0. Introduction

In 1973, Drinfeld introduced the notion of elliptic modules, which is now known as Drinfeld modules. After that the analogies between number fields and function fields have many interesting new aspects. Drinfeld modular function theory is one of these.

Drinfeld modular functions were studied by several mathematicians and known to have many properties analogous to those of classical elliptic functions, such as the generators of modular function fields, Galois groups between them. ([5],[8],[9]).

In the first part of this note, we establish some more properties of Drinfeld modular functions in analogy with those obtained by Shimura. In [12], Shimura proved his exact-sequence and his reciprocity law. Lang proved the exact-sequence another way in [11] using the isogeny theory. Shimura's proof of the reciprocity law is not easy. For example he used the parametrizations of the models of a modular function field over $Q$. In [11], Lang avoided Shimura's method using the decomposition groups which is well-known in algebraic number theory. In this article, we will follow Shimura's method to prove the exact-sequence, and Lang's method to prove the reciprocity law in the function field case.

In the second part, we go on to study two variable Drinfeld modular functions in analogy with two variable elliptic functions studied by Berndt. ([1]). In [2], he also generalized Shimura's exact-sequence and his reciprocity law corresponding to this extended modular function fields. We discuss the analogies of these in the Drinfeld setting.

1. Definitions and basic facts

Let $A=F_q[T]$, $k=F_q(T)$, $k_\infty$ be the completion of $k$ at $\infty=(\frac{1}{T})$, and $C$ the completion of the algebraic closure of $k_\infty$. Then $C$ has an absolute value extending that of $k_\infty$. By an $A$-lattice in $C$, we mean a projective $A$-submodule $\Lambda$ of $C$ which is discrete in the topology of $C$. A meromorphic function $f$ on $C$ is said to be even if $f(\mu z)=f(z)$ for every $\mu \in F_q^*$. A meromorphic function $f$ on $C$ is

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called a lattice function for \( \Lambda \) if \( f(z + \lambda) = f(z) \) for every \( \lambda \in \Lambda \). For an \( A \)-lattice \( \Lambda \), we define the lattice function

\[
e_{\Lambda}(z) = z \prod_{\lambda \in \Lambda} (1 - z/\lambda).
\]

The basic properties of the function \( e_{\Lambda} : \mathbb{C} \to \mathbb{C} \) are ([8], 1.2)

(i) \( e_{\Lambda} \) is entire, i.e. it converges uniformly on bounded sets.
(ii) \( e_{\Lambda} \) has simple zeros at the points of \( \Lambda \) and no further zeros.
(iii) \( e_{\Lambda} \) is unique up to constant multiple with properties (i) and (ii).
(iv) \( e_{\Lambda} \) is \( F_q \)-linear and surjective.
(v) For \( c \in \mathbb{C} \), \( e_{c\Lambda}(cz) = ce_{\Lambda}(z) \).

**Lemma 1.1.**

a) If a meromorphic function \( f \) on \( \mathbb{C} \) is even and has a zero of order \( m \) at \( u \), then \( f \) has a zero of the same order at \( \mu u \) for every \( \mu \in F_q^* \).

b) If, moreover, \( f \) is a lattice function for \( \Lambda \), then \( (q-1) \) divides \( \text{ord}_u(f) \) for \( u \in \Lambda \).

**Proof.** a) Let

\[
f(z) = a(z-u)^m + \text{higher terms}
\]

be the expansion of \( f \) around \( u \). Then

\[
f(\mu^{-1} z) = a(\mu^{-1} z - u)^m + \text{higher terms}
\]

\[= \mu^{-m} a(z - \mu u)^m + \text{higher terms}.
\]

Hence we get the result.

b) We may assume \( u = 0 \). Write

\[
f(z) = az^m + \text{higher terms}.
\]

Then

\[f(\mu z) = \mu^m az^m + \text{higher terms}.
\]

Thus \( \mu^m = 1 \), so that \( (q-1) \mid m \).

**Proposition 1.2.** The field of even lattice functions for \( \Lambda \) is generated by \( e_{\Lambda}^{-1}(z) \) over \( \mathbb{C} \).

**Proof.** Exactly the same proof as in the classical case replacing \( p(z) \) by \( e_{\Lambda}^{-1}(z) \) and using Lemma 1.1 would give the result.
By a morphism of lattices $\Lambda$ and $\Lambda'$ we mean a number $c \in C$ with $c\Lambda \subset \Lambda'$. Two lattices $\Lambda$ and $\Lambda'$ are said to be similar if $\Lambda' = c\Lambda$ for some nonzero $c \in C$. For each lattice $\Lambda$, it is well-known that we associate a Drinfeld module $\phi^\Lambda$ so that $e_\Lambda(az) = \phi^\Lambda(z)$ and the association $\Lambda \mapsto \phi^\Lambda$ defines an equivalence of the categories of rank $r$ $A$-lattices in $C$ to the category of Drinfeld modules of rank $r$ over $C$, which maps a similarity class to an isomorphism class.

From now on we only consider Drinfeld modules of rank 2 over $C$, so that we do not need to distinguish them from rank 2 $A$-lattices. $GL_2(k_\infty)$ acts on the rank 2 lattices in the usual manner. Then $\Lambda = \gamma \Lambda$ for $\gamma \in GL_2(k_\infty)$ if and only if $\gamma \in GL_2(A)$. The similarity classes of rank 2-lattices can be represented by $\Omega = C - k^\times$ if we identify $\omega \in \Omega$ to the lattice $\Lambda_\omega = [\omega, 1]$ generated by $\omega$ and 1. Therefore the set of the isomorphism classes of Drinfeld modules of rank 2 is parametrized by $GL_2(A) \backslash \Omega$.

For a Drinfeld module $\phi$ of rank 2 over $C$ and $N \in A$, let $D(\phi, N)$ be the $A$-submodule $\text{Ker} \phi^N$. We call $D(\phi, N)$ the set of $N$-division points.

By a level $N$-structure we mean an isomorphism $\alpha: (N^{-1} / A)^2 \to D(\phi, N)$.

An isomorphism between two Drinfeld modules with level $N$-structures is defined as in the classical case. Let $\Gamma(N) = \{ \gamma \in \Gamma : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \}$. Then the isomorphism classes of Drinfeld modules of rank 2 with level $N$-structure are parameterized by $\Gamma(N) \backslash \Omega$. Let $\rho$ be the Carlitz module which is defined by $\rho_{TX} = TX + X^q$. Then $\rho$ corresponds to the lattice $\pi A$ for some $\pi \in C$. Let $N \in A$ and $\Lambda_N = \text{Ker} \rho_N$. Let $k_N$ be the field extension of $k$ generated by $\Lambda_N$, which we call the $N$-th cyclotomic function field. Then we have

**Theorem 1.3** ([10]) a). $k_N$ is a Galois extension of $k$ with Galois group isomorphic to $(A / N)^\times$. The action of $a \in (A / N)^\times$ on $\Lambda_N$ is that of $\rho_a$.

b) $k_N$ is ramified only on the divisors of $N$ and $\infty$. The inertia group at $\infty$ is $F_q^* \subset (A / N)^\times$.

c) Let $k_N^+$ be the fixed field of $F_q^*$. Then $k_N^+$ is generated by $\lambda^{q^{-1}}$, $\lambda \in \Lambda_N$, over $k$.

**Theorem 1.4** ([8]). $\Gamma(N) \backslash \Omega$ can be given a structure of an affine curve $Y(N)$ over $C$. If we add some cusps to $Y(N)$, we get a projective curve $X(N)$. $X(N)$ can be defined over $k_N^+$.

**Definition 1.5.** Let $\Gamma$ be a congruence subgroup of $GL_2(A)$. A function $f: \Omega \to C$ is a modular form of weight $k$, if the following conditions are satisfied;
(i) For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $z \in \Omega$, we have

$$f(\gamma z) = (cz + d)^k f(z)$$

(ii) $f$ is meromorphic on $\Omega$ in the rigid analytic sense.

(iii) $f$ is meromorphic at the cusps of $\Gamma$, that is, $f$ has a Laurent series expansion in $t_N = e^{-\frac{1}{2} (\ell_N^2)}$, where $\Gamma(N) \subset \Gamma$.

In (ii) and (iii) if we replace meromorphic by holomorphic, we call such a modular form a holomorphic modular form.

We say that a holomorphic modular form $f$ is a cusp form if $f$ vanishes at all the cusps. A modular form of weight 0 is called a modular function.

Let $\phi$ be the Drinfeld module of rank 2 associated to the lattice $\Lambda_z = [z,1]$. For each $a \in A$ with $\deg a = d$,

$$\phi_a = \sum_{i=0}^{2d} l_i(a,z) X_i.$$

Then $l_i(a,z)$ is a holomorphic modular form of weight $q^i - 1$. In particular, let $a = T$. Then

$$\phi_T = T X + g(z) X^{q^1} + \Delta(z) X^{q^2}.$$

The functions $g(z)$ and $\Delta(z)$ are the most important modular forms, and $\Delta(z)$ is called the discriminant function. In fact, $\Delta(z)$ is a cusp form. $j(z) = g(z)^{q+1} / \Delta(z)$ is a modular function.

Let $M_k$ be the set of all holomorphic modular forms of weight $k$ for $GL_2(A)$, and $M = \bigoplus_k M_k$. Then as in the classical case, we have;

**Theorem 1.6 ([9])**. $M = C[g, \Delta]$.

The function $j$ gives a bijection

$$GL_2(A) \backslash \Omega \overset{j}{\sim} \mathbb{C}.$$

Therefore $X(1)$ is the projective $j$-line and $C(X(1)) = C(j)$. Since $X(N)$ is the projective model of the affine curve $\Gamma(N) \backslash \Omega$, the group of ramified covering for $X(N)$ over $X(1)$ is $\Gamma(1) / \Gamma(N)$, which is isomorphic to $\overline{GL_2(A/N)} / Z(F_q)$ where

$$\overline{GL_2(A/N)} = \{ \gamma \in GL_2(A/N) : \det \gamma \in F_q^* \}.$$

For $a = (a_1, a_2) \in (N^{-1} A / A)^2$, define
where $\Lambda_z = [z, 1]$. Then $e_a$ is a modular form of weight $-1$ for $\Gamma(N)$. Define the Fricke function

$$h_a(z) = g(z)e_a(z)^{q-1}.$$ 

Then $\gamma \in GL_2(A)$ acts on $h_a$ via $h_a^\gamma(z) = h_a(\gamma z)$. It is easy to see that $h_a^\gamma(z) = h_a(\gamma z)$ and that $h_a^\gamma = h_a$ for $\gamma \in \Gamma(N)$.

**Theorem 1.7** ([5]). $C(X(N))$ is a Galois extension of $C(X(1)) = C(j)$ generated by the Fricke functions $h_a$, $a \in (N^{-1} A / A)^2$. The Galois group is $GL_2(A / N) / Z(F_q)$ and its action on $h_a$ is that given above.

We now consider the field 

$$F_N = k(j, h_a : a \in (N^{-1} A / A)^2)$$

Then we have

**Theorem 1.8** ([8]).

(i) The algebraic closure of $k$ in $F_N$ is $k_N^\ast$.

(ii) $\text{Gal}(F_N / k(j)) = GL_2(A / N) / Z(F_q)$ with its action on $h_a$ given by $h_a^\gamma = h_a^\gamma$.

(iii) The elements in $F_N$ have their coefficients of $t_N$-expansions in $k_N$.

(iv) The subgroup $\{(0, 1)^a : a \in (A / N)^*\} \simeq (A / N)^*$ of $GL_2(A / N)$ acts on the coefficients as the action of $(A / N)^*$ on $k_N$.

If one follows the methods in [11], pp66–67, one gets

**Corollary 1.9.** The action of $\gamma \in GL(2, A / N)$ on $k_N^\ast$ is given by

$$(\lambda^{q-1})^\gamma = [\rho_{det}(\lambda)]^{q-1}$$

for $\lambda \in \Lambda(N)$.

2. Shimura exact sequence

Let $F = \cup_N F_N$. We discuss the structure of automorphism group $\text{Aut}_k(F)$ of $F$ over $k$. For each finite place $v$ of $k$, let $G_v = GL_2(k_v)$. Define

$$U_F = \prod_{v \text{ finite}} GL_2(A_v)$$

and
\[ G(A_f) = \prod_{\text{finite}} GL_2(k_v) \]

where \( \Pi' \) means the restricted product with respect to \( U_f \). Let \( \tilde{K}^{ab} \) be the maximal abelian extension of \( k \) where \( \infty \) splits completely. Then we have a natural map

\[ \sigma: G(A_f) \to \text{Gal}(\tilde{K}^{ab}/k) \]

given by

\[ \sigma(x) = \left( (\text{det} x)^{-1}, \tilde{K}^{ab} \right) / k \]

**Remark.** We use \( \tilde{K}^{ab} \), instead of the maximal abelian extension of \( k \), because the algebraic closure of \( k \) in \( F_N \) is \( k_N^+ \).

For \( u \in U_f \), define \( \tau(u) \in \text{Gal}(F/F_1) \) by \( h^{(u)}_a = h_{au} \) for every \( a \in k^2 / A^2 \). Then we get

**Proposition 2.1.**

(i) The sequence

\[ 1 \to F_q^* \to U_f \to \text{Gal}(F/F_1) \to 1 \]

is exact.

(ii) \( \tau(u) = \sigma(u) \) on \( \tilde{K}^{ab} \).

(iii) \( h^{(u)} = h \circ \gamma \) for every \( h \in F \) and \( \gamma \in GL_2(A) \).

Now it is easy to see that \( GL_2(A_f) = GL_2(k) \cdot U_f \). Hence we have to define the action of \( GL_2(k) \) on \( F \). For \( \gamma \in GL_2(k) \) and \( h \in F \), define

\[ h^{(u)} = h \circ \gamma. \]

Then on \( GL_2(k) \cap U_f = GL_2(A) \), the two definitions coincide from the fact that \( h_{au}(z) = h_u(yz) \) and \( f(yz) = f(z) \). To show that \( \tau \) is a well-defined homomorphism, we need the following proposition.

**Proposition 2.2.**

(i) For every \( \gamma \in GL_2(k) \) and for every \( h \in F \), the function \( h \circ \gamma \) belongs to \( F \).

(ii) If \( \gamma_1, \gamma_2 \in GL_2(k) \), \( u_1, u_2 \in U \) and \( \gamma_1 u_1 = u_2 \gamma_2 \), then \( (j \circ \gamma_1)^{(u_1)} = j \circ \gamma_2 \), and \( (h \circ \gamma_1)^{(u_1)} = h_{au_2} \circ \gamma_2 \) for every \( a \in k^2 / A^2 \), \( \neq 0 \).

Proof. Let \( F' = k(h \circ \gamma_1): h \in F, \gamma \in GL_2(k) \). Choose a point \( z_0 \in \Omega \) such that the specialization map \( f \to f(z_0) \) defines an isomorphism of \( F' \) onto \( F'_0 = \{ f(z_0) : f \in F' \} \). Taking suitable scalar multiples of \( \gamma_1 \) and \( \gamma_2 \) instead of \( \gamma_1 \) and \( \gamma_2 \), we
may assume that $\gamma_1^{-1}$ and $\gamma_2^{-1}$ belong to $M_2(A)$. Define a Drinfeld module $\phi$ of rank 2 such that

$$\phi_T = TX + X^q + \frac{1}{j(z_0)}X^q.$$ 

Let $\Lambda_{z_0} = [z_0, 1]$ and $\Lambda$ be the corresponding lattice for $\phi$. Then $\Lambda = c\Lambda_{z_0}$ for some $c \in C$. Define

$$\xi : C \rightarrow C$$

by $\xi(x) = c_1 e_{\Lambda z_0}(x)$. Since $c$ is isomorphism of $\Lambda_{z_0}$ to $\Lambda$, it is easy to see that $g(z_0) = c^{-1}$. Let $z_i = \gamma_i z_0$ for $i = 1, 2$. Replacing $z_0$ by $z_i$, we can define $\phi_i$, $c_i$ and $\eta_i$ corresponding to $\phi$, $c$ and $\xi$. Then we have

$$\eta_i \circ a = \phi_i \circ \eta_i$$

for all $a \in A$.

We also have that $g(z_i) = c_i^{-1}$. If $\alpha = (a \ b) \in GL_2(k)$, define $\mu_2 = cz_0 + d$. Let $\mu_i = \mu_{y_i}$ for $i = 1, 2$. Then we have

$$\mu_i^{-1}(z_0) = \gamma_i^{-1}(z_i).$$

It follows that the multiplication by $c_i^{-1} \mu_i^{-1} c_i$ defines an isogeny

$$c\Lambda_{z_0} \rightarrow c_i \Lambda_{z_i},$$

hence it induces an isogeny

$$\lambda_i : \phi \rightarrow \phi_i.$$ 

Then we have the following commutative diagram.

$$\begin{array}{ccc}
\Lambda_{z_0} & \rightarrow & C \\
\mu_i^{-1} \downarrow & & \eta_i \downarrow \\
\Lambda_{z_i} & \rightarrow & C \\
\end{array}$$

Let $\sigma$ be the automorphism of $k(h(z_0) : h \in F)$ over $k(j(z_0))$ such that $h_\sigma(z_0) = h_{su_1}(z_0)$ for all $a \in k^2 / A^2$, $\neq 0$. Extend it to an automorphism of $C$, and denote it again by $\sigma$. Then we see that

$$\xi(a\left(\begin{array}{c}
z_0 \\
1 \end{array}\right))^\sigma = e\xi(au_1\left(\begin{array}{c}
z_0 \\
1 \end{array}\right)).$$
for some $\varepsilon \in F_q^*$, because

$$
((\bar{\xi}(a(z_0)))^{\sigma})^{q-1} = ((ce_{\Lambda_{\alpha}}(a(z_0)))^{q-1})^\sigma \\
= (g(z_0)e_{\Lambda_{\alpha}}(a(z_0)))^{q-1} \\
= h_d(z_0)^\sigma \\
= h_{au_1}(z_0) \\
= (ce_{\Lambda_{\alpha}}(au_1(z_0)))^{q-1} \\
= \bar{\xi}(au_1(z_0))^{q-1}.
$$

Then following the method of Proposition 6.22 of [12], our proposition follows. The only thing to note in (i) is that any field extension is separable in the classical case, but we need to check it in our case. So we need to show that $F'$ is separable over $F$. Then it reduces to showing that $j(\gamma z)$ and $h_d(\gamma z)$ are separable over $F$. Let $z' = \gamma z$. Since the coefficients of the polynomial

$$\prod_{a \in (N^{-1}A/A)^2} (X - h_a(z'))
$$

are invariant under $GL_2(A)$ and holomorphic on $\Omega$, they lie in $k(\Lambda_\alpha)[j(z')]$. In fact, they lie in $k[j(z')]$, because they are fixed by

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} : d \in (A/N A)^* \right\}.
$$

(e.g. Theorem 1.8 (ii), (iv)). Then it reduces to showing that $j(\gamma z)$ is separable over $F$ since $h_a(z')$'s are distinct. Following the method in [11, pp54–55], we have $j(\gamma z)$ is integral over $A[j]$ and separable over $k(j)$. This completes the proof.

**Theorem 2.3.** The sequence

$$1 \to k^* \to G(A_j) \overset{\tau}{\to} \text{Aut}_k(F) \to 1
$$

is exact.

Proof. We claim that
$k^* : U_N = \{ x \in G(A_f) : \tau(x) = id \text{ on } F_N \}$,

where

$$U_N = \{ u = (u_v) \in U_f : u_v \equiv 1 \mod N \cdot M_2(A_v) \}.$$ 

Let $x \in G(A_f)$ be in the kernel of $\tau$. Write $x = u\gamma$ with $u \in U_f$, $\gamma \in GL_2(k)$. Since $j(x) = j \circ \gamma = j$, it is easy to see that $\gamma = \gamma' \gamma'$ for some $y \in k^* \gamma' \in GL_2(A)$. Then we have

$$h_{u}^{\gamma} = h_{u} \circ \gamma = h_{u} \circ \gamma' = h_{u} \gamma'.$$

It follows that

$$h_a = h_{u} \gamma' \text{ for all } a \in (N^{-1}/A)^2.$$ 

We regard $u\gamma'$ as an element in $GL_2(A/N)$. Putting $a$ equal to $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2})$ respectively, it is easy to see that

$$u\gamma' \equiv \left( \begin{array}{c} \varepsilon \\ 0 \end{array} \right) \mod NA,$$

for some $\varepsilon \in F_4^*$. Thus our claim follows. Then it is easy to see that

$$\ker \tau = \bigcap_{N} k^* U_N = k^*.$$ 

Exactly the same proof as in the classical case in [12] gives that $\tau$ is surjective.

3. Shimura's reciprocity law for $F$

Let $L$ be an imaginary quadratic field, that is, $L$ is a quadratic extension of $k$ where $\infty$ does not split completely. Let $I = I_L$ be the group of the ideles in $L$ without $\infty$-component, and let $L^{ab}$ the maximal abelian extension of $L$ where the infinite place splits completely. Let

$$( , L) : I \rightarrow \text{Gal}(L^{ab}/L)$$

be the Artin-homomorphism in class field theory. Then we have

\textbf{Theorem 3.1} [6,(4.5)]. \textit{For any } $\omega_0 \in L \cap \Omega$, \textit{we have}

(i) $j(\omega_0)$ \textit{lies in } $L^{ab}$ \textit{and } $j(\omega_0)(s^{-1}, L) = j(s[\omega_0, 1])$ \textit{for } $s \in L$.

(ii) $L^{ab}$ \textit{is generated by } $\{ j(\omega_0), g(\omega_0)e_{\alpha}(\omega_0)^{s^{-1}} : a \in (k/A)^2 \}$ \textit{over } $L$ \textit{if } $j(\omega_0) \neq 0$ \textit{and generated by } $\{ \Delta(\omega_0)e_{\alpha}(\omega_0)^{s^{-1}} : a \in (k/A)^2 \}$ \textit{over } $L$ \textit{if } $j(\omega_0) = 0$.

Fix a point $\omega_0 \in L \cap \Omega$. We set the following notations;
\[ R = k[j] \]
\[ S = \text{the integral closure of the ring } R \text{ in } F \]
\[ m = \{ f \in R : f(\omega_0) = 0 \} \]
\[ M = \{ f \in S : f(\omega_0) = 0 \}. \]

Then \( M \) is a maximal ideal of \( S \) lying above \( m \). If \( \eta \) is an automorphism of \( S \) which maps \( M \) onto \( M \) then \( \eta \) induces an automorphism on the residue class field, denoted by

\[ \tilde{\eta} : S \to S. \]

We identify \( S \) with the set of all elements \( \tilde{f} = f(\omega_0) \) \( f \in S \). Let \( G_m \) be the decomposition group of \( \text{Gal}(F/F_1) \). If \( \sigma \in G_m \), then we will denote by \( \tilde{\sigma} \) its image in the Galois group of \( S/M \) over \( R/m \).

**Lemma 3.2.** Suppose \( j(\omega_0) \neq 0 \). If \( \tau(u) \in \text{Gal}(F/F_1) \) satisfies

\[ h_a^{\tau(\omega)}(\omega_0) = h_a(\omega_0) \]

for all \( a \in (k/A)^2 \), then \( \tilde{f}^{\tau(\omega)} = \tilde{f} \) for all \( f \in S \).

Proof. It suffices to show that \( \tilde{f}^{\tau(\omega)} = \tilde{f} \) for \( f \in S \cap \mathcal{F}_m \) for any given \( N \). Then we can view \( u \) as an element in \( GL_2(A/NA) \). Since \( h_a^{\tau(\omega)}(\omega_0) = h_a(\omega_0) \) for \( a = (0,0), (0,\hat{k}) \) and \( (\frac{1}{k},\frac{1}{k}) \), it is easy to see that

\[ u = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \in GL_2(A/NA) \]

for some \( \epsilon \in F^*_q \). Thus \( \tilde{\tau}(u) = 1 \), hence our Lemma follows.

We will show that Lemma 3.2 still holds in the case that \( j(\omega_0) = 0 \) if we change \( h_a \) slightly. Define

\[ h'_a(z) = \Delta(z)e_{\Delta^2} \left( a \begin{pmatrix} z \\ 1 \end{pmatrix} \right)^{q^2 - 1} \]

Then \( \tau(u) \) acts on \( h'_a \) via \( h'^{\tau(\omega)}_a = h'^{\omega}_a \).

**Lemma 3.3.** Suppose \( j(\omega_0) = 0 \). If \( \tau(u) \in \text{Gal}(F/F_1) \) satisfies

\[ h_a^{\tau(\omega)}(\omega_0) = h'_a(\omega_0) \]

for all \( a \in (k/A)^2 \), then \( \tilde{f}^{\tau(\omega)} = \tilde{f} \) for all \( f \in S \).
Proof. Since $\omega_0$ is $GL_2(A)$-equivalent to an element in $F_{q^2} - F_q$, we assume that $\omega_0 \in F_{q^2} - F_q$. Then we have $\Lambda_{\omega_0} = F_{q^2}[T]$. We may assume that $f \in F_N$ for some $N$. Write

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A/NA).$$

Putting $a$ equal to $(\frac{1}{q}, 0), (0, \frac{1}{q})$ and $(\frac{1}{q}, \frac{1}{q})$ respectively to the equation

$$h^u_\omega(\omega_0) = h_\omega(\omega_0),$$

it is easy to see that $u \in GL_2(F_q)$ and

$$a\omega_0 + b = \epsilon\omega_0,$$

$$c\omega_0 + d = \epsilon$$

for some $\epsilon \in F_{q^2}^\times$, so that $\tau(u)$ lies in the inertia group at $\omega_0$.

Define an embedding

$$q : L^* \rightarrow GL_2(k)$$

$$s \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

satisfying $s\omega_0 = a\omega_0 + b$ and $s = c\omega_0 + d$. By the continuity, we can extend $q$ to an embedding of the idele group $A_L^*$ of $L$, again denoted by

$$q : A_L^* \rightarrow G(A_f).$$

Then we have the following Shimura's reciprocity law.

**Theorem 3.4.** Suppose $f \in F$ is defined at $\omega_0$. If $L^{ab}(f(\omega_0))$ is a separable extension of $L^{ab}$ then $f(\omega_0)$ lies in $L^{ab}$ and $f^{r(s)}(\omega_0) = f(\omega_0)(s^{-1}L)$. If $L^{ab}(f(\omega_0))$ is not separable over $L^{ab}$, then it is a purely inseparable extension of $L^{ab}$. In this case, we can extend $(s^{-1}L)$ uniquely to an embedding of $L^{ab}(f(\omega_0))$ over $L$. We denote it again by $(s^{-1}L)$. Then $f^{r(s)}(\omega_0) = f(\omega_0)(s^{-1}L)$.

Proof. Write $q(s) = u \cdot \alpha$, $u \in U_f$, $\alpha \in GL_2(k)$. First we will show that our Theorem holds in the case that $f(\omega_0) \neq 0$. Let $\phi^{\omega_0}$ be the Drinfeld module satisfying

$$\phi^{\omega_0} = TX + X^q + \frac{1}{f(\omega_0)}X^{q^2}.$$
Replacing \( \omega_0 \) by \( \sigma(\omega_0) \), we can define \( \phi^{\sigma(\omega_0)} \) and find \( d \) corresponding to \( c \) satisfying \( d^{s-1} = g(\omega_0) \). Let \( \mu = \mu_\sigma \) be as in the proof of Proposition 2.2. Following the classical method in [11, pp150–151], we have the following commutative diagram

\[
\begin{array}{ccc}
(k / A)^2 & \rightarrow & k\Lambda_{\omega_0} / \Lambda_{\omega_0} \\
\downarrow & & \downarrow_{s\mu^{-1}} \\
(k / A)^2 & \rightarrow & k\Lambda_{\sigma(\omega_0)} / \Lambda_{\sigma(\omega_0)}
\end{array}
\]

with \( s\Lambda_{\omega_0} = \mu\Lambda_{\sigma(\omega_0)} \). Then we have

\[
j(\omega_0)^{(s^{-1}, L)} = j(\delta(\omega_0, 1)) \text{ by (3.1)}
\]

\[
= j(\sigma^{(s(\omega_0))}(\omega_0)).
\]

Let \( \sigma \) be any automorphism of \( C \) such that \( \sigma = (s^{-1}, L) \) on \( L^{ab} \). Since \( d \) is well-defined up to the multiplication by an element in \( F^- \), the following commutative diagram follows from [6,(1.12)].

\[
\begin{array}{ccc}
k\Lambda_{\omega_0} / \Lambda_{\omega_0} & \rightarrow & \text{Tor}(\phi_{\omega_0}) \\
\downarrow{s\mu^{-1}} & & \downarrow{\sigma} \\
k\Lambda_{\sigma(\omega_0)} / \Lambda_{\sigma(\omega_0)} & \rightarrow & \text{Tor}(\phi^{\sigma(\omega_0)}).
\end{array}
\]

By (*) and (**), we have

\[
de_{\Lambda_{\sigma(\omega_0)}} \left( au \left( \begin{array}{c} \omega_0 \\ 1 \end{array} \right) \right) = \left( ce_{\Lambda_{\omega_0}} \left( a \left( \begin{array}{c} \omega_0 \\ 1 \end{array} \right) \right) \right)^{\sigma}
\]

for \( a \in (k / A)^2 \). Taking the \((q-1)\)-st power, we have

\[
h_a^{(q(\omega_0))}(\omega_0) = h_a(\omega_0)^{(s^{-1}, L)}.
\]

Thus our theorem holds for \( f = j, h_a \).

Next we prove that the relation of the theorem is true for all elements in \( F \). Define

\[
R' = L[j]
\]

\( S' \) the integral closure of the ring \( R' \) in \( L \cdot F \)

\[
m' = \{ f \in R' : f(\omega_0) = 0 \}
\]

\[
\mathcal{M}' = \{ f \in S' : f(\omega_0) = 0 \}
\]
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\[ G_m = \text{the decomposition group of } \text{Gal}(L \cdot F / L(j)). \]

where \( L \cdot F \) is the compositum of \( L \) and \( F \). Since the infinite place does not split in \( L \), it is easy to see that

\[ \text{Gal}(L \cdot F / L(j)) = \text{Gal}(F / k(j)). \]

Thus we will view \( \tau(u) \) as an element in \( \text{Gal}(L \cdot F / L(j)) \).

We claim that there exists an element \( \rho \in \text{Aut}_k(L \cdot F) \) which maps \( S' \) and \( \mathfrak{M}' \) onto \( S' \) and \( \mathfrak{M}' \) respectively satisfying \( \rho = \tau(q(s)) \) on \( k(j) \) and \( \rho = \sigma \) on \( S' \). Extend \( \tau(\alpha) \) to the automorphism of \( L \cdot F \) such that \( f^{\tau(\alpha)} = f \circ \alpha \) for \( f \in L \cdot F \), and denote it again by \( \tau(\alpha) \). Since \( j \circ \alpha \) is integral over \( L[j] \), \( \tau(\alpha) \) induces an automorphism of \( S' \). The formula \( j(\omega_0) = j(\omega_0)^{(s^{-1}, L)} \) shows that \( m'^{\tau(\alpha)} \subset \mathfrak{M}' \) because \( \tau(\alpha) \) leaves the constants fixed. Consequently \( m' \subset \mathfrak{M}'^{\tau(\alpha^{-1})} \). Since \( \mathfrak{M}'^{\tau(\alpha^{-1})} \) and \( \mathfrak{M}' \) are prime ideals lying above \( m' \), there exists an element \( \pi \in \text{Gal}(L \cdot F / L(j)) \) such that \( \mathfrak{M}'^\pi = \mathfrak{M}'^{\tau(\alpha^{-1})} \), whence we obtain \( \mathfrak{M}'^{\pi \circ \tau(\alpha)} = \mathfrak{M}' \). Clearly \( \pi \circ \tau(\alpha) = \tau(q(s)) \) on \( k(j) \). Then \( \sigma \circ (\pi \circ \tau(\alpha))^{-1} \) lies in Galois group of \( S' \) over \( \overline{R} \). By the surjectivity of \([11, \text{pp364}]\), there exists an automorphism \( \lambda \in G_m \) satisfying \( \lambda \circ \pi \circ \tau(\alpha) = \sigma \) on \( S' \). Put \( \rho = \lambda \circ \pi \circ \tau(\alpha) \). Then \( \rho \) satisfies all the requirements of our claim.

Then \( \tau(q(s)) \rho^{-1} \in \text{Gal}(L \cdot F / L(j)) = \text{Gal}(F / k(j)) \) satisfies the requirements of Lemma 3.2, whence

\[ f^{\tau(q(s))} = f^\rho = f^\sigma \]

for all \( f \in S \). Since \( S \) is a normal extension of \( \overline{R} \) containing \( L_{ab} \) and the equation \( f^{\tau(q(s))} = f^\sigma \) holds for any extension \( \sigma \) of \((s^{-1}, L)\), our theorem follows.

The case that \( j(\omega_0) = 0 \) follows similarly by taking Lemma 3.3 into account.

4. Two variable modular functions

Define

\[ X = C \times \Omega. \]

Let

\[ G = \left\{ g = (r_1, r_2; \alpha) = \begin{pmatrix} 1 & r_1 & r_2 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} : r_1, r_2 \in k_\infty, \ ad - bc \neq 0 \right\} = k_\infty^2 \cdot \text{GL}(2, k_\infty). \]

We fix the following notations;

\[ G(1) = A^2 \cdot \text{GL}(2, A) \]
\[ G^0(N) = A^2 \cdot \Gamma(N) \]
\[ G(N) = (NA)^2 \cdot \Gamma(N) \]

We let \( g = (r_1, r_2, \omega) \) in \( G \) act on \( X \) via
\[ g \cdot (v, \omega) = \left( \frac{v + r_1 \omega + r_2}{c \omega + d}, \frac{a \omega + b}{c \omega + d} \right). \]

Let
\[ t = t(\omega) = e^{-1}(\pi \omega) \quad \text{and} \quad \epsilon(v) = e^{-1}(\pi v). \]

Put
\[ e(v, \omega) = e_{(\omega, 1)}(v) = e_{\omega}(v). \]

Then from [7]
\[ e(v, \omega) = \pi^{-1} \epsilon^{-1} \prod_{\omega \in A \setminus \{0\}} \frac{f_0(t) - \epsilon^{-1} t^{|a|}}{f_0(t)}, \]
where \( f_0(t) = \rho_0(t^{-1}) t^{|a|}, |a| = q^{\text{deg} a} \). Thus
\[ \pi e(v, \omega) \in A(\epsilon)((t)), \]
because \( f_0(t) \in A[[t]] \) with the constant term in \( F_q^* \). We know from [9] that
\[ \pi^{-1 - q} g(\omega) \in A[[[t]]], \quad \pi^{-1 - q^2} \Delta(\omega) \in A[[[t]]], \]
and
\[ j(\omega) = \frac{1}{t^q - 1} + h(t), \quad \text{for some} \quad h(t) \in A[[[t]]]. \]

Define the first Weber function \( z(v, \omega) \) by
\[ z(v, \omega) = g(\omega) e(v, \omega)^{q^{-1}}. \]

Then \( z(v, \omega) \) has an expansion in \( k(\omega)((t)) \).

Let \( K \) be a subfield of \( C \) containing \( k \). Let \( R_N(K) \) (resp. \( R_N^0(K) \)) be the field of functions \( f \) on \( X \) such that
(1) \( f \) is meromorphic on \( X \) in the rigid analytic sense.
(2) \( f \) is \( G(N) \) (resp. \( G^0(N) \))-invariant, that is,
\[ a) \quad f(v + r_1 \omega + r_2, \omega) = f(v, \omega) \quad \text{if} \quad r_1, r_2 \in NA \quad \text{(resp.} \quad r_1, r_2 \in A) \]
\[ b) \quad f\left( \frac{v}{c \omega + d}, \omega(\omega) \right) = f(v, \omega). \]
(3) \( f \) has a meromorphic \( t_N = t((\omega)) \)-expansion with coefficients in \( K \), that is,
there exist \( R > 0, n > 0 \) and \( L \in \mathbb{N} \) such that for \( 0 < |\varepsilon| < R, 0 < |t| < n|\varepsilon| L \), and all \( g \in G(1) \), we have

\[
f(g(v, \omega)) = \sum_{r > -\infty} b^*_r(f) t^*_N
\]

with \( b^*_r \in K(e_N), e_N = e(\frac{r}{N}) \).

Put

\[
\mathcal{R}_N(C) = \mathcal{R}_N \text{ and } \mathcal{R}_N(k_N) = \mathcal{R}_N.
\]

Then imitating the proof of Theorem 1 of [1] and using Proposition 1.2 we get;

**Theorem 4.1.** \( \mathcal{R}_t(K) = K(j(\omega), z(v, \omega)) \).

For \((r,s) \in J^2 - \{(0,0)\}\), put

\[
e_{r,s}(\omega) = e\left(\frac{rw+s}{N}, \omega\right)
\]

and

\[
z_{r,s}(\omega) = z\left(\frac{rw+s}{N}, \omega\right)
\]

Then it is not hard to see that \( z_{r,s}(\omega), w_{r,s}(v, \omega) \in \mathcal{R}^0_N \).

**Theorem 4.2.** Assume that \( K \) contains \( k_N \). Then we have

a) \( \mathcal{R}^0_N(K) = K(j_z(z_{r,s}), z_{r,s}(w_{r,s})) \)

b) \( \text{Gal}(\mathcal{R}^0_N(K) / \mathcal{R}_t(K)) = \widetilde{GL}(2, A / N) \)

c) \( \text{Gal}(\mathcal{R}^0_N / \mathcal{R}_t) = GL(2, A / N) \).

Proof. Note that \( G^0 = G(1) / G^0(N) = GL(2, A) / \Gamma(N) = \widetilde{GL}(2, A / N) \), and that \( \{\sigma_d = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} : d \in (A / N)^*\} \) acts on the \( N \)-th root \( \lambda_N \) of \( \rho \) by \( \sigma_d(\lambda_N) = \rho_d(\lambda_N) \). Then exactly the same proof as in the classical case ([1], Theorem 2) works replacing \( \{\pm 1\} \) by \( F^*_q \) and \( SL(2, Z / N) \) by \( GL(2, A / N) \).

Now let, for \((r,s) \in J^2\),

\[
Z_{r,s}(v, \omega) = z\left(\frac{v+rw+s}{N}, \omega\right).
\]
Then we easily get

1. \( Z_{r,s}(v + r_1 \omega + s, \omega) = Z_{r+r_1,s+r_2}(v, \omega) \)
2. \( Z_{r,s}(v + r_1 \omega + r_2, \omega) = Z_{r,s}(\omega) \) if \((r_1, r_2) \equiv (0, 0) \text{ mod } N.\)
3. \( Z_{r,s}(v, \omega) = \alpha(\omega) \) if \(\alpha \equiv 1 \text{ mod } N.\)

It is straightforward to see that \( Z_{r,s}(g(v, \omega)) \) lies in \( k_N(e_N)((t_N)) \), for any \( g \in G(1) \). Following the methods in \cite{1} we get;

**Theorem 4.3.** Assume that \( K \) contains \( k_N \).

a) \( \mathcal{R}_N(K) = K(j, (z_{r,s}), z_{r,s}, (w_{r,s})) \)

b) \( \text{Gal}(\mathcal{R}_N(K) / \mathcal{R}_1(K)) = (A / N)^2 \cdot GL(2, A / N) \)

c) \( \text{Gal}(\mathcal{R}_N / \mathcal{R}_1) = (A / N)^2 \cdot \overline{GL}(2, A / N). \)

Let \( \phi^0 = T + g(\omega)\tau + \Delta(\omega)\tau^2 \)

be the Drinfeld module associated to the lattice \([\omega, 1]\). Then \( \Delta(\omega) = \frac{\epsilon^{1 + \lambda(\omega)}}{\sigma(\omega)}. \)

For \( M \in A \) we have

\[
z(Mv, \omega) = g(\omega)e_\omega(Mv)^{q-1} = g(\omega)(\phi^0_M(e_\omega(v)))^{q-1} = g(\omega)(\sum h_i g(\omega)^{q_i-1} e_\omega(v)^{q_i})^{q-1} h_i \in k(j) = h_M(g(\omega)e_\omega(v)^{q-1}),
\]

for some homogeneous polynomial \( h_M \in k(j)[X] \). For \( r, s \in A \), we have

\[
e_\omega^r N^{(\omega)} = \sum_{i=0}^{2 \text{deg} r} h_i g(\omega)^{q_i-1} e_\omega^{(\omega)} N^{q_i} = e_\omega^{(\omega)} N^{(\omega)} \cdot \text{polynomial in } z_{1,0} \text{ with coefficients in } k(j),
\]

and

\[
e_\omega^s N^{(\omega)} = \sum_{i=0}^{2 \text{deg} s} h_i g(\omega)^{q_i-1} e_\omega^{(\omega)} N^{q_i} = e_\omega^{(\omega)} N^{(\omega)} \cdot \text{polynomial in } z_{0,1} \text{ with coefficients in } k(j).
\]
Then

\[ z_{r,s}(\omega) = g(\omega)(e_{\omega}(\frac{r\omega}{N}) + e_{\omega}(\frac{s}{N}))^{q-1} \]

\[ = g(\omega)(\sum_{r=0}^{q-1} a_r e_{\omega}(\frac{r\omega}{N}) e_{\omega}(\frac{s}{N})^{q-1-r}) \]

\[ = \sum_{r=0}^{q-1} \left( \frac{e_{1,0}(\omega)}{e_{0,1}} \right)^r \cdot z_{0,1} \cdot \text{polynomial in } z_{1,0} \text{ and } z_{0,1}. \]

Therefore we have

\[ z_{r,s} \cdot \frac{e_{r,s}}{e_{0,1}} \in k(j, z_{1,0}, z_{0,1}, \frac{e_{1,0}}{e_{0,1}}) \]

Note that

\[ w_{r,s} = \frac{e_{0,1}}{e_{r,s}} \cdot w_{0,1}. \]

Therefore we have

\[ \mathcal{H}_N(K) = K(j, z_{1,0}, z_{0,1}, \frac{e_{1,0}}{e_{0,1}}, z, w_{0,1}). \]

Also

\[ Z_{r,s}(v, \omega) \]

\[ = z(\frac{v + r\omega + s}{N}, \omega) \]

\[ = g(\omega)(e_{\omega}(\frac{v}{N}) + e_{r,s}(\omega))^{q-1} \]

\[ = g(\omega) \sum_{r=0}^{q-1} a_r e_{\omega}(\frac{v}{N})^{q-1-r} e_{r,s}(\omega)^r \]

\[ = \sum a_r Z_{0,0} \cdot \left( \frac{e_{r,s}(\omega)}{e_{\omega}(\frac{k}{N})} \right)^r \]

\[ = \sum a_r Z_{0,0} \left( \frac{e_{1,0}(\omega)}{e_{\omega}(\frac{k}{N})} \right) \cdot \text{polynomials in } z_{1,0} + \frac{e_{0,1}(\omega)}{e_{\omega}(\frac{k}{N})} \cdot \text{polynomial in } z_{0,1}. \]

Therefore

\[ \mathcal{H}_N(K) = K(j, z_{1,0}, z_{0,1}, \frac{e_{1,0}}{e_{0,1}}, z, w_{0,1}, \frac{e_{1,0}}{e_{0,1}}, Z_{0,0}). \]
5. The Automorphism of $\mathcal{R}$ over $k$

Let $\mathcal{R} = \bigcup \mathcal{R}_N$. For any $L$, define $G_L = L^2 \cdot GL_2(L)$, as $G$ in section 4. By Theorem 4.3 (c), we have

$$\text{Gal}(\mathcal{R} / \mathcal{R}_1) = \lim(A / NA)^2 \cdot GL(A / NA)$$

$$= \prod_{v: \text{finite}} (A_v)^2 \cdot GL_2(A_v)$$

$$=: \overline{U}.$$

Since the level is not fixed in $\mathcal{R}$, write $z_{r,s} = z_a$, $w_{r,s} = w_a$, $Z_{r,s} = Z_a$ if $a = (\ell, k)$. For any element $\bar{u} = (m, u) \in \overline{U}$, let $\tau(\bar{u})$ be the corresponding element in $\text{Gal}(\mathcal{R} / \mathcal{R}_1)$. Then $\tau(\bar{u})$ acts on $\mathcal{R}_N$ via

$$j, z \mapsto j, z$$
$$z_a w_a \mapsto z_{au} w_{au}$$
$$Z_a \mapsto Z_{au} + \frac{m}{2}.$$

We need some explanation about the notation $au + \frac{m}{2}$. There exists a canonical isomorphism

$$\varphi : (k / A)^2 \rightarrow \prod_{v: \text{finite}} (k_v / A_v)^2.$$ 

Then $\varphi^{-1}(au + \frac{m}{2}) \in (N^{-1} A / A)^2$. Denote it by $au + \frac{m}{2}$. Define $f^* : \text{Aut}_k(\mathcal{R})$ for $g \in G_k$ and $f \in \mathcal{R}$. Following the method in [2], $\tau(g) \in \text{Aut}_k(\mathcal{R})$ for all $g \in G_k$, and two definitions of $\tau$ are the same on the intersection of their defined domains, $G_k \cap \overline{U} = G(1)$.

Define

$$G(A_f) = \prod' k_v^2 \cdot GL_2(k_v),$$

where $\prod'$ means the restricted product with respect to $\overline{U}$. Then we have

$$G(A_f) = G_k \cdot \overline{U} = \overline{U} \cdot G_k.$$

Under this decomposition, we can define the action of $G(A_f)$ on $\mathcal{R}$ via $\tau$. To show that

$$\tau : G(A_f) \rightarrow \text{Aut}_k(\mathcal{R})$$
is a well-defined homomorphism, we need the following proposition.

**Proposition 5.1.** If \( g\tilde{u}=\tilde{u}g' \) for \( g,g'\in G_k \) and \( \tilde{u},\tilde{u}'\in \tilde{U} \), then \( \tau(g)\tau(\tilde{u})=\tau(\tilde{u}')\tau(g') \).

**Proof.** Write \( g=(l,\alpha), g'=(l',\alpha') \), \( \tilde{u}=(m,u) \) and \( \tilde{u}'=(m',u') \). Since \( g\tilde{u}=\tilde{u}g' \), we have \( au=u'a' \) and \( m+lu=l'+m'a' \). It suffices to show that two actions are the same on the generators of \( R_N \). By Proposition 2.2, two actions are the same on \( j, z_b \). If the level \( N=1 \), \( Z_b \) is equal to \( z \). Thus it suffices to show that these actions are the same on \( Z_b \) and \( w_b \). Choose a point \( z_0=(v_0,\omega_0)\in C\times\Omega \) such that \( f(z_0) \) defines an isomorphism of \( R \) to \( R_0=\{f(z_0)|f\in R\} \).

Under this specialization map, it suffices to show our assertions for \( Z_b(z_0), w_b(z_0) \). We will view \( \tau(\tilde{u}) \) as an automorphism of \( R_0 \). Extend it to an automorphism of \( C \), and denote it by \( \sigma \). Throughout this proof, we will denote \( v_0,\omega_0 \) simply by \( v,\omega \). Let \( a=bx+\frac{1}{b} \). By definition, we have

\[
Z_b^{\tau(g)\tau(\tilde{u})}=z\left(\frac{1}{\mu}+a\left(\frac{\omega}{1}\right),x(\omega)\right)^{\tau(\tilde{u})}.
\]

On the other hand,

\[
Z_b^{\tau(\tilde{u})\tau(g')}=z\left(\frac{v+m'(\omega)}{\mu N}+\left(\frac{bu'+m'}{\mu N}\right)\left(\frac{x'(\omega)}{1}\right),x'(\omega)\right)
\]

Thus we are going to show that two last terms of above equations are equal. First we assume that \( x^{-1}, x'^{-1} \in M_2(A) \). Let \( \phi_\omega \) be the Drinfeld module satisfying

\[
\phi_\omega^\omega=TX+X^q+1/f(\omega).
\]

In the proof of Proposition 2.2, we can find \( c(\omega) \) such that \( c(\omega)\Lambda_\omega \) is the corresponding lattice to \( \phi_\omega \) satisfying \( g(\omega)=c(\omega)^\omega=1 \). Let \( c=c(\omega), d=c(x(\omega)) \) and \( d'=c(x'(\omega)) \). Then the multiplication by \( c^{-1}\mu^{-1}d \) defines an isogeny

\[
C/c\Lambda_\omega \to C/d\Lambda_{x(\omega)},
\]

and it induces the isogeny

\[
\lambda: \phi_\omega \to \phi_{x(\omega)}.
\]

Similarly we can have the isogeny

\[
\lambda': \phi_\omega \to \phi_{x'(\omega)}.
\]
Following the proof of Proposition 2.2, we can find $\varepsilon \in F_q^*$ such that

(1) \[ \lambda^\varepsilon = \varepsilon \lambda'. \]

(2) \[ de_{\Lambda_{\omega}(\omega)} \frac{1}{\mu} = \lambda \circ ce_{\Lambda_{\omega}}. \]

(3) \[ d' e_{\Lambda_{\omega}^*(\omega)} \frac{1}{\mu'} = \lambda' \circ ce_{\Lambda_{\omega}}. \]

We claim that there exists $\varepsilon' \in F_q^*$ such that

(4) \[ \left( ce_{\Lambda_{\omega}} \left( \frac{v}{N} + a \left( \frac{\omega}{1} \right) \right) \right)^\varepsilon = \varepsilon' ce_{\Lambda_{\omega}} \left( \frac{v}{N} + \left( au + \frac{m}{N} \right) \left( \frac{\omega}{1} \right) \right). \]

Taking $(q-1)$-th power of left hand side of (4), we have

\[ \left[ g(\omega) e_{\Lambda_{\omega}} \left( \frac{v}{N} + a \left( \frac{\omega}{1} \right) \right) ^{q-1} \right]^{\tau(\omega)}. \]

By definition of $\tau(\omega)$, this is equal to

\[ g(\omega) e_{\Lambda_{\omega}} \left( \frac{v}{N} + \left( au + \frac{m}{N} \right) \left( \frac{\omega}{1} \right) \right)^{q-1} = \left[ ce_{\Lambda_{\omega}} \left( \frac{v}{N} + \left( au + \frac{m}{N} \right) \left( \frac{\omega}{1} \right) \right) \right]^{q-1}. \]

This proves our claim. Now we have

\[ z \left( \left( \frac{v}{N} + a \left( \frac{\omega}{1} \right) \right)^\frac{1}{\mu}, \omega(\omega) \right)^\sigma \]

\[ = \left\{ g(\omega) e_{\Lambda_{\omega}} \left( \left( \frac{v}{N} + a \left( \frac{\omega}{1} \right) \right)^\frac{1}{\mu} \right)^{q-1} \right\}^\sigma \]

\[ = \left\{ \left( de_{\Lambda_{\omega}} \left( \left( \frac{v}{N} + a \left( \frac{\omega}{1} \right) \right)^\frac{1}{\mu} \right) \right)^{q-1} \right\}^\sigma \]

\[ = \left\{ \lambda \circ ce_{\Lambda_{\omega}} \left( \frac{v}{N} + a \left( \frac{\omega}{1} \right) \right)^{q-1} \right\}^\varepsilon \quad \text{by (2)} \]

\[ = \left\{ \lambda^\varepsilon \circ \left( ce_{\Lambda_{\omega}} \left( \frac{v}{N} + a \left( \frac{\omega}{1} \right) \right) \right)^{q-1} \right\} \]

\[ = \left\{ \varepsilon \varepsilon' \lambda' \circ \left( ce_{\Lambda_{\omega}} \left( \frac{v}{N} + \left( au + \frac{m}{N} \right) \left( \frac{\omega}{1} \right) \right) \right)^{q-1} \right\} \quad \text{by (1), (4)} \]
This proves our assertion for $Z_b$ in the case of $\alpha^{-1}, \alpha'^{-1} \in M_2(A)$. In general case, write $\alpha = r\beta$ and $\alpha' = r'\beta'$, with $r \in A$ and $\beta^{-1}, \beta'^{-1} \in M_2(A)$. Then the multiplication by $c^{-1}r^{-1}d$ defines an isogeny

$$C/c\Lambda_\omega \rightarrow C/d\Lambda_{s(o)},$$

and it induces the isogeny

$$\lambda: \phi \rightarrow \phi_{s(o)}.$$

Then equation (1) is unchanged whereas equations (2) and (4) are changed as follows.

(2')

$$de_{\Lambda_{s(o)}(\omega,\mu)} = \lambda \circ ce_{\Lambda_\omega} \circ r^{-1}.$$

(4')

$$\left(ce_{\Lambda_\omega} \left( \frac{1}{r} \frac{v}{N} + a \left( \frac{\omega}{1} \right) \right) \right)^\sigma = \varphi \circ ce_{\Lambda_\omega} \left( \frac{1}{r} \frac{v}{N} + a \left( \frac{m}{N} \right) \left( \frac{\omega}{1} \right) \right).$$

Similarly we can show that two actions are same on $Z_b$. Now it remains to show that two actions are same on $w_b$. We will only prove it under the assumption that $\alpha^{-1}, \alpha'^{-1} \in M_2(A)$. The general case can be shown similarly. By definition, we have

$$w_{d(v,\omega)}^{\varphi(\omega),\sigma} = \frac{e_{\Lambda_{s(o)}(\omega,\mu)}(\varphi(\omega))}{e_{\Lambda_{s(o)}(\omega,\mu)}(\varphi(\omega))},$$

On the other hand,

$$w_{d(v,\omega)}^{\varphi(\omega),\sigma} = \frac{\{e_{\Lambda_{s(o)}(\omega,\mu)}(\varphi(\omega))\}^\sigma}{\{e_{\Lambda_{s(o)}(\omega,\mu)}(\varphi(\omega))\}^\sigma} = \frac{\{de_{\Lambda_{s(o)}(\omega,\mu)}(\varphi(\omega))\}^\sigma}{\{de_{\Lambda_{s(o)}(\omega,\mu)}(\varphi(\omega))\}^\sigma} = \frac{\lambda \circ ce_{\Lambda_\omega}(\varphi(\omega))}{\lambda \circ ce_{\Lambda_\omega}(\varphi(\omega))} \text{ by (2).}$$
Taking the \((q - 1)\)-th power and using the definition of \(\tau(\bar{u})\), we can prove followings as we did in (4).

\[
\left\{ c e_{\Lambda_0} \left( v + l \left( \begin{array}{c} \omega \\ 1 \end{array} \right) \right) \right\}^\sigma = e_1 \left( c e_{\Lambda_0} \left( v + (lu + m) \left( \begin{array}{c} \omega \\ 1 \end{array} \right) \right) \right)
\]

and

\[
\left\{ c e_{\Lambda_0} \left( ax \left( \begin{array}{c} \omega \\ 1 \end{array} \right) \right) \right\}^\sigma = e_2 \left( c e_{\Lambda_0} \left( axu \left( \begin{array}{c} \omega \\ 1 \end{array} \right) \right) \right).
\]

for some \(\varepsilon_1, \varepsilon_2 \in F_q^*\). By definition of \(\tau(\bar{u})\), the following equation holds, which implies \(\varepsilon_1 = \varepsilon_2\).

\[
\left( \frac{e_{\Lambda_0}(v + l(\omega))}{e_{\Lambda_0}(ax(\omega))} \right)^\sigma = \left( \frac{e_{\Lambda_0}(v + (lu + m)(\omega))}{e_{\Lambda_0}(axu(\omega))} \right).
\]

Together with these facts and by (1), we have

\[
w_a(v, \omega)^{\tau(\bar{u}(\bar{\omega}))} = \frac{e_{\Lambda_0}(v + (lu + m)(\omega))}{e_{\Lambda_0}(mxu(\omega))} = \frac{d' e_{\Lambda_0}(v + (lu + m)(\omega))}{d' e_{\Lambda_0}(axu(\omega))} \text{ by (3)}
\]

\[
= \frac{e_{\Lambda_0}(v + (l' + m'x)(\omega))}{e_{\Lambda_0}(ax(\omega))} = \frac{e_{\Lambda_0}(v + (l(\omega)))}{e_{\Lambda_0}(ax(\omega))}.
\]

This proves our assertion for \(w_a\), hence our proposition follows.

**Proposition 5.2.** \(\tau : G(A_f) \to \text{Aut}_k(\mathfrak{H})\) is injective.

Proof. Let \(x = (n, x) \in G(A_f)\) be a kernel of \(\tau\). By Theorem 2.3, \(\alpha\) equals to \(\left( \begin{array}{cc} y & 0 \\ 0 & y \end{array} \right)\) for some \(y \in k^*\). Write \(n = l + ma\), \(l \in k^2\), \(m \in \Pi(A_0)^2\). Then \(x = \bar{u} : g\) with \(\bar{u} = (n, 1) \in \bar{U}, g = (l, x) \in G_k\). Since \(z^{\tau(x)} = z\), it's easy to see that \(y \in F_q^*\) and \(l \in A^2\). Since \(\omega_{\xi(x)} = \omega_a\), it follows that \(y = 1\). Thus \(x = (n, 1) \in \bar{U}\). Finally, \(n = 0\) because \(Z_a^{\xi(x)} = Z_a\).

6. Shimura's reciprocity law for \(\mathfrak{H}\)

Let \(L, L^{ab}, I\) and \((L, L)\) be as in section 3. Fix a point \(z_0 = (v_0, \omega_0) \in (L \times L) \cap X\).
We set the following notations;

$$R = k[j, z]$$

$S$ = the integral closure of the ring $R$ in $\mathfrak{R}$

$$m = \{ f \in R : f(z_0) = 0 \}$$

$$\mathfrak{M} = \{ f \in S : f(z_0) = 0 \}$$

$G_{\text{fin}}$ = the decomposition group of $\text{Gal}(\mathfrak{R}/\mathfrak{R}_1)$.

Under this fixed point $z_0$, we can define $S$, $\sigma$ and $\mathfrak{M}$ for $\sigma \in G_{\text{fin}}$, $f \in S$ as we did in section 3.

**Lemma 6.1.** Suppose $j(\omega_0) \neq 0$. If $\tau(\bar{u}) \in \text{Gal}(\mathfrak{R}/\mathfrak{R}_1)$ satisfies

$$z_{a}^{\omega_0}(z_0) = z_{d}(z_0)$$

$$Z_{a}^{\omega_0}(z_0) = Z_{d}(z_0)$$

for all $a \in (k/A)^2$, then $\tilde{\mathfrak{M}} = \mathfrak{M}$ for all $f \in S$.

Proof. It suffices to show that $\mathfrak{M} = \mathfrak{M}$ for all $f \in S \cap \mathfrak{R}_N$, where $\deg N$ is sufficiently large. Then $\bar{u} = (m, u)$ can be viewed as an element in $(A/NA)^2 \cdot GL_2(A/NA)$. Since $z_{a}^{\omega_0}(z_0) = z_{d}(z_0)$ for $a = (\bar{k}, 0), (0, \bar{k})$ and $(\bar{k}, \bar{k})$, it is easy to see that

$$u = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} \in GL_2(A/NA)$$

for some $\varepsilon \in F^*_q$. We claim that

$$v_0 + m \begin{pmatrix} \omega_0 \\ 1 \end{pmatrix} \equiv \varepsilon v_0 \mod N \Lambda_{\omega_0}$$

Putting $a$ equal to $(\bar{k}, 0), (0, \bar{k})$ and $(\bar{k}, \bar{k})$, respectively, to the equation $Z_a^{\omega_0}(z_0) = Z_d(z_0)$, we have

1. $v_0 + \varepsilon \omega_0 + m \begin{pmatrix} \omega_0 \\ 1 \end{pmatrix} \equiv \varepsilon_1(v_0 + \omega_0) \mod N \Lambda_{\omega_0}$

2. $v_0 + \varepsilon + m \begin{pmatrix} \omega_0 \\ 1 \end{pmatrix} \equiv \varepsilon_2(v_0 + 1) \mod N \Lambda_{\omega_0}$

3. $v_0 + \varepsilon T \omega_0 + m \begin{pmatrix} \omega_0 \\ 1 \end{pmatrix} \equiv \varepsilon_3(v_0 + T \omega_0) \mod N \Lambda_{\omega_0}$
for some \( \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \in F_q^* \). Write \( v_0 = x\omega_0 + y \) with \( x, y \in k \). Then (1)-(2) gives that
\[
(\varepsilon_1 - \varepsilon_2)x + (\varepsilon_1 - \varepsilon) \equiv 0 \pmod{NA}
\]
(4)
\[
(\varepsilon_1 - \varepsilon_2)y + (\varepsilon - \varepsilon_2) \equiv 0 \pmod{NA}.
\]
If \( x \notin F_q \) (respectively \( y \notin F_q \)), then (4) (respectively (5)) gives that \( \varepsilon_1 = \varepsilon_2 = \varepsilon \) because \( \deg N \) is large. Then our claim follows from (1). If both \( x \) and \( y \) lie in \( F_q \) then the equation (1)-(3) gives that \( \varepsilon_3 = \varepsilon \), whence our claim follows from (3). Thus we can write
\[
\frac{v_0 + m(1)}{\varepsilon} = v_0 + N_1\omega_0 + N_2.
\]
for some \( N_1, N_2 \in NA \). It follows that
\[
\overline{f(\omega)} = f(v_0 + N_1\omega_0 + N_2, \omega_0)
\]
\[
= f(v + N_1\omega + N_2, \omega)
\]
\[
= f(v, \omega)
\]
\[
= \overline{f}.
\]
When \( j(\omega_0) = 0 \), we change \( z_a \) and \( Z_a \) slightly by
\[
z'_a = \Delta(\omega)e\left(a\begin{pmatrix} \omega \\ 1 \end{pmatrix}, \omega\right)^{q^2-1}
\]
\[
Z'_a = \Delta(\omega)e\left(\frac{v}{N} + a\begin{pmatrix} \omega \\ 1 \end{pmatrix}, \omega\right)^{q^2-1}.
\]
Then we have

**Lemma 6.2.** Suppose \( j(\omega_0) = 0 \). If \( \tau(\omega) \in \text{Gal}(R/R_1) \) satisfies
\[
z'_{a(\tau)}(z_0) = z'_a(z_0)
\]
\[
Z'_{a(\tau)}(z_0) = Z'_a(z_0)
\]
for all \( a \in (k/A)^2 \), then \( \overline{f(\omega)} = \overline{f} \) for all \( f \in S \).

Define an embedding
\[
\hat{q}: L^* \rightarrow k^2 \cdot GL_2(k)
\]
\[
s \rightarrow (l(s), q(s))
\]
with the properties;
By the continuity, we can extend $q$ to an embedding of the idele groups $A_L^*$ of $L$, and denote it again by

$$
q : A_L^* \rightarrow G(A_f).
$$

Then Shimura's reciprocity law in our case is given by

**Theorem 6.3.** Suppose $f \in \mathcal{M}$ is defined at $z_0$. If $L_{ab}(f(z_0))$ is a separable extension of $L_{ab}$ then $f(z_0)$ lies in $L_{ab}$ and $f^{(q(\omega_0))}(z_0) = f(z_0)^{q^{-1},L}$. If $L_{ab}(f(z_0))$ is not separable over $L_{ab}$, then it is a purely inseparable extension of $L_{ab}$. In this case, we can extend $(s^{-1}, L)$ uniquely to an embedding of $L_{ab}(f(z_0))$ over $L$. We denote it again by $(s^{-1}, L)$. Then $f^{(q(\omega_0))}(z_0) = f(z_0)^{q^{-1},L}$.

Proof. Write $q(s) = \tilde{u}g$ with $\tilde{u} = (m, u) \in \tilde{U}$, $g = (l, \alpha) \in G_k$. Then we have

$$
q(s) = u\alpha
$$

and

$$
l(s) = l + m\alpha.
$$

We will show that our Theorem holds for $j(\omega_0) \neq 0$. By Theorem 3.4, our theorem holds for $f = j, z_a$. For $f = Z_a$, we have

$$
Z_a^{(\omega_0)}
$$

where $\sigma$ is any automorphism of $C$ such that $\sigma = (s^{-1}, L)$ on $L_{ab}$. Since $z = Z_a$ if the level $N$ is equal to 1, we have proved our theorem for the special functions
\[ f = j, z, z_\alpha, Z_\alpha. \] Define
\[ R' = L[j, z], \]
\[ m' = \{ f \in R' : f(z_0) = 0 \}. \]

Note that \( L \) is contained in \( R \), so we need not define \( S', \mathfrak{M} \) as in the proof of Theorem 3.4 because they are the same with \( S, \mathfrak{M} \) in this case. It is not hard to see that \( j \circ g \) and \( z \circ g \) are integral over \( k[j, z] \). Then the proof of the following claim is mostly the same as that of the claim in the proof of Theorem 3.4.

Claim: There exists an element \( \rho \in \text{Aut}_k(\mathfrak{M}) \) which maps \( S \) and \( \mathfrak{M} \) onto \( S \) and \( \mathfrak{M} \) respectively satisfying \( \rho = \tau(\bar{q}(s)) \) on \( k(j, z) \) and \( \bar{\rho} = \sigma \) on \( \bar{S} \).

The rest are mostly the same as the proof of Theorem 3.4 by taking Lemma 6.1 into account. In the case that \( f(\omega_0) = 0 \), we can argue similarly by taking Lemma 6.2 into account and by replacing \( z, z_\alpha \) and \( Z_\alpha \) by \( z' = \Delta(\omega)e(\nu, \omega)e^{2-1}, z'_\alpha \) and \( Z'_\alpha \) respectively.

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