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MODULES WITH EVERY SUBGENERATED MODULE LIFTING

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It was shown in Dung-Smith [2] that, for a module $M$, every module in $\sigma[M]$ is extending (CS module) if and only if every module in $\sigma[M]$ is a direct sum of indecomposable modules of length 2 or, equivalently, every module in $\sigma[M]$ is a direct sum of $M$-injective module and a semisimple module. Here we characterize these modules by the fact that every module in $\sigma[M]$ is lifting or, equivalently, decompose as a direct sum of a semisimple module and a projective module in $\sigma[M]$. They are also determined by the functor ring of $\sigma[M]$ being a $QF$-2 ring with Jacobson radical square zero.

As a corollary we obtain a result of Vanaja-Purav [8]: All (left) $R$-modules are lifting if and only if $R$ is a generalizad uniserial ring with Jacobson radical square zero.

1. Preliminaries

Let $R$ denote an associative ring with unit, $R$-Mod the category of unital left $R$-modules, and $M$ a left $R$-module. We call $M$ locally artinian, noetherian, of finite length every finitely generated submodule of $M$ has the corresponding property. The notation $K\ll M$ means that $K$ is a small (superfluous) submodule of $M$.

By $\sigma[M]$ we denote the full subcategory of $R$-Mod whose objects are submodules of $M$-generated modules.

For any $R$-module $N$, $E(N)$ will denote the injective hull of $N$ in $R$-Mod. For $N \in \sigma[M]$, $\tilde{N}$ is the injective hull of $N$ in $\sigma[M]$. $\tilde{N}$ is also called the $M$-injective hull of $N$ and is isomorphic to the trace of $M$ in $E(N)$.

$N \in \sigma[M]$ is injective in $\sigma[M]$ if and only if $N$ is $M$-injective hull.

**Proposition 1.1 (Functor ring).** Denote by $\{U_i\}_\Lambda$ a representing set of all finitely generated modules in $\sigma[M]$ and $U = \bigoplus U_i$.

$T := \hat{\text{End}}(U_R) = \{ f \in \text{End}_R(U) | (U_i)f = 0 \text{ almost every where} \}$ is called the functor ring of $\sigma[M]$. $T$ has no unit but has enough idempotents. The following hold:
(1) \( T \) is left perfect if and only if every module in \( \sigma[M] \) is a direct sum of finitely generated modules. In this case \( M \) is called pure semisimple ([10], 53.4]).

(2) Assume \( M \) is locally of finite length. Then \( T \) is semiperfect ([10], 51.7).

(3) Assume for every primitive idempotent \( e \in T \), \( Te \) is finitely cogenerated. Then \( M \) is locally artinian ([10], 52.1).

A ring \( T \) with enough idempotents is called semiperfect if every simple \( T \)-modules has projective covers (see [10], 49.10). \( T \) is said to be a left (right) \( QF-2 \) ring if it is a semiperfect and, for every primitive idempotent \( e \in T \), \( Te \) (resp. \( eT \)) has a simple essential socle (e.g., [3], section 4).

Theorem 1.2. For an \( R \)-module \( M \) with functor ring \( T \) the following are equivalent:

(a) For some \( k \in \mathbb{N} \), every module in \( \sigma[M] \) is a direct sum of uniserial modules of length \( \leq k \);

(b) \( T \) is a left and right \( QF-2 \) ring and \( \text{Jac}(T) \) is nilpotent.

Proof. Consider a representing set \( \{U_\lambda\}_\Lambda \) of all finitely generated modules in \( \sigma[M] \), \( U = \bigoplus U_\lambda \) and \( T = \text{End}_R(U) \).

(a) \( \Rightarrow \) (b) By condition (a), \( U \) is a direct sum of indecomposable modules of bounded length. Hence, by the Haraba-Sai Lemma (e.g., [10], 54.1), \( T \) is semiperfect and \( \text{Jac}(T) \) is nilpotent.

Since \( M \) is locally of finite length, we know from [10], 53.5 that \( U_T \) is \( T \)-injective. Now we can use the conclusions (a) \( \Rightarrow \) (b) \( \Rightarrow \) (c) of [10], 55.15 to derive that \( T \) is left and right \( QF-2 \).

(b) \( \Rightarrow \) (a) Assume \( T \) is a left and right \( QF-2 \) ring and \( \text{Jac}(T)^n = 0 \), for some \( n \in \mathbb{N} \). Then \( M \) is pure semisimple and locally artinian (see 1.1) and hence locally of finite length. With the proof of (c) \( \Rightarrow \) (a) in [10], 55.15 we see that indecomposable modules in \( \sigma[M] \) are uniserial.

It remain to show that for every uniserial module \( N \in \sigma[M] \), length \( N \leq n \). Assume \( N \) has composition series

\[ 0 \neq N_1 \subset \cdots \subset N_n \subset N_{n+1} = N. \]

From this we obtain a sequence of \( n \) morphisms in \( \text{Jac}(T) \),

\[ N_n \to N \to N/N_1 \to \cdots \to N/N_{n-1}, \]

whose product is not zero, contradicting \( \text{Jac}(T)^n = 0 \).

2. Lifting modules

An \( R \)-module \( M \) is called extending of \( CS \) module if every submodule is

\[ \cdots \]
essential in a direct summand of \( M \).

\( M \) is said to be lifting if every submodule \( K \subset M \) lies above a direct summand, i.e., there is a direct summand \( X \subset M \) with \( X \subset K \) and \( K/X \cong M/X \). For characterizations of this condition refer to [10], 41.11 and 41.12.

A family \( \{N_\lambda\}_\lambda \) of independent submodules of \( M \) is said to be a local direct summand of \( M \) if finite (direct) sum of \( N_\lambda \)'s is a direct summand in \( M \), and we say it is a direct summand if \( \bigoplus_\lambda N_\lambda \) is a direct summand in \( M \) (see [4], Definition 2.15).

A module is called continuous if it is extending and direct injective. In particular, self-injective modules are continuous.

Recall two results about these modules:

**Lemma 2.1.** Let \( M \) be an \( R \)-module.

1. Assume every local direct summand of \( M \) is a direct summand. Then \( M \) is a direct sum of indecomposable submodules.
2. Assume \( M \) is lifting and continuous. Then every local direct summand of \( M \) is a direct summand.

Proof. (1) See [5], Lemma 2.4 or [4], Theorem 2.17.

(2) This is shown in [5], Lemma 2.5.

A ring \( R \) is called a left \( H \)-ring if every injective module is \( R \text{-Mod} \) is lifting. Some of the characterizations of \( H \)-rings (see [5], Theorem 1) can be extended to modules. For this we need the

**Definition.** A module \( K \in \sigma[M] \) is said to be small in \( \sigma[M] \) if it is small submodule in its \( M \)-injective hull, i.e., \( K \preccurlyeq \hat{K} \).

**Theorem 2.2.** For any \( R \)-module \( M \), the following are equivalent:

(a) Every injective module in \( \sigma[M] \) is lifting.

(b) \( M \) is locally noetherian and every non-small module in \( \sigma[M] \) contains an \( M \)-injective submodule.

(c) Every module in \( \sigma[M] \) is a direct sum of an \( M \)-injective module and a small module.

Proof. (a) \( \Rightarrow \) (b) By 2.1, every injective module in \( \sigma[M] \) is a direct sum of indecomposable submodules. This implies that \( M \) is locally noetherian (see [10], 27.5).

Assume \( N \) is not small in its \( M \)-injective hull \( \hat{N} \). Since \( \hat{N} \) is lifting there is a direct summand \( X \subset \hat{N} \) with \( X \subset N \) and \( N/X \preccurlyeq \hat{N}/X \). By assumption, \( X \) is not zero.

(b) \( \Rightarrow \) (a) Referring to [10], 27.3, apply the proof of Proposition 2.7 in [5].

(a) \( \Rightarrow \) (c) Consider \( N \in \sigma[M] \) with \( M \)-injective hull \( N \). Since \( \hat{N} \) is lifting, by [10],
41.11, a direct summand $X \subset \hat{N}$ is contained in $N$ and $N = X + Y$ with $Y \ll \hat{N}$. This implies that $Y$ is small in $\sigma[M]$.

(c) $\Rightarrow$ (a) With respect to [10], 41.11, this is obvious.

It was pointed out in Osofsky [6], Lemma B (also in the proof (1) $\Rightarrow$ (3) of Vanaja-Purav, Proposition 2.13) that, for a uniserial module $M$ with composition series $0 \neq V \subset U \subset M$, $M \oplus U/V$ is not an extending module. For the same situation we observe:

**Lemma 2.3.** Assume $M$ is a uniserial module with composition series $0 \neq V \subset U \subset M$. Then the module $M \oplus U/V$ is not lifting.

**Proof.** Assume $M \oplus U/V$ is lifting. Then, by Theorem 1 in [1], $U/V$ is $M$-projective. However, the diagram

$$
\begin{array}{c}
{\textstyle U/V} \\
\downarrow
\end{array}
\begin{array}{c}
{\textstyle M \\ \rightarrow M/V \rightarrow 0}
\end{array}
$$

can not be extended commutatively by any $h: U/V \rightarrow M$, since the image of such a morphism always is contained in $V$.

The main purpose of this note is to prove:

**Theorem 2.4.** For any $R$-module $M$ the following are equivalent:

(a) Every module in $\sigma[M]$ is lifting;

(b) every module in $\sigma[M]$ is direct sum of a semisimple module and a projective module in $\sigma[M]$;

(c) every module in $\sigma[M]$ is direct sum of modules of length $\leq 2$

(d) $T$ is left and right OF-2 ring and $\text{Jac}(T)^2 = 0$.

If this conditions hold, there is a projective generator in $\sigma[M]$ and all indecomposable modules of length $\leq 2$ are $M$-projective.

**Proof.** (a) $\Rightarrow$ (d) Assume every module in $\sigma[M]$ is lifting. Then by Theorem 2.2, $M$ is locally noetherian. It is easy to see that finitely generated uniform lifting module are local modules, i.e., their factor modules are indecomposable.

Consider an indecomposable injective module $Q \in \sigma[M]$. Then for any finitely generated submodule $K \subset Q$, $K/\text{Rad}(K)$ is simple and hence $Q$ is uniserial (see [10], 55.1). In particular, every uniform module in $\sigma[M]$ is uniserial of length $\leq 2$ (by Lemma 2.3). So the $M$-injective hull $\hat{M}$ of $M$ is a direct sum of modules of length $\leq 2$ and hence $\hat{M}$ (and $M$) is locally of finite length. This implies that every finitely generated module in $\sigma[M]$ is a direct sum of indecomposable module (of
Denote by \( \{ U_\lambda \}_A \) a representing set of all finitely generated modules in \( \sigma[M] \) and \( U = \oplus_A U_\lambda \). By the Harada-Sai Lemma, the functor ring \( T := \text{End}_R(U) \) has the properties that \( T/Jac(T) \) is semisimple and \( Jac(T) \) is nilpotent.

In particular, \( M \) is pure-semisimple, i.e., every module in \( \sigma[M] \) is a direct sum of finitely generated modules and these are direct sums of uniserial submodules of length \( \leq 2 \). Now the assertion follows from Theorem 1.2.

Since \( T \) is right perfect, there exists a projective generator in \( \sigma[M] \) by [10], 51.13.

Consider an indecomposable module \( N \) of length 2. This is a factor module of a supplemented projective module in \( \sigma[M] \) and hence has a projective cover \( P \) (see [10], 42.1), which again is indecomposable and hence of length \( \leq 2 \). This implies \( P = N \), i.e., \( N \) is \( M \)-projective.

(c) \( \Rightarrow \) (d) This is clear by Theorem 1.2.

(c) \( \Rightarrow \) (a) Consider any module \( N = \oplus_A N_\lambda \) in \( \sigma[M] \), with \( N_\lambda \) uniserial of length \( \leq 2 \). By Theorem 1 in [1], \( N \) is lifting if and only if \( \{ N_\alpha \}_A \) is locally semi-\( T \)-nilpotent and \( N_\alpha \) is almost \( N_\beta \)-projective for any \( \alpha \neq \beta \) in \( \Lambda \).

The first condition is satisfied by the Harada-Sai Lemma (see [10], 54.1]. Any \( N_\lambda \) of length 2 is projective in \( \sigma[M] \) (as noted above) and hence is almost \( K \)-projective for any \( K \in \sigma[M] \).

Assume \( N_\lambda \) has length 1 and consider any diagram with exact line

\[
\begin{array}{c}
N_\beta \\
\downarrow^f \\
N_\beta \to L \to 0,
\end{array}
\]

with length \( N_\beta \leq 2 \). If \( f \) is not an isomorphism and \( f \neq 0 \), there exists an epimorphism \( g : N_\beta \to N_\alpha \) with \( p = gf \). From this we see that \( N_\lambda \) is almost \( N_\beta \)-projective and \( N \) is lifting.

(c) \( \Rightarrow \) (b) It is clear from the above that modules of length 2 are \( M \)-projective. Recall that finitely generated \( M \)-projective modules are projective in \( \sigma[M] \). From this the assertion is obvious.

(b) \( \Rightarrow \) (c) Consider a finitely generated \( N \in \sigma[M] \). Then any factor module of \( N \) is a direct sum of a projective module and a noetherian module and hence \( N \) is noetherian by [7], section 3. This implies that \( M \) is locally noetherian.

Now let \( K \in \sigma[M] \) be any indecomposable \( M \)-injective module. Assume \( K \) is not semisimple. Then it is projective in \( \sigma[M] \). Since \( \text{End}_R(K) \) is local, \( K \) is a local module, i.e., every factor module is indecomposable (see [10], 19.7) and hence simple. From this we deduce that \( k \) has length \( \leq 2 \).

Since every \( M \)-injective module in \( \sigma[M] \) is a direct sum of indecomposables, the assertions follows.
From Theorem 2.4 together with Theorem 11 in Dung-Smith [2] we obtain a characterization of rings with all modules lifting which extends Proposition 2.13 in Vanaja-Purvav [8]:

**Corollary 2.5.** For any ring $R$ the following are equivalent:

(a) Every left $R$-module is lifting;

(b) Every left $R$-module is extending;

(c) Every left $R$-module is a direct sum of a semisimple module and a projective module;

(d) Every left $R$-module is a direct sum of modules of length $\leq 2$;

(e) $R$ is a generalized uniserial ring with $\text{Jac}(R)^2 = 0$

It follows from (e) that the conditions (a)-(d) are left right symmetric.

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**References**


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