ZERO MEAN CURVATURE SURFACES IN LORENTZ–MINKOWSKI 3-SPACE WHICH CHANGE TYPE ACROSS A LIGHT-LIKE LINE


<table>
<thead>
<tr>
<th>Citation</th>
<th>Osaka Journal of Mathematics. 52(1): 285-297</th>
</tr>
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<tr>
<td>Issue Date</td>
<td>2015-01</td>
</tr>
<tr>
<td>Textversion</td>
<td>Publisher</td>
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<tr>
<td>Right</td>
<td>©Departments of Mathematics of Osaka University and Osaka City University.</td>
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<tr>
<td>DOI</td>
<td>10.18910/57676</td>
</tr>
<tr>
<td>Is Identical to</td>
<td><a href="https://doi.org/10.18910/57676">https://doi.org/10.18910/57676</a></td>
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<td>Relation</td>
<td>The OJM has been digitized through Project Euclid platform</td>
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ZERO MEAN CURVATURE SURFACES IN LORENTZ–MINKOWSKI 3-SPACE WHICH CHANGE TYPE ACROSS A LIGHT-LIKE LINE


(Received October 2, 2013)

Abstract

It is well-known that space-like maximal surfaces and time-like minimal surfaces in Lorentz–Minkowski 3-space $\mathbb{R}^{3}_{1}$ have singularities in general. They are both characterized as zero mean curvature surfaces. We are interested in the case where the singular set consists of a light-like line, since this case has not been analyzed before. As a continuation of a previous work by the authors, we give the first example of a family of such surfaces which change type across a light-like line. As a corollary, we also obtain a family of zero mean curvature hypersurfaces in $\mathbb{R}^{n+1}$ that change type across an $(n-1)$-dimensional light-like plane.

Introduction

Many examples of space-like maximal surfaces containing singular curves in the Lorentz–Minkowski 3-space $\mathbb{R}^{3}_{1}$; $t, x, y)$ of signature $(-++)$ have been constructed in [11], [1], [12], [8], [4] and [5].

In this paper, we are interested in the zero mean curvature surfaces in $\mathbb{R}^{3}_{1}$ changing their causal type: Klyachin [10] showed under a sufficiently weak regularity assumption that a zero mean curvature surface in $\mathbb{R}^{3}_{1}$ changes its causal type only on the following two subsets:

- null curves (i.e., regular curves whose velocity vector fields are light-like) which are non-degenerate (i.e., their projections into the $xy$-plane are locally convex plane curves), or
- light-like lines, which are degenerate everywhere.

Given a non-degenerate null curve $\gamma$ in $\mathbb{R}^{3}_{1}$, there exists a zero mean curvature surface which changes its causal type across this curve from a space-like maximal surface to...
a time-like minimal surface (cf. [6], [10], [9] and [7]). This construction can be accomplished using the Björling formula for the Weierstrass-type representation formula of maximal surfaces. (The reference [3] is an expository article on this subject.) However, if \( \gamma \) is a light-like line, the aforementioned construction fails, since the isothermal coordinates break down at the light-like singular points. Locally, such a surface is the graph of a function \( t = f(x, y) \) satisfying

\[
(1 - f_y^2)f_{xx} + 2f_xf_yf_{xy} + (1 - f_x^2)f_{yy} = 0,
\]

where \( f_x = \partial f / \partial x, \quad f_{xy} = \partial^2 f / \partial y \partial x, \) etc. We call this and its graph the zero mean curvature equation and a zero mean curvature surface, respectively. Until now, zero mean curvature surfaces which actually change type across a light-like line were unknown. As announced in [2], the main purpose of this paper is to construct such an example. In Section 1, we give a formal power series solution of the zero mean curvature equation describing all zero mean curvature surfaces which contain a light-like line. Using this, we give the precise statement of our main result and show how the statement can be reduced to a proposition (cf. Proposition 1.3). In Section 2, we then prove it. As a consequence, we obtain the first example of (a family of) zero mean curvature surfaces which change type across a light-like line.

1. The main theorem

We discuss solutions of the zero mean curvature equation (*) which have the following form

\[
f(x, y) = b_0(y) + \sum_{k=1}^{\infty} \frac{b_k(y)}{k} x^k,
\]

where \( b_k(y) \) \((k = 1, 2, \ldots)\) are \( C^\infty \)-functions. When \( f \) contains a singular light-like line, we may assume without loss of generality that (cf. [2])

\[
b_0(y) = y, \quad b_1(y) = 0.
\]

As was pointed out in [2], there exists a real constant \( \mu \) called the characteristic of \( f \) such that \( b_2(y) \) satisfies the following equation

\[
b_2'(y) + b_2(y)^2 + \mu = 0 \quad (\prime = d/dy).
\]

Now we derive the differential equations satisfied by \( b_k(y) \) for \( k \geq 3 \) assuming (1.2). If we set

\[
Y := f_y - 1 = \sum_{k=2}^{\infty} \frac{b_k(y)}{k} x^k
\]
and
\[ P := 2(Y f_{xx} - f_x f_{xy}), \quad Q := Y^2 f_{xx} - 2 f_x f_{xy} Y, \quad R := f_x^2 f_{yy}, \]
then, by straightforward calculations, we see that
\[ P = -b_2 b'_2 x^2 - \frac{4}{3} b_2 b'_3 x^3 - \sum_{k=4}^{\infty} \left( P_k + \frac{2(k-1)}{k} b_2 b'_k + (3 - k) b'_2 b_k \right) x^k, \]
\[ Q = -\sum_{k=4}^{\infty} Q_k x^k, \quad R = \sum_{k=4}^{\infty} R_k x^k, \]
where
\[
P_k := \sum_{m=3}^{k-1} \frac{2(k-2m+3)}{k-m+2} b_m b'_{k-m+2},
\]
\[
Q_k := \sum_{m=2}^{k-2} \sum_{n=2}^{k-m} \frac{3n-k+m-1}{mn} b'_m b'_n b_{k-m-n+2},
\]
\[
R_k := \sum_{m=2}^{k-2} \sum_{n=2}^{k-m} \frac{b_m b'_n b'_{k-m-n+2}}{k-m-n+2},
\]
for \( k \geq 4 \), and that the zero mean curvature equation (*) reduces to
\[
\sum_{k=2}^{\infty} \frac{b''_k}{k} x^k = f_{yy} = P + Q + R.
\]
It is now immediate, by comparing the coefficients of \( x^k \) from both sides, to see that each \( b_k \) \((k \geq 3)\) satisfies the following ordinary differential equation
\[
b''_k(y) + 2(k-1) b_2(y) b'_k(y) + (3 - k) b'_2(y) b_k(y) = -k(P_k + Q_k - R_k),
\]
where \( P_3 = Q_3 = R_3 = 0 \) and \( P_k, Q_k \) and \( R_k \) are as in (1.4) for \( k \geq 4 \). Note that \( P_k, Q_k \) and \( R_k \) are written in terms of \( b_j \) \((j = 1, \ldots, k - 1)\) and their derivatives.

Now, we consider the case that \( 1 - f_x^2 - f_y^2 \) changes sign across the light-like line \( \{ t = y, x = 0 \} \). This case occurs only when the characteristic \( \mu \) as in (1.3) of \( f \) vanishes [2]. If we set
\[ b_2(y) = 0 \quad (y \in \mathbb{R}), \]
then (1.3) holds for \( \mu = 0 \). So we assume
\[
b_0(y) = y, \quad b_1(y) = 0, \quad b_2(y) = 0, \quad b_3(y) = 3 cy,
\]
where \( c \) is a non-zero constant. Then \( f(x, y) \) in (1.1) can be rewritten as

\[
f(x, y) = y + cy^3 + \sum_{k=4}^{\infty} \frac{b_k(y)}{k}x^k.
\]

In this situation, we will find a solution satisfying

\[
b_k(0) = b_k'(0) = 0 \quad (k \geq 4).
\]

Then (1.5) reduces to

\[
\frac{b_k''(y)}{b_k(y)} = -k(P_k + Q_k - R_k), \quad b_k(0) = b_k'(0) = 0, \quad (k \geq 4),
\]

\[
P_k = \sum_{m=3}^{k-1} \frac{2(k-2m+3)}{k-m+2} b_m(y)b_{k-m+2}(y) \quad (k \geq 4),
\]

\[
Q_k = \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{3n-k+m-1}{mn} b_m(y)b_n(y)b_{k-m-n+2}(y) \quad (k \geq 7),
\]

\[
R_k = \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{b_m(y)b_n(y)b_{k-m-n+2}(y)}{k-m-n+2} \quad (k \geq 7),
\]

and \( Q_k = R_k = 0 \) for \( 4 \leq k \leq 6 \), where the fact that \( b_2(y) = 0 \) has been extensively used. For example,

\[
b_0 = y, \quad b_1 = b_2 = 0, \quad b_3 = 3cy, \quad b_4 = -4c^2y^3, \quad b_5 = 9c^3y^5,
\]

\[
b_6 = -24c^2y^7, \quad b_7 = 70c^5y^9 - 14c^3y^3, \ldots.
\]

In this article, we show the following assertion:

**Theorem 1.1.** For each positive number \( c \), the formal power series solution \( f(x, y) \) uniquely determined by (1.9), (1.10), (1.11) and (1.12) gives a real analytic zero mean curvature surface on a neighborhood of \((x, y) = (0, 0)\). In particular, there exists a non-trivial 1-parameter family of real analytic zero mean curvature surfaces each of which changes type across a light-like line (see Fig. 1).

As a consequence, we get the following:

**Corollary 1.2.** There exists a family of zero-mean curvature hypersurfaces in Lorentz–Minkowski space \( \mathbb{R}^{n+1}_1 \) each of which changes type across an \((n - 1)\)-dimensional light-like plane.

**Proof.** Let \( f \) be as in the theorem. The graph of the function defined by

\[
\mathbb{R}^n \ni (x_1, \ldots, x_n) \mapsto f(x_1, x_2) \in \mathbb{R}
\]
Fig. 1. The graph of $t = f(x, y)$ for $c = 1/2$ and $|x|, |y| < 0.8$ (The range of the graph is wider than the range used in our mathematical estimation. However, this figure still has a sufficiently small numerical error term in the Taylor expansion.)

gives the desired hypersurface. In this case, the zero mean curvature equation

\[
\left(1 - \sum_{j=1}^{n} f_{x_j}^2\right) \sum_{i=1}^{n} f_{x_i} f_{x_j} + \sum_{i,j=1}^{n} f_{x_i x_j} f_{x_i} f_{x_j} = 0 \quad (f_{x_i} := \partial f / \partial x_i, \ f_{x_i x_j} := \partial^2 f / \partial x_j \partial x_i)
\]

reduces to (*) in the introduction.

To prove Theorem 1.1, it is sufficient to show that for arbitrary positive constants $c > 0$ and $\delta > 0$ there exist positive constants $n_0$, $\theta_0$, and $C$ such that

(1.13)\[ |b_k(y)| \leq \theta_0 C^k \quad (|y| \leq \delta) \]

holds for $k \geq n_0$. In fact, if (1.13) holds, then the series (1.7) converges uniformly over the rectangle $[-C^{-1}, C^{-1}] \times [-\delta, \delta]$.

The key assertion to prove (1.13) is the following
Proposition 1.3. For each $c > 0$ and $\delta > 0$, we set
\begin{equation}
M := 3 \max \{144c\tau|\delta|^{3/2}, \sqrt[4]{192c^2\tau}\},
\end{equation}
where $\tau$ is the positive constant given by (A.3) in the appendix, such that
\begin{equation}
t \int_{t}^{t+1} \frac{du}{u^2(1-u)^2} \leq \tau \quad (0 < t < 1/2).
\end{equation}
Then the function $\{b_l(y)\}_{l \geq 3}$ formally determined by the recursive formulas (1.9)--(1.12) satisfies the inequalities
\begin{align}
|b_l''(y)| &\leq c|y|^{l^*}M^{l-3}, \\
|b_l'(y)| &\leq \frac{3c|y|^{l^*+1}}{l^* + 2}M^{l-3}, \\
|b_l(y)| &\leq \frac{3c|y|^{l^*+2}}{(l^* + 2)^2}M^{l-3}
\end{align}
for any
\begin{equation}
y \in [-\delta, \delta],
\end{equation}
where
\begin{equation}
l^* := \frac{1}{2}(l - 1) - 2 \quad (l \geq 3).
\end{equation}
Once this proposition is proven, (1.13) follows immediately. In fact, if we set
\begin{equation}
\theta_0 = \frac{3}{c}(\delta M)^3, \quad C := \delta M
\end{equation}
and $n_0 \geq 7$, then $1 \leq l^* + 2 < l - 3$ and (1.13) follows from
\begin{equation}
\frac{3c|y|^{l^*+2}}{(l^* + 2)^2}M^{l-3} \leq \theta_0 C^l.
\end{equation}

2. Proof of Proposition 1.3
We prove the proposition using induction on the number $l \geq 3$. If $l = 3$, then
\begin{align}
|b_3''(y)| &\geq 0 = \frac{c}{|y|} = c|y|^3 M^0, \\
|b_3'(y)| &\leq 3c = \frac{3c|y|^{3^*+1}}{3^* + 2}M^0, \\
|b_3(y)| &\leq 3c|y| = \frac{3c|y|^{3^*+2}}{(3^* + 2)^2}M^0.
\end{align}
hold, using that \(b_3(y) = 3y\), \(M^0 = 1\) and \(3^* = -1\). So we prove the assertion for \(l \geq 4\). Since (1.17), (1.18) follow from (1.16) by integration, it is sufficient to show that (1.16) holds for each \(l \geq 4\). (In fact, the most delicate case is \(l = 4\). In this case \(l^* = -1/2\) and we can use the fact that \(\int_0^{\gamma_0} 1/\sqrt{y} \, dy\) for \(\gamma_0 > 0\) converges.)

The inequality (1.16) follows if one shows that, for each \(k \geq 4\)

\[
|k P_k(y)|, |k Q_k(y)|, |k R_k(y)| \leq \frac{c}{3} |y|^{k^*} M^{k-3} \quad (|y| \leq \delta)
\]

under the assumption that (1.16), (1.17) and (1.18) hold for all \(3 \leq l \leq k - 1\). In fact, if (2.1) holds, (1.16) for \(l = k\) follows immediately. Then by the initial condition (1.9) (cf. (1.8)), we have (1.17) and (1.18) for \(l = k\) by integration.

**The estimation of \(|k P_k|\) for \(k \geq 4\).** By (1.10) and using the fact that (1.17), (1.18) hold for \(l \leq k - 1\), we have for each \(|y| < \delta\) that

\[
|k P_k| \leq \sum_{m=3}^{k-1} \frac{2k[k - 2m + 3]}{k - m + 2} \cdot |b_m(y)| \cdot |b_{k-m+2}(y)|
\]

\[
\leq \sum_{m=3}^{k-1} \frac{2k[k - 2m + 3]}{k - m + 2} \cdot \left( \frac{3cM^{m-3}|y|^{m^*+2}}{(m^* + 2)^2} \right) \cdot \left( \frac{3cM^{k-m+2-3}|y|^{(k-m+2)^*+1}}{(k-m+2)^* + 2} \right)
\]

\[
= c M^{k-3} |y|^{k^*} \frac{144c|y|^{3/2}}{M} \sum_{m=3}^{k-1} \frac{k[k - 2m + 3]}{(m - 1)^2(k-m+1)(k-m+2)}
\]

\[
\leq c M^{k-3} |y|^{k^*} \frac{144c|y|^{3/2}}{M} \sum_{m=3}^{k-1} \frac{k[k - 2m + 3]}{(m - 1)^2(k-m+1)(k-m+2)}
\]

\[
\leq \frac{c}{3\tau} M^{k-3} |y|^{k^*} \sum_{m=3}^{k-1} \frac{k[k - 2m + 3]}{(m - 1)^2(k-m+1)^2}.
\]

Here, we used (1.14). Since

\[
\max_{m=3, \ldots, k-1} |k - 2m + 3| = \max_{m=3, k-1} |k - 2m + 3| = \max\{|k - 3|, |k + 5|\},
\]

by setting \(q = m - 1\), we have that

\[
|k P_k| \leq \frac{c}{3\tau} M^{k-3} |y|^{k^*} \sum_{m=3}^{k-1} \frac{k^2}{(m - 1)^2(k-m+1)^2} = \frac{c}{3\tau} M^{k-3} |y|^{k^*} \frac{1}{k} \sum_{q=2}^{k-2} \frac{k^3}{q^2(q-k)^2}
\]

\[
\leq \frac{c}{3\tau} M^{k-3} |y|^{k^*} \cdot \frac{1}{k} \int_{1/k}^{\gamma_{1/k}} \frac{du}{u^2(1-u)^2} \leq \frac{c}{3} M^{k-3} |y|^{k^*},
\]

where we applied Lemma A.1 and (1.15) at the last step of the estimations. Hence, we get (2.1) for \(k P_k\).
The estimation of $|kQ_k|$ for $k \geq 7$. By (1.11) and the induction assumption, we have that

$$|kQ_k| \leq \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k|3n-k+m-1|}{mn} |b'_m(y)| |b'_n(y)| |b_{k-m-n+2}(y)|$$

By (1.11) and the induction assumption, we have that

$$|kQ_k| \leq \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k|3n-k+m-1|}{mn} \left( \frac{3cM^{m-3}|y|^{n+1}}{m^*+2} \right)$$

$$\times \left( \frac{3cM^{n-3}|y|^{n*+1}}{n^*+2} \right) \left( \frac{3cM^{k-m-n+2-3}|y|^{(k-m-n+2)^*+2}}{((k-m-n+2)^*+2)^2} \right)$$

$$= cM^{k-3}|y|^k \frac{432c^2}{M^4} \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k|3n-k+m-1|}{(m-1)^2(n-1)^2(k-m-n+2)^2}.$$

Now we apply the inequalities

$$\max_{3 \leq m \leq k-4, 3 \leq n \leq k-m-1} |3n-k+m-1| = \max_{(m,n)=(3,3),(3,k-4),(k-4,3)} |3n-k+m-1|$$

$$= \max(\lfloor -k+11 \rfloor, 4, |2k-10|) \leq 2k,$$

and also

$$\frac{432c^2}{M^4} \leq \frac{1}{36\tau},$$

which follows from (1.14). Setting $p := m - 1$, $q = n - 1$, we have that

$$|kQ_k| \leq \frac{c}{36\tau} M^{k-3}|y|^k \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{2k^2}{(m-1)^2(n-1)^2(k-m-n+2)^2}$$

$$= \frac{c}{18\tau} M^{k-3}|y|^k \sum_{p=2}^{k-5} \sum_{q=2}^{k-p-2} \frac{k^2}{p^2q^2(k-p-q)^2}.$$

Now applying Lemma A.2, we have that

$$|kQ_k| \leq \frac{c}{18\tau} M^{k-3}|y|^k \times 6\tau \leq \frac{c}{3} M^{k-3}|y|^k,$$

which proves (2.1) for $kQ_k$. 
The estimation of $|kR_k|$ for $k \geq 7$. As in the case of $|kQ_k|$, we have that

$$|kR_k| \leq \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k|b_m(y)| |b_n(y)| |b_{m-n+2}(y)|}{k-m-n+2}$$

$$\leq \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k}{k-m-n+2} \left( \frac{3cM^{m-3} |y|^{m+2}}{(m^*+2)^2} \right) \times \left( \frac{3cM^{n-3} |y|^{n+2}}{(n^*+2)^2} \right) (cM^{k-m-n+2-3} |y|^{(k-m-n+2)^*})$$

$$= 144c^3M^{k-7} |y|^k \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k}{k-m-n+2(m-1)^2(n-1)^2}$$

$$= cM^{k-3} |y|^k \sum_{m=3}^{k-4} \sum_{n=3}^{k-m-1} \frac{k^2}{(k-m-n+2)(m-1)^2(n-1)^2}.$$ 

Now we set $p = m-1$, $q = n-1$, and using the inequality

$$3^4 \times 144c^2 \tau \leq 3^4 \times 192c^2 \tau < M^4,$$

we have that

$$|kR_k| \leq \frac{c}{3^4 \tau} M^{k-3} |y|^k \sum_{p=2}^{k-5} \sum_{q=2}^{k-p-2} \frac{k^2}{p^2q^2(k-p-q)^2}.$$ 

By applying Lemma A.2, we have that

$$|kR_k| \leq \frac{c}{3^4 \tau} M^{k-3} |y|^k \times 6\tau \leq \frac{c}{3} M^{k-3} |y|^k,$$

which proves (2.1) for $kR_k$. This completes the proof of Proposition 1.3.

**Appendix A. Inequalities used in the proof of Theorem 1.1**

For $a > 0$, it holds that

$$\frac{1}{u^2(a-u)^2} = \frac{1}{a^3} \left( \frac{a}{u^2} + \frac{2}{u} + \frac{a}{(a-u)^2} + \frac{2}{a-u} \right).$$

Therefore,

$$\int_{t}^{a-t} \frac{du}{u^2(a-u)^2} = \frac{2}{a^3} \left( \frac{a(a-2t)}{t(a-t)} + 2 \log \frac{a-t}{t} \right) \quad (0 < t < a/2).$$
In particular, one can show that there exists a positive constant \( \tau \) such that

\[
A.3 \quad t \int_t^{1-t} \frac{du}{u^2(1-u)^2} \leq \tau \quad (0 < t < 1/2).
\]

The following assertion is needed to prove (2.1) for \( kP_k(y) \):

**Lemma A.1.** Let \( p \) be a non-negative integer and \( k \) an integer satisfying \( k \geq p + 4 \). Then the inequality

\[
\sum_{q=2}^{k-p-2} \frac{k^3}{q^2(k-p-q)^2} \leq \int_{1/k}^{a-1/k} \frac{du}{u^2(a-u)^2} \quad (a := 1 - p/k)
\]

holds.

**Proof.** In fact, if we set \( a := 1 - p/k \), then (A.1) yields that

\[
\frac{k^3}{q^2(k-p-q)^2} = \frac{1}{k} \frac{1}{(q/k)^2(a-q/k)^2}
\]

\[
= \frac{1}{a^3} \left[ \frac{1}{k} \left( \frac{a}{(q/k)^2} + \frac{2}{q/k} \right) + \frac{1}{k} \left( \frac{a}{(a-q/k)^2} + \frac{2}{a-q/k} \right) \right].
\]

Since \( x \mapsto (a + 2x)/x^2 \) is a monotone decreasing function and the function \( x \mapsto (a + 2(a - x))/(a - x)^2 \) is monotone increasing on the interval \((0, a/2)\), we have that

\[
\frac{k^3}{q^2(k-p-q)^2} \leq \frac{1}{a^3} \left[ \int_{q-1/k}^{q/k} \left( \frac{a}{u^2} + \frac{2}{u} \right) du + \int_{q/k}^{(q+1)/k} \left( \frac{a}{(a-u)^2} + \frac{2}{a-u} \right) du \right],
\]

which yields that

\[
\sum_{q=2}^{k-p-2} \frac{a^3 k^3}{q^2(k-p-q)^2} \leq \int_{1/k}^{a-1/k} \left( \frac{a}{u^2} + \frac{2}{u} \right) du + \int_{a-1/k}^{a/k} \left( \frac{a}{(a-u)^2} + \frac{2}{a-u} \right) du
\]

\[
\leq \int_{1/k}^{a-1/k} \left( \frac{a}{u^2} + \frac{2}{u} \right) du + \int_{1/k}^{a-1/k} \left( \frac{a}{(a-u)^2} + \frac{2}{a-u} \right) du
\]

\[
= \int_{1/k}^{a-1/k} \frac{du}{u^2(a-u)^2}.
\]

This proves the assertion.

The following assertion is needed to prove (2.1) for \( kQ_k(y) \) and \( kR_k(y) \):
Lemma A.2. For any integer $k \geq 7$, the following inequalities holds:

\[
\sum_{p=2}^{k-5} \sum_{q=2}^{k-p-2} \frac{k^2}{p^2 q^2 (k - p - q)^2} \leq 6 \int_{1/k}^{1-1/k} \frac{du}{u^2 (1 - u)^2} \leq 6 \tau,
\]

where $\tau$ is a constant satisfying (A.3).

Proof. We set $a = a(p) := 1 - (p/k)$. Applying Lemma A.1 and the identity (A.2), we have that

\[
\sum_{p=2}^{k-5} \sum_{q=2}^{k-p-2} \frac{k^2}{p^2 q^2 (k - p - q)^2} = \sum_{p=2}^{k-5} \left[ \frac{1}{kp^2} \sum_{q=2}^{k-p-2} \frac{k^3}{q^2 (k - p - q)^2} \right] \\
\leq \sum_{p=2}^{k-5} \left[ \frac{1}{kp^2} \int_{1/k}^{a-1/k} \frac{du}{u^2 (a - u)^2} \right] = \sum_{p=2}^{k-5} \left[ \frac{2}{p^2 a^2} \left( 1 + 2 \frac{\log ka}{ka} \right) \right] \leq \sum_{p=2}^{k-5} \frac{6}{p^2 a^2},
\]

where we used the fact that $(\log ka)/(ka) < 1$. By applying Lemma A.1 and by using the property (A.3) of the constant $\tau$, it holds that

\[
\sum_{p=2}^{k-5} \sum_{q=2}^{k-p-2} \frac{k^2}{p^2 q^2 (k - p - q)^2} \leq \sum_{p=2}^{k-5} \frac{1}{p^2 (1 - p/k)^2} = \frac{6}{k} \sum_{p=2}^{k-5} \frac{k^3}{p^2 (k - p)^2} \\
\leq \frac{6}{k} \sum_{p=2}^{k-2} \frac{k^3}{p^2 (k - p)^2} \leq \frac{6}{k} \int_{1/k}^{1-1/k} \frac{du}{u^2 (1 - u)^2} < 6 \tau,
\]

which proves the assertion. \( \square \)

References


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