CONSTRUCTION OF VERSAL GALOIS COVERINGS USING TORIC VARIETIES

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Abstract

In this article we give an explicit construction of versal Galois coverings for any given finite subgroup of $GL(n, \mathbb{Z})$. By this construction we give a positive answer to Question 1.4 in [5].

Introduction

Let $X$ and $Y$ be normal projective varieties. Let $\pi : X \to Y$ be a finite surjective morphism. We denote the rational function fields of $X$ and $Y$ by $\mathbb{C}(X)$ and $\mathbb{C}(Y)$, respectively. Under these circumstances, one can regard $\mathbb{C}(Y)$ as a subfield of $\mathbb{C}(X)$ by $\pi^* : \mathbb{C}(Y) \to \mathbb{C}(X)$.

DEFINITION 0.1. $\pi$ is said to be a Galois covering if $\mathbb{C}(X)/\mathbb{C}(Y)$ is a Galois extension. We call $\pi$ a $G$-covering when the Galois group of the field extension is isomorphic to a finite group $G$.

REMARK 0.2. Note that there exists a natural $G$-action on $X$ such that $Y = X/G$.

In [2], Namba gave a method for constructing new $G$-coverings from a given $G$-covering as follows: Let $\pi : X \to Y$ be a $G$-covering. Let $W$ be a normal projective variety.

NOTATION 0.3. We denote the stabilizer of $x \in X$ by $G_x$. Also we define $\text{Fix}(X, G)$ by

$$\text{Fix}(X, G) = \{x \in X \mid G_x \neq [1]\}.$$ 

DEFINITION 0.4. A rational map $\nu : W \dashrightarrow Y$ is called a $G$-indecomposable rational map to $Y$ if $\nu(W) \not\subseteq \pi(\text{Fix}(X, G))$ and $\nu$ does not factor through $\pi_H : X/H \to Y$ for any $H$, where $X/H$ is the quotient variety of $X$ by a subgroup $H \subset G$ and $\pi_H$ is the quotient morphism.

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Fix a $G$-indecomposable rational map $v: W \rightarrow Y$. Let $W_0$ be the graph of $v$. Then we can obtain a $G$-covering $Z$ over $W$ by taking the $\mathbb{C}(W_0 \times_Y X)$ normalization of $W$. We also obtain a $G$-equivariant rational map from $\mu: Z \rightarrow X$ such that $\mu(Z) \not\in \text{Fix}(X,G)$. We can construct many new $G$-coverings in this manner. However, we may not be able to construct every $G$-covering by this method, as the construction depends on the existence of a $G$-indecomposable rational map. This leads us to the notion of a versal $G$-covering introduced in [5] and [6].

**Definition 0.5.** $\varphi: X \rightarrow Y$ is called a versal $G$-covering if, for any $G$-covering $\pi': Z \rightarrow W$, there exists a $G$-equivariant rational map $\mu: Z \rightarrow X$ such that $\mu(Z) \not\in \text{Fix}(X,G)$.

**Remark 0.6.** $\mu$ induces a $G$-indecomposable rational map $v$ from $W$ to $Y$, and $Z$ coincides with the $G$-covering constructed by the method above by using $v$. Note that the versal $G$-covering here is not unique.

By the definition any $G$-covering can be obtained as a “rational pullback” from a versal $G$-covering. As for the existence of versal $G$-coverings, Namba proved the following.

**Theorem 0.7** (Namba [2]). For any finite group $G$, there exists a versal $G$-covering.

Namba explicitly constructed a versal $G$-covering for each finite group $G$. However his method of construction gave versal coverings with dimensions equal to the order of the given group $G$, and it does not seem to be practical to use it in order to construct new Galois coverings. In [6], Tsuchihashi constructed versal $G$-coverings over the projective space $\mathbb{P}^m$ for the symmetric groups and for a generalization of the symmetric groups using toric varieties. In this paper we generalize Tsuchihashi’s result partially and construct versal coverings of dimension $n$ for any subgroup $G$ of $\text{GL}(n,\mathbb{Z})$. Our result is the following.

**Theorem 0.8.** Let $N$ be a free $\mathbb{Z}$-module, $\Delta$ a projective fan in $N_{\mathbb{R}}$. Let $X(\Delta)$ be the toric variety associated to the fan $\Delta$. Let $G$ be a subgroup of $\text{Aut}_\mathbb{Z}(N)$ which keeps $\Delta$ invariant. Then $G$ acts naturally on $X(\Delta)$ and

$$\varphi: X(\Delta) \rightarrow X(\Delta)/G$$

is a versal $G$-covering.
1. Construction and proof of versality

In this section we will prove Theorem 0.8. We will first construct projective toric varieties with $G$-action and construct $G$-coverings by taking the quotient variety and the quotient morphism. Then we prove that the $G$-coverings that we have constructed are versal.

We will mostly follow Fulton [1] for notations concerning toric varieties. Let $N$ be a free $\mathbb{Z}$-module of rank $n$. Let $M$ be the dual module of $N$. We denote the dual pairing by $\langle u, v \rangle$ for $u \in M$ and $v \in N$. We denote a fan by $\Delta$, and denote the toric variety associated to the fan $\Delta$ by $X(\Delta)$. We will say a fan to be a projective fan when $X(\Delta)$ is a projective variety. For basic properties of toric varieties, we refer the reader to Fulton [1] and Oda [3, 4].

A toric variety $X(\Delta)$ with $G$-action for a given finite subgroup $G$ of $GL(n, \mathbb{Z})$ can be constructed as follows.

Suppose that $\Delta$ is a complete $G$-invariant fan (i.e. for any $g \in G$ and any $\sigma \in \Delta$ there exists $\sigma' \in \Delta$ such that $g(\sigma) = \sigma'$). Then $g : N \to N$, for any $g \in G$, induces an automorphism of varieties $g_* : X(\Delta) \to X(\Delta)$. Thus we can define a $G$-action on $X(\Delta)$. We will abuse notation and denote $g_* : X(\Delta) \to X(\Delta)$ by $g$. By the following lemma there exists a complete projective invariant fan for any finite subgroup $G$ of $GL(n, \mathbb{Z})$.

**Lemma 1.1.** For any finite subgroup $G$ of $GL(n, \mathbb{Z})$, there exists a complete projective $G$-invariant fan.

**Proof.** Take a fan $\Delta'$ of $N_\mathbb{R}$ corresponding to $(\mathbb{P}^1)^n$. It is a fan obtained by decomposing $N_\mathbb{R}$ with hyperplanes. By taking the images of these hyperplanes by $G$ and by decomposing $N_\mathbb{R}$ with this new set of hyperplanes, we obtain a $G$-invariant fan $\Delta$ of $N_\mathbb{R}$. By the proof of Proposition 2.17 in [3], a complete fan obtained as a hyperplane decomposition is projective, hence $\Delta$ is projective.

By taking the quotient variety $X/G$ of $X$ by $G$, and taking the quotient morphism $\varphi : X \to X/G$ we obtain a $G$-covering. We will now prove some lemmas in order to show that the $G$-coverings constructed in the fashion above are versal.

**Lemma 1.2.** Let $X(\Delta)$ be a complete projective toric variety with $G$-action. Then there exists a $G$-invariant $T_N$-invariant very ample divisor on $X(\Delta)$.

**Proof.** Since $X(\Delta)$ is projective, there exists a $T_N$-invariant very ample divisor $D$ on $X(\Delta)$. Let $D'$ be

$$D' = \frac{1}{|G_D|} \sum_{g \in G} g(D)$$
where \( G_D = \{ g \in G \mid g(D) = D \} \). Then \( D' \) is a \( G \)-invariant \( T_N \)-invariant divisor. It remains to show that \( D' \) is ample.

For any \( T_N \)-invariant ample divisors \( D_1 \) and \( D_2 \) the sum \( D_1 + D_2 \) is also ample. This is true since if \( D_1 \) and \( D_2 \) are ample, the piecewise linear functions \( \psi_{D_1} \) and \( \psi_{D_2} \) corresponding to \( D_1 \) and \( D_2 \) respectively are strictly convex. Then \( \psi_{D_1 + D_2} \) is also strictly convex which implies the ampleness of \( D_1 + D_2 \).

Each \( g(D) \) is ample so \( D' \) is an ample divisor and for some \( m, mD' \) is a very ample \( G \)-invariant \( T_N \)-invariant divisor.

Let \( \Delta(1) \) be the set of one dimensional cones of \( \Delta \). Let \( D_{\tau_i} \) be the \( T_N \)-invariant divisor corresponding to \( \tau_i \in \Delta(1) \). Let \( v_i \) be a primitive generator of \( \tau_i \). Let \( D = \sum_{\tau_i \in \Delta(1)} a_i D_{\tau_i} \) be a \( G \)-invariant \( T_N \)-invariant cartier divisor (wich implies \( a_i = a_j \) if there exists \( g \in G \) such that \( g(\tau_i) = \tau_j \)). Then \( P_D = \{ u \in \mathbb{R} \mid \langle u, v_i \rangle \geq -a_i, \forall v_i \in \Delta(1) \} \subset \mathbb{R} \) is also \( G \)-invariant. From [1] p.66, the global sections of the sheaf \( \mathcal{O}(D) \) is generated by \( \omega^u, u \in P_D \cap M \).

\[
\mathcal{H}^0(X(\Delta), \mathcal{O}(D)) = \bigoplus_{u \in P_D \cap M} \mathbb{C} \cdot \omega^u.
\]

Hence we can define a (right) \( G \)-action on the global sections of the sheaf \( \mathcal{O}(D) \) by \( (\omega^u) \cdot g^s \mapsto \omega^{(u)g^s} \).

Define \( u(\sigma) \in M \) by \( \langle u(\sigma), v \rangle = \psi_D(u)|_\sigma \). Then from [1] p.62, \( \Gamma(U_{\sigma}, \mathcal{O}(D)) = \chi^{u(\sigma)} \cdot A_\sigma \). Thus we have local trivialization isomorphisms \( \eta_\sigma : \Gamma(U_{\sigma}, \mathcal{O}(D)) \cong A_\sigma \) given by \( \omega^u \mapsto \chi^{u(\sigma)} \). Let \( \sigma \) and \( \sigma' \) be maximal cones of \( \Delta \) and suppose there exists \( g \in G \) such that \( g(\sigma) = \sigma' \). Since \( D \) is \( G \)-invariant we have \( (u(\sigma'))g^s = u(\sigma) \).

Then
\[
\eta_\sigma(u^u \cdot g^s) = \chi^{(u^u)g^s - u(\sigma)} = \chi^{(u - u(\sigma))g^s} = \eta_{\sigma'}(\omega^u) \cdot g^s.
\]

Hence this action on the global sections of \( \mathcal{O}(D) \) coincides with the geometric action of \( G \) on \( X(\Delta) \).

**Lemma 1.3.** For a finite set of vectors \( \{ u_1, \ldots, u_s \in M \} \), there exists \( v \in N \) such that \( \{ \langle u_i, v \rangle \}_{i=1,\ldots,s} \) are mutually distinct.

Proof. We prove this by induction on the rank on \( M \). For \( \text{rank}(M) = 1 \) take any \( u \neq 0 \).

Let \( \text{rank}(M) = k \). Fix a basis for \( M \) and let \( u_i = (a_{i1}, \ldots, a_{ik}) \). Define a projection \( p \) onto a lattice of rank \( k - 1 \) by \( (a_{i1}, \ldots, a_{ik}) \mapsto (a_{i2}, \ldots, a_{ik}) \). Then by the hypothesis of induction there exists \( v' = (b_2, \ldots, b_k) \) such that \( \langle p(u_i), v' \rangle \) are distinct for distinct \( p(u_i) \). Let \( b_1 = 2 \max ||p(u_i), v'||_{i=1,\ldots,s} + 1 \). Then \( v = (b_1, \ldots, b_s) \) satisfies the desired condition. This can be checked directly.
Let \( u_i = (a_{i_1}, \ldots, a_{i_k}) \), \( u_j = (a_{j_1}, \ldots, a_{j_k}) \), \( i \neq j \). If \( a_{i_1} > a_{j_1} \) then
\[
\langle u_i, v \rangle - \langle u_j, v \rangle = (a_{i_1} - a_{j_1})b_1 + \left( \sum_{i=2}^{k} a_{i_1}b_i \right) - \left( \sum_{i=2}^{k} a_{j_1}b_i \right)
\]
\[
> b_1 + \left( \sum_{i=2}^{k} a_{i_1}b_i \right) - \left( \sum_{i=2}^{k} a_{j_1}b_i \right)
\]
\[
\geq 1 \quad \text{(by the choice of } b_1). \]

If \( a_{i_1} = a_{j_1} \) then \( p(u_i) \neq p(u_j) \) and
\[
\langle u_i, v \rangle - \langle u_j, v \rangle = 0 + \left( \sum_{i=2}^{k} a_{i_1}b_i \right) - \left( \sum_{i=2}^{k} a_{j_1}b_i \right)
\]
\[
\neq 0 \quad \text{(by the choice of } v'). \]

Hence \( \{\langle u_i, v \rangle\}_{i=1, \ldots, n} \) are distinct. \( \square \)

**Lemma 1.4.** Let \( \pi' : Z \to W \) be a \( G \)-covering. Let \( G = \{g_1, \ldots, g_{|G|}\} \).

1. There exists \( z \in Z \) such that \( z_i = g_i(z) \) \((i = 1, \ldots, |G|) \) are mutually distinct.
2. For any \( \alpha_1, \ldots, \alpha_{|G|} \in \mathbb{C} \) there exists a rational function \( f \) on \( Z \) such that \( f(z_i) = \alpha_i \).
3. If \( \alpha_i \neq 0 \) for all \( i \), then there exists a \( G \)-invariant affine open set \( U \) such that there exists a point \( z \) in \( U \) satisfying (1) and a function \( f \) satisfying (2) and in addition \( f \) and \( f^{-1} \) are regular on \( U \).

Proof. Let \( U' = \text{Spec}(R) \) be an \( G \)-invariant affine open set of \( Z \) where \( G \) acts freely. Then clearly any point \( z \) of \( U' \) satisfies (1).

For any finite number of distinct points \( z_i \in U' \), \( i = 1, \ldots, s \) and for any \( \alpha_i \in \mathbb{C} \), \( i = 1, \ldots, s \), there exists a regular function \( f \) on \( U \) satisfying \( f(z_i) = \alpha_i \). This is proved by induction on the number of points. The case where \( s = 1 \) is trivial. Let \( s = k \) and let \( m_i \subset R \) be the maximal ideal corresponding to the point \( z_i \). Then \( m_i \setminus \bigcup_{j \neq i} m_j \neq \emptyset \). For each \( i \) take a regular function \( f_i \in m_i \setminus \bigcup_{j \neq i} m_j \). Then
\[
\begin{aligned}
f_1 \cdots f_{k-1}(z_i) &= 0, &i = 1, \ldots, k - 1 \\
&\neq 0, &i = k
\end{aligned}
\]

By the hypothesis of induction, there exists regular functions \( h, h' \) satisfying \( h(z_i) = \xi_i \) for \( i = 1, \ldots, k - 1 \), and \( h'(z_k) = (\alpha_k - h(z_k))/(f_1 \cdots f_{k-1}(z_k)) \). Then \( f = h + f_1 \cdots f_k \cdot h' \) satisfies \( f(z_i) = \alpha_i \) for \( i = 1, \ldots, k \). Hence we have a regular function satisfying (2).

Let \( V = \text{Spec}(R_f) \). Then \( U = \bigcap_{g \in G} g(V) \) satisfies (3). \( \square \)
Let $\pi': Z \to W$ be any $G$-covering. Let $f$ be a rational function on $Z$. For $f$, $u \in M$, $v \in N$, define $f^{u,v}$ by

$$f^{u,v} = \prod_{g \in G} f^{u,g(v)} \cdot (g^{-1}).$$

Then $f^{u,v}$ satisfies the following properties (1) and (2) for any $u_1$ and $u_2 \in M$ and any $g' \in G$.

$$(1) \quad f^{u_1,v} \cdot f^{u_2,v} = \prod_{g \in G} f^{u_1+u_2,g(v)} \cdot (g^{-1})$$

$$(2) \quad f^{u,v} \cdot (g') = \prod_{g' \in G} f^{g(g(v))} \cdot (g'^{-1}(g'))$$

Let $V = \text{Spec}(R)$ be a $G$-invariant affine open set of $Z$ where $f$ and $1/f$ are regular. Define a ring homomorphism $\mu_f^\circ: R \to \mathbb{C}[M]$ by $\mu_f^\circ(\chi^u) = f^{u,v}$. Then from equations (1) and (2) above, $\mu_f^\circ$ is a $G$-equivariant ring homomorphism. Thus we obtain a $G$-equivariant morphism of varieties $\mu_f^\circ: V \to T_N = \text{Spec}(\mathbb{C}[M])$.

We will show that we can choose a rational function $f$ of $Z$ and $v \in N$ so that $\mu_f^\circ(Z) \not\subseteq \text{Fix}(X(\Delta), G)$.

Let $D$ be a $G$-invariant very ample divisor of $X(\Delta)$. Then

$$H^0(X(\Delta), \mathcal{O}(D)) = \bigoplus_{u \in P_D \cap M} \mathbb{C} \cdot \omega^u$$

as before. Put $h = \dim(H^0(X(\Delta), \mathcal{O}(D)))$, and $\{u_1 = 0, u_2, \ldots, u_h\} = P_D \cap M$. Put $g(i) = j$ when $(u_i)g = u_j$. Note again that $P_D$ is $G$-invariant.

Let $\Phi|D|$ be the morphism associated to the divisor $D$ and embed $X(\Delta)$ into $\mathbb{P}^{h-1}$. For $x \in X(\Delta)$, $\Phi|D|(x)$ is given by

$$\Phi|D|(x) = [\omega^0(x) : \omega^{u_2}(x) : \cdots : \omega^{u_h}(x)].$$

Restricting to $T_N$,

$$\Phi|D|_{T_N}(x) = [1 : \chi^{u_2}(x) : \cdots : \chi^{u_h}(x)]$$

since $\omega^0 \neq 0$ on $T_N$. 
Take $z \in Z$ so that $\{g, z \mid g \in G\}$ are distinct. Take $f \in \mathbb{C}(Z)$ so that $|f(z)| \neq 1, 0$ and $f(gz) = 1$ for $g \neq 1_G$. Let $V = \text{Spec}(R)$ be an affine $G$-invariant open set where $f, f^{-1}$ are regular. Take $v \in N$ so that $\{|u_i, v\} = c_i\}_{i=1, \ldots, n}$ are distinct. Then

$$\Phi|_{\partial} \circ \mu^v_j(z) = [1 : f(z)^{c_2} : \cdots : f(z)^{c_n}]$$

$$\Phi|_{\partial} \circ \mu^v_j(gz) = [1 : f(gz)^{c_2} : \cdots : f(gz)^{c_n}]$$

$$= [1 : f(z)^{c_{g2}} : \cdots : f(z)^{c_{gn}}]$$

and we can see that $\{\mu^v_j(gz)\}_{g \in G}$ are distinct so $\mu^v_j(z) \notin \text{Fix}(X(\Delta), G)$.

Thus we have proved Theorem 0.8.

2. Examples

Here we give some examples of versal $G$-coverings. Generally it is difficult to compute the quotient, but in some cases it is possible.

**Example 2.1** (Namba). We will restate Namba’s construction of versal $G$-coverings from our point of view. Let $G = \{g_1, \ldots, g_n\}$ be any finite group of order $n$. Let $N$ be a lattice of rank $n$ and let $\{e_{g_1}, \ldots, e_{g_n}\}$ be a basis of $N$. Then $G$ can be identified to a subgroup of $\text{Aut}(N)$. The action of $G$ on $N$ is defined by $g(e_{g_i}) = e_{g_{g_i}}$.

Let $\Delta$ be the complete fan of $N$ consisting of cones generated by $\{\pm e_{g_1}, \ldots, \pm e_{g_n}\}$. Then $\Delta$ is a complete projective $G$-invariant fan and $X(\Delta) \cong (\mathbb{P}^1)^n$. Then $\sigma : X(\Delta) \to X(\Delta)/G$ is a versal galois covering from Theorem 0.8. Thus a versal $G$-covering exists for any finite group.

**Example 2.2.** Let $N$ be a lattice of rank 2 and $\{e_1, e_2\}$ be a basis of $N$. Let $\Delta$ be the complete fan of $N$ generated by $v_1 = v_2 = e_2, v_3 = -e_1 + e_2, v_4 = -e_1, v_5 = -e_2, v_6 = e_1 - e_2$, as in the figure above. Thus $X(\Delta)$ is isomorphic to $\mathbb{P}^2$ blown-up along three points.
Let $G$ be the subgroup of $\text{Aut}(N)$ generated by

$$\alpha = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then $G = \langle \alpha, \beta \mid \alpha^6 = \beta^2 = (\alpha \beta)^2 = 1 \rangle \cong D_{12}$ where $D_{12}$ is the dihedral group of order 12. $\Delta$ is an invariant fan of $G$ and by Theorem 0.8, $\sigma : X(\Delta) \to X(\Delta)/G$ is a versal Galois covering. One can compute the quotient as the weighted projective space $\mathbb{P}(1,1,2)$. This is done by taking the very ample divisor $D = \sum_{i=1}^{6} D_{v_i}$ and compute the $D_{12}$-invariant ring of

$$\bigoplus_{i=1}^{\infty} H^0(X, O(iD)).$$

It is generated by algebraically independent elements of weight 1, 1, and 2.

**Proposition 2.3.** Example 2.2 gives a positive answer to Question 1.4 in [5].

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