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THE DIVISIBILITY IN THE CUT-AND-PASTE GROUP OF *G*-MANIFOLDS AND FIBRING OVER THE CIRCLE WITHIN A COBORDISM CLASS

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Abstract

We prove a divisibility theorem for elements in the cut-and-paste group, or the SK-group of G-manifolds, G a finite abelian group of odd order. As an application we obtain necessary and sufficient conditions for that a closed G-manifold is equivariantly cobordant to the total space of G-fibration over the circle.

1. Introduction

All manifolds considered in this paper are in the smooth category, and those are all unoriented, with or without boundary. G always denotes a finite abelian group of odd order unless otherwise stated, and m a fixed nonnegative integer.

Let $\mathfrak{M}_m^G$ denote the set of m-dimensional closed G-manifolds. We define an equivalence relation on $\mathfrak{M}_m^G$ which is called a *cut-and-paste equivalence* or an *SK-equivalence*. The quotient set by this relation is denoted by $\mathfrak{M}_m^G/SK$, and this becomes a semigroup with the addition induced from the disjoint union of manifolds. The Grothendiek group of $\mathfrak{M}_m^G/SK$ is called the *cut-and-paste group* or the *SK-group* of m-dimensional closed G-manifolds, and is denoted by SK_m^G.

In this paper we will consider the divisibility for elements in SK_m^G, i.e., for a given $x \in SK_m^G$ and an integer $t \geq 0$ the existence of $y \in SK_m^G$ such that $x = ty$. We will obtain a necessary and sufficient condition for the dibisibility in terms of the Euler characteristics of manifolds (Theorem 4.2).

The following is an old result proved by Conner-Floyd [4]:

A closed manifold M is cobordant to the total space of a fibration over the circle S^1 if and only if the Euler characteristic $\chi(M)$ of M is even.

To obtain an equivariant version of this result we will apply our divisibility theorem, and obtain a necessary and sufficient condition for that a closed G-manifold is equivariantly cobordant to the total space of a G-fibration over S^1 such that the G-action takes place within the fibres, i.e., the action is trivial on S^1 (Theorem 6.3). We will also have some variants of the condition (Theorem 7.1), and remark that one of those variants is essentailly the same as the one which is obtained in a different way by Hara [6].

2. G-manifolds of type $\mathcal{F}$

Let G be a finite abelian group of odd order, and M a G-manifold. For a subgroup H of G, M^H denotes the fixed point set of M by the restricted H-action. G_x denotes the isotropy subgroup at $x \in M$, and Gx the orbit of x. There is a G-invariant neighbourhood of Gx in M which is G-diffeomorphic to $G \times_{G_x} V$ for some representation V of G_x (see for example Bredon [2, Corollary VI.2.4] or Kawakubo [9, Theorem 4.10]). Let V_x denote the nontrivial part of V, i.e., $V = V_x \oplus V^{G_x}$. In this paper we call (G_x, V_x) a *slice type at x in M*, though (G_x, V) is usually called a slice type in literature. More generally, let H be a subgroup of G and V a representation of H such that $V^H = \{0\}$, then (H, V) is called a *slice type for G*. Here V can be zero-dimensional. Since G is of odd order, the dimension of V is always even.

A *family* $\mathcal{F}$ of slice types for G is a set of slice types satisfying the condition that for any $(H, V) \in \mathcal{F}$ and any $x \in G \times_H V$, the slice type (G_x, V_x) at x in $G \times_H V$ belongs to $\mathcal{F}$. For a family $\mathcal{F}$ of slice types for G, if $(G_x, V_x) \in \mathcal{F}$ for any $x \in M$ then M is called a *G-manifold of type $\mathcal{F}$*.

We give a partial order $\preceq$ on the set of slice types for G in such a way that $(H', V') \preceq (H, V)$ if and only if H' is a subgroup of H and $V = V^{H'} \oplus V'$ as representations of H'. For a slice type (H, V), define

$$M^{(H,V)} = \left\{x \in M \mid (H, V) \preceq (G_x, V_x)\right\}.$$

This is a G-invariant submanifold of M of codimension even.

3. SK-group of G-manifolds

Let P and Q be m-dimensional compact G-manifolds. If $\varphi\colon \partial P \to \partial Q$ is a G-diffeomorphism between the boundaries of P and Q, then we obtain a closed G-manifold $P \cup_\varphi Q$ by pasting P and Q with each other along the boundary by φ. If P and Q are of type $\mathcal{F}$, then so is $P \cup_\varphi Q$. If $\psi\colon \partial P \to \partial Q$ is a second G-diffeomorphism, we obtain a second closed G-manifold $P \cup_\psi Q$. Then $P \cup_\varphi Q$ is said to be *obtained from $P \cup_\psi Q$ by equivariant cutting and pasting* (or *Schneiden und Kleben* in German), and vice versa.

Let $\mathfrak{M}_m^G(\mathcal{F})$ be the set of m-dimensional closed G-manifolds of type $\mathcal{F}$. For M, $N \in \mathfrak{M}_m^G(\mathcal{F})$, M and N are said to be *equivariantly SK-equivalent*, if there is $L \in \mathfrak{M}_m^G(\mathcal{F})$ such that the disjoint unions $M + L$ and $N + L$ are obtained from each other by a finite sequence of equivariant cuttings and pastings. The SK-equivalence is an equivalence relation on $\mathfrak{M}_m^G(\mathcal{F})$. The quotient set by this relation is denoted by $\mathfrak{M}_m^G(\mathcal{F})/SK$. This becomes a semigroup with the addition induced from disjoint union of G-manifolds. The Grothendieck group of $\mathfrak{M}_m^G(\mathcal{F})/SK$ is called *the SK-group of m-dimensional closed G-manifolds of type $\mathcal{F}$*, and is denoted by $SK_m^G(\mathcal{F})$.

Let $\mathcal{F}' \subset \mathcal{F}$ be families of slice types for G. If M is a G-manifold of type $\mathcal{F}'$, then M is automatically of type $\mathcal{F}$. So we obtain an inclusion $\mathfrak{M}_m^G(\mathcal{F}') \subset \mathfrak{M}_m^G(\mathcal{F})$, and this induces a homomorphism $\iota\colon SK_m^G(\mathcal{F}') \to SK_m^G(\mathcal{F})$.

If $M \in \mathfrak{M}_m^G(\mathcal{F})$, and if $(H, V) \in \mathcal{F}$ is maximal with $n = \dim V$, then $M^{(H,V)}$ is a G-manifold with one orbit type and the orbit space $M^{(H,V)}/G$ is a manifold of dimension $m - n$. Assigning $M^{(H,V)}/G$ to M, we have a correspondence $\mathfrak{M}_m^G(\mathcal{F}) \to \mathfrak{M}_{m-n}$, where $\mathfrak{M}_{m-n}$ is the set of $(m-n)$-dimensional closed manifolds. An equivariant cutting and pasting operation on M restricts to one on $M^{(H,V)}$. So the correspondence $\mathfrak{M}_m^G(\mathcal{F}) \to \mathfrak{M}_{m-n}$ induces a homomorphism $\rho\colon SK_m^G(\mathcal{F}) \to SK_{m-n}$, where SK_{m-n} is the SK-group of $(m - n)$-dimensional (nonequivariant) closed manifolds (cf. Karras–Kreck–Neumann-Ossa [8]).

Theorem 3.1. *Let $\mathcal{F}$ be a family of slice types for G with a maximal element (H, V), and let $\mathcal{F}' = \mathcal{F} - \{(H, V)\}$ and $n = \dim V$. Then*

$$0 \longrightarrow SK_m^G(\mathcal{F}') \xrightarrow{\iota} SK_m^G(\mathcal{F}) \xrightarrow{\rho} SK_{m-n} \longrightarrow 0$$

is a split short exact sequence.

For a proof of this theorem, see the proof of Komiya [11, Theorem 6.5], and also see Kosniowski [12, Corollary 2.6.3].

4. Divisibility theorem

Let $\mathcal{F}$ be a family of slice types (H, V) for G with $\dim V \leq m$, and give a partial order $\preceq$ on $\mathcal{F}$ as in Section 2. Note that $\mathcal{F}$ is finite. Let the elements of $\mathcal{F}$ be indexed by an indexing set $I(\mathcal{F})$, i.e., $\mathcal{F} = \{(H_i, V_i) \mid i \in I(\mathcal{F})\}$. The partial order on $\mathcal{F}$ induces a partial order on $I(\mathcal{F})$. We denote this order by the same symbol $\preceq$. Let $\mu_{I(\mathcal{F})}(\ ,\ )$ be the Möbius function on the partially ordered set $I(\mathcal{F})$, which is inductively defined as follows (cf. Aigner [1]): For any $i, j \in I(\mathcal{F})$ with $i \preceq j$,

$$\begin{aligned} &\mu_{I(\mathcal{F})}(i, i) = 1, \\ &\mu_{I(\mathcal{F})}(i, j) = -\sum_{i \preceq \dot{h} \prec j} \mu_{I(\mathcal{F})}(i, h) = -\sum_{i \prec \dot{h} \preceq j} \mu_{I(\mathcal{F})}(h, j) \quad \text{if } i \prec j, \end{aligned}$$

where the dot $\cdot$ means the sum is taken over the letters under $\cdot$. As we remarked in Komiya [11, Lemma 7.2] we obtain the following proposition in a similar way to Komiya [10].

Proposition 4.1. *For any $M \in \mathfrak{M}_m^G(\mathcal{F})$ and $i \in I(\mathcal{F})$ we have*

$$\sum_{i \preceq j} \mu_{I(\mathcal{F})}(i, j)\chi(M^{(H_j,V_j)}) \equiv 0 \mod \chi(G/H_i),$$

where $\chi(G/H_i)$ is the cardinality of G/H_i.

An element $x \in SK_m^G(\mathcal{F})$ is written in the form $x = [M] - [N]$ for some M, $N \in \mathfrak{M}_m^G(\mathcal{F})$, where $[M]$, $[N]$ denote the SK-equivalence class of M, N, respectively. Define $\chi(x) = \chi(M) - \chi(N)$ and $\chi_j(x) = \chi(M^{(H_j,V_j)}) - \chi(N^{(H_j,V_j)})$ for $(H_j, V_j) \in \mathcal{F}$. This is well-defined since cutting and pasting operation keeps the Euler characteristic invariant. Note that if m is odd then $\chi(x) = 0$ and $\chi_j(x) = 0$ for any $x \in SK_m^G(\mathcal{F})$.

If $\mathcal{F}$ is the family of all slice types (H, V) for G with $\dim V \leq m$, then any m-dimensional G-manifold is of type $\mathcal{F}$, and we denote $\mathfrak{M}_m^G(\mathcal{F})$ and $SK_m^G(\mathcal{F})$ by $\mathfrak{M}_m^G$ and SK_m^G, respectively. In this case we also denote by I the indexing set $I(\mathcal{F})$ of the elements of $\mathcal{F}$.

We obtain the following divisibility theorem.

Theorem 4.2. *An element $x \in SK_m^G$ is divisible by an integer $t \geq 0$, i.e., there is $y \in SK_m^G$ such that $x = ty$, if and only if*

$$\sum_{i \preceq j} \mu_I(i, j)\chi_j(x) \equiv 0 \mod t\chi(G/H_i)$$

for any $i \in I$.

The "only if" part of this theorem is easily shown from Proposition 4.1. The "if" part will be shown in the next section. Before we proceed to the next section, we note that if m is odd then $SK_m^G = 0$ (see Kosniowski [12, Chapter 5]) and hence the theorem is trivially valid.

5. Proof of the divisibility theorem

In this section we prove the "if" part of Theorem 4.2 in the even dimensional case.

Let $\mathcal{F}$ be the family of all slice types (H, V) for G with $\dim V \leq 2m$. Let the elements of $\mathcal{F}$ be indexed by integers in such a way that

$$\mathcal{F} = \{(H_i, V_i) \mid i \in I\}, \qquad I = \{0, 1, \dots, k\}$$

and $i \preceq j \Rightarrow i \leq j$, where $\leq$ is the ordinary order for integers. Then (H_0, V_0) is the minimal element of $\mathcal{F}$ with respect to the order $\preceq$ such that $H_0 = \{1\}$ the trivial group, and $V_0 = \{0\}$. We have a filtration of $\mathcal{F}$,

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k = \mathcal{F},$$

where

$$\mathcal{F}_j = \{(H_i, V_i) \mid i \in I(\mathcal{F}_j)\}, \qquad I(\mathcal{F}_j) = \{0, 1, \dots, j\}.$$

Then (H_j, V_j) is a maximal element of $\mathcal{F}_j$. Consider the following assertion:

$A(j)$. An element $x \in SK^G_{2m}(\mathcal{F}_j)$ is divisible by an integer $t \geq 0$, if for any $i \in I(\mathcal{F}_j)$

$$\sum_{i \preceq h} \mu_{I(\mathcal{F}_j)}(i, h)\chi_h(x) \equiv 0 \mod t\chi(G/H_i),$$

where $\mu_{I(\mathcal{F}_j)}(\ ,\ )$ is the Möbius function on the partially ordered set $I(\mathcal{F}_j)$.

$A(k)$ is the "if" part of Theorem 4.2. For any j $(0 \leq j \leq k)$ we prove $A(j)$ by induction.

To prove $A(0)$, note that $SK^G_{2m}(\mathcal{F}_0)$ is the SK-group of $2m$-dimensional closed free G-manifolds, and is isomorphic to SK_{2m} by the isomorphism which assigns to a free G-manifold M its orbit space M/G. Note also that Euler characteristic detects the elements of SK_{2m}. See for these facts Karras–Kreck–Neumann–Ossa [8] and Kosniowski [12]. Assume $\chi(x) \equiv 0 \mod t\chi(G)$ for $x \in SK^G_{2m}(\mathcal{F}_0)$. If $m \neq 0$ and we take $N \in \mathfrak{M}_{2m}$ with $\chi(N) = \chi(x)/t\chi(G)$, then we see $x = t[G \times N]$ in $SK^G_{2m}(\mathcal{F}_0)$. We can also take such an $N \in \mathfrak{M}_0$, if $m = 0$ and $\chi(x) \geq 0$. If $\chi(x) < 0$, then consider $-x$. This proves $A(0)$.

To proceed the induction step, consider the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & SK^G_{2m}(\mathcal{F}_{j-1}) & \xrightarrow{\iota} & SK^G_{2m}(\mathcal{F}_j) & \xrightarrow{\rho} & SK_{2m-n} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda_t} & & \downarrow{\scriptstyle \lambda_t} & & \downarrow{\scriptstyle \lambda_t} & & \\
0 & \longrightarrow & SK^G_{2m}(\mathcal{F}_{j-1}) & \xrightarrow{\iota} & SK^G_{2m}(\mathcal{F}_j) & \xrightarrow{\rho} & SK_{2m-n} & \longrightarrow & 0,
\end{array}$$

where $n = \dim V_j$, and λ_t is the homomorphism given by the multiplication by t. The horizontal sequences are exact from Theorem 3.1.

Assume $A(j-1)$ as an induction hypothesis, and assume that $x \in SK^G_{2m}(\mathcal{F}_j)$ satisfies the congruence in $A(j)$. From the congruence we see

$$\chi\big(\rho(x)\big) = \chi_j(x)/\chi(G/H_j) \equiv 0 \mod t,$$

and hence $\rho(x)$ is divisible by t, i.e., is in the image of λ_t. By diagram chasing and the exactness of the sequences, we have $z \in SK^G_{2m}(\mathcal{F}_j)$ and $w \in SK^G_{2m}(\mathcal{F}_{j-1})$ such that $\iota(w) = x - tz$ in $SK^G_{2m}(\mathcal{F}_j)$. Since $\chi_h(w) = \chi_h(x) - \chi_h(tz)$, we see that w satisfies the congruence in $A(j-1)$. Thus the induction hypothesis assures that w is in the image of λ_t. This implies that there is $u \in SK^G_{2m}(\mathcal{F}_{j-1})$ such that $x = t(\iota(u) + z)$. This proves $A(j)$, and completes the proof of Theorem 4.2.

6. Fibring over the circle

If there is an $(m+1)$-dimensional compact G-manifold W such that ∂W is the disjoint union $M + N$ of M, $N \in \mathfrak{M}^G_m$, then M and N are said to be *equivariantly* *cobordant with a G-cobordism W*. This is an equivalence relation on $\mathfrak{M}_m^G$, and gives rise to the m-dimensional (unoriented) cobordism group $\mathfrak{N}_m^G$. We denote by $[M]_{\mathfrak{N}}$ the cobordism class of M. It is clear that $2[M]_{\mathfrak{N}} = 0$ in $\mathfrak{N}_m^G$.

Lemma 6.1 (cf. Karras–Kreck–Neumann–Ossa [8, Lemma 1.9]). *For M, $N \in \mathfrak{M}_m^G$, if M is obtained from N by equivariant cutting and pasting, then $M+N$ is equivariantly cobordant to the total space of a G-fibration over S^1 such that the G-action takes place within the fibres.*

Proof. Let $M = P \cup_\varphi Q$ and $N = P \cup_\psi Q$, where $\varphi, \psi \colon \partial P \to \partial Q$ are G-diffeomorphisms between the boundaries of m-dimensional compact G-manifolds P, Q. Let W be the union of $P \times [0,1]$ and $Q \times [0,1]$ with the following identifications: for $x \in \partial P$, identify $(x,t) \in \partial P \times [0,1/3]$ with $(\varphi(x),t) \in \partial Q \times [0,1/3]$ and $(x,t) \in \partial P \times [2/3,1]$ with $(\psi(x),t) \in \partial Q \times [2/3,1]$. After smoothing, we obtain an $(m+1)$-dimensional compact G-manifold W such that $\partial W = M + N + L$, where L is the total space of a required fibration. □

Lemma 6.2. *Given $M \in \mathfrak{M}_m^G$, if $[M] = 2x$ in SK_m^G for some $x \in SK_m^G$, then M is equivariantly cobordant to the total space of a G-fibration over S^1 such that the G-action takes place within the fibres.*

Proof. Let $x = [N_1] - [N_2]$ for N_1, $N_2 \in \mathfrak{M}_m^G$. Then $[M + 2N_2] = [2N_1]$ in SK_m^G, and this implies that for some $L \in \mathfrak{M}_m^G$, $M + 2N_2 + L$ is obtained from $2N_1 + L$ by a finite sequence of equivariant cuttings and pastings. From Lemma 6.1 we see that M is G-cobordant to the total space of a required fibration. □

Theorem 6.3. *$M \in \mathfrak{M}_m^G$ is equivariantly cobordant to the total space of a G-fibration over S^1 such that the G-action takes place within the fibres, if and only if $\chi(M^{(H,V)})$ is even for any slice type (H, V).*

Proof. If M is equivariantly cobordant to the total space of a G-fibration as above, then for any slice type (H, V), $M^{(H,V)}$ is also cobordant to the total space of a fibration over S^1. Conner-Floyd [4] implies $\chi(M^{(H,V)})$ is even.

Assume, conversely, that $\chi(M^{H_i,V_i)})$ is even for any $i \in I$, where $\{(H_i, V_i) \mid i \in I\}$ is the family of all slice types (H, V) with $\dim V \leq m$. Then we see from Proposition 4.1,

$$\sum_{i \preceq j} \mu_I(i, j)\chi(M^{(H_j,V_j)}) \equiv 0 \mod 2\chi(G/H_i)$$

for any $i \in I$, since $\chi(G/H_i)$ is odd. Theorem 4.2 implies that $[M] = 2x$ in SK_m^G for some $x \in SK_m^G$, and Lemma 6.2 implies that M is equivariantly cobordant to the total space of a G-fibration over S^1 such that the G-action takes place within the fibres. $\square$

7. Some variants

In this section we obtain some variants of the condition obtained in Theorem 6.3.

Let $\mathcal{F} = \{(H_i, V_i) \mid i \in I\}$ be the family of all slice types (H, V) with $\dim V \leq m$. An *Euler function* $\varphi_I(\ ,\ )$ on I is defined as follows: for $i, j \in I$ with $i \preceq j$,

$$\varphi_I(i, j) = \sum_{i \preceq h \preceq j} \mu_I(h, j)\chi(H_h/H_i).$$

Theorem 7.1. *For $M \in \mathfrak{M}_m^G$, the following* (i)–(v) *are equivalent to each other*:

(i) *M is equivariantly cobordant to the total space of a G-fibration over S^1 such that the G-action takes place within the fibres,*

(ii) *$\chi(M^{(H,V)})$ is even for any slice type (H, V),*

(iii) *$[M]$ is divisible by 2 in SK_m^G,*

(iv) *for any $i \in I$,*

$$\sum_{i \preceq j} \mu_I(i, j)\chi(M^{(H_j,V_j)}) \equiv 0 \mod 2\chi(G/H_i),$$

(v) *for any $i \in I$,*

$$\sum_{i \preceq j} \varphi_I(i, j)\chi(M^{(H_j,V_j)}) \equiv 0 \mod 2\chi(G/H_i).$$

Proof. (i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv) are already shown as Theorem 6.3 and Theorem 4.2, respectively. (ii) $\Rightarrow$ (iv) is also already noted in the proof of Theorem 6.3. (iv) $\Rightarrow$ (ii) and (v) $\Rightarrow$ (ii) are inductively and easily shown, since $\mu_I(i, i) = 1$ and $\varphi_I(i, i) = 1$.

Finally (iv) $\Rightarrow$ (v) is shown as follows: If we put for any $i \in I$,

$$\sum_{i \preceq j} \mu_I(i, j)\chi(M^{(H_j,V_j)}) = 2l_i\chi(G/H_i)$$

for some integer l_i, then we have

$$\begin{aligned}
&\sum_{i \preceq j} \varphi_I(i, j)\chi(M^{(H_j,V_j)}) \\
&= \sum_{i \preceq j} \left(\sum_{i \preceq h \preceq j} \mu_I(h, j)\chi(H_h/H_i) \right) \chi(M^{(H_j,V_j)})
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i \preceq \dot{h}} \left(\sum_{h \preceq \dot{j}} \mu_I(h, j)\chi(M^{(H_j, V_j)}) \right) \chi(H_h/H_i) \\
&= \sum_{i \preceq \dot{h}} 2l_h \chi(G/H_h)\chi(H_h/H_i) \\
&= 2\chi(G/H_i) \sum_{i \preceq \dot{h}} l_h \\
&\equiv 0 \mod 2\chi(G/H_i). \qquad \square
\end{aligned}
$$

CONCLUDING REMARKS. (1) The Euler characteristic of an odd dimensional closed manifold is zero, and $M^{(H,V)}$ is even codimensional. Thus, if $M \in \mathfrak{M}_m^G$ is odd dimensional, then the statements (ii)–(v) in Theorem 7.1 are trivially valid and hence (i) always holds.
(2) (v) is essentially the same as the condition obtained in Hara [6, Theorem 3.10].
(3) If G is of even order, the situation is somewhat different. When $G = \mathbb{Z}_{2^r}$, the cyclic group of order 2^r, there is obtained in Hara [5] a condition for $M \in \mathfrak{M}_m^G$ to be equivariantly cobordant to a G-manifold which is equivariantly fibred over S^1.
(4) There are also corresponding results in the oriented case, for which the signature of manifold is needed instead of the Euler characteristic. See Burdick [3] and Neumann [13] for the nonequivariant oriented case, and Hermann-Kreck [7] for the $\mathbb{Z}_2$-equivariant oriented case.

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