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FEYNMAN–KAC FUNCTIONALS

MASAYOSHI TAKEDA and YOSHIHIRO TAWARA

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A LARGE DEVIATION PRINCIPLE FOR SYMMETRIC MARKOV PROCESSES NORMALIZED BY FEYNMAN–KAC FUNCTIONALS

MASAYOSHI TAKEDA and YOSHIHIRO TAWARA

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Abstract

We establish a large deviation principle for the occupation distribution of a symmetric Markov process normalized by Feynman–Kac functional. The obtained theorem means a large deviation from a ground state, not from an invariant measure.

1. Introduction

Let \( M = (\Omega, X, \mathbb{P}, \zeta) \) be an \( m \)-symmetric irreducible Markov process on a locally compact separable metric space \( X \). Here \( \zeta \) is the lifetime and \( m \) is a positive Radon measure with full support. Let \( (\mathcal{E}, \mathcal{D}(\mathcal{E})) \) be the Dirichlet form on \( L^2(X; m) \) generated by \( M \) (for the definition, see (2.1)). We denote by \( \mathcal{P} \) the set of probability measures with the weak topology, and for a positive Green-tight Kato measure \( \mu \) (Definition 2.1) define the function \( I^\mu \) on the set \( \mathcal{P} \) by

\[
I^\mu(v) = \begin{cases} 
\mathcal{E}(\sqrt{f} \cdot \sqrt{f}) & \text{if } v = f \cdot m, \sqrt{f} \in \mathcal{D}(\mathcal{E}), \\
\infty & \text{otherwise},
\end{cases}
\]

where \( \mathcal{E}^\mu = \mathcal{E} - (\cdot, \cdot)_\mu \). Given \( \omega \in \Omega \) with \( 0 < t < \zeta(\omega) \), let \( L_t(\omega) \in \mathcal{P} \) be the normalized occupation distribution: for a Borel set \( A \) of \( X \)

\[
L_t(\omega)(A) = \frac{1}{t} \int_0^t 1_A(X_s(\omega)) \, ds,
\]

where \( 1_A \) is the indicator function of the set \( A \). We denote by \( A^t_\mu \) the positive continuous additive functional with Revuz measure \( \mu \). One of authors proved Donsker–Varadhan type large deviation principle with rate function \( I^\mu \).

**Theorem 1.1** ([24]). Assume that the Markov process \( M \) possesses the strong Feller property and the tightness property (see (III) in Section 2).
(i) For each open set $G \subset \mathcal{P}$

$$
\liminf_{t \to \infty} \frac{1}{t} \log \mathbb{E}_x(e^{\lambda t}; L_t \in G, \ t < \zeta) \geq -\inf_{v \in G} I^\mu(v).
$$

(ii) For each closed set $K \subset \mathcal{P}$

$$
\limsup_{t \to \infty} \frac{1}{t} \log \sup_{x \in \mathcal{X}} \mathbb{E}_x(e^{\lambda t}; L_t \in K, \ t < \zeta) \leq -\inf_{v \in K} I^\mu(v).
$$

Varadhan [29] gave an abstract formulation for the large deviation principle. The statement in Theorem 1.1 is slightly different from his formulation. In fact, the rate function $I^\mu$ is not always non-negative because it is defined by the Schrödinger form $\mathcal{E}^\mu$, not by the Dirichlet form $\mathcal{E}$. Furthermore, Theorem 1.1 does not represent a large deviation from a invariant measure because the Markov process is allowed to be explosive. By this reason, we consider the normalized probability measure $Q_{x,t}$ on $\mathcal{P}$ defined by, for a Borel set $B \subset \mathcal{P}$,

$$
Q_{x,t}(B) = \frac{\mathbb{E}_x(e^{\lambda t}; L_t \in B, \ t < \zeta)}{\mathbb{E}_x(e^{\lambda t}; \ t < \zeta)},
$$

and prove that the family of probability measures $\{Q_{x,t}\}_{t \geq 0}$ obeys the large deviation principle as $t \to \infty$ in the sense of Varadhan’s formulation. In other words, $\{Q_{x,t}\}_{t \geq 0}$ satisfies the full large deviation principle with a good rate function in the sense of [11, Section 2.1]. This is the main theorem of this paper (Theorem 4.1). The rate function is given by

\begin{equation}
J(v) := I^\mu(v) - \lambda_2(\mu), \quad v \in \mathcal{P}.
\end{equation}

Here $\lambda_2(\mu)$ is the bottom of the spectrum of the Schrödinger type operator $\mathcal{L} + \mu$, where $\mathcal{L}$ is the generator of the Markov process:

$$
\lambda_2(\mu) = \inf\{\mathcal{E}^\mu(u, u); u \in \mathcal{D}(\mathcal{E}), \|u\|_2 = 1\}.
$$

To obtain the main theorem, we need to show that the rate function $J$ is good, that is, enjoys the properties (i)–(iv) in Lemma 4.1. In particular, we must show that $J$ has a unique zero point, that is, the existence of a ground state $\phi_0$ of the operator $\mathcal{L} + \mu$. In order to show the existence of a ground state, we usually use the $L^2$-weak compactness of the set $\{u \in \mathcal{D}(\mathcal{E}); \mathcal{E}^\mu(u, u) \leq l\}$ ($l \in \mathbb{R}$) and the lower semi-continuity of the Schrödinger form $\mathcal{E}^\mu$ with respect to the $L^2$-weak topology (e.g. [17]); however we can not derive these properties from our general setting. Hence we here use the following properties instead, the tightness of the level set $\{v \in \mathcal{P}; I^\mu(v) \leq l\}$ and the lower semi-continuity of the function $I^\mu$ with respect to the weak topology. This is a key to the proof of the
goodness of the rate function \( J \). We would like to emphasize that the tightness follows from the condition (III) and the Green-tightness of \( \mu \), and the lower semi-continuity of \( I^\mu \) follows from a variational formula for the Schrödinger form (Proposition 2.1), that is, the identification of the Schrödinger form with the modified \( I \)-function defined in (2.8). The latter is an extension of a well-known fact due to Donsker and Varadhan that for a symmetric Markov process, the \( I \)-function is identical with the Dirichlet form. On account of Lemma 4.1, we can regard the main theorem as a large deviation from the ground state of the Schrödinger operator.

In [16], they prove that if a Markov semigroup is intrinsically ultracontractive, then the resolvent by \( \varphi \) is the so-called Yaglom limit and a unique quasi-stationary distribution. In the last section, we will give an extension of this fact to generalized Feynman–Kac semigroups by employing Fukushima’s ergodic theorem.

2. Symmetric Markov processes with non-local Feynman–Kac functionals

Let \( X \) be a locally compact separable metric space and \( \mathcal{B}(X) \) the Borel \( \sigma \)-field. adjoining an extra point \( \infty \) to the measurable set \( (X, \mathcal{B}(X)) \), we set \( X_\infty = X \cup \{ \infty \} \) and \( \mathcal{B}(X_\infty) = \mathcal{B}(X) \cup \{ \{ \infty \} ; B \in \mathcal{B}(X) \} \). Let \( \mathbf{M} = (\Omega, \mathcal{F}, \mathbb{P}, \xi) \) be a right Markov process on \( X \) with lifetime \( \xi := \inf \{ t > 0 ; X_t = \infty \} \). We define the semigroup and the resolvent by

\[
p_t f(x) = \mathbb{E}_x(f(X_t)); \quad t < \xi, \quad \mathcal{R}_\beta f(x) = \int_0^\infty e^{-\beta t} p_t f(x) \, dt
\]

for a bounded Borel function \( f \) on \( X \). We assume that the Markov process \( \mathbf{M} \) is \( m \)-symmetric, \((p_t, g)_m = (f, p_t g)_m\), where \( m \) is a positive Radon measure with full support. Let \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) be the Dirichlet form on \( L^2(X; m) \) generated by \( \mathbf{M} \):

\[
\begin{cases}
\mathcal{D}(\mathcal{E}) = \left\{ u \in L^2(X; m) : \lim_{t \to 0} \frac{1}{t}(u - p_t u, u)_m < \infty \right\}, \\
\mathcal{E}(u, v) = \lim_{t \to 0} \frac{1}{t}(u - p_t u, v)_m.
\end{cases}
\]

(2.1)

For basic materials on right processes and associated Dirichlet forms (quasi-regular Dirichlet forms), we refer to [7], [18].

We impose three assumptions on \( \mathbf{M} \).

(1) (Irreducibility) If a Borel set \( A \) is \( p_t \)-invariant, i.e., \( p_t(1_A f)(x) = 1_A p_t f(x) \) \( m \)-a.e. for any \( f \in L^2(X; m) \cap \mathcal{B}_b(X) \) and \( t > 0 \), then \( A \) satisfies either \( m(A) = 0 \) or \( m(X \setminus A) = 0 \). Here \( \mathcal{B}_b(X) \) is the space of bounded Borel functions on \( X \).
(II) \textbf{(Strong Feller property)} For each $t$, $p_t(B_b(X)) \subseteq C_b(X)$, where $C_b(X)$ is the space of bounded continuous functions on $X$.

(III) \textbf{(Tightness)} For any $\epsilon > 0$, there exists a compact set $K$ such that

$$\sup_{x \in X} R_1 1_{K^c}(x) \leq \epsilon.$$ 

Here $1_{K^c}$ is the indicator function of the complement of the compact set $K$.

The assumption (II) implies that $\mathcal{M}$ satisfies the \textit{absolute continuity condition}, that is, its transition probability $p_t(x, \cdot)$ is absolutely continuous with respect to $m$ for each $t > 0$ and $x \in X$. As a result, the resolvent kernel is also absolutely continuous with respect to $m$, $R_\beta(x, dy) = R_\beta(x, y)m(dy)$. By [14, Lemma 4.2.4] the density $R_\beta(x, y)$ is assumed to be a non-negative Borel function such that $R_\beta(x, y)$ is symmetric and $\beta$-excessive in $x$ and in $y$. Under the absolute continuity condition, “quasi everywhere” statements are strengthened to “everywhere” ones. Moreover, we can defined notions without exceptional set, for example, smooth measures in the strict sense or positive continuous additive functional in the strict sense (cf. [14, Section 5.1]). Here we only treat the notions in the strict sense and omit the phrase “in the strict sense”.

We denote $S_{00}$ the set of positive Borel measures $\mu$ such that $\mu(X) < \infty$ and $R_1 \mu(x) \equiv \int X R_1(x, y)\mu(dy))$ is uniformly bounded in $x \in X$. A positive Borel measure $\mu$ on $X$ is said to be \textit{smooth} if there exists a sequence $\{E_n\}_{n=1}^\infty$ of Borel sets increasing to $X$ such that $1_{E_n} \cdot \mu \in S_{00}$ for each $n$ and

$$\mathbb{P}_x \left( \lim_{n \to \infty} \sigma_{X \setminus E_n} \geq \xi \right) = 1, \quad \forall x \in X,$$

where $\sigma_{X \setminus E_n}$ is the first hitting time of $X \setminus E_n$. The totality of smooth measures is denoted by $S_1$.

If an additive functional $\{A_t\}_{t \geq 0}$ is positive and continuous with respect to $t$ for each $\omega \in \Lambda$, it is said to be a \textit{positive continuous additive functional} (PCAF in abbreviation). By [14, Theorem 5.1.7], there exists a one-to-one correspondence between positive smooth measures and PCAF’s \textit{(Revuz correspondence)}: for each smooth measure $\mu$, there exists a unique PCAF $\{A_t\}_{t \geq 0}$ such that for any positive Borel function $f$ on $X$ and $\gamma$-excessive function $h$ ($\gamma \geq 0$), that is, $e^{-\gamma t} p_t h \leq h$,

$$\lim_{t \to 0} \frac{1}{t} \mathbb{E}_{h_{\mu}} \left( \int_0^t f(X_s) \, dA_s \right) = \int_X f(x) h(x) \mu(dx).$$

Here $\mathbb{E}_{h_{\mu}}(\cdot) = \int_X \mathbb{E}_x(\cdot) h(x)m(dx)$. We denote by $A_{t,\mu}$ the PCAF corresponding to the smooth measure $\mu$. For a signed Borel measure $\mu = \mu^+ - \mu^-$, let $|\mu| = \mu^+ + \mu^-$. When $|\mu|$ is a smooth measure, we define $A_{t,\mu} = A_{t,\mu^+} - A_{t,\mu^-}$ and $A_{t,|\mu|} = A_{t,\mu^+} + A_{t,\mu^-}$.

Following Chen [4], we introduce classes of potentials.
DEFINITION 2.1. (i) A signed Borel measure \( \mu \) is said to be the Kato measure (in notation, \( \mu \in \mathcal{K} \)), if \( |\mu| \in S_1 \) and
\[
\lim_{t \to 0} \sup_{x \in X} E_x(A_t^{[\mu]}) = 0.
\]
(ii) A measure \( \mu \in \mathcal{K} \) is said to be in the class \( \mathcal{K}_\infty \), if for any \( \epsilon > 0 \) there exist a compact subset \( K \) and a positive constant \( \delta > 0 \) such that for all measurable set \( B \subseteq K \) with \( |\mu|(B) < \delta \),
\[
\sup_{x \in X} \int_{K \cup B} R_1(x, y)|\mu|(dy) \leq \epsilon.
\]
(iii) A signed Borel measure \( \mu \) is said to be in the class \( \mathcal{S}_\infty \), if for any \( \epsilon > 0 \) there exist a compact subset \( K \) and a positive constant \( \delta > 0 \) such that for all measurable set \( B \subseteq K \) with \( |\mu|(B) < \delta \),
\[
\sup_{(x, z) \in X \times X \setminus d} \int_{K \cup B} \frac{R_1(x, y)R_1(y, z)}{R_1(x, z)}|\mu|(dy) \leq \epsilon.
\]
It is known in [2] that \( \mu \) belongs to \( \mathcal{K} \) if and only if
\[
(2.3) \quad \lim_{\beta \to \infty} \sup_{x \in X} \int_X R_\beta(x, y)|\mu|(dy) = 0,
\]
and in [4] that
\[
(2.4) \quad \mathcal{S}_\infty \subset \mathcal{K}_\infty \subset \mathcal{K}.
\]
We denote that \( (N, H) = (N(x, dy), H_t) \) is the Lévy system of \( \mathbf{M} \), that is, \( N \) is a kernel on \( (X_\infty, \mathcal{B}(X_\infty)) \) with \( N(x, \{x\}) = 0 \) and \( H \) is a positive continuous additive functional of \( \mathbf{M} \) such that for any non-negative measurable function \( F \) on \( X \times X \) vanishing on the diagonal set and any \( x \in X \),
\[
E_x \left( \sum_{0 < s \leq t} F(X_{s-}, X_s); t < \zeta \right) = E_x \left( \int_0^t \int_X F(X_s, y)N(X_s, dy) dH_s \right).
\]
We denote by \( \mu^H \) be the smooth measure corresponding to \( H_t \).

DEFINITION 2.2. Let \( F \) be a bounded measurable function on \( X \times X \) vanishing on the diagonal set.
(i) \( F \) is said to be in the class \( \mathcal{A}_\infty \), if for any \( \epsilon > 0 \) there exist a compact subset \( K \) and a positive constant \( \delta > 0 \) such that for all measurable set \( B \subseteq K \) with \( |\mu|(B) < \delta \),
\[
\sup_{(x, z) \in X \times X \setminus d} \int_{(K \setminus B) \times (K \setminus B)^c} \frac{R_1(x, y)|F(y, z)|R_1(z, w)}{R_1(x, w)}N(y, dz)\mu^H(dy) \leq \epsilon.
\]
(ii) $F$ is said to be in the class $\mathcal{A}_2$, if $F \in \mathcal{A}_\infty$ and

$$
\mu_{F}(dx) = \left( \int_X |F(x, y)| N(x, dy) \right) \mu^H(dx) \in \mathcal{S}_\infty.
$$

For properties and examples of $\mathcal{A}_\infty$ and $\mathcal{A}_2$, see [4], [5]. In the remainder of this paper, we assume that $F$ is symmetric, $F(x, y) = F(y, x)$. We write $\mu + F \in \mathcal{K}_\infty + \mathcal{A}_2$ if $\mu \in \mathcal{K}_\infty$ and $F \in \mathcal{A}_2$

For $\mu + F \in \mathcal{K}_\infty + \mathcal{A}_2$ define the AF $A^{\mu + F}_t$ by

$$
A^{\mu + F}_t = A^\mu_t + \sum_{0 < s \leq t} F(X_{s-}, X_s),
$$

and the generalized Feynman–Kac semigroup $\{p^{\mu + F}_t\}_{t \geq 0}$ by

$$
p^{\mu + F}_t f(x) = \mathbb{E}_x \left( e^{\int_0^t F(X_s) \mu^H(dx)} f(X_t); t < \zeta \right), \quad f \in \mathcal{B}_B(X).
$$

For $F \in \mathcal{A}_2$, we define the symmetric Dirichlet form ($\mathcal{E}_F, \mathcal{D}(\mathcal{E})$) as follows: for $u, v \in \mathcal{D}(\mathcal{E})$

$$
\mathcal{E}_F(u, v) = \mathcal{E}^{(c)}(u, v) + \mathcal{E}^{(k)}(u, v)
+ \frac{1}{2} \int_{X \times X} (u(x) - u(y))(v(x) - v(y)) e^{F(x, y)} N(x, dy) \mu^H(dx),
$$

where $\mathcal{E}^{(c)}$ and $\mathcal{E}^{(k)}$ are the local part and the killing part of the Dirichlet form ($\mathcal{E}, \mathcal{D}(\mathcal{E})$) in Beurling–Deny formula ([14, Theorem 3.2.1]). Fundamental properties of non-local Feynman–Kac transforms were earlier studied by J. Ying [31], [32]. It is known in [8] that $\{p^{\mu + F}_t\}_{t \geq 0}$ is the semigroup generated by the Schrödinger form ($\mathcal{E}^{\mu + F}, \mathcal{D}(\mathcal{E})$):

$$
\mathcal{E}^{\mu + F}(u, v) = \mathcal{E}_F(u, v) - \int_X u(x)v(x) d\mu_{F_1}(x) - \int_X u(x)v(y) d\mu(x),
$$

where $F_1 = \exp(F) - 1$. The form $\mathcal{E}^{\mu + F}$ is also written as

$$
\mathcal{E}^{\mu + F}(u, v) = \mathcal{E}(u, v) - \int_{X \times X} u(x)v(x)F_1(x, y) N(x, dy) d\mu^H(y)
- \int_X u(x)v(x) d\mu(x), \quad u, v \in \mathcal{D}(\mathcal{E}).
$$

Let $\mathcal{P}$ be the set of probability measures on $X$ equipped with the weak topology. We define the function $I^{\mu + F}$ on $\mathcal{P}$ by

$$
I^{\mu + F}(\nu) = \begin{cases} 
\mathcal{E}^{\mu + F}(\sqrt{f}, \sqrt{f}) & \text{if } \nu = f \cdot m, \sqrt{f} \in \mathcal{D}(\mathcal{E}), \\
\infty & \text{otherwise}.
\end{cases}
$$
Let $\mu + F \in \mathcal{K}_\infty + A_2$ and define $\kappa(\mu + F)$ by

$$
\kappa(\mu + F) = \lim_{t \to \infty} \frac{1}{t} \log \| p_t^{\mu + F} \|_{\infty, \infty}.
$$

We see from [1] that $\kappa(\mu + F)$ is finite. If $\alpha > \kappa(\mu + F)$ and $f \in \mathcal{B}_b(X)$, we define the resolvent $R_a^{\mu + F}$ by

$$
R_a^{\mu + F} f(x) = \mathbb{E}_x \left( \int_0^\infty e^{-\alpha t + A_t^{\mu + F}} f(X_t) \, dt \right).
$$

We set

$$
\mathcal{D}_+(\mathcal{H}^{\mu + F}) = \left\{ R_a^{\mu + F} f : \alpha > \kappa(\mu + F), \ f \in L^2(X; m) \cap C_b(X), \ f \geq 0 \mbox{ and } f \neq 0 \right\}.
$$

Each function $\phi = R_a^{\mu + F} f \in \mathcal{D}_+(\mathcal{H}^{\mu + F})$ is strictly positive because $\mathbb{P}_x(\sigma_O < \zeta) > 0$ for any $x \in X$ by the assumption (I). Here $O$ is a non-empty open set $\{ x \in X : f(x) > 0 \}$ and $\sigma_O = \inf\{ t > 0 : X_t \in O \}$. We define the generator $\mathcal{H}^{\mu + F}$ by

$$
\mathcal{H}^{\mu + F} u = \alpha u - f, \ u = R_a^{\mu + F} f \in \mathcal{D}_+(\mathcal{H}^{\mu + F}).
$$

Let $h$ be the function defined by $h(x) = \mathbb{E}_x(\exp(A_\zeta^{\mu + F}))$. We may assume that $\mu + F$ is gaugeable, that is, $\sup_{x \in X} h(x) < \infty$. In fact, it is enough to prove Theorem 2.1 and Theorem 4.1 below for the $\beta$-subprocess, $\mathbb{E}_x^{(\beta)} = e^{-\beta t} \mathbb{P}_x$. Moreover, we see that every $\mu + F \in \mathcal{K}_\infty + A_2$ becomes gaugeable with respect to the $\beta$-subprocess of $\mathbf{M}$ for a large enough $\beta$. In fact, we see from [5, Theorem 3.4] that $\mu + F \in \mathcal{K}_\infty + A_2$ is gaugeable with respect to the $\beta$-subprocess if and only if

$$
\inf \left\{ \mathcal{E}_F(u, u) + \int_X u(x)^2 (\mu^- + \mu_{F^-}) (dx) + \beta \int_X u(x)^2 m(dx) : \right. \\
\left. \int_X u(x)^2 (\mu^+ + \mu_{F^+}) (dx) = 1 \right\} > 1,
$$

(2.7)

where $F^+_1$ and $F^-_1$ is the positive and negative part of $F_1$. Since by (3.1)

$$
\mathcal{E}_F(u, u) + \int_X u(x)^2 (\mu^- + \mu_{F^-}) (dx) + \beta \int_X u(x)^2 m(dx) \\
\geq e^{-1F^-_1\infty} \left( \mathcal{E}(u, u) + \beta \int_X u(x)^2 m(dx) \right) \geq \frac{e^{-1F^-_1\infty}}{\| R_p(\mu^+ + \mu_{F^+}) \|_{\infty}},
$$

and the right hand side tends to $\infty$ as $\beta \to \infty$ because of $\mu^+ + \mu_{F^+_1} \in \mathcal{K}$, (2.7) holds for a large $\beta$. 

We define the function $I_h$ on $\mathcal{P}$ by
\begin{equation}
I_h(v) = -\inf_{\phi \in \mathcal{D}_+(H^{\mu+F})} \int_X H^{\mu+F} \phi d\nu.
\end{equation}

The gauge function $h(x)$ satisfies $0 < c \leq h(x) \leq C < \infty$. Indeed, it follows from Proposition 2.2 in [4] and (2.4) that for $\mu \in \mathcal{K}_\infty$ and $F \in A_2$, $\sup_{x \in X} \mathbb{E}_x(A_{\xi}^{[\mu|\nu|F]}_c) < \infty$. Hence, by Jensen’s inequality,
\[
\inf_{x \in X} \mathbb{E}_x(\exp(A^{\mu+F}_{\xi})) \geq \exp \left( -\sup_{x \in X} \mathbb{E}_x(A^{\mu+F}_{\xi}) \right) > 0.
\]

Let us define the function $I_\alpha$ on $\mathcal{P}$ by
\[
I_\alpha(v) = -\inf_{\phi \in \mathcal{D}_+(H^{\mu+F})} \int_X \log \left( \frac{\alpha R^{\mu+F}_\alpha u + \epsilon h}{u + \epsilon h} \right) d\nu.
\]

**Lemma 2.1.** It holds that
\[
I_\alpha(v) \leq \frac{I_h(v)}{\alpha}, \quad v \in \mathcal{P}.
\]

**Proof.** For $u = R^{\mu+F}_\alpha f \in \mathcal{D}_+(H^{\mu+F})$ and $\epsilon > 0$, set
\[
\phi(\alpha) = -\int_X \log \left( \frac{\alpha R^{\mu+F}_\alpha u + \epsilon h}{u + \epsilon h} \right) d\nu.
\]

Then, noting that $(d/d\alpha)(R^{\mu+F}_\alpha u) = -R^{\mu+F}_\alpha (R^{\mu+F}_\alpha u) = -(R^{\mu+F}_\alpha)^2 u$, we have
\[
\frac{d\phi}{d\alpha} = -\int_X \frac{R^{\mu+F}_\alpha u - \alpha (R^{\mu+F}_\alpha)^2 u}{\alpha R^{\mu+F}_\alpha u + \epsilon h} d\nu = \int_X \frac{H^{\mu+F}_\alpha (R^{\mu+F}_\alpha)^2 u}{\alpha R^{\mu+F}_\alpha u + \epsilon h} d\nu.
\]

Since
\[
(\alpha (R^{\mu+F}_\alpha)^2 u - R^{\mu+F}_\alpha u)(\alpha (R^{\mu+F}_\alpha)^2 u + \epsilon h) - (\alpha (R^{\mu+F}_\alpha)^2 u - R^{\mu+F}_\alpha u)(\alpha R^{\mu+F}_\alpha u + \epsilon h)
\]
equals $\alpha (\alpha (R^{\mu+F}_\alpha)^2 u - R^{\mu+F}_\alpha u)^2 \geq 0$, we have
\[
\frac{\alpha (R^{\mu+F}_\alpha)^2 u - R^{\mu+F}_\alpha u}{\alpha R^{\mu+F}_\alpha u + \epsilon h} \geq \frac{\alpha (R^{\mu+F}_\alpha)^2 u - R^{\mu+F}_\alpha u}{\alpha^2 (R^{\mu+F}_\alpha)^2 u + \epsilon h}.
\]
and thus
\[
\int_X \frac{\mathcal{H}^{\mu+F}(R_{\alpha}^{\mu+F})^2 u}{\alpha R_{\alpha}^{\mu+F} u + \epsilon h} \, dv \geq \frac{1}{\alpha^2} \left( - \int_X \frac{\mathcal{H}^{\mu+F}(R_{\alpha}^{\mu+F})^2 u}{(R_{\alpha}^{\mu+F})^2 u + \epsilon h / \alpha^2} \, dv \right)
\]
\[
\geq - \frac{1}{\alpha^2} I_h(v).
\]
Therefore
\[
\phi(\infty) - \phi(\alpha) = \int_X \log \left( \frac{\alpha R_{\alpha}^{\mu+F} u + \epsilon h}{u + \epsilon h} \right) \, dv \geq - \frac{I_h(v)}{\alpha},
\]
which implies
\[
- \inf_{u \in D_{\alpha}(\mathcal{H}^r)} \int_X \log \left( \frac{\alpha R_{\alpha}^{\mu+F} u + \epsilon h}{u + \epsilon h} \right) \, dv \leq \frac{I_h(v)}{\alpha}.
\]
Since \( \| \beta R_{\beta}^{\mu+F} f \|_\infty \leq C \| f \|_\infty \), \( \beta > 0 \), and \( \beta R_{\beta}^{\mu+F} f(x) \to f(x) \) as \( \beta \to \infty \),
\[
(2.9) \quad \int_X \log \left( \frac{\alpha R_{\alpha}^{\mu+F} (\beta R_{\beta}^{\mu+F} f) + \epsilon h}{\beta R_{\beta}^{\mu+F} f + \epsilon h} \right) \, dv \xrightarrow{\beta \to \infty} \int_X \log \left( \frac{\alpha R_{\alpha}^{\mu+F} f + \epsilon h}{f + \epsilon h} \right) \, dv.
\]
Define the measure \( \nu_\alpha \) by
\[
\nu_\alpha(A) = \int_X \alpha R_{\alpha}^{\mu+F} (x, A) \, dv(x), \quad A \in \mathcal{B}(X).
\]
Given \( v \in B_{\beta}^+(X) \), take a sequence \( \{ g_n \}_{n=1}^{\infty} \subset C_{\beta}^+(X) \cap L^2(X; m) \) such that
\[
\int_X |v - g_n| \, d(\nu_\alpha + v) \to 0 \quad \text{as} \quad n \to \infty.
\]
We then have
\[
\int_X |\alpha R_{\alpha}^{\mu+F} v - \alpha R_{\alpha}^{\mu+F} g_n| \, dv \leq \int_X \alpha R_{\alpha}^{\mu+F} (|v - g_n|) \, dv = \int_X |v - g_n| \, d\nu_\alpha \to 0
\]
as \( n \to \infty \), and so
\[
(2.10) \quad \int_X \log \left( \frac{\alpha R_{\alpha}^{\mu+F} g_n + \epsilon h}{g_n + \epsilon h} \right) \, dv \xrightarrow{n \to \infty} \int_X \log \left( \frac{\alpha R_{\alpha}^{\mu+F} v + \epsilon h}{v + \epsilon h} \right) \, dv.
\]
Hence, combining (2.9) and (2.10)

\[
\inf_{u \in D_+(H^{F})} \int_X \log \left( \frac{\alpha R^\mu_{u} F u + \epsilon h}{u + \epsilon h} \right) \, dv = \inf_{u \in B^+_U(X)} \int_X \log \left( \frac{\alpha R^\mu_{u} F u + \epsilon h}{u + \epsilon h} \right) \, dv,
\]

which implies the lemma.

\[\square\]

**Lemma 2.2.** If \( I_h(v) < \infty \), then \( v \) is absolutely continuous with respect to \( m \).

*Proof.* By a similar argument in the proof of [12, Lemma 4.1], we obtain this lemma. Indeed, for \( a > 0 \) and \( A \in B(X) \), set \( u(x) = a1_A(x) + 1 \in B^+_U(X) \). Then

\[
\int_X \log \left( \frac{\alpha R^\mu_{a} + F u + \epsilon h}{u + \epsilon h} \right) \, dv = \int_X \log \left( \frac{a\alpha R^\mu_{a} + F (x, A) + \alpha R^\mu_{a} + F (x, X) + \epsilon h}{a1_A(x) + 1 + \epsilon h} \right) \, dv.
\]

Define the measure \( \nu_\alpha \) as in the proof of Lemma 2.1. Put

\[c_\alpha = \int_X \alpha R^\mu_{a} + F (x, X) \, dv(x) \quad (= \nu_\alpha(X)).\]

We see from Lemma 2.1 and Jensen’s inequality that

\[\log(a \nu_\alpha(A) + c_\alpha + \epsilon h) \geq \nu(A) \log(a + 1 + \epsilon h) + \nu(A^c)(1 + \epsilon h) - \frac{I_h(v)}{\alpha},\]

and by letting \( \epsilon \to 0 \)

\[\log(a \nu_\alpha(A) + c_\alpha) \geq \nu(A) \log(a + 1) - \frac{I_h(v)}{\alpha}.\]

Since \( \log x \leq x - 1 \) for \( x > 0 \), we have

\[a \nu_\alpha(A) + c_\alpha - 1 \geq \nu(A) \log(a + 1) - \frac{I_h(v)}{\alpha},\]

and so

\[\nu_\alpha(A) - \nu(A) \geq \frac{-I_h(v)/\alpha + \nu(A)(\log(a + 1) - a) + 1 - c_\alpha}{a}.
\]

Noting that \( \log(a + 1) - a < 0 \), we have

\[\nu_\alpha(A) - \nu(A) \geq \frac{-I_h(v)/\alpha + (\log(a + 1) - a) + 1 - c_\alpha}{a}\]

for all \( A \in B(X) \) and

\[\nu(A) - \nu_\alpha(A) = 1 - c_\alpha + (\nu_\alpha(A^c) - \nu(A^c)) \geq \frac{-I_h(v)/\alpha + (\log(a + 1) - a) + (1 - c_\alpha)(a + 1)}{a}\]
for all \( A \in \mathcal{B}(X) \). Therefore we can conclude that

\[
\sup_{A \in \mathcal{B}(X)} |\nu(A) - \nu_\alpha(A)| \leq \frac{a - \log(a + 1) + I_h(v)/\alpha + (1 - c_\alpha)(a + 1)}{a}.
\]

Note that \( c_\alpha \to 1 \) as \( \alpha \to \infty \). Then since

\[
\limsup_{\alpha \to \infty} \sup_{A \in \mathcal{B}(X)} |\nu(A) - \nu_\alpha(A)| \leq \frac{a - \log(a + 1)}{a}
\]

and the right-hand side converges to 0 as \( a \to 0 \), the lemma follows.

\[ \square \]

**Proposition 2.1.** It holds that for \( v \in \mathcal{P} \)

\[
I_h(v) = I^{\mu + F}(v).
\]

Proof. We follow the argument of the proof of [12, Theorem 5]. Suppose that \( I_h(v) = l < \infty \). By Lemma 2.2, \( \nu \) is absolutely continuous with respect to \( m \). Let us denote by \( f \) its density and let \( f^n = \sqrt{f} \land n \). Since \( \log(1 - x) \leq -x \) for \( -\infty < x < 1 \) and

\[
-\infty < \frac{f^n - \alpha R_a^{\mu + F} f^n}{f^n + \epsilon h} < 1,
\]

we have

\[
\int_X \log \left( \frac{\alpha R_a^{\mu + F} f^n + \epsilon h}{f^n + \epsilon h} \right) f \, dm = \int_X \log \left( 1 - \frac{f^n - \alpha R_a^{\mu + F} f^n}{f^n + \epsilon h} \right) f \, dm 
\]

\[
\leq - \int_X \frac{f^n - \alpha R_a^{\mu + F} f^n}{f^n + \epsilon h} f \, dm,
\]

and then

\[
\int_X \frac{f^n - \alpha R_a^{\mu + F} f^n}{f^n + \epsilon h} f \, dm \leq I_a(f \cdot m).
\]

By letting \( n \to \infty \) and \( \epsilon \to 0 \), we have

\[
\int_X \sqrt{f} \left( \sqrt{f} - \alpha R_a^{\mu + F} \sqrt{f} \right) dm \leq I_a(f \cdot m) \leq \frac{I_h(f \cdot m)}{\alpha},
\]

which implies that \( \sqrt{f} \in \mathcal{D}(\mathcal{E}) \) and \( \mathcal{E}_a^{\mu + F} (\sqrt{f}, \sqrt{f}) \leq I_h(f \cdot m) \).

Let \( \phi \in \mathcal{D}_+(\mathcal{H}^{\mu + F}) \) and define the semigroup \( P_t^\phi \) by

\[
P_t^\phi f(x) = \mathbb{E}_x \left( e^{\Lambda_{t-\epsilon}^\phi} (\phi + \epsilon h)(X_t) (\phi + \epsilon h)(X_0) \exp \left( - \int_0^t \mathcal{H}^{\mu + F}_s \phi(X_s) ds \right) f(X_t) \right).
\]
Then, $P^\phi_t$ is $(\phi + \epsilon h)^2m$-symmetric and satisfies $P^\phi_t 1 \leq 1$. Given $v = f \cdot m \in \mathcal{P}$ with $\sqrt{f} \in \mathcal{D}(\mathcal{E})$, set

$$S^\phi_t \sqrt{f}(x) = E_x \left( e^{A^{\mu + F}_t} \exp \left( - \int_0^t \frac{\mathcal{H}^{\mu + F}_t \phi}{\phi + \epsilon h} (X_s) \, ds \right) \sqrt{f}(X_t) \right).$$

Then

$$\int_X (S^\phi_t \sqrt{f})^2 \, dm = \int_X (\phi + \epsilon h)^2 \left( P^\phi_t \left( \frac{\sqrt{f}}{\phi + \epsilon h} \right) \right)^2 \, dm \leq \int_X (\phi + \epsilon h)^2 P^\phi_t \left( \left( \frac{\sqrt{f}}{\phi + \epsilon h} \right)^2 \right) \, dm \leq \int_X (\phi + \epsilon h)^2 \left( \frac{\sqrt{f}}{\phi + \epsilon h} \right)^2 \, dm.$$

Hence

$$0 \leq \lim_{t \to 0} \frac{1}{t} (\sqrt{f} - S^\phi_t \sqrt{f}, \sqrt{f})_m = \mathcal{E}^{\mu + F}(\sqrt{f}, \sqrt{f}) + \int_X \frac{\mathcal{H}^{\mu + F}_t \phi}{\phi + \epsilon h} f \, dm,$$

and thus $\mathcal{E}^{\mu + F}(\sqrt{f}, \sqrt{f}) \geq I_h(f \cdot m).$ \hfill \Box

We now obtain a generalization of Theorem 1.1 in exactly the same way as the proof of it (cf. [10], [28]):

**Theorem 2.1** ([24]). Assume (I), (II) and (III). Suppose that $\mu + F \in \mathcal{K}_\infty + \mathcal{A}_2$.

(i) For each open set $G \subset \mathcal{P}$

$$\liminf_{t \to \infty} \frac{1}{t} \log E_x \left( e^{A^{\mu + F}_t} ; L_t \in G, t < \zeta \right) \geq - \inf_{v \in G} I^{\mu + F}(v).$$

(ii) For each closed set $K \subset \mathcal{P}$

$$\limsup_{t \to \infty} \frac{1}{t} \log \sup_{x \in X} E_x \left( e^{A^{\mu + F}_t} ; L_t \in K, t < \zeta \right) \leq - \inf_{v \in K} I^{\mu + F}(v).$$

3. The existence of ground states

We first recall an inequality ([19]): for $\mu \in \mathcal{K}$,

$$\int_X \tilde{u}^2 \, d\mu \leq \| R_\alpha \mu \|_{\infty}(\mathcal{E}(u, u) + \alpha(u, u)_m), \quad u \in \mathcal{D}(\mathcal{E}).$$
Let $\lambda_2(\mu + F)$ be the bottom of the spectrum of $\mathcal{H}^{\mu + F}$:

\begin{equation}
(3.2) \quad \lambda_2(\mu + F) = \inf \{ \mathcal{E}^{\mu + F}(u, u); u \in \mathcal{D}(\mathcal{E}), \|u\|_2 = 1 \}.
\end{equation}

**Proposition 3.1.** Assume (I), (II) and (III). There exists a unique ground state $\phi_0 \in \mathcal{D}(\mathcal{E})$: $\lambda_2(\mu + F) = \mathcal{E}^{\mu + F}(\phi_0, \phi_0)$.

Proof. Let $\{u_n\}$ be a minimizing sequence of the right-hand side of (3.2), i.e., $\|u_n\|_2 = 1$ and $\lambda_2(\mu + F) = \lim_{n \to \infty} \mathcal{E}^{\mu + F}(u_n, u_n)$. Put $\mu' = |\mu| + |\mu_F|$. Since $\mathcal{E}(u_n, u_n) \leq c \cdot \mathcal{E}(u_n, u_n)$ ($c = \exp(-\|F\|_\infty)$) and $\int_X u_n^2 d\mu' \leq \|R_{\alpha} \mu'\|_\infty \cdot (\mathcal{E}(u_n, u_n) + \alpha)$,

$\mathcal{E}^{\mu + F}(u_n, u_n) = \mathcal{E}_F(u, u) - \int_X u_n^2 d\mu'$

$\geq \frac{1}{c} \mathcal{E}(u_n, u_n) - \|R_{\alpha} \mu'\|_\infty (\mathcal{E}(u_n, u_n) + \alpha)$

$= \left( \frac{1}{c} - \|R_{\alpha} \mu'\|_\infty \right) \mathcal{E}(u_n, u_n) - \alpha \|R_{\alpha} \mu'\|_\infty.$

Taking $\alpha$ large enough so that $c \|R_{\alpha} \mu'\|_\infty < 1$ on account of (2.3), we have

$$\sup \mathcal{E}(u_n, u_n) \leq \frac{c(\sup \mathcal{E}^{\mu + F}(u_n, u_n) + \alpha \|R_{\alpha} \mu'\|_\infty)}{1 - c \|R_{\alpha} \mu'\|_\infty} < \infty.$$  

We see from the assumption (III) that for any $\epsilon > 0$ there exists a compact set $K$ such that

$$\sup \int_K u_n^2 dm \leq \|R_1 1_K\|_\infty \cdot \left( \sup \mathcal{E}(u_n, u_n) + 1 \right) < \epsilon.$$  

As a result, the subset $\{u_n^2 \cdot m\}$ of $\mathcal{P}$ is tight. Hence there exists a subsequence $u_{n_k} \cdot m$ which converges to a probability measure $\nu$ weakly. Since the function $I^{\mu + F}$ is lower semi-continuous by Proposition 2.1,

$$I^{\mu + F}(\nu) \leq \lim \inf_{k \to \infty} I^{\mu + F}(u_{n_k}^2 \cdot m) = \lim \inf_{k \to \infty} \mathcal{E}^{\mu + F}(u_{n_k}, u_{n_k}) < \infty.$$  

Therefore $\nu$ can be written as $\nu = \phi_0^2 m$, $\phi_0 \in \mathcal{D}(\mathcal{E})$ by Proposition 2.1 and $\lambda_2(\mu + F) = \mathcal{E}^{\mu + F}(\phi_0, \phi_0)$, that is, $\phi_0$ is the ground state. The uniqueness of the ground state follows from the irreducibility (I) (e.g. [9, Proposition 1.4.3]).

We also know from the proof above that the level set $\{ \nu \in \mathcal{P}; I^{\nu + F}(\nu) \leq l \}$ is compact.
4. Large deviations from ground states

Given $\omega \in \Omega$ with $0 < t < \zeta(\omega)$, we define the occupation distribution $L_t(\omega) \in \mathcal{P}$ by

$$L_t(\omega)(A) = \frac{1}{t} \int_0^t 1_A(X_s(\omega)) \, ds$$

for a Borel set $A$ of $X$, where $1_A$ is the indicator function of the set $A$.

Define the probability measure $Q_{x,t}$ on $\mathcal{P}$ by

$$(4.1) \quad Q_{x,t}(B) = \frac{\mathbb{E}_x(e^{A^{x,F}}; L_t \in B, t < \zeta)}{\mathbb{E}_x(e^{A^{x,F}}; t < \zeta)}, \quad B \in \mathcal{B}(\mathcal{P}).$$

We define the function $J$ on $\mathcal{P}$ by

$$(4.2) \quad J(v) = I^{\mu+F}(v) - \lambda_2(\mu + F).$$

We then have the next lemma by Proposition 2.1 and Proposition 3.1.

**Lemma 4.1.** The function $J$ satisfies:

(i) $0 \leq J(v) \leq \infty$.

(ii) $J$ is lower semicontinuous.

(iii) For each $l < \infty$, the set $\{v \in \mathcal{P} : J(v) \leq l\}$ is compact.

(iv) $J(\phi_0^2 \cdot m) = 0$ and $J(v) > 0$ for $v \neq \phi_0^2 \cdot m$.

**Remark 4.1.** Let $(\mathcal{E}^{\phi_0}, \mathcal{D}(\mathcal{E}^{\phi_0}))$ the bilinear form on $L^2(X; \phi_0^2 m)$ defined by

$$\begin{cases}
\mathcal{E}^{\phi_0}(u, v) = \mathcal{E}^{\mu+F}(u\phi_0, u\phi_0) - \lambda_2(\mu + F)(u\phi_0, u\phi_0)_m, \\
\mathcal{D}(\mathcal{E}^{\phi_0}) = \{u \in L^2(X; \phi_0^2 m) : u\phi_0 \in \mathcal{D}(\mathcal{E})\}.
\end{cases}$$

We then see that $(\mathcal{E}^{\phi_0}, \mathcal{D}(\mathcal{E}^{\phi_0}))$ is a Dirichlet form and $\mathcal{E}^{\phi_0}$ is expressed by

$$\mathcal{E}^{\phi_0}(u, v) = \int_X \phi_0^2 d\mu^{c}_{[u,v]} + \int_{X \times X \setminus \Delta} (u(x) - u(y))(v(x) - v(y))\phi_0(x)\phi_0(y) J(dx, dy).$$

Here $\mu^{c}_{[u,v]}$ is the local part of energy measure ([6]). We then see that

$$J(v) = I_{\mathcal{E}^{\phi_0}}(v),$$

where $I_{\mathcal{E}^{\phi_0}}$ is defined by

$$(4.3) \quad I_{\mathcal{E}^{\phi_0}}(v) = \begin{cases}
\mathcal{E}^{\phi_0}(\sqrt{\mathcal{I}}, \sqrt{\mathcal{I}}) & \text{if } v = f \cdot \phi_0^2 m, \sqrt{\mathcal{I}} \in \mathcal{D}(\mathcal{E}^{\phi_0}), \\
\infty & \text{otherwise}.
\end{cases}$$
We then have the main theorem:

**Theorem 4.1.** Assume (I), (II) and (III), let \( Q_{x,t} \) be a family of probability measures defined in (4.1). Then \( Q_{x,t} \) obeys a large deviation principle with rate function \( J \):

1. For each open set \( G \subset \mathcal{P} \)
   \[
   \liminf_{t \to \infty} \frac{1}{t} \log Q_{x,t}(G) \geq - \inf_{v \in G} J(v).
   \]

2. For each closed set \( K \subset \mathcal{P} \)
   \[
   \limsup_{t \to \infty} \frac{1}{t} \log Q_{x,t}(K) \leq - \inf_{v \in K} J(v).
   \]

**Corollary 4.1.** The measure \( Q_{x,t} \) converges to \( \delta_{\phi_0^2 \cdot m} \) weakly.

Proof. If a closed set \( K \) does not contain \( \phi_0^2 \cdot m \), then \( \inf_{x \in K} J(x) > 0 \) by Lemma 4.1 (iv). Hence Theorem 4.1 (ii) says that \( \lim_{t \to \infty} Q_{x,t}(K) = 0 \) and \( \lim_{t \to \infty} Q_{x,t}(K^c) = 1 \). For a positive constant \( \delta \) and a bounded continuous function \( f \) on the set of \( \mathcal{P} \), define the closed set \( K \subset \mathcal{P} \) by \( K = \{ v \in \mathcal{P} : |f(v) - f(\phi_0^2 \cdot m)| \geq \delta \} \). Then we have

\[
\left| \int_{\mathcal{P}} f(v)Q_{x,t}(dv) - f(\phi_0^2 \cdot m) \right| \leq \int_{\mathcal{P}} |f(v) - f(\phi_0^2 \cdot m)| Q_{x,t}(dv)
= \int_K |f(v) - f(\phi_0^2 \cdot m)| Q_{x,t}(dv) + \int_{K^c} |f(v) - f(\phi_0^2 \cdot m)| Q_{x,t}(dv)
\leq \delta Q_{x,t}(K^c) + 2 \| f \|_\infty Q_{x,t}(K) \to \delta
\]

as \( t \to \infty \). Since \( \delta \) is arbitrary, the weak convergence follows.

On account of Corollary 4.1, we can regard Theorem 4.1 as a genuine large deviation principle from the ground state.

5. **Quasistationary distribution**

In this section, we consider the existence of quasi-stationary distributions as an application of the existence of ground states. We continue with the setting of the preceding section.

Define the semigroup \( \{ p^\phi_t \}_{t \geq 0} \) on \( L^2(X; \phi_0^2 m) \) generated by \( (E^\phi, D(E^\phi)) \), that is

\[
p^\phi_t f(x) = e^{\frac{1}{2}t(\mu + F)} \frac{1}{\phi_0(x)} E_x \left( e^{\frac{1}{2}A^\phi_t} \phi_0(X_t) f(X_t) \right).
\]

(5.1)
Let \( M^{\phi_0} = (\Omega, X, \mathbb{E}_t^{\phi_0}) \) be the \( \phi_0^2 m \)-symmetric Markov process generated by the Markov semigroup \( p_t^{\phi_0} \) in (5.1).

Set
\[
\mathcal{P}_0 = \left\{ \nu \in \mathcal{P} : \int_X \sqrt{p_1^{\mu+F}(x, x)} \, d\nu(x) < \infty, \quad \int_X \phi_0(x) \, d\nu(x) < \infty \right\}.
\]

We then have

**Theorem 5.1.** Assume that \( m(X) < \infty \). Then for \( \nu \in \mathcal{P}_0 \) and \( B \in \mathcal{B}(X) \)
\[
\lim_{t \to \infty} e^{\lambda t (\mu+F) x} \mathbb{E}_t^{\phi_0}(e^{A_t^{\mu+F}} ; X_t \in B) = \int_X \phi_0 \, d\nu \int_B \phi_0 \, dm.
\]

**Proof.** Note that
\[
e^{\lambda t (\mu+F) x} \mathbb{E}_t^{\phi_0}(e^{A_t^{\mu+F}} ; X_t \in B) = \int_X \phi_0(x) \mathbb{E}_t^{\phi_0} \left( \frac{1_B}{\phi_0}(X_t) \right) \, d\nu(x).
\]

Let \( \{ E_\lambda, 0 \leq \lambda < \infty \} \) be the spectral family of \((\mathcal{E}^{\phi_0}, \mathcal{F}^{\phi_0})\). Then \( \lim_{t \to \infty} p_t^{\phi_0} f = E_0 f \) in \( L^2(X; \phi_0^2 m) \). Since \( \mathbb{E}^{\phi_0}(E_0 f, E_0 f) = 0 \), \( E_0 f \) equals \( \int_X f \phi_0^2 \, dm \), m.a.e. by the irreducibility of \((\mathcal{E}^{\phi_0}, \mathcal{F}^{\phi_0})\) (cf. [7, Theorem 5.2.13]). Note that \( p_t^{\phi_0}(x, \cdot) \in L^2(X; \phi_0^2 m) \) because \( \int_X p_t^{\phi_0}(x, y)^2 \phi_0^2(y) \, dm(y) = p_{2t}^{\phi_0}(x, x) < \infty \). Put \( c = \int_B \phi_0 \, dm \). We then have

\[
\left| \int_X \phi_0(x) \mathbb{E}_t^{\phi_0} \left( \frac{1_B}{\phi_0}(X_t) \right) \, d\nu(x) - \int_X \phi_0 \, d\nu \int_B \phi_0 \, dm \right| \\
= \left| \int_X \phi_0(x) \left( \frac{1_B}{\phi_0}(X_t) \right) - \frac{1_B}{\phi_0}(X_t - \frac{1}{2}) \right) \phi_0(y)^2 \, dm(y) \right| \, d\nu(x).
\]

The right-hand side is dominated by

\[
\int_X \phi_0(x) \sqrt{\int_X p_{t/2}^{\phi_0}(x, y)^2 \phi_0^2(y) \, dm(y)} \, d\nu(x) \cdot \sqrt{\int_X \left( \frac{1_B}{\phi_0}(X_t - \frac{1}{2}) \right) - c} \phi_0^2(y) \, dm(y).
\]

Since
\[
p_t^{\phi_0}(x, y) = e^{\lambda t (\mu+F) x} \frac{p_t^{\mu+F}(x, y)}{\phi_0(x) \phi_0(y)},
\]
the first factor is equal to

\[
\int_X \phi_0(x) \sqrt{p_1^{\phi_0}(x, x)} \, d\nu(x) = e^{(1/2) \lambda (\mu+F)} \int_X \sqrt{p_1^{\mu+F}(x, x)} \, d\nu(x)
\]
and is finite by the assumption that \( \nu \in \mathcal{P}_0 \). Hence the right-hand side of (5.2) converges to zero as \( t \to \infty \) because \( 1_B/\phi_0 \in L^2(X; \phi_0^2 m) \). \( \square \)
Let $\eta$ and $R_{v, t}$ be probability measures on $X$ defined by

$$
(5.3) \quad \eta(B) = \frac{\int_B \phi_0(x) \, dm(x)}{\int_X \phi_0(x) \, dm(x)} \quad \text{and} \quad R_{v, t}(B) = \frac{E_v(\mathbf{e}^{A_t \phi}; X_t \in B)}{E_v(\mathbf{e}^{A_t \phi}; t < \xi)} \quad \text{for} \quad B \in \mathcal{B}(X).
$$

**Corollary 5.1.** For $v \in \mathcal{P}_0$ and $B \in \mathcal{B}(X)$

$$
(5.4) \quad \lim_{t \to \infty} R_{v, t}(B) = \eta(B).
$$

Note that the Dirac measure $\delta_x$ belongs to $\mathcal{P}_0$ and so the distribution $R_{v, t}$ converges to $\eta$ for all $x \in X$. Hence Corollary 5.1 says that the semigroup $\{p_t^\phi\}_{t \geq 0}$ is conditionally ergodic and $\eta$ is a quasi-stationary distribution of the semigroup $\{p_t^\phi\}_{t \geq 0}$: for any $t > 0$

$$
(5.5) \quad R_{v, t} = \eta
$$

(e.g. [16]). If the semigroup $\{p_t^\phi\}_{t \geq 0}$ is ultracontractive, $p_t^\phi(x, y) \leq c_t$, then $p_t^\phi(x, x)$ and $\phi_0(x)$ are bounded and $\mathcal{P}_0$ equals $\mathcal{P}$. Consequently, for any $v \in \mathcal{P}$, the distribution $R_{v, t}$ converges to $\eta$.

When the measure $m$ is not finite, we assume the intrinsic ultracontractivity of $\{p_t^\phi\}_{t \geq 0}$, that is,

$$
(5.6) \quad p_t^\phi(x, y) \leq C_t \phi_0(x) \phi_0(y).
$$

In [16], they proved that for a (not necessary symmetric) Markov process, the intrinsic ultracontractivity is a sufficient condition for the measure $\eta$ being a unique quasi-stationary distribution, and the equation (5.4) holds for any initial distribution. We would like to give another proof of this fact by using the next theorem due to Fukushima [13].

**Theorem 5.2.** Assume that $m(X) < \infty$ and $\mathbf{M}$ is conservative, $p_t 1 = 1$, $t > 0$. Then for $f \in L^1(X; m)$,

$$
\lim_{t \to \infty} p_t f(x) = \frac{1}{m(X)} \int_X f(x) \, dm(x), \quad m\text{-a.e. and in } L^1(X; m).
$$

Note that $\mathbf{M}^\phi_0$ satisfies the assumptions in Theorem 5.2.

**Theorem 5.3.** Assume that $\{p_t^\phi\}_{t \geq 0}$ is intrinsically ultracontractive. Then for any $v \in \mathcal{P}$ and any $B \in \mathcal{B}(X)$

$$
\lim_{t \to \infty} e^{\lambda t} \mathbf{E}_v(\mathbf{e}^{A_t \phi}; X_t \in B) = \int \phi_0 \, dv \int \phi_0 \, dm.
$$
Consequently, the equation (5.4) follows.

Proof. First note that the upper bound (5.6) implies the lower bound ([9, Theorem 4.2.5]):

\[ c_t \phi_0(x) \phi_0(y) \leq p_t^{\mu+F}(x, y). \]

As a result,
\[
\sup_{x \in X} \phi_0(x) \int_X \phi_0(y) \, dm(y) \leq \frac{1}{c_t} \| p_t^{\mu+F} \|_\infty < \infty.
\]

Hence \( \phi_0 \) belongs to \( L^1(X; m) \cap L^\infty(X; m) \) and \( 1_B / \phi_0 \in L^1(X; \phi_0^2 m) \). Applying Theorem 5.2 to \( M \phi_0 \), we have
\[
\mathbb{E}_y^{\phi_0} \left( \frac{1_B}{\phi_0}(X_t) \right) \to \int_B \phi_0 \, dm, \quad \text{m-a.e. } y \text{ and } L^1(X; \phi_0^2 m)
\]
as \( t \to \infty \). Since \( p_{1/2}^{\phi_0}(x, \cdot) \) is bounded by the ultracontractivity, it follows from the equation (5.2) that
\[
\lim_{t \to \infty} \int_X \phi_0(x) \mathbb{E}_t^{\phi_0} \left( \frac{1_B}{\phi_0}(X_t) \right) \, dv(x) = \int_X \phi_0 \, dv \int_B \phi_0 \, dm.
\]

We finally consider the exponential integrability of hitting times of compact sets. Let \( K \subset X \) be a compact set and \( D \) the complement of \( K \), \( D = X \setminus K \). We define the part (or absorbing) process \( X^D \) on \( D \) by
\[
X^D_t = \begin{cases} X_t & t < \tau_D, \\ \Delta & t \geq \tau_D, \\ \tau_D = \inf\{ t \geq 0 ; X_t \notin D \}. \end{cases}
\]

Define the regular Dirichlet form \((\mathcal{E}^D, \mathcal{D}(\mathcal{E}^D))\) on \( D \) by
\[
\begin{cases}
\mathcal{E}^D = \mathcal{E}, \\
\mathcal{D}(\mathcal{E}^D) = \{ u \in \mathcal{D}(\mathcal{E}) : u = 0 \text{ q.e. on } K \}.
\end{cases}
\]

By [14, Theorem 4.4.3] the part process \( X^D \) is regarded as a Hunt process generated by \((\mathcal{E}^D, \mathcal{D}(\mathcal{E}^D))\). We see from [4, Theorem 4.2] that \( m \) is in \( \mathcal{K}_\infty \). We write \( \mathcal{K}_\infty(R_1) \) for \( \mathcal{K}_\infty \) to show the dependence. Let \( R^D_1 \) be the \( 1 \)-resolvent of \( X^D \). The restriction \( m^D \) of \( m \) on \( D \) is in \( \mathcal{K}_\infty(R^D_1) \). Indeed, let a compact set \( \tilde{K} \) and a positive constant \( \delta \) in the definition of \( \mathcal{K}_\infty \) (Definition 2.1). We can suppose \( K \subset \tilde{K} \). Let \( G \) be a relatively compact open set such that \( K \subset G \subset \tilde{G} \subset \tilde{K} \) and \( m(G \setminus K) < \delta \). Then \( \tilde{K} \cap \tilde{G}^c \) is a compact subset of \( D \) and
\[
R^D_1 1_{\tilde{K} \cap \tilde{G}^c} = R^D_1 1_{\tilde{G} \cup (G \setminus K)} \leq R_1 1_{\tilde{K}^c} + R_1 1_{G \setminus K} \leq 2 \epsilon.
\]
Moreover, $R^D_11_B \leq R_1 1_B$ for any Borel set $B \subset \overline{K} \cap G^c$. Hence we have $m^D \in K_\infty(R^D_1)$.

If $X^D$ satisfies the irreducibility (I), it follows from [4, Theorem 4.1] that

$$\sup_{x \in D} \mathbb{E}_x(e^{\lambda x}) < \infty \iff \lambda < \lambda^D,$$

where $\lambda^D$ is the bottom of the spectrum of $(E^D, D(E^D))$.

Noting that by (3.1)

$$1 \leq \|R_1 1_D\|_\infty(\lambda^D + 1),$$

we see from (III) that

$$(5.8) \quad \lambda_D \uparrow \infty \quad \text{as} \quad K \uparrow X.$$  

We can conclude that if for any compact set $K$, the part process $X^D$ ($D = X \setminus K$) is irreducible, then for any $\lambda > 0$ there exists a compact set $K$ such that

$$(5.9) \quad \sup_{x \in X} \mathbb{E}_x(e^{\lambda x}) < \infty.$$  

If $M$ is conservative, $\tau_D$ equals the first hitting time $\tau_K$ of $K$, $\tau_K = \inf\{t > 0 : X_t \in K\}$. Then the property (5.9) is called the uniform hyper-exponential recurrence in [30].

**Example 5.1** (One-dimensional diffusion processes). Let us consider a one-dimensional diffusion process $M = (X_t, \mathbb{P}_x, \xi)$ on an open interval $I = (r_1, r_2)$ such that $\mathbb{P}_x(X_{\xi^-} = r_1$ or $r_2, \xi < \infty) = \mathbb{P}_x(\xi < \infty, x \in I$, and $\mathbb{P}_a(\tau_b < \infty) > 0$ for any $a, b \in I$. The diffusion $M$ is symmetric with respect to its canonical measure $m$ and it satisfies (I) and (II). The boundary point $r_1$ of $I$ is classified into four classes: regular boundary, exit boundary, entrance boundary and natural boundary ([15, Chapter 5]):

(a) If $r_2$ is a regular or exit boundary, then $\lim_{x \to r_2} R_1 1_{(x,r)} = 0$.
(b) If $r_2$ is an entrance boundary, then $\lim_{r \to r_2} \sup_{x \in (r_1, r_2)} R_1 1_{(r,r_2)}(x) = 0$.
(c) $r_2$ is a natural boundary, then $\lim_{x \to r_2} R_1 1_{(r,r_2)}(x) = 1$ and thus $\sup_{x \in (r_1, r_2)} R_1 1_{(r,r_2)}(x) = 1$.

Therefore, (III) is satisfied if and only if no natural boundaries are present. As a corollary of the equation (5.8), if $r_2$ is entrance, for any $\lambda > 0$ there exists $r_1 < r < r_2$ such that

$$\sup_{x > r} \mathbb{E}_x(\exp(\lambda \sigma_r)) < \infty,$$

where $\sigma_r$ is the first hitting time of $\{r\}$. The statement above implies a uniqueness of quasi-stationary distributions ([3]).
References


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