Mohamed and Müller showed in [5] that continuous modules have the exchange property. And, recently, they also showed in [6] that for nonsingular quasi-continuous modules, the finite exchange property implies the exchange property. However, it is still open whether this is true or not for any quasi-continuous modules ([5, Problem 2]). The purpose of this paper is to answer this problem in the affirmative. This, then also provides another instance of modules for which the existence of the finite exchange property implies that of the exchange property in reference to the longstanding open question posed in Crawley-Jonsson [1].

Discrete (= semiperfect) modules and quasi-discrete (= quasi-semiperfect) modules are dual to continuous modules and quasi-continuous modules, respectively. We note that discrete modules have the exchange property and, for quasi-discrete modules, the finite exchange property implies the exchange property. These results follows by summarizing following results:

1. (Oshiro [10]) Every quasi-discrete module $M$ has an indecomposable decomposition $M = \sum_{i} \oplus M_i$ such that $M' = \sum \{M_i | M_i \text{ completely indecomposable}\}$ satisfies the finite exchange property. So, if $M$ is discrete then it satisfies the finite exchange property since $M = M'$.

2. (Harada-Ishii [2], Yamagata [12], [13]) If a module has an indecomposable decomposition and satisfies the finite exchange property, then it satisfies the exchange property in direct sums of completely indecomposable modules.

3. (Zimmermann-Huisgen and Zimmermann [11]) A module $M$ satisfies the exchange property if and only if for any $P = M \oplus X = \sum_{i} \oplus M_i$ with each $M_i \cong M$, there exists $M_i < \oplus M_i$ for each $i \in I$ such that $P = M \oplus \sum_{i} \oplus M_i'$.

The reader is referred to Mohamed and Müller 'Book [5] for the background of these results.
1. Preliminaries. Throughout this paper $R$ will denote a ring with identity and all $R$-modules will be unital right $R$-modules. For two $R$-modules $X$ and $Y$, we use $X \subseteq_e Y$, $X \subseteq Y$, $X \subseteq_0 Y$, $X \vartriangleleft Y$ and $X \triangleleft Y$ to mean that $X$ is an essential submodule of $Y$, $X$ is isomorphic to a submodule of $Y$, $X$ is isomorphic to an essential submodule of $Y$, $X$ is a direct summand and $X$ is isomorphic to a direct summand of $Y$, respectively. For a set $I$, $|I|$ stands for the cardinal of $I$.

An $R$-module $M$ is called an extending module (or a CS-module) if it satisfies

(C1): for any submodule $X$ of $M$, there exists a direct summand $X^*$ of $M$ such that $X \subseteq_e X^*$

$M$ is called continuous if it satisfies (C1) and

(C2): Every submodule of $M$ which is isomorphic to a direct summand of $M$ is a direct summand.

$M$ is called a quasi-continuous module if it satisfies (C1) and

(C3): If $X$ and $Y$ are direct summands of $M$ with $X \cap Y = 0$, then $X \oplus Y$ is a direct summand.

For an $R$-module $M$ with a decomposition $M = \sum_i \oplus M_i$, we use the following condition:

(A) For any choice of $x_i \in M_i (i \in I, \alpha_i$ distinct) such that $(0 : x_1) \subseteq (0 : x_2) \subseteq \ldots$, the sequence becomes stationary, where $(0 : x)$ denotes the annihilator right ideal of $x$.

This condition appeared in [8] (cf.[5], [?]). One of interesting results on this condition is (2) of the following

Proposition 1.1. Let $M$ be a quasi-continuous module. Then the following hold:

1. for any decomposition $M = \sum_i \oplus M_i$ and any $J \subseteq I$, $\sum_i \oplus M_i$ is $\sum_j \oplus M_j$-injective

2. any decomposition $M = \sum_i \oplus M_i$ satisfies the condition (A).

3. for any decomposition $M = \sum_i \oplus M_i$ and any direct summand $X$ of $M$, there exists $N_i \subset \oplus M_i$ such that $M = X \oplus \sum_i \oplus N_i$.

Proof. (1) follows from [9, Proposition 1.5]. (2) follows from [5, Proposition 2.13] or from (1) above and [5, Proposition 1.9]. And see [3] for (3).

A module $M$ is called a square module if $M \cong X \oplus X$ for some module $X$, and a module is called a square free if it does not contain non-zero square modules.

The following results are important and useful in the study of quasi-continuous modules.
Lemma 1.1 ([5, Theorem 2.37]). *Any quasi-continuous module is a direct sum of a quasi-injective module and a square free module.*

Proposition 1.2. For an R-module X, the following conditions are equivalent:

1. X is quasi-continuous.
2. Any decomposition \( E(X) = \bigoplus_i M_i \) implies \( X = \bigoplus_i (M_i \cap X) \), where \( E(X) \) is the injective hull of X.
3. For any R-module Y with \( X \subseteq_e Y \), any decomposition \( Y = \bigoplus_i Y_i \) implies \( X = \bigoplus_i (Y_i \cap X) \).

Proof. The equivalence of 1) and 2) is well known [5, Theorem 2.8]. We may show the implication 2) \( \Rightarrow \) 3). Let \( Y \) be an R-module with \( X \subseteq_e Y \) and consider a decomposition \( Y = \bigoplus_i Y_i \). Let \( x \in X \). Then there exists a finite subset \( F = \{1, 2, \ldots, n\} \) of I such that \( x \in \bigoplus_{i \in F} Y_i \). Let \( x = y_1 + \ldots + y_n \), where \( y_i \in Y_i \). Consider \( E(X) = \bigoplus_{i \in F} E(Y_i) \). By 2), we have \( X = \bigoplus_{i \in F} (E(Y_i) \cap X) \cap (\bigoplus_{i \in F} Y_i) \cap X \). Express the element \( x \) as \( x = p_1 + p_2 + \ldots + p_n + q \) where \( p_i \in E(Y_i) \cap X \) and \( q \in E(\bigoplus_{i \in F} Y_i) \cap X \). Since \( x = y_1 + \ldots + y_n \) with \( y_i \in E(Y_i) \), we see \( p_i = y_i \), \( i = 1, 2, \ldots, n \). Hence \( y_i \in Y_i \cap X \), \( i = 1, 2, \ldots, n \), so \( x \in \bigoplus_{i \in F} (Y_i \cap X) \). Accordingly we see \( X = \bigoplus_{i \in F} (Y_i \cap X) \).

For a cardinal \( \alpha \), an R-module X has the \( \alpha \)-exchange property if for any R-module M and any two decompositions \( M = X \oplus N = \bigoplus_i M_i \) with \( |I| \leq \alpha \), there exists \( M_i \subset \oplus M \) for each \( i \in I \) such that

\[ M = X \oplus (\bigoplus_i M_i) \]

X has the exchange property if this holds for any cardinal \( \alpha \) and has the finite exchange property if this holds whenever the index set I is finite. We note that, in these definitions, we may assume that \( M_i \approx X \) for all \( i \in I \) (by Zimmermann-Huisgen and Zimmermann [11]).

2. A key lemma. In this section we show the following which is a key lemma of this paper. We note that this lemma is also used for the study of direct sums of relative continuous modules [3], [4].

Lemma 2.1. Let P be an R-module with a decomposition \( P = \bigoplus_i M_i \) such that each \( M_i \) is extending. We consider the index set I as an well ordered set:
$I = \{0, 1, \ldots, w, w+1, \ldots\}$, and let $X$ be a submodule of $M$. Then there are submodules $T(i) \subseteq e T(i)^* \oplus \bigoplus M_i$, decompositions $M_i = T(i)^* \oplus N_i$ and a submodule $\sum T(i) \subseteq e X$ for which the following properties hold:

1) $X(0) = T(0) \subseteq e T(0)^*$.
2) $X(k) \subseteq T(k) \oplus \sum_{i < k} N_i$ for all $k \in I$.
3) $\sigma(X(k)) = T(k) \subseteq e T(k)^*$, $X(k) = \sigma(X(k))(\text{by } \sigma|X(k))$ for all $k \in I$, where $\sigma$ is the projection: $P = \sum_{i} T(i)^* \oplus \sum_{i} N_i \to \sum_{i} T(i)^*$.
4) $X \cong \sigma(X)(\text{by } \sigma|X)$.

For a proof of this result, we need two lemmas

**Lemma 2.2.** Let $M$ be an $R$-module with a decomposition $M = M_1 \oplus M_2$ and let $X$ a submodule of $M$. If there is a decomposition $M_i = M_i^* \oplus M_i^{**}$ such that $M_i \cap X \subseteq e M_i^*$ for $i = 1, 2, \ldots$, then $X \supseteq (M_i^* \cap X) \oplus (M_i^{**} \cap X)$.

Proof. Let $(x+y) \in X$ and express $x$ in $M = M_1^* \oplus M_1^{**} \oplus M_2^* \oplus M_2^{**}$ as $x = x_1^* + x_2^* + x_3^* + x_4^*$ where $x_1^* \in M_1^*$ and $x_2^* \in M_1^{**}$. If $x_3^* + x_4^* \in (M_1^* \cap X) \oplus (M_1^{**} \cap X)$, then $x_3^* + x_4^* \in (M_1^* \cap X) \oplus (M_1^{**} \cap X)$ and hence $x \in (M_1^* \cap X) \oplus (M_1^{**} \cap X) \oplus (M_2^* \cap X) \oplus (M_2^{**} \cap X)$.

**Lemma 2.3.** Let $M$ be an $R$-module with a decomposition $M = A \oplus B \oplus C \oplus D$ and let $X$ be a submodule of $M$. If $Y$ is a submodule of $X$ such that $Y \subseteq (A \oplus B) \cap X$, then $Y = \sigma(Y) \subseteq e A$, where $\sigma$ is the projection: $M = A \oplus B \oplus C \oplus D \to A$. Then $Y \subseteq (B \oplus C) \cap X \subseteq e (A \oplus B \oplus C) \cap X$.

Proof. Let $(x+y) \in X$ and express $x$ in $M = A \oplus B \oplus C \oplus D$ as $x = a + b + c \in (A \oplus B \oplus C) \cap X$, where $a \in A$, $b \in B$ and $c \in C$. If $a = 0$, then $x = b + c \in (B \oplus C) \cap X$. If $a \neq 0$, then $0 = \sigma(xr) \subseteq \sigma(Y)$ for some $r \in R$; so there exists $y \in Y$ such that $\sigma(xr) = \sigma(y)$. Since $xry \in \ker \sigma \cap (B \oplus C)$, we see $xry \in Y \subseteq (B \oplus C) \cap X$. Hence we see $Y \subseteq (B \oplus C) \cap X \subseteq e (A \oplus B \oplus C) \cap X$.

Proof of Lemma 2.1. We put $X_i = M_i \cap Y$ for all $i \in I$. Since $M_i$ is extending, we have a decomposition
such that $X_i \subseteq eX_i^*$ for all $i \in I$. By Lemma 2.2,

$$(M_0 \oplus M_1) \cap X_e \supseteq X_0 \oplus X_1 \oplus (X_0^{**} \oplus X_1^{**}) \cap X.$$

We put

$$X(0) = X_0, \quad X(1) = X_1 \oplus (X_0^{**} \oplus X_1^{**}) \cap X.$$

Let $\pi_0, \pi_1$ be the projections:

$$X_0^{**} \oplus X_1^{**} \to X_0^{**}, \quad X_0^{**} \oplus X_1^{**} \to X_1^{**}$$

respectively. Since $X_i^{**} \cap X = 0$, we see that

$$(X_0^{**} \oplus X_1^{**}) \cap X \simeq \pi_0((X_0^{**} \oplus X_1^{**}) \cap X) \simeq \pi_1((X_0^{**} \oplus X_1^{**}) \cap X).$$

canonically. Put

$$T(0) = X(0), \quad T(0)^* = X(0)^* = X_0^*, \quad T(1) = X_1 \oplus \pi_1((X_0^{**} \oplus X_1^{**}) \cap X).$$

Since $M_1$ is extending, we have a decomposition

$$M_1 = T(1)^* \oplus N_1$$

with $T(1) \subseteq eT(1)^*$. Putting $N_0 = X_0^{**}$, we have

$$P = T(0)^* \oplus T(1)^* \oplus N_0 \oplus N_1 \oplus \sum_{i \in I} M_i$$

such that

$$X(0) = T(0), \quad X(1) \subseteq T(1) \oplus N_0,$$

$$\sigma_i(X(i)) = T = (i), \quad X(i) \simeq T(i) \text{ by } (\sigma_i|X(i)) \text{ (cf. (*) above)}$$

for $i = 1, 2$, where $\sigma_i$ is the projection:

$$P = T(0)^* \oplus T(1)^* \oplus N_0 \oplus N_1 \oplus \sum_{i \in I} M_i \to T(0)^* \oplus T(1)^*$$

Next consider $(M_0 \oplus M_1 \oplus M_2) \cap X$. Put $A = T(0)^* \oplus T(1)^*$, $B = N_0 \oplus N_1$, $C = M_2$ and $D = \sum_{i \in I} M_i$, and $Y = X(0) \oplus X(1)$. Then $X \subseteq P = A \oplus B \oplus C \oplus D$, $A \oplus B \oplus C = M_0 \oplus M_1 \oplus M_2$ and $Y \simeq \sigma_i(Y) \subseteq eA$. So we see from Lemma 2.3 that

$$(M_0 \oplus M_1 \oplus M_2) \cap X_e \supseteq X(0) \oplus X(1) \oplus (N_0 \oplus N_1 \oplus N_2) \cap X.$$

Further, since $(N_0 \oplus N_1) \cap X = 0$, we see from Lemma 2.2 that

$$(N_0 \oplus N_1 \oplus M_2) \cap X_e \supseteq X_2 \oplus (N_0 \oplus N_1 \oplus X_2^{**}) \cap X.$$

Let $\pi_0, \pi_2$ be the projections:

$$N_0 \oplus N_1 \oplus X_2^{**} \to N_0 \oplus N_1, \quad N_0 \oplus N_1 \oplus X_2^{**} \to X_2^{**}.$$
respectively. Since \((N_0 \oplus N_1) \cap X = 0\) and \(X_{2*} \cap X = 0\), we see
\[
(N_0 \oplus N_1 \oplus X_{2*}) \cap X \cong \pi_0((N_0 \oplus N_1 \oplus X_{2*}) \cap X) \cong \pi_0((N_0 \oplus N_1 \oplus X_{2*}) \cap X)
\]
canonicaly...(**)
Put
\[
X(2) = X_2 \oplus (N_0 \oplus N_1 \oplus X_{2*}) \cap X,
\]
\[
T(2) = X_2 \oplus \pi_0((N_0 \oplus N_1 \oplus X_{2*}) \cap X).
\]
Then
\[
(M_0 \oplus M_1 \oplus M_2) \cap X \cong X(0) \oplus X(1) \oplus X(2).
\]
Since \(M_2\) is extending, we have a decomposition
\[
M_2 = T(2)^* \oplus N_2
\]
with \(T(2) \subseteq e T(2)^*.\) Here we see
\[
P = T(0)^* \oplus T(1)^* \oplus T(2)^* \oplus N_0 \oplus N_1 \oplus N_2 \oplus \sum_{i \leq 1} M_i,
\]
\[
X(2) \subseteq T(2)^* \oplus N_0 \oplus N_1,
\]
and for the projection :
\[
\sigma_2 : P = T(0)^* \oplus T(1)^* \oplus T(2)^* \oplus N_0 \oplus N_1 \oplus N_2 \oplus \sum_{i \leq 1} M_i \rightarrow T(0)^* \oplus T(1)^* \oplus T(2)^*.
\]
We see that
\[
\sigma_2(X(i)) = T(i),\ X(i) = T(i)(\text{by } \sigma_2|X(i))
\]
for \(i = 0, 1, 2, (\text{cf (**)}).
Now we proceed our argument by transfinite induction on \(a \in I\). Let \(a \in I\) and put \(J = \{i \in I | i < a\}\)
Assume that there are submodules \(T(i) \subseteq e T(i)^* \oplus M_i\), decompositions \(M_i = T(i)^* \oplus N_i\) for which the following hold :
1) \(X(0) = T(0),\)
2) \(X(k) \subseteq T(k) \oplus \sum_{i \leq k} N_i \forall k \in J,\)
3) \((\sum_{i \leq k} M_i) \cap X_2 \cong \sum_{i \leq k} X(i) \forall k \in J\) ; \(so(\sum_{i \leq k} M_i) \cap X_2 \cong \sum_{i \leq k} X(i),\)
4) \(X(k) \cong T(k)\) by \((\sigma_j|X(k))\) \(\forall k \in J\)
where \(\sigma_j\) is the projection :
\[
P = \sum_{j} T(i)^* \oplus \sum_{j} N(i) \oplus \sum_{j} M(i) \rightarrow \sum_{j} T(i)^*
\]
So
Consider \((\sum_f \oplus M_f) \cap X\). We note that

\[
\sum_f \oplus X(i) \subseteq (\sum_f \oplus M_f) \cap X = (\sum_f \oplus T(i)^* \oplus \sum_f \oplus N_f) \cap X,
\]

\[
\sum_f \oplus X(i) \simeq \sum_f \oplus T(i) \text{ (by } \sigma_f|\sum_f \oplus X(i))
\]

Considering

\[
\sum_f \oplus T(i)^* \oplus \sum_f \oplus N(i) \oplus M_a \oplus \sum_{f \neq a} \oplus M_f \rightarrow \sum_f \oplus T(i)^*
\]

we infer from Lemma 2.3 that

\[
(\sum_f \oplus M_f) \cap X_i \equiv \sum_f \oplus X(i) \oplus (\sum_f \oplus N_f \oplus M_a) \cap X.
\]

Since \(\sum_f \oplus N_f \cap X = 0\), we see by Lemma 2.2 that

\[
(\sum_f \oplus N_f \oplus M_a) \cap X_i \equiv X_i \oplus (\sum_f \oplus N_f \oplus X^*_a) \cap X
\]

(where \(X_a = M_a \cap X \subseteq X_a^*\), \(M_a = X^*_a \oplus X_a^**\))

Let \(\pi_f\) and \(\pi_a\) be the projections:

\[
\sum_f \oplus N_f \oplus X^*_a \rightarrow \sum_f \oplus N_f, \quad \sum_f \oplus N_f \oplus X^**_a \rightarrow X^*_a
\]

respectively. We see that

\[
(\sum_f \oplus N_f \oplus X^*_a) \cap X \simeq \pi_f((\sum_f \oplus N_f \oplus X^*_a) \cap X) \simeq \pi_a((\sum_f \oplus N_f \oplus X^**_a) \cap X)
\]

canonically. We put

\[
X(a) = X_a \oplus (\sum_f \oplus N_f \oplus X^*_a) \cap X,
\]

\[
T(a) = X_a \oplus \pi_a((\sum_f \oplus N_f \oplus X^**_a) \cap X).
\]

Since \(M_a\) is extending, we have a decomposition

\[
M_a = T(a)^* \oplus N_a
\]

with \(T(a) \subseteq T(a)^*\). Now we see

\[
X(a) \subseteq T(a) \oplus \sum_f \oplus N_f,
\]

\[
\sigma(X(a)) = T(a), \quad X(a) \simeq T(a) \text{ (by } \sigma|X(a))
\]

where \(\sigma\) is the projection:

\[
P = \sum_f \oplus T(i)^* \oplus \sum_{f \neq a} \oplus N(i) \oplus \sum_{f \neq a} \oplus M(i) \rightarrow \sum_f \oplus T(i)^*.
\]
Furthermore we see
\[(\sum_{f \in J} \oplus M_f) \cap X_e \supseteq \sum_{f \in J} \oplus X(f)\]
Thus 1)-5) above hold for \( J \cup a \), and this completes the proof by transfinite induction.

3. The exchange property. Using Lemma 2.1 we shall give a proof of the exchange property of continuous modules from our point of view.

**Proposition 3.1.** Let \( P \) be an \( R \)-module and \( X \) a submodule of \( P \). If \( X \) is continuous and \( P \) has a decomposition \( P=\sum_i \oplus M_i \) with each \( M_i \cong X \), then there exists direct summand \( N_i < \oplus M_i \) for each \( i \in I \) such that \( P=X \oplus \sum_i \oplus N_i \).

So, \( X \) is a direct summand of \( P \).

**Proof.** By Lemma 2.1, we have
\[P=\sum_i \oplus T(i)^* \oplus \sum_i \oplus N_i, \quad X_e \supseteq \sum_i \oplus X(i)\]
such that, for each \( i \in I \),
1) \( T(i) \subseteq \varepsilon T(i)^* \),
2) \( M_i = T(i)^* \oplus N_i \),
3) \( \sigma(X(i)) = T(i), \quad X(i) \cong T(i) (\text{by } \sigma|X(i)) \)
where \( \sigma \) is the projection :
\[P=\sum_i \oplus T(i)^* \oplus \sum_i \oplus N_i \rightarrow \sum_i \oplus T(i)^* .\]
Since \( X \) is quasi-continuous and \( X \cong \sigma(X) \subseteq \varepsilon \sum_i \oplus T(i)^* \), we obtain, by Proposition 1.2,
\[\sigma(X) = \varepsilon \sum_i \oplus (T(i)^* \cap \sigma(X)) .\]
Putting \( X(i)^* = \sigma^{-1}(T(i)^* \cap \sigma(X)) \), we see
\[X=\sum_i \oplus X(i)^*, \]
\[X(i) \subseteq \varepsilon X(i)^* \forall i \in I , \]
\[T(i) \subseteq \varepsilon \sigma(X(i)^*) \subseteq \varepsilon T(i)^* \forall i \in I .\]
Since \( X \cong M_i \) and \( X(i)^* < \oplus X \), we see from the condition \((C_2)\) for \( X \) that \( \sigma(X(i)^*) < \oplus T(i)^* ; \) whence \( \sigma(X(i)^*) = T(i)^* \) for all \( i \in I \).

As a result
Hence it follows \( P = X \oplus \sum_{i} T(i) \) (by \( \sigma(X) \)).

As an immediate consequence we have

**Theorem 3.1** ([5, Theorem 3.24]). Continuous module have the exchange property.

**Remark.** We note that, in the proof above, the exchange property of quasi-injective modules is not used. (Compare our proof to the proof of [5, Theorem 3.24])

Now, we are in a position to show our main result

**Theorem 3.2.** Any quasi-continuous module with the finite exchange property has the exchange property.

Proof. Let \( X \) be a quasi-continuous module with the finite exchange property. We may assume \( X \) to be a square free by Lemma 1.1. In order to show our result by transfinite induction, let \( \alpha \) be an infinite cardinal and assume that \( X \) satisfies \( \beta \)-exchange property for any cardinal \( \beta < \alpha \). To show that \( X \) satisfies the \( \alpha \)-exchange property, consider the situation of \( R \)-modules:

\[
P = \sum_{i} M_{i} = X \oplus Y
\]

where \( |I| = \alpha \) and \( M_{i} \cong X \) for all \( i \in I \). We may consider \( I \) as a well ordered set; \( I = \{0, 1, \ldots, \omega, \ldots\} \), whose ordinal is an initial ordinal; so, for any \( \beta \in I \), the cardinal of \( \{i \in I | i < \beta\} < \alpha \). By Lemma 2.1 and, as in the proof of Proposition 3.1, we have decompositions:

\[
P = \sum_{i} T(i)^{*} \oplus \sum_{i} N_{i},
X = \sum_{i} X(i)^{*} \geq \sum_{i} X_{i}
\]

such that, for all \( k \in I \),

\[
M_{k} = T(k)^{*} \oplus N_{k}, \quad T(k) \subseteq_{e} T(k)^{*}, \quad X(k) \subseteq_{e} X(k)^{*},
X(k) \subseteq T(k) \oplus \sum_{i \leq k} N_{i},
X(k)^{*} \subseteq T(k)^{*} \oplus \sum_{i} N_{k},
\sigma(X(k)) = T(k) \subseteq_{e} T(k)^{*} \subseteq_{e} T(k)^{*},
X(k) \cong T(k) \quad \text{(by } \sigma[X(k)]\text{)},
X(k)^{*} \cong \sigma(X(k)^{*}) \quad \text{(by } \sigma[X(k)^{*}]\text{)}
\]
where $\sigma$ is the projection:

$$P=\sum T(i)^*+\sum N_i \mathcal{R} \sum T(i)^*.$$

Since $N_k \subset \bigoplus M_k \approx X=\sum X^*$, by Proposition 1.1, we have a decomposition $N_k = \sum_{i \in I} N_k(i)$ for each $k \in I$ with $N_k(i) \subset \bigoplus X(i)^*$. We note that $\sum T(i)^*$ is square free, since $X \subset \sum T(i)^*$. Now, using the finite exchange property of $X(0)^*$ for

$$X(0)^* \subset \sum T(0)^*+\sum T(i)^* \sum N_i,$$

we have decompositions

$$T(0)^* = T(0)^* + T(0)^*$$

such that

$$P = X(0)^* + T(0)^* + \sum_{i=0}^n T(i)^* + \sum N_i.$$

We denote, by $\pi_0$, the projection:

$$P = X(0)^* + T(0)^* + \sum_{i=0}^n T(i)^* + \sum N_i \rightarrow X(0)^*.$$

Then

$$T(0)^* + \sum N_i \approx X(0)^* \text{ (by } \pi_0 \text{)}.$$

Assume $0 \neq \sum N_i$ and take $0 \neq n'' \in \sum N_i$. We express $n''$ in $P = X(0)^* + T(0)^* + \sum_{i=0}^n T(i)^* + \sum N_i$ as $n'' = a + b + n'$, where $a \in X(0)^*$, $b \in T(0)^* + \sum_{i=0}^n T(i)^*$, and $n' \in \sum N_i$.

Since $0 \neq \pi_0(n'') = a \in X(0)^* \subset X(0)$, there exists $r \in R$ such that $0 \neq ar \in X(0)$. Since $X(0) = T(0)$ and $n''r = ar + br + n'r$, we see from $n''r - n'r = ar + br \in \sum T(i)^* \cap \sum N_i = 0$ that $n''r - n'r = 0$; whence $n''r = 0$, so $ar = 0$, a contradiction. Accordingly, $\sum N_i = 0$ and hence

$$P = X(0)^* + T(0)^* + \sum_{i=0}^n T(i)^* + \sum N_i,$$

so
EXCHANGE PROPERTY

\[ X(0)^* \oplus T(0)^* \simeq T(0)^* \]

Since \( X(0)^* \oplus T(0)^* \) is square free and \( X(0)^* \subseteq T(0)^* \), we also see that \( T(0)^* = 0 \). Therefore

\[ P = X(0)^* \oplus \sum_{0} T(i)^* \oplus \sum \oplus N_i, \]

Next using the finite exchange property of \( X(1)^* \) in

\[ P = X(0)^* \oplus T(1)^* \oplus \sum_{i=0} T(i)^* \oplus N_0(1) \oplus \sum \oplus N_i \]

where \( W = X(0)^* \oplus \sum_{i=0} T(i)^* \oplus \sum \oplus N_0(i) \), we have decompositions

\[ W = \overline{W} \oplus W \]
\[ T(1)^* = \overline{T(1)^*} \oplus T(1)^* \]
\[ N_0(1) = N_0(1) \oplus N_0(1) \]
\[ \sum \oplus N_i = \sum \oplus N_i \oplus \sum \oplus N_i \]

such that

\[ P = \overline{W} \oplus T(1)^* \oplus N_0(1) \oplus \sum \oplus N_i. \]

Since \( \sum \oplus T(i)^* \) is square free, we see from \( \overline{W} \subseteq X(1)^* \subseteq T(1)^* \) that \( \overline{W} = 0 \). So

\[ P = X(0)^* \oplus X(1)^* \oplus T(1)^* \oplus \sum_{i=0} T(i)^* \oplus N_0(1) \oplus \sum \oplus N_i. \]

In order to show \( \sum \oplus N_i = 0 \), consider the projection \( \pi_1 : P = X(0)^* \oplus X(1)^* \oplus T(1)^* \oplus \sum_{i=0} T(i)^* \oplus N_0(1) \oplus \sum \oplus N_i \to X(1)^* \). Assuming \( \sum \oplus N_i \neq 0 \) we take \( 0 \neq n'' \in \sum \oplus N_i \). We express \( n'' \) in \( P = X(0)^* \oplus X(1)^* \oplus T(1)^* \oplus \sum_{i=0} T(i)^* \oplus N_0(1) \oplus \sum \oplus N_i \) as \( n'' = a + b + n' \), where \( a \in X(0)^* \oplus X(1)^* \), \( b \in T(1)^* \oplus \sum \oplus T(i)^* \oplus N_0(1) \oplus \sum \oplus N_i \). Since \( X(0)^* \oplus X(1)^* \subseteq X(0)^* \oplus X(1)^* \), we can take \( r \in R \), such that \( 0 \neq ar \in X(0)^* \oplus X(1)^* \). Note that \( 0 \neq n'' r = ar + br + n'' r \). Since \( X(0)^* \oplus X(1)^* \subseteq T(0)^* \oplus N_0 \), this implies \( ar + br = 0 \) and \( n'' r = n'' r \); whence \( n'' r = 0 \), a contradiction. Thus we get

\[ P = X(0)^* \oplus X(1)^* \oplus T(1)^* \oplus \sum_{i=0} T(i)^* \oplus N_0(1) \oplus \sum \oplus N_i \]

We proceed with the same argument for \( X(2)^* \). We consider
\[
P = X(0)^* \oplus X(1)^* \oplus T(1)^* \oplus T(2)^* \oplus \sum_{i \in \{0,1,2\}} \oplus T(i)^* \\
\oplus N_0(1) \oplus N_0(2) \oplus \sum_{i \in \{0,1\}} \oplus N_0(i) \\
\oplus N_1(2) \oplus \sum_{i \in \{0,1\}} \oplus N_1(i) \\
\oplus \sum_{i \in \{0,1\}} \oplus N_i.
\]

where

\[
W = X(0)^* \oplus X(1)^* \oplus T(1)^* \oplus \sum_{i \in \{0,1,2\}} \oplus T(i)^* \oplus N_0(1) \\
\oplus \sum_{i \in \{0,1,2\}} \oplus N_0(i) \oplus \sum_{i \in \{0,1,2\}} \oplus N_1(i).
\]

And using the finite exchange property of \(X(2)^*\) in this decomposition, we have decompositions

\[
\begin{align*}
W &= W \oplus \overline{W}, \\
T(2)^* &= T(2)^* \oplus T(2)^*, \\
N_0(2) &= \overline{N_0(2)} \oplus \overline{N_0(2)}, \\
N_1(2) &= \overline{N_1(2)} \oplus \overline{N_1(2)}, \\
\sum_{i \in \{0,1\}} \oplus N_i &= \sum_{i \in \{0,1\}} \oplus N_i \oplus \sum_{i \in \{0,1\}} \oplus N_i.
\end{align*}
\]

such that

\[
P = X(2)^* \oplus \overline{W} \oplus T(2)^* \oplus N_0(2) \oplus N_1(2) \oplus \sum_{i \in \{0,1\}} \oplus N_i.
\]

But, as \(W \subseteq X(2) \subseteq T(2)^*\) and as \(\sum_{i \in \{0,1\}} \oplus T(i)^*\) is square free, we obtain \(\overline{W} = 0\). So

\[
P = W \oplus T(2)^* \oplus N_0(2) \oplus N_1(2) \oplus \sum_{i \in \{0,1\}} \oplus N_i \\
= X(0)^* \oplus X(1)^* \oplus X(2)^* \oplus T(1)^* \oplus T(2)^* \oplus \sum_{i \in \{0,1,2\}} \oplus T(i)^* \\
\oplus N_0(1) \oplus N_0(2) \oplus \sum_{i \in \{0,1,2\}} \oplus N_0(i) \\
\oplus N_1(2) \oplus \sum_{i \in \{0,1,2\}} \oplus N_1(i) \\
\oplus \sum_{i \in \{0,1\}} \oplus N_i.
\]

We denote by \(\pi_0\) the projection:

\[
P = X(0)^* \oplus X(1)^* \oplus X(2)^* \oplus T(1)^* \oplus T(2)^* \oplus \sum_{i \in \{0,1,2\}} \oplus T(i)^* \\
\oplus N_0(1) \oplus N_0(2) \oplus \sum_{i \in \{0,1,2\}} \oplus N_0(i) \\
\oplus N_1(2) \oplus \sum_{i \in \{0,1,2\}} \oplus N_1(i) \oplus \sum_{i \in \{0,1\}} \oplus N_i \longrightarrow X(2)^*.
\]

Then note that
\[ T(2)^* \oplus N_0(2) \oplus N_1(2) \oplus \sum_{i=1,i} N_i \simeq X(2)^* \ldots (*) \]

(by \( \pi_2 \)).

We shall show \( \sum_{i=1,i} N_i = 0 \). Assume not, and take \( 0 \neq n'' \in \sum_{i=1,i} N_i \). We express \( n'' \) as \( n'' = a + b + n' \)

where \( a \in X(0)^* \oplus X(1)^* \oplus X(2)^* \), \( b \in T(1)^* \oplus T(2)^* \oplus \sum_{i=1,i} T(i)^* \oplus N_0(1) \)

\( \oplus N_0(2) \oplus \sum_{i=1,i} N_0(i) \oplus \sum_{i=2} N_1(i) \oplus N_1(2) \), \( n' \in \sum_{i=1,i} N_i \). Note that \( a \neq 0 \) by (\( \ast \)). Since \( X(0)^* \oplus X(1)^* \oplus X(2)^* \), we can take \( r \in \mathbb{R} \) such that \( 0 = ar \in X(0)^* \oplus X(1)^* \oplus X(2)^* \); so \( 0 \neq n'' r \). As \( X(0)^* \oplus X(1)^* \oplus X(2)^* \), \( T(0)^* \oplus T(1)^* \oplus T(2)^* \oplus N_0 \oplus N_1 \), we see that \( ar + br = 0 \) and \( n'' r - n' r = 0 \). But \( n'' r - n' r = 0 \) implies \( n'' r = 0 \), a contradiction. Hence \( \sum_{i=1,i} N_i = 0 \). As a result, we have

\[ P = X(0)^* \oplus X(1)^* \oplus X(2)^* \oplus T(1)^* \oplus T(2)^* \oplus \sum_{i=1,i} T(i)^* \oplus \sum_{i=1,i} \oplus N_0(i) \]

\( \oplus N_0(1) \oplus N_0(2) \oplus \sum_{i=1,i} N_1(i) \)

\( \oplus \sum_{i=1,i} N_i \).

We transfinitely proceed with this argument. For the sake of convenience, for any \( k \) in \( I \), we put

\[ I(k) = \{ i \in I \mid i < k \}. \]

Now, let \( \beta \in I \) and assume that we have obtained decompositions:

\[ T(i)^* = T(i)^* \oplus T(i)^* \ \forall i \in I(\beta), \]

\[ N_0(i) = N_0(i) \oplus N_0(i) \ \forall i \in I(\beta) - 0, \]

\[ N_1(i) = N_1(i) \oplus N_1(i) \ \forall i : 1 < i < \beta, \]

\[ N_k(i) = N_k(i) \oplus N_k(i) \ \forall k < i < \beta, \]

\[ \ldots \]

such that, for any \( k \in I(\beta), \)

\[ P = \sum_{0 \leq i \leq k} \oplus X(i)^* \sum_{0 < r \leq i} \oplus T(i)^* \sum_{k < i} \oplus T(i)^* \]

\( \oplus N_0(0) \oplus \sum_{0 < r \leq k} \oplus N_0(i) \oplus \sum_{k < i} N_0(i) \)

\( \oplus N_1(0) \oplus N_1(1) \oplus \sum_{1 < r \leq k} \oplus N_1(i) \oplus \sum_{k < i} N_1(i) \)

\( \oplus N_1(0) \oplus N_1(0) \oplus N_2(2) \oplus \sum_{2 < r \leq k} \oplus N_2(i) \oplus \sum_{k < i} N_2(i) \)
For $k \in I(\beta)$, we put
\[ Q(k) = \frac{T(k)^*}{*} \oplus \sum_{0 \leq i < k} \oplus N_i(k) \]
\[ Q = \sum_{I(\beta) = 0} \oplus Q(k). \]
Since $Q(k) \simeq X(k)^* \subset T(k)^*$ for all $k \in I(\beta) - 0$ and $\sum \oplus T(k)^*$ satisfies the condition $A$, $Q = \sum \oplus Q(k)$ satisfies the condition $A$. We note that, for any $q_k \in Q_k$ and $q_i \in Q_i$, $(0 : q_k) - (0 : q_i)$, since $Q$ is square free.

Putting
\[ \tilde{N}_0 = N_0(0) \oplus \sum_{0 < i < \beta} \oplus N_0(i) \oplus \sum_{\beta \leq i} \oplus N_0(i) \subseteq N_0, \]
\[ \tilde{N}_k = N_0(k) \oplus N_0(1) \oplus \ldots \oplus N_0(k) \oplus \sum_{0 < i < \beta} \oplus N_0(i) \oplus \sum_{\beta \leq i} \oplus N_0(i) \subseteq N_k \]
for $0 \neq k \in I(\beta)$, we claim that
\[ P = \sum_{i < \beta} \oplus X(i)^* \oplus \sum_{0 < i < \beta} \oplus T(i)^* \oplus \sum_{\beta \leq i} \oplus T(i)^* \oplus \sum_{0 < i < \beta} \oplus \tilde{N}(i) \oplus \sum_{\beta \leq i} \oplus N_i. \]
To show this we may show that $Q$ is contained in
\[ Z = \sum_{i < \beta} \oplus X(i)^* \oplus \sum_{0 < i < \beta} \oplus T(i)^* \oplus \sum_{\beta \leq i} \oplus T(i)^* \oplus \sum_{0 < i < \beta} \oplus \tilde{N}(i) \oplus \sum_{\beta \leq i} \oplus N_i. \]
Assume $Q \notin Z$. Since $Q = \sum \oplus Q(k)$ satisfies the condition $A$, we can take $Q_k$ and $q_k \in Q_k$ such that $q_k \notin Z$ and, for any $k < l$ and $q_i \in Q_i$,
\[ (0 : q_k) \subset (0 : q_i) \implies q_i \in Z. \ldots (*) \]
We express $q_k$ in
\[ P = \sum_{0 \leq i \leq k} \oplus X(i)^* \oplus \sum_{0 < i \leq k} \oplus T(i)^* \oplus \sum_{k < i} \oplus T(i)^* \]
\[ \oplus N_0(0) \oplus \sum_{0 < i \leq k} \oplus N_0(i) \oplus \sum_{k < i} \oplus N_0(i) \]
\[ \oplus N_0(1) \oplus N_0(1) \oplus \sum_{1 < i \leq k} \oplus N_0(i) \oplus \sum_{k < i} \oplus N_0(i) \]
\[ \oplus N_0(2) \oplus N_0(2) \oplus \sum_{2 < i \leq k} \oplus N_0(i) \oplus \sum_{k < i} \oplus N_0(i) \]
\[ \oplus \sum_{k < i} \oplus N_i \]
as $q_k = a + b$, where
\[ a \in \sum_{0 \leq i \leq k} \oplus X(i)^* \oplus \sum_{0 < i \leq k} \oplus T(i)^* \oplus \sum_{k < i} \oplus T(i)^* \oplus \sum_{\beta \leq i} \oplus T(i)^* \]
\[ \bigoplus N_0(0) \oplus \sum_{0 < i \leq k} \bigoplus N_0(i) \oplus \sum_{k < i < \beta} \bigoplus N_0(i) \oplus \sum_{\beta \leq i} \bigoplus N_0(i) \]
\[ \bigoplus N_1(0) \oplus \bigoplus N_1(1) \oplus \sum_{1 < i \leq k} \bigoplus N_1(i) \oplus \sum_{k < i < \beta} \bigoplus N_1(i) \oplus \sum_{\beta \leq i} \bigoplus N_1(i) \]
\[ \bigoplus N_2(0) \oplus \bigoplus N_2(1) \oplus \bigoplus N_2(2) \oplus \sum_{2 < i < k} \bigoplus N_2(i) \oplus \sum_{k < i < \beta} \bigoplus N_2(i) \oplus \sum_{\beta \leq i} \bigoplus N_2(i) \]

\[ \sum_{\beta \leq i} \bigoplus N_i \]

and

\[ b \in \sum_{k < i < \beta} \bigoplus T(i) \oplus \sum_{k < i < \beta} \bigoplus N_0(i) \oplus \sum_{k < i < \beta} \bigoplus N_1(i) \oplus \sum_{k < i < \beta} \bigoplus N_2(i) \oplus \sum_{k < i < \beta} \bigoplus N_3(i) \]

\[ = \sum_{k < i < \beta} \bigoplus Q(i). \]

Then \( a \in Z \) and (*) shows \( b \in Z \), so \( q_k = a + b \in Z \), a contradiction. Thus we get

\[ P = \sum_{i < \beta} \bigoplus X(i) \oplus \sum_{0 < i < \beta} \bigoplus T(i) \oplus \sum_{\beta \leq i} \bigoplus T(i) \]

\[ = \sum_{0 < i < \beta} \bigoplus X(i) \oplus \sum_{0 < i < \beta} \bigoplus T(i) \oplus \sum_{\beta \leq i} \bigoplus T(i) \]

\[ \bigoplus N_0(0) \oplus \sum_{0 < i < \beta} \bigoplus N_0(i) \oplus \sum_{\beta \leq i} \bigoplus N_0(i) \]

\[ \bigoplus N_1(0) \oplus \bigoplus N_1(1) \oplus \sum_{1 < i < \beta} \bigoplus N_1(i) \oplus \sum_{\beta \leq i} \bigoplus N_1(i) \]

\[ \bigoplus N_2(0) \oplus \bigoplus N_2(1) \oplus \bigoplus N_2(2) \oplus \sum_{2 < i < \beta} \bigoplus N_2(i) \oplus \sum_{\beta \leq i} \bigoplus N_2(i) \]

\[ \sum_{\beta \leq i} \bigoplus N_i \]

We put

\[ W = \sum_{i < \beta} \bigoplus X(i) \oplus \sum_{0 < i < \beta} \bigoplus T(i) \oplus \sum_{\beta \leq i} \bigoplus T(i) \]

\[ \bigoplus N_0(0) \oplus \sum_{0 < i < \beta} \bigoplus N_0(i) \oplus \sum_{\beta \leq i} \bigoplus N_0(i) \]

\[ \bigoplus N_1(0) \oplus \bigoplus N_1(1) \oplus \sum_{1 < i < \beta} \bigoplus N_1(i) \oplus \sum_{\beta \leq i} \bigoplus N_1(i) \]

\[ \bigoplus N_2(0) \oplus \bigoplus N_2(1) \oplus \bigoplus N_2(2) \oplus \sum_{2 < i < \beta} \bigoplus N_2(i) \oplus \sum_{\beta \leq i} \bigoplus N_2(i) \]

\[ \sum_{\beta \leq i} \bigoplus N_i \]

And we consider the decomposition : \( P = W \oplus T(\beta) \oplus \sum_{i < \beta} \bigoplus N_i(\beta) \oplus \sum_{\beta \leq i} \bigoplus N_i \) Here using the \(|I(\beta)|\)-exchange property of \( X(\beta) \) in this decomposition, we get decompositions :

\[ W = \overline{W} \oplus \overline{W} \]
\[ T(\beta)^* = \overline{T(\beta)^*} \oplus \overline{T(\beta)^*} \]
\[ N_i(\beta) = \overline{N_i(\beta)} \oplus \overline{N_i(\beta)} \]
for $i < \beta$

$$\sum_{\beta \leq i} \oplus N_i = \sum_{\beta \leq i} \oplus N_i \oplus \sum_{\beta \leq i} \oplus N_i$$

such that

$$P = X(\beta)^* \oplus W \oplus \sum_{0 < i \leq \beta} \oplus T(i)^* \oplus \sum_{\beta < i} \oplus N_i(\beta) \oplus \sum_{\beta \leq i} \oplus N_i$$

But, by the same argument above, we can that

$$\overline{W} = 0, \sum_{\beta \leq i} \oplus N_i = 0$$

so, we have

$$P = \sum_{0 \leq i < \beta} \oplus X(i)^*$$

$$\oplus \sum_{0 < i \leq \beta} \oplus T(i)^*$$

$$\oplus \sum_{\beta < i} \oplus T(i)^*$$

$$\oplus N_0(0) \oplus \sum_{0 < i \leq \beta} \oplus N_0(i) \oplus \sum_{\beta < i} \oplus N_0(i)$$

$$\oplus N_1(0) \oplus N_1(1) \oplus \sum_{0 < i \leq \beta} \oplus N_1(i) \oplus \sum_{\beta < i} \oplus N_1(i)$$

$$\oplus N_2(0) \oplus N_2(1) \oplus N_2(2) \oplus \sum_{2 < i \leq \beta} \oplus N_2(i) \oplus \sum_{\beta < i} \oplus N_2(i)$$

Thus, by transfinite induction, we have decompositions

$$T(i)^* = \overline{T(i)^*} \oplus \overline{T(i)^*} \forall i \in I$$

$$N_0(i) = N_0(i) \oplus N_0(i) \forall i \in I - 0,$$

$$N_1(i) = N_1(i) \oplus N_1(i) \forall i \in I - (I(2),$$

$$N_k(i) = N_k(i) \oplus N_k(i) \forall i \in I - (I(k),$$

such that, for any $k \in I$,

$$P = \sum_{0 \leq i \leq k} \oplus X(i)^*$$

$$\oplus \sum_{0 < i \leq k} \oplus T(i)^* \oplus \sum_{k < i} \oplus T(i)^*$$

$$\oplus N_0(0) \oplus \sum_{0 < i \leq k} \oplus N_0(i) \oplus \sum_{k < i} \oplus N_0(i)$$

$$\oplus N_1(0) \oplus N_1(1) \oplus \sum_{0 < i \leq k} \oplus N_1(i) \oplus \sum_{k < i} \oplus N_1(i)$$

$$\oplus N_2(0) \oplus N_2(1) \oplus N_2(2) \oplus \sum_{2 < i \leq k} \oplus N_2(i) \oplus \sum_{k < i} \oplus N_2(i)$$
So, by the quite similar argument above for $X(1)^*$ or $X(2)^*$, we have

$$P = \sum_{i} X(i)^* \oplus \sum_{i} T(i)^*$$

$$\oplus N_0(0) \oplus \sum_{i} \oplus N_0(i)$$

$$\oplus N_1(0) \oplus N_1(1) \oplus \sum_{i} \oplus N_1(i)$$

$$\oplus N_2(0) \oplus N_2(1) \oplus N_2(2) \oplus \sum_{i} \oplus N_2(i)$$

$$\oplus N_k(0) \oplus N_k(1) \oplus N_k(2) \oplus \sum_{i} \oplus N_k(i)$$

This completes the proof, as $X = \sum_{i} X(i)^*$.

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