hom}_R(M, E(_RR))$ such that $f(x)\neq 0$. Since $E(_RR)$ is flat there is a finitely generated free left module F [13, Théorème 1.2], $v\in\mathrm{Hom}_R(M, F)$, $w\in\mathrm{Hom}_R(F, E(_RR))$ such that $f=wv$. It is clear that $v(x)\neq 0$. Hence, $x\notin\mathrm{Ker}\, v$. This proves that $x\notin\cap\{\mathrm{Ker} f: f\in\mathrm{Hom}_R(M, R)\}$. In other words, we have the inclusion $\lambda(M)\supseteq\cap\{\mathrm{Ker} f: f\in\mathrm{Hom}_R(M, R)\}$ and this gives the desired equality.

If M is λ-finitely presented, then there is an exact sequence

$$0\to K\to F\to M\to 0$$

where F is a finitely generated free left R-module and K is λ-finitely generated. This means that K contains a λ-dense finitely generated submodule K_0. Since F/K_0 is finitely presented we have that $\lambda(F/K_0)=\cap\{\mathrm{Ker} f: f\in\mathrm{Hom}_R(F/K_0, R)\}$.

Now consider the epimorphism $p\colon F/K_0 \to F/K \cong M$. Since $\mathrm{Ker}\, p = K/K_0$ is λ-torsion, $g|_{\mathrm{Ker}\, p}=0$ for every $g \in \mathrm{Hom}_R(F/K_0, R)$. Therefore, given $g \in \mathrm{Hom}_R(F/K_0, R)$ there is $f \in \mathrm{Hom}_R(M, R)$ such that $g = f \circ p$. Let $x \in \cap \{\mathrm{Ker}\, f\colon f \in \mathrm{Hom}_R(M, R)\}$ and consider $y \in K/K_0$ such that $p(y)=x$. Let $g \in \mathrm{Hom}_R(F/K_0, R)$ and let $f \in \mathrm{Hom}_R(M, R)$ such that $g = f \circ p$. Then $g(y) = (f \circ p)(y) = f(x) = 0$. Therefore $y \in \cap \{\mathrm{Ker}\, g\colon g \in \mathrm{Hom}_R(F/K_0, R)\} = \lambda(F/K_0)$. But $p(\lambda(F/K_0)) \subseteq \lambda(M)$. Hence, $x = p(y) \in \lambda(M)$ and we deduce that $\cap \{\mathrm{Ker}\, f\colon f \in \mathrm{Hom}_R(M, R)\} \subseteq \lambda(M)$. Since R is λ-torsionfree we have also that $\lambda(M) \subseteq \cap \{\mathrm{Ker}\, f\colon f \in \mathrm{Hom}_R(M, R)\}$ and (1) holds.

(2) We will prove that if R is not λ-noetherian then there is a strictly ascending chain of left annihilators. Assume that R is not λ-noetherian. Then there exists a strictly ascending chain of λ-closed left ideals

$$I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots$$

Choose $x_{i+1} \in I_{i+1} \setminus I_i$ for each natural number i and consider $C_n = Cl_\lambda^R(Rx_1 + \cdots + Rx_n)$, where x_1 is any element of I_1. It is clear that

$$C_1 \subset C_2 \subset \cdots \subset C_n \subset \cdots$$

is a strictly ascending chain of left ideals of R. Consider the finitely presented left R-module $M_n = R/(Rx_1 + \cdots + Rx_n)$. It is clear that $\lambda(M_n) = Cl_\lambda^R(Rx_1 + \cdots + Rx_n)/(Rx_1 + \cdots + Rx_n)$. By part (1), $M_n/\lambda(M_n) \cong R/Cl_\lambda^R(Rx_1 + \cdots + Rx_n)$ is torsionless and, thus, $C_n = Cl_\lambda^R(Rx_1 + \cdots + Rx_n)$ is a left annihilator ideal. Therefore, R has not A.C.C. on left annihilators.

To see that R is left SQF-3 observe that, since R is λ-noetherian, every finitely generated left R-module is λ-finitely presented. But this is the case for any finitely generated submodule of $E(_RR)$. The part (1) assures now that R is left SQF-3.

The assertion that R is right FTF is a consequence of Propostion 2.2 since every λ-noetherian ring is λ-coherent. ■

REMARK 2.5. Propositions 2.2 and 2.4 improve [16, Theorem 1].

Proposition 2.6. *The following conditions are equivalent for a ring R.*

(i) *R is left FTF and τ_0-noetherian.*

(ii) *Every finitely generated $E(_RR)$-torsionless left R-module embeds in a free left R-module and R satisfies A.C.C. on left annihilators.*

(iii) *$E(_RR)$ is π-flat and R satisfies A.C.C. on left annihilators.*

In such a case, $\tau_0 = \lambda$.

Proof. (i)$\Rightarrow$(ii) It is clear that R satisfies A.C.C. on left annihilators. If M is a finitely generated $E(_RR)$-torsionless left R-module, then M is τ_0-torsionfree and, since R is τ_0-noetherian, M is τ_0-finitely presented. A slight modification of the proof of [7, Proposition 4.5.(3)] gives that $\tau_0(M)=\mathrm{Ker} f_1 \cap \cdots \cap \mathrm{Ker} f_n$ for some $f_1, \cdots, f_n \in \mathrm{Hom}_R(M, R)$. Since $\tau_0(M)=0$, we have that M embeds in the free left R-module R^m via the f_i's.

(ii)$\Rightarrow$(iii) By [17, Lemma 2], $E(_RR)$ is π-flat.

(iii)$\Rightarrow$(i) By Proposition 2.4, R is λ-noetherian and Proposition 2.2 implies that R is left (and right) FTF. Now, $\tau_0=\lambda$ by [7, Theorem 4.6]. ■

Now we are ready to show the main result of this section.

Theorem 2.7. *The following conditions are equivalent for a ring R.*

(i) *R is left FTF and τ_0-artinian.*

(ii) *R is left SQF-3 and satisfies D.C.C. on left annihilators.*

(iii) *R is left MQF-3 and satisfies A.C.C. and D.C.C. on left annihilators.*

(iv) *R is left MQF-3, R satisfies D.C.C. on left annihilators and every cyclic $E(_RR)$-torsionless left R-module has finite left Goldie dimension.*

(i′) *R is right FTF and τ_0'-artinian.*

(ii′) *R is right SQF-3 and satisfies D.C.C. on right annihilators.*

(iii′) *R is right MQF-3 and satisfies A.C.C. and D.C.C. on right annihilators.*

(iv′) *R is right MQF-3, R satisfies D.C.C. on right annihilators and every cyclic $E(R_R)$-torsionless right R-module has finite right Goldie dimension.*

Proof. (iii)$\Leftrightarrow$(iii′) Assume that R is left MQF-3 and satisfies A.C.C. and D.C.C. on left annihilators. It is immediate that R satisfies A.C.C. and D.C.C. on right annihilators. By Proposition 2.4, R is right FTF. In particular, $E(R_R)$ is flat. Thus, R is right MQF-3. We have proved that (iii)$\Rightarrow$(iii′). The converse is clear by symmetry.

(i)$\Rightarrow$(ii) If R is left FTF and τ_0-artinian, then R is τ_0-noetherian by [15, Theorem 1.4]. By Proposition 2.6, R is left SQF-3. Since R is τ_0-atrinian, R satisfies D.C.C. on left annihilators.

(ii)$\Rightarrow$(iii) First, observe that every rationally closed left ideal is a left annihilator. For, given I a rationally closed left ideal of R, there is an index set S such that $R/I \hookrightarrow E(_RR)^S$. For each $s \in S$ let f_s denote the composition of the embedding $R/I \hookrightarrow E(_RR)^S$ with the s-th canonical projection from $E(_RR)^S$ onto $E(_RR)$, and let $I_s/I=\mathrm{Ker} f_s$. It is clear that R/I_s embeds in $E(_RR)$. Since R is left SQF-3, R/I_s is torsionless and, thus, I_s is a left annihilator. But $I=\cap\{I_s: s\in S\}$. Hence, I is a left annihilator. Since R has D.C.C. on left annihilators, we have that R is λ-artinian. Therefore, every finitely generated submodule M of $E(_RR)$ is λ-artinian and λ-torsionfree. By the assumption, M is torsionless. Since M is λ-artinian this implies that M embeds in a free left R-module. By [17, Lemma 2], $E(_RR)$ is flat, that is, R is left MQF-3. Moreover, R is λ-noetherian and this implies that R satisfies A.C.C. on left annihilators.

(iii)$\Rightarrow$(i) By Proposition 2.4, R is right FTF. Since (iii) is equivalent to (iii′), we have by symmetry that R is left FTF. By Proposition 2.6, R is τ_0-noetherian. It remains to prove that R is τ_0-artinian. Again, the key point is to prove that every τ_0-closed left ideal is a left annihilator. Let I be a τ_0-closed left ideal. Since R is τ_0-noetherian, I must be τ_0-finitely generated, i.e., there is a finitely generated left ideal D contained in I such that I/D is τ_0-torsion. It is clear that $\tau_0(R/D)=I/D$. But R/D is finitely presented and this implies, by [7, Proposition 4.5.(3)], that $I/D=\mathrm{Ker} f_1\cap\cdots\cap\mathrm{Ker} f_m$ for some $f_1, \cdots, f_n\in \mathrm{Hom}_R(R/D, R)$. Hence, I is the left annihilator of a finite subset of R. Finally, since R has D.C.C. on left annihilators and every τ_0-closed left ideal is an an nihilator, it follows that R is τ_0-artinian.

(i)$\Rightarrow$(iv) It is clear that R is left MQF-3 and that R has D.C.C. on left annihilators. By [15, Theorem 1.4], R is τ_0-noetherian. By Proposition 2.6, $\lambda=\tau_0$ and R is λ-artinian. Thus every cyclic λ-torsionfree left R-module is λ-artinian and, therefore, every cyclic λ-torsionfree left R-module has finite left Goldie dimension.

(iv)$\Rightarrow$(iii) We will prove that R is λ-artinian. In view of the Proposition of [8] we only need to prove that for every descending chain of principal left ideals of R

$$Ra_1\supseteq Ra_2\supseteq\cdots\supseteq Ra_n\supseteq\cdots$$

there is a natural number n_0 such that for every $n\geq n_0$, Ra_n/Ra_{n+1} is λ-torsion. By Proposition 2.4.(1), $(R/Ra_i)/\lambda(R/R_i)$ is torsionless for every $i\in\boldsymbol{N}$. But $(R/Ra_i)/\lambda(R/Ra_i)\cong R/Cl^R_\lambda(Ra_i)$. Hence, $Cl^R_\lambda(Ra_i)$ is a left annihilator for every $i\in\boldsymbol{N}$. In this way we obtain a descending chain of left annihilators

$$Cl^R_\lambda(Ra_1)\supseteq Cl^R_\lambda(Ra_2)\supseteq\cdots\supseteq Cl^R_\lambda(Ra_n)\supseteq\cdots$$

that must stop. Therefore Ra_n/Ra_{n+1} is λ-torsion for $n\geq n_0$ for some natural number n_0, and [8, Proposition] assures that R is λ-artinian. By [15, Theorem 1.4], R is λ-noetherian and, thus, R has A.C.C. on left annihilators. ■

Corollary 2.8. *Let R be a ring with D.C.C. on rationally closed left ideals. The following conditions are equivalent:*

(i) *R is left MQF-3.*
(ii) *R is right MQF-3.*
(iii) *R is left SQF-3.*
(iv) *R is right SQF-3.*
(v) *R is left FTF.*
(vi) *R is right FTF.*

Corollary 2.9. *Let R be a ring with left Krull dimension. R is left FTF and τ_0-artinian if and only if R is left MQF-3 and satisfies D.C.C. on left annihi-*

lators.

REMARKS 2.10. (1) H. Sato showed [18, Theorem 1.1] that a left and right noetherian ring is left or right MQF-3 if and only if it is left or right SQF-3. On the other hand, T. Sumioka proved [20, Lemma 7] that a right SQF-3 ring with D.C.C. on right annihilators is right MQF-3. Theorem 2.7 extends both results.
(2) Masaike showed [14, Theorem 2] that a ring satisfies condition (ii) of Theorem 2.7 if and only if it has a semiprimary QF-3 two sided maximal quotient ring. On the other hand, it is not hard to deduce by combining results of [24], [3], and Theorem 2.7 that a semiprimary (or perfect) ring is QF-3 if and only if it is FTF.

Corollary 2.11. *A perfect ring R is QF-3 if and only if R is FTF. Moreover, in such a case, R is semiprimary.*

Proof. Assume that R is FTF. If P is any projective left R-module then $E(P)$ is flat and, since R is perfect, projective. By [3, Theorem 1.3 and Theorem 1,2] R is semiprimary and contains a faithful injective left ideal and a faithful injective right ideal. By [24, Proposition 3.1], R is QF-3.

Conversely, assume that R is QF-3. By [3, Theorem 1.3], R contains a Σ-injective and π-projective left ideal I. Therefore $E({}_RR)$ embeds in a direct product of copies of I. Hence $E({}_RR)$ is π-projective and Σ-injective. Hence R is λ-noetherian. Therefore R has A.C.C. on left annihilators. Since the conditions on R are symmetric, R has A.C.C. on right annihilators and, thus, D.D.C. on left annihilators. By Theorem 2.7, R is FTF. ■

EXAMPLE 2.12. The following example shows that the A.C.C. on annihilators in conditions (iii) and (iii′) of Theorem 2.7 cannot be deleted. Also, the condition on the λ-torsionfree cyclic left R-modules in (iv) is not negligible. Let A be a principal left and right ideal domain with a simple injective left A-module $S=A/Aa$. Consider

$$R=\begin{pmatrix} A & S \\ 0 & C \end{pmatrix}$$

where $C=\mathrm{End}_A(S)$. It is possible to show [23, p. 78] that

$$E({}_RR)=\begin{pmatrix} D & S \\ 0 & C \end{pmatrix}$$

where D is the division ring of fractions of A. It is easy to prove that $E({}_RR)$ is flat and, hence, R is an example of a left noetherian left MQF-3 ring. By Proposition 2.4, R is also right MQF-3, since it is right FTF, We will prove that

R is not right SQF-3. On the contrary, R must be τ_0'-artinian by Theorem 2.7. Therefore, R is τ_0'-noetherian by [15, Theorem 1.4]. In particular, R has A.C.C. on right annihilators. But this is not the case of R: Consider for each $n \in N$ the left ideal of R

$$I_n = \begin{pmatrix} Aa^n & S \\ 0 & C \end{pmatrix}$$

If A is not a division ring, then the chain

$$I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$$

is strictly descending. By using the injectivity of $S=A/Aa$ it is possible to prove that $lr(I_n)=I_n$ for every natural number n. Therefore, the chain of right annihilators

$$r(I_1) \subset r(I_2) \subset \cdots \subset r(I_n) \subset \cdots$$

is strictly ascending.

In conclusion, R is a (two sided) MQF-3 ring with D.C.C. on right annihilators that it is not right SQF-3.

3. LOCALIZATION IN FTF RINGS

To show the first theorem of this section, we need some technical facts collected in Lemma 3.1. The proof of this lemma runs over standard torsion-theoretic arguments. A left module is said to be $\aleph_0$-injective if it satisfies the Baer's criterion for finitely generated left ideals.

Lemma 3.1. *Let τ be a faithful hereditary torsion theory on R-Mod, Q a ring extension of R such that Q/R is τ-torsion and M a left Q-module such that ${}_RM$ is τ-torsionfree. The following assertions are true.*

(1) *${}_RM$ is injective if and only if ${}_QM$ is injective.*

(2) *If ${}_QM$ is $\aleph_0$-injective then ${}_RM$ is $\aleph_0$-injective.*

(3) *The structure of left R-module induced over $E({}_QM)$ provides an injective envelope of ${}_RM$ in R-Mod, that is $E({}_RM)={}_RE({}_QM)$.*

Theorem 3.2. *If R is a left FTF ring then $Q_{\tau_0}(R)$ is a left FTF ring. In such a case, a left Q-module embeds in a flat left Q-module if and only if it embeds in a flat left R-module.*

Proof. Put $Q=Q_{\tau_0}(R)$. It suffices to prove for Q the conditions (1) and (2) of Proposition 2.1.

(1) Let M be a flat left Q-module. We claim that ${}_RM$ is τ_0-torsionfree. By [13, Théorème 1.2] ${}_QM$ is isomorphic to a direct limit of finitely generated

free left Q-modules. Since the functor restriction of scalars commutes with direct limits, we obtain that ${}_RM$ is a direct limit of left R-modules isomorphic to a direct sum of finitely many copies of ${}_RQ$. In particular, ${}_RM$ is a direct limit of τ_0-torsionfree left R-modules. Since τ_0 is of finite type by [7, Proposition 4.5], we conclude that ${}_RM$ is τ_0-torsionfree. By Lemma 3.1, $E({}_RM)={}_RE({}_QM)$. This implies that ${}_RE({}_QM)$ is flat. Then $Q\otimes_R E({}_QM)$ is a flat left Q-module. Consider the R-monomorphism θ: $E({}_QM)\to Q\otimes_R E({}_QM)$ given by $\theta(x)=1\otimes x$ for all $x\in E({}_QM)$. Since $Q\otimes_R E({}_QM)$ is τ_0-torsionfree, θ is Q-linear and, thus, $E({}_QM)$ is isomorphic as left Q-module to a direct summand of the flat left Q-module $Q\otimes_R E({}_QM)$. Hence, $E({}_QM)$ is flat.
(2) Let $\{M_i: i\in I\}$ be a family of injective flat left Q-modules and put $M=\Pi\{M_i: i\in I\}$. It is possible to argue as in part (1) to obtain that M_i is an injective flat left R-module for every $i\in I$. Therefore, M is an injective flat left R-module. Now, the R-monomorphism θ: $M\to Q\otimes_R M$ is Q-linear and we have again that M is isomorphic as left Q-module to a direct summand of the flat left Q-module $Q\otimes_R M$. Hence, M is a flat left Q-module.

By Proposition 2.1, Q is a left FTF ring. Moreover, a consequence of the foregoing proof is that a left Q-module embeds in a flat left Q-module if and only if it embeds in a flat left R-module. ■

The following Corollary shows that there is a nice relation between left FTF and left IF rings.

Corollary 3.3. *Let R be a left FTF ring. $Q_{\tau_0}(R)$ is left IF if and only if τ_0 is perfect.*

Proof. Assume that $Q_{\tau_0}(R)$ is left *IF*. Then every left Q-module embeds in a flat left Q-module. By Theorem 3.2 every left Q-module embeds in a flat left R-module, that is, every left Q-module is τ_0-torsionfree and, hence, τ_0 is perfect.

Conversely, if we assume that τ_0 is perfect, then every left Q-module is τ_0-torsionfree. Therefore, every left Q-module embeds in a flat left Q-module. That is, Q is left *IF*. ■

We will prepare the proof of the second main result of this section. Proposition 3.4 can be used to construct FTF rings, like we make to obtain the Example 3.5.

Proposition 3.4. *Let ρ: $R\to S$ be an injective ring homomorphism such that ${}_RS$ and S_R are flat R-modules. If S is an IF ring, then R is an FTF ring. Moreover, if S/R is τ_0-torsion, then $S=Q_{\tau_0}(R)$ and τ_0 is perfect.*

Proof. If we denote by $\mathcal{F}_S$ the class of all left R-modules that are R-submodules of left S-modules, it is not hard to see that $\mathcal{F}_0=\mathcal{F}_S$. It is clear that

$\mathcal{F}_S$ is closed under submodules and direct products. We claim that $\mathcal{F}_S$ is closed under injective hulls. Given $M \in \mathcal{F}_S$, there is an R-monomorphism $M \to N$, where N is a left S-module. We can assume that N is injective as left S-module. Since S_R is flat, ${}_RN$ is injective. Thus, ${}_RN$ contains an injective hull $E({}_RM)$ of ${}_RM$. Hence, $E({}_RM) \in \mathcal{F}_S$. Therefore $\mathcal{F}_0 = \mathcal{F}_S$ is closed under submodules, direct products and injective hulls and this assures that $\mathcal{F}_0$ is the torsionfree class for some hereditary torsion theory τ_0 on R-Mod.

Since S is an *IF* ring, S is left $\aleph_0$-injective as left S-module [1, Theorem 2]. By Lemma 3.1, ${}_RS$ is $\aleph_0$-injective. Since τ_0 is of finite type [7, Proposition 4.5], ${}_RS$ is τ_0-injective. Moreover, S/R is τ_0-torsion. In these circumstances, the only possibility is that $Q_{\tau_0}(R) = S$. By Corollary 3.3, τ_0 is perfect. ■

By $Q^l_{\max}(R)$ (rep. $Q^r_{\max}(R)$) we denote the left (resp. right) maximal quotient ring of R. When both quotient ring coincide, we use the notation $Q_{\max}(R)$.

EXAMPLE 3.5. An example of a commutative FTF ring R with τ_0 perfect, $Q_{\max}(R)$ regular but $Q_{\tau_0}(R) \neq Q_{\max}(R)$. Let D be a commutative domain with infinitely many elements and K its field of fractions. Let Ω be an infinite set. We denote by D^Ω (resp. K^Ω) the direct product indexed by Ω of copies of D (resp. K). Let R (resp. S) be the subring of D^Ω (resp. K^Ω) consisting of those maps f from Ω to D (resp. K) such that the set $\{f(\omega)\colon \omega \in \Omega\}$ is finite. Using [13, Théorème 1.2] it can be shown that ${}_RS$ is flat. It is clear that S is a regular ring. By Proposition 3.4, R is FTF. By using [7, Proposition 4.5.(4)] it can be shown that S/R is τ_0-torsion. By Proposition 3.4, τ_0 is perfect and $Q_{\tau_0}(R) = S$. It is easy to show that $Q_{\max}(R) = K^\Omega$. Of course, $R \neq S \neq K^\Omega$ unless D is a field.

Proposition 3.6. *A ring R has a QF two sided maximal quotient ring if and only if R is a τ_0-artinian left FTF ring with τ_0 perfect.*

Proof. Assume that R has a QF two sided maximal quotient ring Q. By [16, Theorem 4] ${}_RQ$ and Q_R are flat R-modules. By Proposition 3.4, R is an FTF ring. Moreover, since Q is artinian, R satisfies A.C.C. and D.C.C. on left annihilators. Theorem 2.7 assures that R is τ_0-artinian and that $\tau_0 = \lambda$. Therefore, Q/R is λ-torsion and Proposition 3.4 implies that τ_0 is perfect.

If R is a τ_0-artinian left FTF ring with τ_0 perfect, then $Q = Q_{\tau_0}(R)$ is an *IF* ring by Corollary 3.3. But a left artinian *IF* ring is necessarily a *QF* ring. ■

Theorem 3.7. *Let R be a left SQF-3 ring. R has a QF left classical ring of fractions if and only if the following two conditions hold:*

(1) *R has D.C.C. on left annihilators.*

(2) *Every finitely generated left ideal with zero right annihilator contains a regular element.*

Proof. Assume that R has a *QF* classical left ring of fractions Q. Since

${}_RQ$ is injective, $Q=Q^l_{\max}(R)$. By [23, Proposition 4.6], Q is contained in $Q^r_{\max}(R)$. But Q is right self-injective, which implies that $Q=Q^l_{\max}(R)=Q^r_{\max}(R)$. By Proposition 3.6, R is a left FTF ring with τ_0 perfect and R is τ_0-artinian. Moreover, $\lambda=\tau_0$ (Proposition 2.6). It is clear that R has D.C.C. on left annihilators. Consider a finitely generated left ideal I of R such that $r(I)=0$. By [7, Proposition 4.5] I is τ_0-dense in R. Since τ_0 is perfect, $QI=Q$ [19, Proposition XI.3.4]. Thus, $1=s_1^{-1}x_1+\cdots+s_n^{-1}x_n$ for some regular elements $s_i\in R$ and elements $x_i\in I$. It is clear that, if we put $r_1=s_1$, then $r_1=x_1+q_2^1x_2+\cdots+q_n^1x_n$, for certain $q_2^1,\cdots,q_n^1\in Q$. Since Q is the left classical quotient ring for R, there is a regular element r_2 in R such that $q_2^1=(r_2)^{-1}t_2$, for some t_2 in R. Therefore, $r_2r_1=r_2x_1+t_2x_2+q_3^2x_3+\cdots+q_n^2x_n$, for certain elements $q_3^2,\cdots,q_n^2\in Q$. We can repeat this argument until we obtain regular elements $r_n,\cdots,r_1$ in R and $t_n,\cdots,t_2$ in R such that $r=r_n\cdots r_1=r_n\cdots r_2x_1+r_n\cdots r_3t_2x_2+\cdots+t_nx_n$. Hence r is a regular element contained in I.

Conversely, assume that the left SQF-3 ring R satisfies conditions (1) and (2). By Theorem 2.7, R is a τ_0-artinian and τ_0'-artinian FTF ring. Let $Q=Q_{\tau_0}(R)$. We will prove that τ_0 is perfect. Let I be a τ_0-dense left ideal of R. Since τ_0 is of finite type [7, Proposition 4.5], I contains a finitely generated τ_0-dense left ideal D. By [7, Proposition 4.5], $r(D)=0$. Hence, D contains a regular element r. Then Rr is a projective τ_0-dense left ideal of R [7 Proposition 4.5]. Therefore every τ_0-dense left ideal contains a projective τ_0-dense left ideal. By [19, Proposition XI.3.3] τ_0 is perect. Proposition 3.6 assures that Q is a QF two sided maximal quotient ring of R.

Next, we will prove that Q is a left classical ring of fractions of R. For, given $q\in Q$, there is a finitely generated τ_0-dense left ideal I of R such that $Iq\subseteq R$. Since $r(I)=0$ [7, Proposition 4.5], I contains a regular element r. It is clear that $rq\in R$. To finish the proof, we only need to show that every regular element of R is invertible in Q. Let r be a regular element of R. Again by [7, Proposition 4.5], Rr is τ_0-dense in Q and rR is τ_0'-dense in Q. Since τ_0 and τ_0' are perfect torsion theories we deduce [19, Proposition XI.3.4] that $Qr=rQ=Q$. Hence, r is invertible in Q. ■

To finish, we will illustrate how to use Theorem 3.7 to obtain a characterization of QF rings within the class of QF-3 rings.

Corollay 3.8. *Let R be a QF-3 ring. The following conditions are equivalent.*

(i) *R is QF.*

(ii) *R has a QF left classical quotient ring.*

(iii) *R has D.C.C. on left annihilators and every finitely generated left ideal with zero right annihilator contains a regular element.*

Proof. (i)$\Rightarrow$(ii) Evident.

(ii)$\Leftrightarrow$(iii) Apply Theorem 3.7.

(ii) and (iii)$\Rightarrow$(i) By Theorem 2.7, R is left FTF and τ_0-artinian. Since R is right QF-3, there is an injective faithful right ideal I. The left ideal RI has zero right annihilator. Since R is τ_0-noetherian [15, Theorem 1.4], RI is τ_0-finitely generated. Combining this fact with [7, Proposition 4.5.(4)], RI contains a finitely generated ideal with zero right annihilator and, thus, RI contains a regular element r. Therefore, $Q=RIQ$. But I_R is injective and this implies that I is a right Q-submodule of Q, that is, $IQ=I$. We conclude that $Q=RIQ=RI\subseteq R$. Hence, $Q=R$. ■

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